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SYMPLECTIC TORUS ACTIONS WITH COISOTROPIC PRINCIPAL ORBITS

by Johannes Jisse DUISTERMAAT & Alvaro PELAYO (*)

ABSTRACT. — In this paper we completely classify symplectic actions of a torus T on a compact connected symplectic manifold (M, σ) when some, hence every, principal orbit is a coisotropic submanifold of (M, σ). That is, we construct an explicit model, defined in terms of certain invariants, of the manifold, the torus action and the symplectic form. The invariants are invariants of the topology of the manifold, of the torus action, or of the symplectic form.

In order to deal with symplectic actions which are not Hamiltonian, we develop new techniques, extending the theory of Atiyah, Guillemin-Sternberg, Delzant, and Benoist. More specifically, we prove that there is a well-defined notion of constant vector fields on the orbit space M/T. Using a generalization of the Tietze-Nakajima theorem to what we call V-parallel spaces, we obtain that M/T is isomorphic to the Cartesian product of a Delzant polytope with a torus.

We then construct special lifts of the constant vector fields on M/T, in terms of which the model of the symplectic manifold with the torus action is defined.

RÉSUMÉ. — Dans cet article nous donnons une classification complète des actions symplectiques d'un tore T sur une variété compacte connexe symplectique (M, σ) pour laquelle une, et donc toute orbite principale est une variété coïsotrope de (M, σ). Cela veut dire que nous construisons un modèle explicite, défini en termes de certains invariants de la variété, l'action torique et de la forme symplectique.

Pour traiter des actions symplectiques qui ne sont pas hamiltoniennes, nous développons des techniques nouvelles, étendant la théorie d'Atiyah, Guillemin-Sternberg, Delzant et Benoist. En particulier, nous démontrons qu'il y a une notion bien définie de champs de vecteurs constants sur l'espace des orbites M/T. En utilisant une généralisation du théorème de Tietze-Nakayama à ce que nous appelons aussi espaces V-parallèles, nous obtenons que M/T est isomorphe au produit cartésien d'un polytope de Delzant avec un tore.

Nous construisons alors les champs de vecteurs spéciaux dans M qui se projettent sur les champs de vecteurs constants sur M/T, à l'aide desquels le modèle de la variété symplectique avec action torique est défini.

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1. Introduction

Let (M, σ) be a smooth compact and connected symplectic manifold of dimension $2n$ and let T be a torus which acts effectively on (M, σ) by means of symplectomorphisms. We furthermore assume that some principal T-orbit is a coisotropic submanifold of (M, σ), which implies that $d_T \geqslant n$ if d_T denotes the dimension of T. See Lemma 2.3 for alternative characterizations of our assumptions. In this paper we will classify the compact connected symplectic manifolds with such torus actions, by constructing a list of explicit examples to which each of our manifolds is equivariantly symplectomorphic. See Theorem 9.4, Theorem 9.6 and Corollary 9.7 for our main result.

In many integrable systems in classical mechanics, we have an effective Hamiltonian action of an n-dimensional torus on the $2n$-dimensional symplectic manifold, but also non-Hamiltonian actions occur in physics, see for instance Novikov [42].

If the effective action of T on (M, σ) is Hamiltonian, then $d_T = n$ and the principal orbits are Lagrangian submanifolds. Moreover, the image of the momentum mapping is a convex polytope Δ in the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, where $\mathfrak{t}$ denotes the Lie algebra of T. Δ has the special property that at each vertex of Δ there are precisely n codimension one faces with normals which form a $\mathbb{Z}$-basis of the integral lattice $T_{\mathbb{Z}}$ in $\mathfrak{t}$, where $T_{\mathbb{Z}}$ is defined as the kernel of the exponential mapping from $\mathfrak{t}$ to T. The classification of Delzant [11] says that for each such polytope Δ there is a compact connected symplectic manifold with Hamiltonian torus action having Δ as image of the momentum mapping, and the symplectic manifold with torus action is unique up to equivariant symplectomorphisms. For an efficient proof of the uniqueness in the more general setting of orbifolds, see Lerman and Tolman [33]. Such polytopes Δ and corresponding symplectic T-manifolds (M, σ, T) are called *Delzant polytopes* and *Delzant manifolds* in the exposition of this subject by Guillemin [20], after Delzant [11]. Each Delzant manifold has a T-invariant Kähler structure such that the Kähler form is equal to σ.

Because critical points of the Hamiltonian function correspond to zeros of the Hamiltonian vector field, a Hamiltonian action on a compact manifold always has fixed points. Therefore the other extreme case of a symplectic torus action with coisotropic principal orbits occurs if the action is *free*. In this case, M is a principal torus bundle over a torus, hence a nilmanifold for a two-step nilpotent Lie group as described in Palais and Stewart [46]. If the nilpotent Lie group is not commutative, then M does not admit a Kähler structure, *cf.* Benson and Gordon [7]. For four-dimensional manifolds M, these correspond to the third case in the description of Kodaira [30, Th. 19] of the compact complex analytic surfaces which carry a nowhere vanishing holomorphic $(2,0)$-form. These were rediscovered as the first examples of compact symplectic manifolds without Kähler structure by Thurston [51]. See the end of Remark 7.6.

The general case is a combination of the Hamiltonian case and the free case, in the sense that M is an associated G-bundle $G \times_H M_{\mathrm{h}}$ over G/H with a $2d_{\mathrm{h}}$-dimensional Delzant submanifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$ of (M, σ, T) as fiber. Here T_{h} is the unique maximal subtorus of T which acts in Hamiltonian fashion on (M, σ). It has dimension d_{h} and its Lie algebra is denoted by $\mathfrak{t}_{\mathrm{h}}$. G is a two-step nilpotent Lie group, and H is a commutative closed Lie subgroup of G, which acts on M_{h} via $T_{\mathrm{h}} \subset H$. The base space G/H is a torus bundle over a torus, see Remark 7.3. This leads to an explicit model of (M, σ, T) in terms of the ingredients 1) – 6) in Definition 9.1. See Proposition 7.2 and Proposition 7.4. The model allows explicit computations of many aspects of (M, σ, T). As an example we determine the fundamental group of M in Proposition 8.2, and the Chern classes of the normal bundle in M/T_{f} of the fixed point set of the action of T_{h} on M/T_{f} in Proposition 8.1. Here T_{f} is a complementary subtorus to T_{h} in T, which acts freely on M. The main result of this paper is that the compact connected symplectic manifolds with symplectic torus action with coisotropic principal orbits are completely classified by the ingredients 1) – 6) in Definition 9.1, see Theorem 9.4 and Theorem 9.6.

The proof starts with the observation that the symplectic form on the orbits is given by a two-form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$, see Lemma 2.1. Write $\mathfrak{l} := \ker \sigma^{\mathfrak{t}}$. The inner product of the symplectic form σ with the infinitesimal action of T defines a closed basic $\mathfrak{l}^*$-valued one-form $\widehat{\sigma}$ on M, which turns the orbit space M/T into a locally convex polyhedral $\mathfrak{l}^*$-parallel space, as defined in Definition 10.1. The locally convex polyhedral $\mathfrak{l}^*$-parallel space M/T is isomorphic to $\Delta \times (N/P)$, in which Δ is a Delzant polytope in $(\mathfrak{t}_{\mathrm{h}})^*$ and P is a cocompact discrete additive subgroup of the space N of all linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{t}_{\mathrm{h}}$. See Proposition 3.8.

The main step in the proof of the classification is the construction of lifts to M of the constant vector fields on the $\mathfrak{l}^*$-parallel manifold M/T with the simplest possible Lie brackets and symplectic products of the lifts. See Proposition 5.5. This construction uses calculations involving the de Rham cohomology of M/T.

All the proofs become much simpler in the case that the action of T on M is free. We actually first analyzed the free case with Lagrangian orbits. Next we treated the case with Lagrangian principal orbits where M is fibered by Delzant manifolds, and only after we became aware of the article of Benoist [6], we generalized our results to the case with coisotropic principal orbits. In [6], Theorem 6.6 states that every compact connected symplectic manifold with a symplectic torus action with coisotropic principal orbits is isomorphic to the Cartesian product of a Delzant manifold and a compact connected symplectic manifold with a free symplectic torus action. However, even in the special case that the principal orbits are Langrange submanifolds of M, this conclusion appears to be too strong, if the word "isomorphic" implies "equivariantly diffeomorphic", see Remark 9.8 and Benoist [5].

The paper is organized as follows. In Section 2 we discuss the condition that some (all) principal orbits are coisotropic submanifolds of (M, σ). In Section 3 we analyze the space of T-orbits in all detail, where we use the definitions and theorems in the appendix Section 10 concerning what we call "V-parallel spaces". Section 4 contains a lemma about basic differential forms and one about equivariant diffeomorphisms which preserve the orbits. In Section 5 we construct our special lifts of constant vector fields on the orbit space. These are used in Section 6 in order to construct the Delzant submanifolds of (M, σ) and in Section 7 for the normal form of the symplectic T-manifold. The classification is completed by means of the theorems in Section 9. In the first appendix, Section 10, we prove that every straight line complete, connected and locally convex V-parallel space is isomorphic to the Cartesian product of a closed convex subset of a finite-dimensional vector space and a torus. See Theorem 10.13 for the precise statement. This result is a generalization of the theorem of Tietze [52] and Nakajima [41], which states that every closed and connected locally convex subset of a finite-dimensional vector space is convex. In the second appendix, Section 11, we describe the local model of Benoist [6, Prop. 1.9] and Ortega and Ratiu [44] for a proper symplectic action of an arbitrary Lie group on an arbitrary symplectic manifold.

There are many other texts on the classification of symplectic torus actions on compact manifolds which in some way are related to ours. The book of Audin [2] is on Hamiltonian torus actions, with emphasis on the topological aspects. Orlik and Raymond [43] and Pao [47] classified actions of two-dimensional tori on four-dimensional compact connected smooth manifolds. Because they do not assume an invariant symplectic structure, our classification in the four-dimensional case forms only a tiny part of theirs. On the other hand the completely integrable systems with local torus actions of Kogan [31] form a relatively close generalization of torus actions with Lagrangian principal orbits. The classification of Hamiltonian circle actions on compact connected four-dimensional manifolds in Karshon [27], and of centered complexity one Hamiltonian torus actions in arbitrary dimensions in Karshon and Tolman [28], are also much richer than our classification in the case that $n - d_{\mathrm{h}} \leqslant 1$. McDuff [38] and McDuff and Salamon [39] studied non-Hamiltonian circle actions, and Ginzburg [18] non-Hamiltonian symplectic actions of compact groups under the assumption of a "Lefschetz condition". In another direction Symington [50] and Leung and Symington [34] classified four-dimensional compact connected symplectic manifolds which are fibered by Lagrangian tori where however the fibration is allowed to have elliptic or focus-focus singularities. In the book of Mukherjee [40, Def. 3.4.2 and Lem. 3.4.3] there is a definition of "locally toric manifolds" and a characterization of their orbit spaces which is related to our characterization of the the orbit spaces as the cartesian product of a Delzant polytope with a torus.

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2. Coisotropic principal orbits

Let (M, σ) be a smooth compact and connected symplectic manifold and let T be a torus which acts effectively on (M, σ) by means of symplectomorphisms. In this section we show that some principal T-orbit is a coisotropic submanifold of (M, σ) if and only if the Poisson brackets of any pair of smooth T-invariant functions on M vanish if and only if every principal T-orbit is a coisotropic submanifold of (M, σ). See Lemma 2.3, Remark 2.5 and Remark 2.11 below.

This follows from the local model of Benoist [6, Prop. 1.9], see Theorem 11.1, which in the case of symplectic torus actions with coisotropic principal orbits assumes a particularly simple form, see Lemma 2.10.

If X is an element of the Lie algebra $\mathfrak{t}$ of T, then we denote by X_M the infinitesimal action of X on M. It is a smooth vector field on M, and the invariance of σ under the action of T implies that

$$\mathrm{d}(\mathrm{i}_{X_M}\,\sigma) = \mathrm{L}_{X_M}\,\sigma = 0. \tag{2.1}$$

Here L_v denotes the Lie derivative with respect to the vector field v, and $\mathrm{i}_v\,\omega$ the inner product of a differential form ω with v, obtained by inserting v in the first slot of ω. The first identity in (2.1) follows from the homotopy identity $\mathrm{L}_v = \mathrm{d} \circ \mathrm{i}_v + \mathrm{i}_v \circ \mathrm{d}$ combined with $\mathrm{d}\sigma = 0$.

If f is a smooth real-valued function on M, then the unique vector field v on M such that $-\,\mathrm{i}_v\,\sigma = \mathrm{d}f$ is called the *Hamiltonian vector field* of f, and will be denoted by Ham_f. Given v, the function f is uniquely determined up to and additive constant, which implies that f is T-invariant if and only if v is T-invariant. If $X \in \mathfrak{t}$, then X_M is Hamiltonian if and only if the closed two-form $\mathrm{i}_{X_M}\,\sigma$ is exact.

The following lemma says that the pull-back to the T-orbits of the symplectic form σ on M is given by a constant antisymmetric bilinear form on the Lie algebra $\mathfrak{t}$ of T.

Lemma 2.1. — *There is a unique antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$, such that*

$$\sigma_x(X_M(x), Y_M(x)) = \sigma^{\mathfrak{t}}(X, Y)$$

for every $X, Y \in \mathfrak{t}$ and every $x \in M$.

Proof. — It follows from Benoist [6, Lemme 2.1] that if u and v are smooth vector fields on M such that $\mathrm{L}_u\,\sigma = 0$ and $\mathrm{L}_v\,\sigma = 0$, then $[u, v] = \mathrm{Ham}_{\sigma(u,v)}$. We repeat the proof.

$$\mathrm{i}_{[u,v]}\,\sigma = \mathrm{L}_u(\mathrm{i}_v\,\sigma) = \mathrm{i}_u(\mathrm{d}(\mathrm{i}_v\,\sigma)) + \mathrm{d}(\mathrm{i}_u(\mathrm{i}_v\,\sigma)) = -\,\mathrm{d}(\sigma(u, v)).$$

Here we used $\mathrm{L}_u\,\sigma = 0$ in the first equality, the homotopy formula for the Lie derivative in the second identity, and finally $\mathrm{d}\sigma = 0$, the homotopy identity and $\mathrm{L}_v\,\sigma = 0$ in the third equality. Applying this to $u = X_M$, $v = Y_M$ for $X, Y \in \mathfrak{t}$, and using that $[X, Y] = 0$, hence $[X_M, Y_M] = -[X, Y]_M = 0$, it follows that $\mathrm{Ham}_{\sigma(X_M, Y_M)} = 0$. Thus $\mathrm{d}(\sigma(X_M, Y_M)) = 0$, and the function $x \mapsto \sigma_x(X_M(x), Y_M(x))$ is constant on M, because M is connected. □

In the further discussion we will need some basic facts about proper actions of Lie groups, see for instance [15, Sec. 2.6–2.8]. For each $x \in M$ we write $T_x := \{t \in T \mid t \cdot x = x\}$ for the *stabilizer subgroup* of the T-action at the point x. T_x is a closed Lie subgroup of T, it has finitely many components and its identity component is a torus subgroup of T. The Lie algebra $\mathfrak{t}_x$ of T_x is equal to the space of all $X \in \mathfrak{t}$ such that $X_M(x) = 0$. In other words, $\mathfrak{t}_x$ is the kernel of the linear mapping $\alpha_x : X \mapsto X_M(x)$ from $\mathfrak{t}$ to $\mathrm{T}_x M$. The image of α_x is equal to the tangent space at x of the T-orbit through x, and will be denoted by $\mathfrak{t}_M(x)$. The linear mapping $\alpha_x : \mathfrak{t} \to \mathrm{T}_x M$ induces a linear isomorphism from $\mathfrak{t}/\mathfrak{t}_x$ onto $\mathfrak{t}_M(x)$.

For each closed subgroup H of T which can occur as a stabilizer subgroup, the *orbit type* M^H is defined as the set of all $x \in M$ such that T_x is conjugate to H, but because T is commutative this condition is equivalent to the equation $T_x = H$. Each connected component C of M^H is a smooth T-invariant submanifold of M. The connected components of the orbit types in M form a finite partition of M, which actually is a Whitney stratification. This is called the *orbit type stratification of* M. There is a unique open orbit type, called the *principal orbit type of* M, which is the orbit type of a subgroup H which is contained in every stabilizer subgroup T_x, $x \in M$. Because the effectiveness of the action means that the intersection of all the T_x, $x \in M$ is equal to the identity element, this means that the principal orbit type consists of the points x where $T_x = \{1\}$, that is where the action is free. If the action is free at x, then the linear mapping $X \mapsto X_M(x)$ from $\mathfrak{t}$ to $\mathrm{T}_x M$ is injective. The points $x \in M$ at which the T-action is free are also called the *regular points of* M, and the principal orbit type, the set of all regular points in M is denoted by M_{reg}. The principal orbit type M_{reg} is a dense open subset of M, and connected because T is connected, see [15, Th. 2.8.5]. The *principal orbits* are the orbits in M_{reg}, the principal orbit type. In our situation, the principal orbits are the orbits on which the action of T is free.

LEMMA 2.2. — *Let $\mathfrak{l}$ be the kernel in $\mathfrak{t}$ of the two-form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ defined in Lemma 2.1, the set of all $X \in \mathfrak{t}$ such that $\sigma^{\mathfrak{t}}(X, Y) = 0$ for every $Y \in \mathfrak{t}$. Then $\mathfrak{t}_x \subset \mathfrak{l}$ for every $x \in M$.*

Proof. — If $X \in \mathfrak{t}_x$, then $X_M(x) = 0$, hence $\sigma^{\mathfrak{t}}(X, Y) = \sigma_x(X_M(x), Y_M(x)) = 0$ for every $Y \in \mathfrak{t}$. □

The linear subspace $\mathfrak{l}$ of $\mathfrak{t}$ will play an important part in the classification of the symplectic torus actions with coisotropic principal orbits.

A submanifold C of M is called *coisotropic*, if for every $x \in C$, $v \in \mathrm{T}_x M$, the condition that $\sigma_x(u, v) = 0$ for every $u \in \mathrm{T}_x C$ implies that $v \in \mathrm{T}_x C$. In other words, if the σ_x-orthogonal complement $(\mathrm{T}_x C)^{\sigma_x}$ of $\mathrm{T}_x C$ in $\mathrm{T}_x M$ is contained in $\mathrm{T}_x C$. Every symplectic manifold has an even dimension, say $2n$, and if C is a coisotropic submanifold of dimension k, then

$$2n - k = \dim(\mathrm{T}_x C)^{\sigma_x} \leqslant \dim(\mathrm{T}_x C) = k$$

shows that $k \geqslant n$. C has the minimal dimension n if and only if $(\mathrm{T}_x C)^{\sigma_x} = \mathrm{T}_x C$, if and only if C is a Lagrangian submanifold of M, an isotropic submanifold of M of maximal dimension n. The next lemma is basically the implication (iv) $\Rightarrow$ (ii) in Benoist [6, Prop. 5.1].

Lemma 2.3. — *Let (M,σ) be a connected symplectic manifold, and T a torus which acts effectively and symplectically on (M,σ). Then every coisotropic T-orbit is a principal orbit. Furthermore, if some T-orbit is coisotropic, then every principal orbit is coisotropic, and* $\dim M = \dim T + \dim \mathfrak{l}$.

Proof. — We use Theorem 11.1 with $G = T$, where we note that the commutativity of T implies that the adjoint action of $H = T_x$ on $\mathfrak{t}$ is trivial, which implies that the coadjoint action of H on the component $(\mathfrak{l}/\mathfrak{h})^*$ is trivial as well.

Let us assume that the orbit $T \cdot x$ is coisotropic, which means that $\mathfrak{t}_M(x)^{\sigma_x} \subset \mathfrak{t}_M(x)$, or equivalently the subspace W defined in (11.5) is equal to zero. This implies that the action of H on $E = (\mathfrak{l}/\mathfrak{h})^*$ is trivial, and the vector bundle $T \times_H E = T \times_H (\mathfrak{l}/\mathfrak{h})^*$ is T-equivariantly isomorphic to $(T/H) \times (\mathfrak{l}/\mathfrak{h})^*$, where T acts by left multiplications on the first factor. It follows that in the model all stabilizer subgroups are equal to H, and therefore $T_y = H$ for all y in the T-invariant open neighborhood U of x in M. Because the principal orbit type is dense in M, there are $y \in U$ such that $T_y = \{1\}$, and it follows that $T_x = H = \{1\}$, that is, $T \cdot x$ is a principal orbit. We note in passing that this implies that $\dim M = \dim T + \dim \mathfrak{l}$.

When $W = \{0\}$, we read off from (11.8) with $\sigma^{G/H}$ given by $\sigma^{\mathfrak{t}}$ in Lemma 2.1, and (11.7), that the symplectic form $\Phi^*\sigma$ is given by

$$(\Phi^*\sigma)_{(tH,\lambda)}((X+\mathfrak{h},\delta\lambda),(X'+\mathfrak{h},\delta'\lambda)) = \sigma^{\mathfrak{t}}(X,X') + \delta\lambda(X'_{\mathfrak{l}}) - \delta'\lambda(X_{\mathfrak{l}})$$

for all $(tH,\lambda) \in (T/H) \times E_0$, and $(X+\mathfrak{h},\delta\lambda),(X'+\mathfrak{h},\delta'\lambda) \in (\mathfrak{t}/\mathfrak{h}) \times (\mathfrak{l}/\mathfrak{h})^*$. In this model, the tangent space of the T-orbit is the set of all $(X'+\mathfrak{h},\delta'\lambda)$ such that $\delta'\lambda = 0$, of which the symplectic orthogonal complement is equal to the set of all $(X+\mathfrak{h},\delta\lambda)$ such that $X \in \mathfrak{l}$ and $\delta\lambda = 0$, which implies that in this model every T-orbit is coisotropic and therefore the orbit $T \cdot y$ is coisotropic for every $y \in U$. This shows that the set of all $x \in M$ such that $T \cdot x$ is coisotropic is an open subset of M. Because for all $x \in M_{\mathrm{reg}}$ the tangent spaces of the orbits $T \cdot x$ have the same dimension, equal to $\dim T$, the set of all $x \in M_{\mathrm{reg}}$ such that $T \cdot x$ is coisotropic is closed in M_{reg}. Because M_{reg} is connected, it follows that $T \cdot x$ is coisotropic for all $x \in M_{\mathrm{reg}}$ as soon as $T \cdot x$ is coisotropic for some $x \in M_{\mathrm{reg}}$. □

Remark 2.4. — In the proof of Lemma 2.3, linear forms on $\mathfrak{l}/\mathfrak{h}$ were identified with linear forms on $\mathfrak{l}$. For any linear subspace F of a finite-dimensional vector space E we have the canonical projection $p : x \mapsto x + F : E \mapsto E/F$, and its dual mapping $p^* : (E/F)^* \to E^*$. Because p is surjective, p^* is injective, and its image $p^*((E/F)^*)$ is equal to the space F^0 of all $\varphi \in E^*$ such that $\varphi|_F = 0$. This leads to a canonical linear isomorphism p^* from $(E/F)^*$ onto F^0, which will be used throughout this paper to identify $(E/F)^*$ with the linear subspace F^0 of E^*.

Remark 2.5. — Let $x \in M_{\mathrm{reg}}$. Because the principal orbit type M_{reg} is fibered by the T-orbits, the tangent space $\mathfrak{t}_M(x)$ at x of $T \cdot x$ is equal to the common kernel of the $\mathrm{d}f(x)$, where f ranges over the T-invariant smooth functions on M. Because $-\mathrm{d}f = \mathrm{i}_{\mathrm{Ham}_f}\,\sigma$, it follows that $\mathfrak{t}_M(x)^{\sigma_x}$ is equal to the set of all $\mathrm{Ham}_f(x)$, $f \in \mathrm{C}^\infty(M)^T$. Here $\mathrm{C}^\infty(M)^T$ denotes the space of all T-invariant smooth functions on M.

Suppose that the principal orbits are coisotropic and let $f \in \mathrm{C}^\infty(M)^T$. Then we have for every $x \in M_{\mathrm{reg}}$ that $\mathrm{Ham}_f(x) \in \mathfrak{t}_M(x)^{\sigma_x} \cap \mathfrak{t}_M(x)$, or $\mathrm{Ham}_f(x) = X(x)_M(x)$ for a uniquely determined $X(x) \in \mathfrak{l}$. It follows that the Ham_f-flow leaves every principal orbit invariant, and because M_{reg} is dense in M, the Ham_f-flow leaves every T-orbit invariant. Because a point $x \in M$ is called a *relative equilibrium of a T-invariant vector field v* if the v-flow leaves $T \cdot x$ invariant, the conclusion is that all points of M are relative equilibria of Ham_f, and the induced flow in M/T is at standstill. Moreover the T-invariance of Ham_f implies that $x \mapsto X(x) \in \mathfrak{l}$ is constant on each principal T-orbit, which implies that the Ham_f-flow in M_{reg} is *quasiperiodic*, in the direction of the infinitesimal action of $\mathfrak{l}$ on M_{reg}.

If $f, g \in \mathrm{C}^\infty(M)^T$ and $x \in M_{\mathrm{reg}}$, then $\mathrm{Ham}_f(x)$ and $\mathrm{Ham}_g(x)$ both belong to $\mathfrak{t}_M(x)^{\sigma_x} \cap \mathfrak{t}_M(x)$, and it follows that the *Poisson brackets* $\{f, g\} := \mathrm{Ham}_f g = \sigma(\mathrm{Ham}_f, \mathrm{Ham}_g)$ of f and g vanish at x. Because M_{reg} is dense in M, it follows that $\{f, g\} \equiv 0$ for all $f, g \in \mathrm{C}^\infty(M)^T$ if the principal orbits are coisotropic.

If conversely $\{f, g\} \equiv 0$ for all $f, g \in \mathrm{C}^\infty(M)^T$, then we have for every $x \in M_{\mathrm{reg}}$ that $\mathfrak{t}_M(x)^{\sigma_x} \subset (\mathfrak{t}_M(x)^{\sigma_x})^{\sigma_x} = \mathfrak{t}_M(x)$, which means that $T \cdot x$ is coisotropic. Therefore the principal orbits are coisotropic if and only if the Poisson brackets of all T-invariant smooth functions vanish.

In Guillemin and Sternberg [23], a symplectic manifold with a Hamiltonian action of an arbitrary compact Lie group is called a *multiplicity-free space* if the Poisson brackets of any pair of invariant smooth functions vanish. Because in [23] the emphasis is on representations of noncommutative compact Lie groups, which do not play a role in our paper, and because on the other hand we allow non-Hamiltonian actions, we did not put the adjective “multiplicity-free” in the title.

The next lemma is statement (1) (a) in Benoist [6, Lemma 6.7]. For general symplectic torus actions the stabilizer subgroups need not be connected. For instance, there exist symplectic torus actions with symplectic orbits and nontrivial finite stabilizer subgroups.

Lemma 2.6. — *Let (M, σ) be a connected symplectic manifold, and T a torus which acts effectively and symplectically on (M, σ), with coisotropic principal orbits. Then, for every $x \in M$, the stabilizer group T_x is connected, that is, a subtorus of T.*

Proof. — As in the proof of Lemma 2.3, we use Theorem 11.1 with $G = T$, where H acts trivially on the factor $(\mathfrak{l}/\mathfrak{h})^*$ in $E = (\mathfrak{l}/\mathfrak{h})^* \times W$. Recall that $t \in T$ acts on $T \times_H E$ by sending $H \cdot (t', e)$ to $H \cdot (tt', e)$. When $t = h \in H$, then

$$H \cdot (ht', e) = H \cdot (ht'h^{-1}, h \cdot e) = H \cdot (t', h \cdot e)$$

because T is commutative, and we see that the action of H on $T \times_H E$ is represented by the linear symplectic action of H on W, where W is defined by (11.5).

Because

$$\dim M = (\dim T + \dim(\mathfrak{l}/\mathfrak{h}) + \dim W) - \dim H$$

and because the assumption that the principal orbits are coisotropic implies that $\dim M = \dim T + \dim \mathfrak{l}$, see Lemma 2.3, it follows that $\dim W = 2 \dim H$.

Write $m = \dim H$. The action of the compact and commutative group H by means of symplectic linear transformations on the $2m$-dimensional symplectic vector space (W, σ^W) leads to a direct sum decomposition of W into m mutually σ^W-orthogonal two-dimensional H-invariant linear subspaces E_j, $1 \leqslant j \leqslant m$.

For $h \in H$ and every $1 \leqslant j \leqslant m$, let $\iota_j(h)$ denote the restriction to $E_j \subset W \simeq \{0\} \times W \subset (\mathfrak{l}/\mathfrak{h})^* \times W$ of the action of h on E. Note that $\det \iota_j(h) = 1$, because $\iota_j(h)$ preserves the restriction to $E_j \times E_j$ of σ^W, which is an area form on E_j. Averaging any inner product in each E_j over H, we obtain an H-invariant inner product β_j on E_j, and ι_j is a homomorphism of Lie groups from H to $\mathrm{SO}(E_j, \beta_j)$, the group of linear transformations of E_j which preserve both β_j and the orientation.

On the other hand, if $h \in H$ and $w \in W_{\mathrm{reg}}$, then

$$h \cdot w = \sum_{j=1}^{m} \iota_j(h) w_j \quad \text{if} \quad w = \sum_{j=1}^{m} w_j, \quad w_j \in E_j.$$

Therefore $\iota_j(h)w_j = w_j$ for all $1 \leqslant j \leqslant m$ implies that $h \cdot w = w$, hence $h = 1$. This implies that the homomorphism of Lie groups ι, defined by

$$\iota : h \mapsto (\iota_1(h), \dots, \iota_m(h)) : H \to \prod_{j=1}^{m} \mathrm{SO}(E_j, \beta_j),$$

is injective. Because both the source group H and the target group are m-dimensional Lie groups, and the target group is connected, it follows that ι is an isomorphism of Lie groups. This implies in turn that H is connected. □

Remark 2.7. — The H-invariant inner product β_j on E_j, introduced in the proof of Lemma 2.6, is unique, if we also require that the symplectic inner product of any orthonormal basis with respect to σ^W is equal to ± 1. In turn this leads to the existence of a unique complex structure on E_j such that, for any unit vector e_j in (E_j, β_j), we have that e_j, $\mathrm{i}\, e_j$ is an orthonormal basis in (E_j, β_j) and $\sigma^W(e_j, \mathrm{i}\, e_j) = 1$. Here $\mathrm{i} := \sqrt{-1} \in \mathbb{C}$. This leads to an identification of E_j with $\mathbb{C}$, which is unique up to multiplication by an element of $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

In turn this leads to an identification of W with $\mathbb{C}^m$, with the symplectic form σ^W defined by

$$\sigma^{\mathbb{C}^m} = \sum_{j=1}^{m} \mathrm{d}\, \overline{z^j} \wedge \mathrm{d} z^j / 2\, \mathrm{i}\,. \tag{2.2}$$

The element $c \in \mathbb{T}^m$ acts on $\mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to the element $c \cdot z$ such that $(c \cdot z)^j = c^j z^j$ for every $1 \leqslant j \leqslant m$. There is a unique isomorphism of Lie groups $\iota : H \to \mathbb{T}^m$ such that $h \in H$ acts on $W = \mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to $\iota(h) \cdot z$.

The identification of W with $\mathbb{C}^m$ is unique up to a permutation of the coordinates and the action of an element of $\mathbb{T}^m$.

In the local model of Lemma 2.10 below, we will use that any subtorus of a torus has a complementary subtorus, in the following sense.

LEMMA 2.8. — *Let U be a d_U-dimensional subtorus of a d_T-dimensional torus T. Let $U_{\mathbb{Z}}$ and $T_{\mathbb{Z}}$ denote the integral lattice, the kernel of the exponential mapping, in the Lie algebra $\mathfrak{u}$ and $\mathfrak{t}$ of U and T, respectively. Let Y_i, $1 \leqslant i \leqslant d_U$, be a $\mathbb{Z}$-basis of $U_{\mathbb{Z}}$. Then there are Z_j, $1 \leqslant j \leqslant d_V := d_T - d_U$, such that the Y_i and Z_j together form a $\mathbb{Z}$-basis of $T_{\mathbb{Z}}$. If we denote by $\mathfrak{v}$ the span of the Z_j, then $V = \exp \mathfrak{v}$ is a subtorus of T with Lie algebra equal to $\mathfrak{v}$. V is a complementary subtorus of U in T in the sense that the mapping $U \times V \ni (u, v) \mapsto uv \in T$ is an isomorphism from $U \times V$ onto T. The Z_j form a $\mathbb{Z}$-basis of the integral lattice $V_{\mathbb{Z}}$ in the Lie algebra $\mathfrak{v}$ of V.*

Proof. — We repeat the well-known argument. If $X \in T_{\mathbb{Z}}$, $c \in \mathbb{Z}$, $c \neq 0$, and $cX \in U_{\mathbb{Z}}$, then $X \in \mathfrak{u}$ and $\exp X = 1$ in T, hence $\exp X = 1$ in U, and it follows that $X \in U_{\mathbb{Z}}$. This means that the finitely generated commutative group $T_{\mathbb{Z}}/U_{\mathbb{Z}}$ is torsion-free, and therefore has a $\mathbb{Z}$-basis $\widetilde{Z}_j$, $1 \leqslant j \leqslant k$, *cf.* Hungerford [26, Th. 6.6 on p. 221]. We have that $\widetilde{Z}_j = Z_j + U_{\mathbb{Z}}$ for some $Z_j \in T_{\mathbb{Z}}$. If $X \in T_{\mathbb{Z}}$, then there are unique $z^j \in \mathbb{Z}$ such that $X + U_{\mathbb{Z}} = \sum_{j=1}^{k} z^j \widetilde{Z}_j$, which means that $X - \sum_{j=1}^{k} z^j Z_j \in U_{\mathbb{Z}}$. But this implies that there are unique $y^i \in \mathbb{Z}$ such that $X - \sum_{j=1}^{k} z^j Z_j = \sum_{i=1}^{d_U} y^i Y_i$, which shows that the Y_i and Z_j together form a $\mathbb{Z}$–basis of $T_{\mathbb{Z}}$, which in turn implies that $k = d_T - d_U = d_V$.

The last statement follows from the fact that the mapping

$$(y, z) \mapsto \exp\left(\sum_{i=1}^{d_U} y^i Y_i + \sum_{j=1}^{d_V} z^j Z_j\right)$$

from $\mathbb{R}^{d_T}$ to T induces an isomorphism from $(\mathbb{R}/\mathbb{Z})^{d_T}$ onto T which maps $(\mathbb{R}/\mathbb{Z})^{d_U} \times \{0\}$ onto U and $\{0\} \times (\mathbb{R}/\mathbb{Z})^{d_V}$ onto V. □

Remark 2.9. — The complementary subtorus V in Lemma 2.8 is by no means unique. The Z_j can be replaced by any

$$Z'_j = Z_j + \sum_{i=1}^{d_U} c_j^i Y_i, \quad 1 \leqslant j \leqslant d_V,$$

in which the c_j^i are integers. This leads to a bijective correspondence between the set of all complementary subtori of a given subtorus U and the set of all $d_U \times d_F$-matrices with integral coefficients.

Let $H = T_x$ be the subtorus of T in Lemma 2.6. Let K be a complementary subtorus of H in T and, for any $t \in T$, let t_H and t_K be the unique elements in H and K, respectively, such that $t = t_H t_K$. Let $X \mapsto X_{\mathfrak{l}}$ be a linear projection from $\mathfrak{t}$ onto $\mathfrak{l}$. We also use the identification of W with $\mathbb{C}^m$ as in Remark 2.7. With these notations, we have the following local model for our symplectic T-space with coisotropic principal orbits.

Lemma 2.10. — *Under the assumptions of Lemma 2.6, there is an isomorphism of Lie groups ι from H onto $\mathbb{T}^m$, an open $\mathbb{T}^m$-invariant neighborhood E_0 of the origin in $E = (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$, and a T-equivariant diffeomorphism Φ from $K \times E_0$ onto an open T-invariant neighborhood U of x in M, such that $\Phi(1, 0) = x$. Here $t \in T$ acts on $K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$ by sending (k, λ, z) to $(t_K k, \lambda, \iota(t_H) \cdot z)$. In addition, the symplectic form $\Phi^* \sigma$*

on $K \times E_0$ is given by

(2.3)
$$(\Phi^*\sigma)_{(k,\lambda,z)}((X,\delta\lambda,\delta z),(X',\delta'\lambda,\delta' z)) = \sigma^{\mathfrak{t}}(X,X') + \delta\lambda(X'_{\mathfrak{l}}) - \delta'\lambda(X_{\mathfrak{l}}) + \sigma^{\mathbb{C}^m}(\delta z,\delta' z)$$

for all $(k,\lambda,z) \in K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$, and $(X,\delta\lambda,\delta z),(X',\delta'\lambda,\delta' z) \in \mathfrak{k} \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$. Here we identify each tangent space of the torus K with $\mathfrak{k}$ and each tangent space of a vector space with the vector space itself. Finally, $\sigma^{\mathbb{C}^m}$ is the symplectic form on $\mathbb{C}^m$ defined in (2.2).

Proof. — As in the proof of Lemma 2.6, we use Theorem 11.1 with $G = T$, where H acts trivially on the factor $(\mathfrak{l}/\mathfrak{h})^*$ and $h \in H$ acts on $W = \mathbb{C}^m$ by sending $z \in \mathbb{C}^m$ to $\iota(h)\cdot z$. Here $\iota : H \to \mathbb{T}^m$ is the isomorphism from the torus H onto the standard torus $\mathbb{T}^m$ introduced in Remark 2.7, and the symplectic form $\sigma^{\mathbb{C}^m}$ on $\mathbb{C}^m$ is given by (2.2).

Because K is a complementary subtorus of H in T, the manifold $K \times E$ is a global section of the vector bundle $\pi_K : T \times_H E \to T/H \simeq K$. Indeed, if $(t,e) \in T \times E$, then $(t_K, t_H \cdot e) = (t{t_H}^{-1}, t_H \cdot e)$ is the unique element in $(K \times E) \cap H \cdot (t,e)$. Furthermore, if $t \in T$ and $(k,e) \in K \times E$, then $(t_K k, t_H \cdot e)$ is the unique element in $(K \times E) \cap H \cdot (tk,e)$. This exhibits $T \times_H E$ as a trivial vector bundle over K, which is a homogeneous T-bundle, where $t \in T$ acts on $K \times E$ by sending (k,e) to $(t_K k, t_H \cdot e)$.

Finally, if in (11.7) we restrict ourselves to $X \in \mathfrak{k}$, then the right hand side simplifies to $\lambda(X_{\mathfrak{l}}) + \sigma^W(w,\delta w)/2$, which leads to (2.3). □

Remark 2.11. — In the local model of Lemma 2.10, we have that $T_{(k,\lambda,z)} = H$ if and only if z is a fixed point of $\iota(H) = \mathbb{T}^m$ if and only if $z = 0$. Because $K \times (\mathfrak{l}/\mathfrak{h})^* \times \{0\}$ is a symplectic submanifold of $K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$, it follows that every orbit type is a smooth symplectic submanifold of (M,σ).

Moreover, $T \cdot (k,\lambda,0) = K \times \{\lambda\} \times \{0\}$ is a coisotropic submanifold of $K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$, and we conclude that every T-orbit is a coisotropic submanifold of its orbit type.

The discussion of the relative equilibria in Remark 2.5, with M_{reg} replaced by any orbit type M^H, leads to the conclusion that for every $f \in \mathrm{C}^\infty(M)^T$ the flow of the Hamiltonian vector field Ham_f in M^H is quasi-periodic, in the direction of the infinitesimal action of $\mathfrak{l}/\mathfrak{h}$ in M^H.

We conclude this section with a discussion of the special case that the two-form $\sigma^{\mathfrak{t}}$ in Lemma 2.1 is equal to zero.

LEMMA 2.12. — *We have $\sigma^{\mathfrak{t}} = 0$ if and only if $\mathfrak{l} := \ker \sigma^{\mathfrak{t}} = \mathfrak{t}$ if and only if some T-orbit is isotropic if and only if every T-orbit is isotropic.*

Also, every principal orbit is a Lagrangian submanifold of (M, σ) *if and only if some principal orbit is a Lagrangian submanifold of* (M, σ) *if and only if* $\dim M = 2 \dim T$ *and* $\sigma^{\mathfrak{t}} = 0$.

Proof. — The equivalence of $\sigma^{\mathfrak{t}} = 0$ and $\ker \sigma^{\mathfrak{t}} = \mathfrak{t}$ is obvious, whereas the equivalence between $\sigma^{\mathfrak{t}} = 0$ and the isotropy of some (every) T-orbit follows from Lemma 2.1.

If $x \in M_{\mathrm{reg}}$ and $T \cdot x$ is a Lagrange submanifold of (M, σ), then $\dim M = 2 \dim(T \cdot x) = 2 \dim T$, and $\sigma^{\mathfrak{t}} = 0$ follows in view of the first statement in the lemma.

Conversely, if $\dim M = 2 \dim T$ and $\sigma^{\mathfrak{t}} = 0$, then every orbit is isotropic and for every $x \in M_{\mathrm{reg}}$ we have $\dim M = 2 \dim T = 2 \dim(T \cdot x)$, which implies that $T \cdot x$ is a Lagrangian submanifold of (M, σ). □

3. The orbit space

In this section we investigate the orbit space of our action of the torus T on the compact connected symplectic manifold (M, σ) with coisotropic principal orbits. The main results are that the closed basic one-form $\widehat{\sigma}$ of Lemma 3.1 exhibits the orbit space as a locally convex polyhedral $\mathfrak{l}^*$-parallel space, see Definition 10.1 and Lemma 3.5, and that as such M/T is isomorphic to the Cartesian product of a Delzant polytope and a torus, see Proposition 3.8. The assumption that the principal orbits are coisotropic will be assumed throughout this section, unless explicitly stated otherwise.

3.1. Canonical local charts on the orbit space

In this subsection we exhibit the space of T-orbits as an $\mathfrak{l}^*$-parallel space in the sense of Definition 10.1.

We denote the space of all orbits in M of the T-action by M/T, and by $\pi : M \to M/T$ the canonical projection which assigns to each $x \in M$ the orbit $T \cdot x$ through the point x. The orbit space is provided with the maximal topology for which the canonical projection is continuous; this topology is Hausdorff.

For each connected component C of an orbit type M^H in M of the subgroup H of T, as introduced in the paragraphs preceding Lemma 2.2, the action of T on C induces a proper and free action of the torus T/H on C, and $\pi(C)$ has a unique structure of a smooth manifold such that

$\pi : C \to \pi(C)$ is a principal T/H-bundle. $(M/T)^H := \pi(M^H)$ is called the *orbit type of H in M/T* and $\pi(C)$ is a connected component of $(M/T)^H$. The connected components of the orbit types in the orbit space form a finite stratification of the orbit space, *cf.* [15, Sec. 2.7].

Although M/T is equal to the union of the finitely many strata of the orbit type stratification in M/T, where each of these strata is a smooth manifold, the orbit space M/T is not a smooth manifold, unless the action of T on M is free. In general the principal orbit type $(M/T)_{\mathrm{reg}} = M_{\mathrm{reg}}/T$ is a smooth manifold of dimension $\dim M - \dim T$, which is an open and dense subset of M/T, and M/T will have singularities at the lower dimensional strata, the strata in the complement of $(M/T)_{\mathrm{reg}}$ in M/T. However, in this section we will obtain a much more explicit description of the orbit space M/T.

A smooth differential form ω on M is called *basic with respect to the T-action* if it is T-invariant, that is $\mathrm{L}_{X_M}\,\omega = 0$ for every $X \in \mathfrak{t}$, and if $\mathrm{i}_{X_M}\,\omega = 0$ for every $X \in \mathfrak{t}$. The basic differential forms constitute a module over the algebra $\mathrm{C}^\infty(M)^T$ of T-invariant smooth functions on M, the basic forms of degree zero on M. A smooth differential form ω on M is basic if and only if the restriction of ω to the principal orbit type is equal to $\pi^*\nu$ for a smooth differential form ν on the principal orbit type in M/T.

A theorem of Koszul [32] says that the Čech (= sheaf) cohomology group $\mathrm{H}^k(M/T, \mathbb{R})$ of M/T is canonically isomorphic to the de Rham cohomology of the basic forms on M, that is, the space of closed basic k-forms on M modulo its subspace consisting of the $\mathrm{d}\nu$ in which ν ranges over the basic $(k-1)$-forms on M. This theorem holds for any proper action of a Lie group on any smooth manifold, and in particularly it does not need the compactness of M.

Lemma 3.1. — *Recall that $\mathfrak{l}$ is the kernel of the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ which had been introduced in Lemma 2.1. For each $X \in \mathfrak{l}$, $\widehat{\sigma}(X) := -\,\mathrm{i}_{X_M}\,\sigma$ is a closed basic one-form on M.*

Proof. — That $\widehat{\sigma}(X)$ is closed follows from (2.1). Because $X \in \mathfrak{l}$, we have for each $Y \in \mathfrak{t}$ that $-\,\mathrm{i}_{Y_M}(\widehat{\sigma}(X)) = \sigma^{\mathfrak{t}}(X, Y) = 0$. Also we have for every $Y \in \mathfrak{t}$ that

$$-\,\mathrm{L}_{Y_M}\,\widehat{\sigma}(X) = \mathrm{i}_{[Y_M, X_M]}\,\sigma + \mathrm{i}_{X_M}(\mathrm{L}_{Y_M}\,\sigma) = 0.$$

Here we have used the Leibniz identity for the Lie derivative, the commutativity of $\mathfrak{t}$, and the T-invariance of σ which implies that $\mathrm{L}_{Y_M}\,\sigma = 0$. □

For each $x \in M$, $\widehat{\sigma}(X)_x$ is a linear form on $\mathrm{T}_x M$ which depends linearly on $X \in \mathfrak{l}$, and therefore $X \mapsto \widehat{\sigma}(X)_x$ is an $\mathfrak{l}^*$-valued linear form on $\mathrm{T}_x M$, which we denote by $\widehat{\sigma}_x$. In this way $x \mapsto \widehat{\sigma}_x$ is an $\mathfrak{l}^*$-valued one-form on M, which we denote by $\widehat{\sigma}$. With these conventions, we have

$$\widehat{\sigma}_x(v)(X) = \widehat{\sigma}(X)_x(v) = \sigma_x(v, X_M(x)), \quad x \in M,\ v \in \mathrm{T}_x M,\ X \in \mathfrak{l}. \tag{3.1}$$

Note that the $\mathfrak{l}^*$-valued one-form $\widehat{\sigma}$ on M is basic and closed.

Let $X \in \mathfrak{t}$ and suppose that $X_M = \mathrm{Ham}_f$ for some $f \in \mathrm{C}^\infty(M)$. Then we have for every $Y \in \mathfrak{t}$ that

$$Y_M(f) = \mathrm{i}_{Y_M}(\mathrm{d}f) = -\,\mathrm{i}_{Y_M}(\mathrm{i}_{X_M}\,\sigma) = \sigma(Y_M, X_M) = \sigma^{\mathfrak{t}}(Y, X),$$

and it follows that $f \in \mathrm{C}^\infty(M)^T$ if and only if $X \in \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$. The T-action on (M, σ) is called a *Hamiltonian T-action* if for every $X \in \mathfrak{t}$ there exists an $f \in \mathrm{C}^\infty(M)^T$ such that $X_M = \mathrm{Ham}_f$ and the assignment $X \mapsto f$ is a Lie algrebra anti-homomorphism from $\mathfrak{t}$ to $\mathrm{C}^\infty(M)$. Note that if $\mathfrak{l} = \mathfrak{t}$, that is, if $\sigma^{\mathfrak{t}} = 0$, then the T-action is Hamiltonian if and only if if for every $X \in \mathfrak{t}$ there exists an $f \in \mathrm{C}^\infty(M)$ such that $X_M = \mathrm{Ham}_f$.

We recall the Delzant manifolds, mentioned in Section 1. Koszul's theorem now implies the following.

Corollary 3.2. — *We do not assume that the principal orbits are coisotropic. Let $X \in \mathfrak{t}$. Then $X_M = \mathrm{Ham}_f$ for some $f \in \mathrm{C}^\infty(M)^T$, if and only if $X \in \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$ and the cohomology class $[\widehat{\sigma}(X)] \in \mathrm{H}^1(M/T, \mathbb{R})$ is equal to zero. If the T-action is Hamiltonian, then $\sigma^{\mathfrak{t}} = 0$. Finally, if $\sigma^{\mathfrak{t}} = 0$ and $\mathrm{H}^1(M/T, \mathbb{R}) = 0$, then the T-action is Hamiltonian and (M, σ, T) is a Delzant manifold.*

Remark 3.3. — In the local model of Lemma 2.10, the T-orbit space of $K \times E_0$ is equal to $E_0/\mathbb{T}^m$, which is contractible by using the radial contractions in E_0. It follows that for every $x_0 \in M$ there is a T-invariant open neighborhood U of x_0 in M such that the open subset $\pi(U)$ of the orbit space M/T is contractible. Because of Koszul's theorem, and because the Čech cohomology of $\pi(U)$ is trivial, it follows that the infinitesimal action of $\mathfrak{l}$ on U is Hamiltonian. Therefore, if $\sigma^{\mathfrak{t}} = 0$, then the T-action is *locally Hamiltonian* in the sense that every element in M has a T-invariant open neighborhood in M on which the T-action is Hamiltonian.

In the local model of Lemma 2.10, we write $z^j = |z^j|\,\mathrm{e}^{\mathrm{i}\,\theta^j}$ with $\theta^j \in \mathbb{R}/2\pi\mathbb{Z}$ for each $1 \leqslant j \leqslant m$. Then the symplectic form σ^W with $W = \mathbb{C}^m$ in (2.2) is equal to

$$\sigma^{\mathbb{C}^m} = \sum_{j=1}^{m} \mathrm{d}\rho_j \wedge \mathrm{d}\,\theta^j, \quad \text{in which} \quad \rho_j := |z^j|^2/2. \tag{3.2}$$

The mapping $(\lambda, \rho) : \overline{M} := K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m \to (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{R}^m$ induces a homeomorphism from the T-orbit space $\overline{M}/T \simeq (\mathfrak{l}/\mathfrak{h})^* \times (\mathbb{C}^m/\mathbb{T}^m)$ onto $(\mathfrak{l}/\mathfrak{h})^* \times \mathbb{R}^m_+$, in which

$$\mathbb{R}^m_+ := \{\rho \in \mathbb{R}^m \mid \rho_j \geqslant 0 \text{ for every } 1 \leqslant j \leqslant m\}.$$

Note that $(\mathrm{e}^{\mathrm{i}\,\alpha^1}, \dots, \mathrm{e}^{\mathrm{i}\,\alpha^m}) \in \mathbb{T}^m$ acts on $\mathbb{C}^m$ by sending θ to $\theta + \alpha$ and leaving ρ fixed. If we identify the Lie algebra of $\mathbb{T}^m$ with $(\mathrm{i}\,\mathbb{R})^m$, then the infinitesimal action of $\beta \in (\mathrm{i}\,\mathbb{R})^m$ in (θ, ρ)-coordinates is equal to the constant vector field $(\beta, 0)$. The tangent mapping at 1 of the isomorphism $\iota : H \to \mathbb{T}^m$ is a linear isomorphism from $\mathfrak{h}$ onto $(\mathrm{i}\,\mathbb{R})^m$, which we we also denote by ι.

For every $Y \in \mathfrak{t}$, the infinitesimal action $Y_{\overline{M}}$ of Y on $\overline{M}$ is equal to the vector field $(Y_{\mathfrak{k}}, 0, \iota(Y_{\mathfrak{h}}) \cdot z)$, see the description of the action of T on the model in Lemma 2.10. Write $Y = Y_{\mathfrak{h}} + Y_{\mathfrak{k}}$ with $Y_{\mathfrak{h}} \in \mathfrak{h}$ and $Y_{\mathfrak{k}} \in \mathfrak{k}$. Here $\mathfrak{k}$ denotes the Lie algebra of the complementary torus K to H in T, which implies that $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{k}$. Because $\mathfrak{h} \subset \mathfrak{l}$, we have $(Y_{\mathfrak{h}})_{\mathfrak{l}} = Y_{\mathfrak{h}}$ and $Y_{\mathfrak{l}} = Y_{\mathfrak{h}} + (Y_{\mathfrak{k}})_{\mathfrak{l}} = Y_{\mathfrak{h}} + (Y_{\mathfrak{l}})_{\mathfrak{k}}$. Because $\delta\lambda \in (\mathfrak{l}/\mathfrak{h})^*$ is a linear form on $\mathfrak{l}$ which is equal to zero on $\mathfrak{h}$, it follows that

$$\delta\lambda((Y_{\mathfrak{k}})_{\mathfrak{l}}) = \delta\lambda(Y_{\mathfrak{l}}), \quad \delta\lambda \in (\mathfrak{l}/\mathfrak{h})^*, \quad Y \in \mathfrak{t}. \tag{3.3}$$

Therefore, if in (2.3) we substitute

$$(X', \delta'\lambda, \delta' z) = Y_{\overline{M}}(k, \lambda, z) = (Y_{\mathfrak{k}}, 0, \iota(Y_{\mathfrak{h}}) \cdot z)$$

with $Y \in \mathfrak{l}$, then we obtain

$$\sigma^{\mathfrak{t}}(X, Y_{\mathfrak{k}}) + \delta\lambda((Y_{\mathfrak{k}})_{\mathfrak{l}}) + \sigma^{\mathbb{C}^m}(\delta z, \iota(Y_{\mathfrak{h}}) \cdot z) = \delta\lambda(Y) + \sum_{j=1}^{m} \iota(Y_{\mathfrak{h}})^j \delta\rho_j / \mathrm{i}\,. \tag{3.4}$$

Here we have used that $Y_{\mathfrak{k}} = (Y_{\mathfrak{l}})_{\mathfrak{k}} = (Y_{\mathfrak{k}})_{\mathfrak{l}} \in \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$ implies that $\sigma^{\mathfrak{t}}(X, Y_{\mathfrak{k}}) = 0$. Furthermore (3.3) with $Y \in \mathfrak{l}$ implies that $\delta\lambda((Y_{\mathfrak{k}})_{\mathfrak{l}}) = \delta\lambda(Y)$. Finally the formula for the $\sigma^{\mathbb{C}^m}$-term follows from (3.2), as $(\mathrm{d}\rho_j)(\iota(Y_{\mathfrak{h}}) \cdot z) = 0$, $(\mathrm{d}\rho_j)(\delta z) = \delta\rho_j$, and $(\mathrm{d}\theta^j)(\iota(Y_{\mathfrak{h}}) \cdot z) = \iota(Y_{\mathfrak{h}})^j / \mathrm{i}$ because the infinitesimal action of $\beta := \iota(Y_{\mathfrak{h}}) \in \mathrm{i}\,\mathbb{R}^m$ is equal to $(\sum_j \beta^j \partial/\partial\theta^j)/\,\mathrm{i}$.

Consider the linear mapping

$$A : (\delta\lambda, \delta\rho) \mapsto [Y \mapsto \delta\lambda(Y) + \sum_{j=1}^{m} \iota(Y_{\mathfrak{h}})^j \delta\rho_j / \,\mathrm{i}] \tag{3.5}$$

from $(\mathfrak{l}/\mathfrak{h})^* \times \mathbb{R}^m$ onto $\mathfrak{l}^*$. A is a linear isomorphism, because the source space and the target space have the same dimension, and $\ker A = 0$: testing with arbitrary $Y \in \mathfrak{h}$ yields that $\delta\rho = 0$, and then testing with arbitrary $Y \in \mathfrak{l}$ yields that $\delta\lambda = 0$.

Let X_j denote the element of $\mathfrak{h} \subset \mathfrak{l}$ such that $\iota(X_j) = 2\pi\,\mathrm{i}\, e_j$, in which e_j denotes the j-th standard basis vector in $\mathbb{R}^m$. Note that the $2\pi\,\mathrm{i}\, e_j$, $1 \leqslant j \leqslant m$, form a $\mathbb{Z}$-basis of the integral lattice of the Lie algebra of $\mathbb{T}^m$, and because $\iota : H \to \mathbb{T}^m$ is an isomorphism of tori, it follows that the X_j, $1 \leqslant j \leqslant m$, form a $\mathbb{Z}$-basis of the integral lattice of the Lie algebra $\mathfrak{h}$ of H. Also note that

$$(A(\lambda, \rho))(X_j) = \sum_{k=1}^{m} \iota(X_j)^k \rho_k / \mathrm{i} = 2\pi \rho_j, \quad 1 \leqslant j \leqslant m.$$

Here it is essential that we use the coordinates ρ_j instead of their infinitesimal displacements $\delta\rho_j$, because in Lemma 3.4 below we are interested in the the consequences of the inequalities $\rho_j \geqslant 0$. This leads to the following conclusion.

Lemma 3.4. — *Let Φ be the T-equivariant symplectomorphism from $K \times E_0 \subset \overline{M}$ onto the open T-invariant neighborhood U of x in M as introduced in Lemma 2.10. Then the smooth mapping $\Psi : U \to \mathfrak{l}^*$, which consists of $\Phi^{-1} : U \to \overline{M}$, followed by the (λ, ρ)-map and then A, induces a homeomorphism χ from U/T onto an open neighborhood of* 0 *in the corner*

$$\{\xi \in \mathfrak{l}^* \mid \xi(X_j) \geqslant 0 \text{ for every } 1 \leqslant j \leqslant m\}$$

in $\mathfrak{l}^*$, such that $\widehat{\sigma} = \mathrm{d}\Psi$. Here the X_j, $1 \leqslant j \leqslant m$, form a $\mathbb{Z}$-basis of the integral lattice of the Lie algebra $\mathfrak{h} \subset \mathfrak{l}$ of H.

Proof. — For every $Y \in \mathfrak{l}$, the right hand side of (3.4) is equal to $-\,\mathrm{i}_{Y_M}\,\sigma$. Combined with the definitions of Ψ and A, this yields that $\widehat{\sigma} = \mathrm{d}\Psi$. Because $(k, \lambda, z) \mapsto (\lambda, \rho)$ is a homeomorphism from $(K \times E)/T$ onto $(\mathfrak{l}/\mathfrak{h})^* \times (\mathbb{R}_+)^m$, χ is a homeomorphism from U/T onto an open neighborhood of 0 in the corner in $\mathfrak{l}^*$ which is determined by the inequalities $\xi(X_j) \geqslant 0$, $1 \leqslant j \leqslant m$. □

If $\widetilde{\Psi} : \widetilde{U} \to \mathfrak{l}^*$ is mapping as in Lemma 3.4, with corresponding chart $\widetilde{\chi} : \widetilde{U}/T \to \mathfrak{l}^*$, then

$$\mathrm{d}(\Psi - \widetilde{\Psi}) = \mathrm{d}\Psi - \mathrm{d}\widetilde{\Psi} = \widehat{\sigma} - \widehat{\sigma} = 0$$

shows that $\Psi - \widetilde{\Psi}$ is locally constant on $U \cap \widetilde{U}$, which implies that $\chi - \widetilde{\chi}$ is locally constant on $(U/T) \cap (\widetilde{U}/T)$. In terms of Definition 10.1, we have proved

Lemma 3.5. — *With the χ of Lemma 3.4 as local charts on M/T, the orbit space M/T is a locally convex polyhedral $\mathfrak{l}^*$-parallel space. The linear forms $v^*_{\alpha,j}$, $1 \leqslant j \leqslant m$, in Definition 10.1 are the $\mathfrak{l}^* \ni \xi \mapsto \xi(X_j)$, where* *the X_j, $1 \leqslant j \leqslant m$, form a $\mathbb{Z}$-basis of the integral lattice of the Lie algebra $\mathfrak{h} \subset \mathfrak{l}$ of a stabilizer group $H = T_x$ of an element $x \in M$.*

In the next lemma we will introduce the subtorus T_{h} of T which later will turn out to be the unique maximal subtorus of T which acts on M in a Hamiltonian fashion. For this reason T_{h} will be called the *Hamiltonian torus.*

LEMMA 3.6. — *There are only finitely many different stabilizer subgroups of T, each of which is a subtorus of T. The product T_{h} of all the different stabilizer subgroups is a subtorus of T, and the Lie algebra $\mathfrak{t}_{\mathrm{h}}$ of T_{h} is equal to the sum of the Lie algebras of all the different stabilizer subgroups of T. It follows from Lemma 2.2 that $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$.*

Proof. — In the local model of Lemma 2.10, the stabilizer subgroup of (k, λ, z) is equal to the set of all $h \in H$ such that $\iota(h)^j = 1$ for every j such that $z^j \neq 0$. It follows that we have 2^m different stabilizer subgroups T_y, $y \in U$, namely one for each subset of $\{1, \dots, m\}$. Because M is compact, is follows that there are only finitely many different stabilizer subgroups of T. For the last statement we observe that the product of finitely many subtori is a compact and connected subgroup of T and therefore a subtorus of T. Also the image under the exponential mapping of the sum of the finitely many different Lie algebras of the stabilizer subgroups of T is equal to T_{h}, which proves that the Lie algebra of T_{h} is equal to the sum of the finitely many different $\mathfrak{t}_x$, $x \in M$. See for instance [15, Sec. 1.12] for the general facts about Lie subgroups of tori, which we have used here. □

Remark 3.7. — The orbit $\pi(x) = T \cdot x \in M/T$ of any $x \in M_{\mathrm{reg}}$ is called a *regular point of M/T*. Recall from the paragraph preceding Lemma 2.2 that $x \in M_{\mathrm{reg}}$ if and only if $T_x = \{1\}$ if and only if $\mathfrak{t}_x = \{0\}$. Therefore the set $(M/T)_{\mathrm{reg}}$ of all regular points in M/T is just the principal orbit type, which is a smooth manifold of dimension $\dim M - \dim T$. In the local model of Lemma 2.10 with $x \in M_{\mathrm{reg}}$, where $\mathfrak{h} = \mathfrak{t}_x = \{0\}$ and $m = 0$, at each point the $\mathfrak{l}^*$-valued one-form $\widehat{\sigma}$ corresponds to the projection $(\delta t, \delta\lambda) \mapsto \delta\lambda : \mathfrak{t} \times \mathfrak{l}^* \to \mathfrak{l}^*$, and $\mathfrak{t} \times \{0\}$ is equal to the tangent space of the T-orbit. It follows that for every $p \in (M/T)_{\mathrm{reg}}$ the induced linear mapping $\widehat{\sigma}_p : \mathrm{T}_p(M/T)_{\mathrm{reg}} \to \mathfrak{l}^*$ is a linear isomorphism.

More generally, the orbit type stratification, introduced in the paragraph preceding Lemma 2.2, leads to a corresponding decomposition of M/T. The strata for the T-action in $\overline{M} = K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$ are of the form $\overline{M}^J$ in which J is a subset of $\{1, \dots, m\}$ and $\overline{M}^J$ is the set of all (k, λ, z) such that $z^j = 0$ if and only if $j \in J$. In terms of the (θ, ρ)-coordinates, this corresponds to $\rho_j = 0$ for all $j \in J$ and $\rho_k > 0$ for $k \notin J$. The Lie algebra of the corresponding stabilizer subgroup of T corresponds to the span of the vector fields $\partial/\partial\theta^j$ with $j \in J$. Therefore, if Σ is a connected component of the orbit type in M/T defined by the subtorus H of T with Lie algebra $\mathfrak{h}$, then for each $p \in \Sigma$ we have $\widehat{\sigma}_p(X) = 0$ for all $X \in \mathfrak{h}$, and $\widehat{\sigma}_p$ may be viewed as an element of $(\mathfrak{l}/\mathfrak{h})^* = \mathfrak{h}^0$, the set of all linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{h}$, see Remark 2.4. The linear mapping $\widehat{\sigma}_p : \mathrm{T}_p\Sigma \to (\mathfrak{l}/\mathfrak{h})^*$ is a linear isomorphism.

3.2. M/T is the Cartesian product of a Delzant polytope and a torus

In the following Proposition 3.8, the orbit space M/T is viewed as a locally convex polyhedral $\mathfrak{l}^*$-parallel space, as in Definition 10.1 with $Q = M/T$ and $V = \mathfrak{l}^*$. See Lemma 3.5. Let the subset D of $\mathfrak{l}^* \times (M/T)$ and the mapping $(\xi, p) \mapsto p+\xi$ from D to M/T be defined as in Definition 10.7. We have the linear subspace N of $V = \mathfrak{l}^*$, which acts on $Q = M/T$ by means of translations, and the period group P of the N-action on Q, as defined in Lemma 10.11 and Lemma 10.12, respectively. With the choice of a base point $p \in M/T$, we write $D_p = \{\xi \in \mathfrak{l}^* \mid (\xi, p) \in D\}$. Let $\mathfrak{t}'_{\mathrm{h}}$ be a linear complement of $\mathfrak{t}_{\mathrm{h}}$ in $\mathfrak{t}$ and let $p \in M/T$. With these definitions, and the identification of $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ with the space of linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{t}_{\mathrm{h}}$, see Remark 2.4, we have the following conclusions.

Proposition 3.8. — *Let C be a linear complement of $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ in $\mathfrak{l}^*$.*

i) *$N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, P is a cocompact discrete subgroup of the additive group N, and N/P is a $\dim N$-dimensional torus.*

ii) *There is a Delzant polytope Δ in $C \simeq (\mathfrak{t}_{\mathrm{h}})^*$, such that $D_p = \Delta + N$.*

iii) *The mapping $\Phi_p : (\eta, \zeta) \mapsto p + (\eta + \zeta)$ is an isomorphism of locally convex polyhedral $\mathfrak{l}^*$-parallel spaces from $\Delta \times (N/P)$ onto M/T.*

Proof. — The linear forms v_j^* which appear in the characterization of N in Theorem 10.13 are equal to the collection of all the $X_i \in \mathfrak{h} \subset \mathfrak{l} = (\mathfrak{l}^*)^*$ which appear in $\mathbb{Z}$-bases of integral lattices of Lie algebras $\mathfrak{h}$ of stabilizer subgroups H of T. Because N is equal to the common kernel of all the v_j^*, N is equal to the set $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ of all elements of $\mathfrak{l}^*$ which vanish on the sum $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l}$ of the finitely many different Lie algebras $\mathfrak{h}$ of stabilizer subgroups of T.

Because C is a linear complement of $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ in $\mathfrak{l}^*$, the mapping $\xi \mapsto \xi|_{\mathfrak{t}_{\mathrm{h}}}$ induces an isomorphism from C onto $(\mathfrak{t}_{\mathrm{h}})^*$. Δ is a Delzant polytope in $(\mathfrak{t}_{\mathrm{h}})^*$

in the sense of Guillemin [20, p. 8], because each $\mathbb{Z}$-basis of the integral lattice of $\mathfrak{t}_x$ can be extended to a $\mathbb{Z}$-basis of the integral lattice of $\mathfrak{t}_{\mathrm{h}}$, see Lemma 2.8.

Because C is a linear complement of $(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^* = N = \mathbb{R}P$ in $\mathfrak{l}^*$, Proposition 3.8 now follows from Lemma 3.5 and Theorem 10.13. $\square$

Corollary 3.9. — *Let (M, σ) be a compact connected $2n$-dimensional symplectic manifold and suppose that we have an effective symplectic action of an n-dimensional torus T on (M, σ), where we do not assume that the principal orbits are coisotropic. Then the following conditions are equivalent.*

i) *The action of T has a fixed point in M.*
ii) *The sum of the Lie algebras of all the different stabilizer subgroups of T is equal to the Lie algebra of T.*
iii) *$\sigma^{\mathfrak{t}} = 0$ and M/T is homeomorphic to a convex polytope.*
iv) *$\sigma^{\mathfrak{t}} = 0$ and $\mathrm{H}^1(M/T, \mathbb{R}) = 0$.*
v) *The action of T is Hamiltonian.*

Proof. — If x is a fixed point, then $T_x = T$, hence $\mathfrak{t}_x = \mathfrak{t}$, which implies ii).

Write $\mathfrak{t}'$ for the sum of the Lie algebras of all the different stabilizer subgroups of T. If $X \in \mathfrak{t}_x$, then $X_M(x) = 0$ and it follows from Lemma 2.1 that $\sigma^{\mathfrak{t}}(X, Y) = 0$ for every $Y \in \mathfrak{t}$. This shows that $\sigma^{\mathfrak{t}}(X, Y) = 0$ for every $X \in \mathfrak{t}'$ and every $Y \in \mathfrak{t}$. Now ii) means that $\mathfrak{t}' = \mathfrak{t}$, hence $\sigma^{\mathfrak{t}} = 0$, and Lemma 2.12 implies that every principal orbit is a Lagrangian submanifold of (M, σ), and therefore coisotropic. It follows that we may apply Proposition 3.8 with $\mathfrak{t}_{\mathrm{h}} = \mathfrak{t}' = \mathfrak{t}$, and conclude that Φ_p is a homeomorphism from the Delzant polytope Δ onto M/T.

iii) $\Rightarrow$ iv) because any convex polytope is contractible.

iv) $\Rightarrow$ v) follows from Corollary 3.2.

Finally v) $\Rightarrow$ i) follows from the fact that the image of the momentum mapping is equal to the convex hull of the images under the momentum mapping of the fixed points, *cf.* Atiyah [1, Th. 1] or Guillemin and Sternberg [22, Th. 4]. $\square$

The implication i) $\Rightarrow$ v) has also been obtained by Giacobbe [17, Th. 3.13].

Note that if the conditions i) – v) in Corollary 3.9 hold, then (M, σ) together with the T-action on M is a Delzant manifold, and M/T is the corresponding Delzant polytope. If a compact Lie group K acts linearly and continuously on a vector space V, then the *average of $v \in V$ over K* is defined as

$$\int_K k \cdot v m(\mathrm{d}k)/m(K),$$

in which m denotes any Haar measure on K.

Corollary 3.10. — *With the notation of Proposition* 3.8*, let* $\pi_{N/P} : M/T \to N/P$ *be the mapping* Φ_p^{-1} *followed by the projection from* $\Delta \times (N/P)$ *onto the second factor. Let* $\iota_p : N/P \to M/T$ *be defined by* $\iota_p(\zeta + P) = p + \zeta$*. Then we have the following conclusions.*

For each nonnegative integer k*, the mapping* $\pi_{N/P}^* : \mathrm{H}^k(N/P, \mathbb{R}) \to \mathrm{H}^k(M/T, \mathbb{R})$ *is an isomorphism, with inverse equal to* ι_p^**.*

The mapping which assigns to any $\lambda \in \Lambda^k N^*$ *the cohomology class of the constant* k*-form* λ *on* N/P *is an isomorphism from* $\Lambda^k N^*$ *onto* $\mathrm{H}^k(N/P, \mathbb{R})$*, and every closed* k*-form on* N/P *is cohomologous to its average over the torus* N/P*.*

Proof. — The first statement follows because Δ is a convex subset of $\mathfrak{t}^*$ and hence it is contractible. The second statement is a well-known characterization of the cohomology of tori. The fact that a closed differential form on a compact connected Lie group is cohomologous to its average goes back to Élie Cartan [9]. □

Any finite-dimensional vector space W carries a positive translation-invariant measure m, which is unique up to a positive factor. For any non-negligible compact subset A of W, the *center of mass of* A is defined as

$$\int_A x m(\mathrm{d}x)/m(A) \in W,$$

which is independent of the choice of the positive translation-invariant measure m on W.

Corollary 3.11. — *Let* $X \in \mathfrak{t}$*. Then* X_M *is Hamiltonian if and only if* $X \in \mathfrak{t}_{\mathrm{h}}$*. Furthermore, the image of any momentum mapping of the Hamiltonian action of* T_{h} *on* M *is equal to a translate of the Delzant polytope* Δ *in Proposition* 3.8*, where we note that any two momentum mappings for the same torus action differ by a constant element of* ${\mathfrak{t}_{\mathrm{h}}}^*$*. The translational ambiguity of* Δ *can be removed by putting the center of mass of* Δ *at the origin. Here a momentum mapping for the Hamiltonian action of* T_{h} *is a smooth* ${\mathfrak{t}_{\mathrm{h}}}^*$*-valued function* μ *on* M *such that for every* $X \in \mathfrak{t}_{\mathrm{h}}$ *the* X*-component of* $\mathrm{d}\mu$ *is equal to* $-\,\mathrm{i}_{X_M}\,\sigma$*.*

Proof. — It follows from Corollary 3.2 and Corollary 3.10 that the vector field X_M is Hamiltonian if and only if $[\widehat{\sigma}(X)] = 0$ if and only if $[\iota_p^*(\widehat{\sigma}(X))] =$

$\iota_p^*[\widehat{\sigma}(X)] = 0$. Now constant one-forms on N/P are canonically identified with linear forms on $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, which are identified with elements of $\mathfrak{l}/\mathfrak{t}_{\mathrm{h}}$. With this identification, $\iota_p^*(\widehat{\sigma}(X))$ corresponds to $X + \mathfrak{t}_{\mathrm{h}}$, which is equal to zero if and only if $X \in \mathfrak{t}_{\mathrm{h}}$.

The second statement in the corollary follows from the fact that if μ is a momentum mapping for the Hamiltonian T_{h}-action, then $\mathrm{d}(\mu(X)) = \widehat{\sigma}(X)$ for every $X \in \mathfrak{t}_{\mathrm{h}}$. In other words, μ differs from the $\mathfrak{t}_{\mathrm{h}}$-component of any canonical local chart on M/T by a constant vector in ${\mathfrak{t}_{\mathrm{h}}}^*$. Therefore the image of μ corresponds to $\Delta \simeq (M/T)/N$, the orbit space of the translational N-action on M/T. Here we use that restriction to $\mathfrak{t}_{\mathrm{h}}$ of linear forms on $\mathfrak{l}$ leads to a canonical identification of $\mathfrak{l}^*/(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ with ${\mathfrak{t}_{\mathrm{h}}}^*$. □

McDuff [38] proved that a symplectic circle action on a four-dimensional compact connected symplectic manifold is Hamiltonian, if and only if it has a fixed point, but that in higher dimensions there exist non-Hamiltonian symplectic circle actions with fixed points. Corollary 3.11 follows from [38] if $\dim M = 4$, but not if $\dim M = 2n > 4$. Our proof of Corollary 3.11 uses in an essential way that X_M is an infinitesimal action of a symplectic action of an n-dimensional torus with a Lagrangian orbit.

Remark 3.12. — Because a Hamiltonian torus action has fixed points, it follows from Corollary 3.11 that the action of T_{h} on M has fixed points, that is, there exist $x \in M$ such that $T_{\mathrm{h}} \subset T_x$, hence $T_{\mathrm{h}} = T_x$ because the definition of T_{h} in Lemma 3.6 implies that $T_x \subset T_{\mathrm{h}}$ for every $x \in M$. In other words, T_{h} can also be characterized as the unique maximal stabilizer subgroup of T.

Actually the fixed points in M for the action of T_{h} are the $x \in M$ such that $\mu(x)$ is a vertex of the Delzant polytope Δ, where $\mu : M \to \Delta \subset {\mathfrak{t}_{\mathrm{h}}}^*$ denotes the momentum map of the Hamiltonian T_{h}-action.

Remark 3.13. — Let $\mathfrak{t}_{\mathrm{h}} \neq \mathfrak{t}$. It follows from Lemma 3.6 that for every $X \in \mathfrak{t} \setminus \mathfrak{t}_{\mathrm{h}}$ the vector field X_M has no zeros in M, and we conclude that the Euler characteristic $\chi(M)$ of M is equal to zero.

Furthermore the localization formula of Berline-Vergne and Atiyah-Bott in equivariant cohomology, in the form of [12, (4.13)], yields for every T-equivariantly closed T-equivariant differential form ω on M that the integral of ω over M is equal to zero, when evaluated at $X \in \mathfrak{t} \setminus \mathfrak{t}_{\mathrm{h}}$. Because $\mathfrak{t} \setminus \mathfrak{t}_{\mathrm{h}}$ is dense in $\mathfrak{t}$, it follows that the integral over M of each T-equivariantly closed T-equivariant differential form is identically equal to zero. If $X \in \mathfrak{t}_{\mathrm{h}}$, then Corollary 3.11 implies that X_M is Hamiltonian, and the zeros of X_M are the critical points of its Hamiltonian function, which form a non-empty subset of M. In this case the localization formula [12, (4.13)] yields that the sum over the connected components F of the zeroset of X_M of the integrals over F of $\omega(X)/\varepsilon(X)$ is equal to zero. The generalization of Ginzburg [18, Th. 6.1] of the Duistermaat-Heckman formula is related to these observations. On the other hand the integral over M of a T_{h}-equivariantly closed T_{h}-equivariant differential form, such as $\mathfrak{t}_{\mathrm{h}} \ni X \mapsto \mathrm{e}^{\mathrm{i}(\mu(X)-\sigma)}$, is usually nonzero.

If $\mathfrak{t}_{\mathrm{h}} = \mathfrak{t}$, then it follows from Corollary 3.9 and Corollary 3.2 that (M, σ, T) is a Delzant manifold, and $\chi(M)$ is equal to the number of vertices of the Delzant polytope Δ. This can be proved by observing that for a generic $X \in \mathfrak{t}$ the momentum map is bijective from the zeroset of X_M to the set of vertices of Δ, and each zero of X_M has Poincaré index equal to one. See also Guillemin [20, Exerc. 4.15].

4. Two lemmas

The following lemmas will be used later in the paper. Lemma 4.1 is used in the proof of Proposition 5.5, whereas Lemma 4.2 is used in the proof of Lemma 5.2 and Lemma 7.1. The proofs of Lemma 4.1 and Lemma 7.1 are based on the local models of Lemma 2.10.

Throughout this section, (M, σ) is a symplectic manifold with an effective symplectic action of a torus T with coisotropic principal orbits.

LEMMA 4.1. — *Let X_j, $1 \leqslant j \leqslant \dim \mathfrak{l}$, be a basis of $\mathfrak{l}$. The basic k-forms on M are the k-forms*

$$\omega = \sum_{j_1 < \ldots < j_k} f_{j_1, \ldots, j_k} \widehat{\sigma}(X_{j_1}) \wedge \ldots \wedge \widehat{\sigma}(X_{j_k}) \tag{4.1}$$

in which $f_{j_1, \ldots, j_k} \in \mathrm{C}^\infty(M)^T$ and $\widehat{\sigma}$ is defined as in Lemma 3.1.

Proof. — Because the one-forms $\widehat{\sigma}(X_j)$ are basic, any ω as in (4.1) with $f_{j_1, \ldots, j_k} \in \mathrm{C}^\infty(M)^T$ is a basic form.

Using partitions of unity with elements of $\mathrm{C}^\infty(M)^T$, it is sufficient to prove the converse statement in a local model as in Lemma 2.10. Let ω be a basic k-form. In the principal stratum where $\rho_j > 0$ for every $1 \leqslant j \leqslant m$, we have that

$$\omega = \sum_{l=0}^{k} \sum_{j_1 < \ldots < j_l} f^{k-l}_{j_1, \ldots, j_l}(\rho) \, \mathrm{d}\rho_{j_1} \wedge \ldots \wedge \mathrm{d}\rho_{j_l},$$

in which the $f^{k-l}_{j_1, \ldots, j_l}(\rho)$ are uniquely determined smooth $(k-l)$-forms on $(\mathfrak{l}/\mathfrak{h})^*$, depending smoothly on ρ. We are done if we can prove that the

$f^{k-l}_{j_1,\ldots,j_l}$ extend smoothly over the boundary where some of the ρ_j are equal to zero.

Recall that $\rho^j = ((p^j)^2 + (q^j)^2)/2$, if $z^j = p^j + \mathrm{i}\, q^j$ with $p^j, q^j \in \mathbb{R}$. Then $\mathrm{d}\rho_j = p^j\,\mathrm{d}p^j + q^j\,\mathrm{d}q^j$ shows that $\mathrm{d}\rho_{j_1} \wedge \ldots \wedge \mathrm{d}\rho_{j_l}$ has the component

$$\Big(\prod_{i=1}^{l} q^{j_i}\Big)\,\mathrm{d}q^{j_1} \wedge \ldots \wedge \mathrm{d}q^{j_l},$$

and therefore the smoothness of ω implies that for each $0 \leqslant l \leqslant k$ and each sequence $j_1, \ldots, j_l$ with $j_1 < \ldots < j_l$ the form

$$\Big(\prod_{i=1}^{l} q^{j_i}\Big) f^{k-l}_{j_1,\ldots,j_l}(\rho)$$

depends smoothly on (p,q). Applying the differential operator $\partial^l/\partial q^{j_1} \ldots \partial q^{j_l}$ and putting $q = 0$, we obtain that

$$f^{k-l}_{j_1,\ldots,j_l}((p^1)^2/2, \ldots, (p^l)^2/2)$$

depends smoothly on p, and moreover is invariant under each of the reflections $p^j \mapsto -p^j$. Whitney [53] proved that this implies that the function $f^{k-l}_{j_1,\ldots,j_l}$ extends smoothly over the boundary where some of the ρ_j are equal to zero. $\square$

For any smooth mapping f from a smooth manifold M to a smooth manifold N, the tangent mapping $\mathrm{T}_x f$ is the linear mapping from $\mathrm{T}_x\, M$ to $\mathrm{T}_{f(x)}\, N$ which in local coordinates corresponds to the Jacobi matrix of f at the point x.

LEMMA 4.2. — *Let $\Phi : M \to M$ be a T-equivariant diffeomorphism which preserves the T-orbits. Then there is a unique smooth T-invariant mapping $\tau : M \to T$ such that $\Phi(x) = \tau(x) \cdot x$ for every $x \in M$.*

If Φ preserves the symplectic form σ, then $(\mathrm{T}_x \tau)(v) \in \mathfrak{l}$ for each $x \in M$ and $v \in \mathrm{T}_x M$. Here $\mathfrak{l}$ is the kernel of the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ introduced in Lemma 2.1, and we identify each tangent space of T with $\mathfrak{t}$.

Proof. — The first statement has been proved for arbitrary torus actions on orbifolds by Haefliger and Salem [25, Th. 3.1], but in our case the proof is elementary. The statement is obvious if we replace M by the set M_{reg} on which the action is free and defines a principal T-fibration, and it remains to be proved that τ has a smooth extension to M. In the local model of Lemma 2.10, we have

$$\Phi : (k, \lambda, z) \mapsto (\tau(\lambda, \rho)_K k, \lambda, \iota(\tau(\lambda, \rho)_H) \cdot z).$$

The smoothness of Φ implies that $(\lambda, \rho) \mapsto \tau(\lambda, \rho)_K$ has a smooth extension. Write $\widehat{\tau}(\lambda, \rho) = \iota(\tau(\lambda, \rho)_H) \in \mathbb{T}^m$. It remains to prove that the fact that $\Psi : (\lambda, z) \mapsto \widehat{\tau}(\lambda, \rho) \cdot z$ has a smooth extension, implies that $\widehat{\tau}$ has a smooth extension, because the fact that $\iota : H \to \mathbb{T}^m$ is an isomorphism of Lie groups then implies that τ_H has a smooth extension.

Now the function

$$f^j(\lambda, z) := \Psi(\lambda, z)^j \overline{z^j} = \widehat{\tau}(\lambda, \rho)^j |z^j|^2$$

has a smooth extension, of which the restriction to the "real domain" $q = 0$ is an even function in each of the variables p^j. It therefore follows from Whitney [53] that there is a smooth function g^j such that $f^j(\lambda, z) = g^j(\lambda, \rho)$. However $g^j(\lambda, \rho) = 0$ when $\rho_j = 0$, and it follows that

$$g^j(\lambda, \rho) = \int_0^1 \partial g^j(\lambda, \rho_1, \ldots, t\rho_j, \ldots, \rho_n)/\partial t \, \mathrm{d}t = h^j(\lambda, \rho)\rho_j,$$

in which

$$(\lambda, \rho) \mapsto h^j(\lambda, \rho) = \int_0^1 \partial g^j(\lambda, \rho_1, \ldots, r_j, \ldots, \rho_n)/\partial r_j\big|_{r_j = t\rho_j} \, \mathrm{d}t$$

is smooth. Because

$$h^j(\lambda, \rho)\rho_j = g^j(\lambda, \rho) = f^j(\lambda, z) = 2\widehat{\tau}(\lambda, \rho)^j \rho_j,$$

it follows that $\widehat{\tau} = h^j/2$ when $\rho_j > 0$, which extends smoothly over the boundary $\rho_j = 0$.

Write, for each $x \in M$, $\tau_x' := \mathrm{T}_x\tau$, viewed as a linear mapping from T_xM to $\mathfrak{t}$, and $\tau(x)_M{}' := \mathrm{T}_x(\tau(x)_M)$, which is a symplectic linear mapping from T_xM to $\mathrm{T}_{\Phi(x)}M$. Then it follows from the sum rule for differentiation of an expression in which a variable occurs at several places, that

$$(\mathrm{T}_x\Phi)v = \tau(x)_M{}'v + (\tau_x'v)_M(\Phi(x)), \quad v \in \mathrm{T}_xM. \tag{4.2}$$

If $X \in \mathfrak{t}$, then the T-equivariance of Φ implies that $(\mathrm{T}_x\Phi)X_M(x) = X_M(\Phi(x))$. On the other hand, the commutativity of T implies that $\tau(x)_M(t \cdot x) = t \cdot \tau(x)_M(x) = t \cdot \Phi(x)$ for every $t \in T$, and differentiating this with respect to t at $t = 1$ in the direction of X, we obtain $\tau(x)_M{}'(X_M(x)) = X_M(\Phi(x))$. The condition $\sigma = \Phi^*\sigma$ implies that we have, for every $x \in M$, $v \in \mathrm{T}_xM$, and $X \in \mathfrak{t}$,

$$\begin{aligned}
\sigma_x(v, X_M(x)) &= \sigma_{\Phi(x)}((\mathrm{T}_x\Phi)v, (\mathrm{T}_x\Phi)X_M(x)) \\
&= \sigma_{\Phi(x)}(\tau(x)_M{}'v + (\tau_x'v)_M(\Phi(x)), X_M(\Phi(x))) \\
&= \sigma_{\Phi(x)}(\tau(x)_M{}'v, \tau(x)_M{}'X_M(x)) + \sigma^{\mathfrak{t}}(\tau_x'v, X),
\end{aligned}$$

which implies that $\sigma^{\mathfrak{t}}(\tau_x' v, X) = 0$ because $\tau(x)_M{}'$ is symplectic. Because $\sigma^{\mathfrak{t}}(X, \tau_x' v) = 0$ for every $X \in \mathfrak{t}$, it follows that $\tau_x' v \in \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$. $\square$

Remark 4.3. — One can prove that Φ is a T-equivariant symplectomorphism of (M, σ) which preserves the T-orbits, if and only if for every $x \in M$ there exists a T-invariant open neighborhood U of x in M, a T-invariant smooth function f on U, and an element $t \in T$, such that $\Phi = \mathrm{e}^{\mathrm{Ham}_f} \circ t_M$ on U. The "if" part follows from Remark 2.5.

5. Lifts

Let (M, σ) be our compact connected symplectic manifold, together with an effective action of the torus T by means of symplectomorphisms of (M, σ), such that some (all) principal orbits of the T-action are coisotropic submanifolds of (M, σ).

If we identify each of the tangent spaces of $(M/T)_{\mathrm{reg}}$ with $\mathfrak{l}^*$ as in Remark 3.7, then any $\xi \in \mathfrak{l}^*$ can be viewed as a constant vector field on $(M/T)_{\mathrm{reg}}$. A vector field L_ξ in M_{reg} is called a *lift of* ξ, if $\mathrm{T}_x \pi(L_\xi(x)) = \xi$ for all $x \in M_{\mathrm{reg}}$. Here the tangent mapping $\mathrm{T}_x\, \pi : \mathrm{T}_x M_{\mathrm{reg}} \to \mathrm{T}_{\pi(x)}(M/T)_{\mathrm{reg}}$ of π is identified with the linear mapping $\widehat{\sigma}_x : \mathrm{T}_x M \to \mathfrak{l}^*$, defined by the $\mathfrak{l}^*$-valued one-form $\widehat{\sigma}$ on M. In view of the definition of $\widehat{\sigma}$ in Lemma 3.1 and (3.1), the condition that L_ξ is a lift of ξ therefore is equivalent to

$$\sigma(L_\xi, X_M) = \xi(X), \quad \xi \in \mathfrak{l}^*, \quad X \in \mathfrak{l}. \tag{5.1}$$

If L_ξ, $\xi \in \mathfrak{l}^*$, is a family of smooth T-invariant vector fields on M_{reg}, which depends linearly on ξ and are lifts in the sense of (5.1), then for each $x \in M_{\mathrm{reg}}$ the vectors $L_\xi(x)$, $\xi \in \mathfrak{l}^*$, span a linear subspace H_x of $\mathrm{T}_x M$ which is complementary to the tangent space $\mathfrak{t}_M(x)$ at x of the orbit $T \cdot x$. The H_x, $x \in M_{\mathrm{reg}}$, are the horizontal spaces for a unique T-invariant *infinitesimal connection* ∇ *for the principal T-bundle* $\pi : M_{\mathrm{reg}} \to (M/T)_{\mathrm{reg}}$. This connection is T-invariant, if and only if each of the lifts L_ξ, $\xi \in \mathfrak{l}^*$, is T-invariant.

Conversely, if we have given a T-invariant infinitesimal connection ∇ for the principal T-fibration in M_{reg}, with horizontal spaces $H_x = H_x^\nabla$, $x \in M_{\mathrm{reg}}$, then we have for each $\xi \in \mathfrak{l}^*$ a unique lift L_ξ of ξ such that $L_\xi(x) \in H_x$ for every $x \in M_{\mathrm{reg}}$. L_ξ is called the *horizontal lift of* ξ *defined by the connection* ∇, and denoted by $\xi_{\mathrm{hor}}^\nabla$ in the literature on connections. Because the mapping $\xi \mapsto \xi_{\mathrm{hor}}^\nabla$ is linear, "lifts L_ξ which depend linearly on ξ" and "connections" are equivalent objects. We will use the somewhat simpler notation L_ξ instead of $\xi_{\mathrm{hor}}^{\nabla}$, because it is the lifts which we will be using to construct our global model.

In this section we construct lifts L_ξ, $\xi \in \mathfrak{l}^*$, depending linearly on ξ, which are admissible in the sense of Definition 5.3, and have Lie brackets and symplectic products which are as simple as we can get them. See Proposition 5.5 below. This construction is based on a computation in the cohomology of the closed basic differential forms on M, which according to the theorem of Koszul [32] is canonically isomorphic to the sheaf (= Čech) cohomology of the orbit space M/T with values in $\mathbb{R}$. The lifts in Proposition 5.5 form the core of the construction of the model for the symplectic T-manifold (M, σ, T), given in Proposition 7.2 and Proposition 7.4.

5.1. Admissible connections

Definition 5.1. — *Let, in the local model of Lemma 2.10 with the diffeomorphism Φ, the lift L_ξ be equal to the image under* $\mathrm{T}\Phi$ *of the vector field $(X_\xi, \delta\lambda_\xi, \delta z_\xi)$. Then, in terms of the (θ, ρ)-coordinates in $\mathbb{C}^m$, we obtain in view of (2.3) and (3.5) that the equation (5.1) is equivalent to $A(\delta\lambda_\xi, \delta\rho_\xi) = \xi$. Let $(\delta\lambda_\xi, \delta\rho_\xi) = A^{-1}(\xi)$, and let L_ξ^{Φ} be the image under* $\mathrm{T}\Phi$ *of the "constant" vector field $(0, \delta\lambda_\xi, (0, \delta\rho_\xi))$, where we use the (θ, ρ)-coordinates in $\mathbb{C}^m$. Then L_ξ^{Φ} is a smooth T-invariant vector field on $U \cap M_{\mathrm{reg}}$, and a lift of ξ. We call L_ξ^{Φ} the* local model lift defined by the local model with the diffeomorphism Φ.

The local model lift L_ξ^{Φ} extends to a smooth T-invariant vector field on U when $\delta\rho_\xi = 0$, that is, when $\xi = 0$ on $\mathfrak{h}$. On the other hand, if we write $r_j = |z_j|$, then $\partial/\partial\rho_j = (1/r_j)\partial/\partial r_j$. This shows that L_ξ^{Φ} has a pole singularity at any point (k, λ, z) for which there exists a $1 \leqslant j \leqslant m$ such that $z^j = 0$ and $\xi(X_j) \neq 0$.

Lemma 5.2. — *Let $\widetilde{\Phi} : \widetilde{K} \times \widetilde{E}_0 \to \widetilde{U}$ be another local model as in Lemma 2.10, where we use the same projection $X \mapsto X_{\mathfrak{l}} : \mathfrak{t} \to \mathfrak{l}$. Then there is a smooth T-invariant mapping $\alpha : U \cap \widetilde{U} \to \mathfrak{l}$, such that $L_\xi^{\widetilde{\Phi}}(x) = L_\xi^{\Phi}(x) + \alpha(x)_M(x)$ for every $x \in U \cap \widetilde{U} \cap M_{\mathrm{reg}}$. Here L_ξ^{Φ} and $L_\xi^{\widetilde{\Phi}}$ are the local model lifts Definition 5.1.*

Proof. — Let $x_0 \in U \cap \widetilde{U}$ and write $(k_0, \lambda_0, (\theta_0, \rho_0)) = \Phi^{-1}(x_0)$, where we use the (θ, ρ)-coordinates in $\mathbb{C}^m$. By permuting the coordinates in $\mathbb{C}^m$, we can arrange that $(\rho_0)_j = 0$ for $1 \leqslant j \leqslant m_0$ and $(\rho_0)_j > 0$ for $m_0 < j \leqslant m$. Then $H_0 := T_{x_0}$ is equal to the subgroup $\iota^{-1}(\mathbb{T}^{m_0} \times \{1\})$ of H, where ι

denotes the isomorphism from H onto $\mathbb{T}^m$, introduced in Remark 2.7. Here $H = T_x$ as in Lemma 2.10. Let $H_0' := \iota^{-1}(\{1\} \times \mathbb{T}^{m-m_0})$. Then H_0' is a complementary subtorus to H_0 in H, and $K_0 := H_0' K$ is a complementary subtorus to H_0 in T which contains K. Let (θ', ρ') and (θ'', ρ'') be the first m_0 and the last $m - m_0$ of the (θ, ρ)-coordinates, respectively. Then the rotation of z^j over $(\theta'')^j$, for each $m_0 < j \leqslant m$, defines an element $R(\theta'')$ of $\{1\} \times \mathbb{T}^{m-m_0}$, and $\iota^{-1}(R(\theta'')) \in H_0'$.

On the other hand

$$\Lambda_0(\rho'') : X \mapsto \sum_{j=m_0+1}^{m} \rho_j \iota(X)/\mathrm{i}$$

is a linear form on $\mathfrak{h}$ which is equal to zero on the Lie algebra $\mathfrak{h}_0$ of H_0. This linear form has a unique extension to a linear form $\Lambda(\rho'')$ on $\mathfrak{l}$ which is equal to zero on $\mathfrak{l} \cap \mathfrak{k}$. In this way we obtain an element $\Lambda(\rho'') \in (\mathfrak{l}/\mathfrak{h}_0)^*$. A straightforward computation shows that the mapping

$$\Psi : (k, \lambda, (\theta, \rho)) \mapsto (\iota^{-1}(R(\theta''))k, \lambda - \lambda_0 + \Lambda(\rho'' - \rho_0''), (\theta', \rho'))$$

when restricted to the the domain where $\rho_j > 0$ for all $m_0 < j \leqslant m$, defines a smooth T-equivariant symplectomorphism from $K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$ to $K_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$. Moreover, $\Psi \circ \Phi^{-1}(x_0)$ belongs to the T-orbit of $(1, 0, 0)$ in $K_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$. Because the tangent mapping of Ψ maps $(0, \delta\lambda, (0, \delta\rho))$ to $(0, \delta\lambda + \Lambda(\delta\rho''), (0, \delta\rho'))$, we have $L_\xi^{\Phi \circ \Psi^{-1}} = L_\xi^{\Phi}$.

Similarly we have a smooth T-equivariant symplectomorphism $\widetilde{\Psi}$ from a T-invariant open neighborhood of $\widetilde{\Phi}^{-1}(x_0)$ in $\widetilde{K} \times (\mathfrak{l}/\widetilde{\mathfrak{h}})^* \times \mathbb{C}^{\widetilde{m}}$ onto a T-invariant open neighborhood of $(1, 0, 0)$ in $\widetilde{K}_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$, such that $\widetilde{\Psi} \circ \widetilde{\Phi}^{-1}(x_0) \in T \cdot (1, 0, 0)$ and $L_\xi^{\widetilde{\Phi} \circ \widetilde{\Psi}^{-1}} = L_\xi^{\widetilde{\Phi}}$. Here $\widetilde{K}_0$ is another complementary subtorus to H_0 in T.

The mapping

$$\Xi : (k, \lambda, z) \mapsto (k_{\widetilde{K}}, \lambda, \iota(k_{H_0}^{\widetilde{K}_0}) \cdot z) : K_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0} \to \widetilde{K}_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$$

is a T-equivariant symplectomorphism which maps $(1, 0, 0)$ to $(1, 0, 0)$. Here we have written, for each $k \in K_0$, $k = k_{\widetilde{K}_0} k_{H_0}^{\widetilde{K}_0}$ with $k_{H_0}^{\widetilde{K}_0} \in H_0$ and $k_{\widetilde{K}_0} \in \widetilde{K}_0$. Because $h = k_{H_0}^{\widetilde{K}_0}$ is the unique element in H_0 such that $k_{\widetilde{K}_0} := kh^{-1} \in \widetilde{K}_0$, the fact that Ξ is a T-equivariant symplectomorphism follows from the proof of Lemma 2.10, with (H, K) replaced by (H_0, K_0) and by $(H_0, \widetilde{K}_0)$, respectively. Because the tangent mapping of Ξ maps $(0, \delta\lambda, \delta z)$ to $(0, \delta\lambda, \delta z)$, we have that

$$L_\xi^{\widetilde{\Phi} \circ \widetilde{\Psi}^{-1} \circ \Xi} = L_\xi^{\widetilde{\Phi} \circ \widetilde{\Psi}^{-1}} = L_\xi^{\widetilde{\Phi}}.$$

The mapping $\Theta := \Psi \circ \Phi^{-1} \circ \widetilde{\Phi} \circ \widetilde{\Psi}^{-1} \circ \Xi$ is a smooth T-equivariant symplectomorphism from an open T-invariant neighborhood of $(1,0,0)$ in $K_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$, onto an open T-invariant neighborhood of $(1,0,0)$ in $K_0 \times (\mathfrak{l}/\mathfrak{h}_0)^* \times \mathbb{C}^{m_0}$, which preserves the T-orbit of $(1,0,0)$. Recall the $\mathfrak{l}^*$-valued one form $\widehat{\sigma}$ defined in Lemma 3.1, which we used to identify all tangent spaces of the orbit space with $\mathfrak{l}^*$. Because every T-equivariant symplectomorphism preserves $\widehat{\sigma}$, its induced transformation of the orbit space has derivative equal to the identity at every point. Therefore Θ is a translation on each connected open subset of the T-orbit space by means of a constant element v of $\mathfrak{l}^*$. Because Θ preserves the T-orbit of $(1,0,0)$, we have $v = 0$ on the connected component of $(1,0,0)$ of the domain of definition Υ of Θ. That is, Θ preserves all the T-orbits in a T-invariant open neighborhood of $(1,0,0)$.

It now follows from Lemma 4.2 that there there is a smooth T-invariant $\mathfrak{l}$-valued function τ on Υ, such that $\Theta(\upsilon) = \tau(\upsilon) \cdot \upsilon$ and $\mathrm{T}_\upsilon \tau(\delta\upsilon) \in \mathfrak{l}$ for every $\upsilon \in \Upsilon$ and $\delta\upsilon \in \mathrm{T}_\upsilon \Upsilon$. It follows from (4.2), with Φ and υ replaced by Θ and $\delta\upsilon := (0, \delta\lambda, \delta z)$, respectively, where $\delta\theta = 0$ and $(\delta\lambda, \delta\rho) = A^{-1}\xi$, that $\mathrm{T}\Theta$ maps the vector field $(0, \delta\lambda, \delta z)$ to the sum of $(0, \delta\lambda, \delta z)$ and

$$((\tau_\upsilon' \delta\upsilon)_{\mathfrak{k}_0}, 0, \iota((\tau_\upsilon' \delta\upsilon)_{\mathfrak{h}_0}).$$

Because $\widetilde{\Phi} \circ \widetilde{\Psi}^{-1} \circ \Xi = \Phi \circ \Psi^{-1} \circ \Theta$, $L_\xi^{\widetilde{\Phi} \circ \widetilde{\Psi}^{-1} \circ \Xi} = L_\xi^{\widetilde{\Phi}}$, and $L_\xi^{\Phi \circ \Psi^{-1}} = L_\xi^{\Phi}$, the conclusion of the lemma follows with $\alpha(\widetilde{x}) = \tau_\upsilon' \delta\upsilon$ if $\widetilde{x} = \Phi \circ \Psi^{-1}(\upsilon)$. $\square$

Definition 5.3. — *We use the atlas of local models as in Lemma 2.10, with a fixed linear projection $X \mapsto X_{\mathfrak{l}}$ from $\mathfrak{t}$ onto $\mathfrak{l}$. For every $\xi \in \mathfrak{l}^*$, an* admissible lift of ξ *is a smooth T-invariant vector field L_ξ on M_{reg} such that for each local model as in Lemma 2.10 there is a smooth T-invariant $\mathfrak{l}$-valued function α_ξ on U, such that $L_\xi(x) = L_\xi^{\Phi}(x) + \alpha_\xi(x)_M(x)$ for every $x \in U$. Here L_ξ^{Φ} is the local model lift introduced in Definition 5.1.*

If we are at an orbit type Σ with stabilizer group H, and ξ is equal to zero on the Lie algebra $\mathfrak{h} \subset \mathfrak{l}$ of H, then L_ξ has a unique smooth T-invariant extension to an open neighborhood of Σ in M, which will also be denoted by L_ξ. In particular, if $\zeta \in N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, the space of linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{t}_{\mathrm{h}}$, then L_ζ is a smooth T-invariant vector field on the whole manifold M.

An admissible connection *for the principal T-bundle $\pi : M_{\mathrm{reg}} \to (M/T)_{\mathrm{reg}}$ is a linear mapping $\xi \mapsto L_\xi$ from $\mathfrak{l}^*$ to the space of smooth vector fields on M_{reg}, such that, for each $\xi \in \mathfrak{l}^*$, L_ξ is an admissible lift of ξ. Because we work with a fixed action of the torus T, we will just write "admissible connection" in the sequel.*

In the literature, the term "admissible connection" has been used in various different frameworks and with correspondingly different meanings. Our usage of the term "admissible connection" continues this.

LEMMA 5.4. — *There exist admissible connections $\xi \mapsto L_\xi$. For each admissible connection $\xi \mapsto L_\xi$, we have*

$$\sigma(L_\xi, X_M) = \xi(X_{\mathfrak{l}}), \quad \xi \in \mathfrak{l}^*, \quad X \in \mathfrak{t}. \tag{5.2}$$

Proof. — If we piece the local model lifts L_ξ^Φ, introduced in Definition 5.1, together by means of a partition of unity consisting of smooth T-invariant functions with supports in the local model neighborhoods U, then it follows from Lemma 5.2 that the resulting connection is admissible.

In the local model of Lemma 2.10, where we use the (θ, ρ)-coordinates in $\mathbb{C}^m$ as in (3.2), (3.5), we have $L_\xi^\Phi = (0, \delta\lambda_\xi, (0, \delta\rho_\xi))$ with $(\delta\lambda_\xi, \delta\rho_\xi) = A^{-1}(\xi)$. Furthermore $X \cdot (k, \lambda, z) = (X_{\mathfrak{k}}, 0, (\iota(X_{\mathfrak{h}})/\,\mathrm{i}, 0))$. It follows that

$$\sigma(L_\xi^\Phi, X_M) = \delta\lambda_\xi((X_{\mathfrak{k}})_{\mathfrak{l}}) + \sum_{j=1}^m \iota(X_{\mathfrak{h}})^j (\delta\rho_\xi)_j/\,\mathrm{i} = \xi(X_{\mathfrak{l}}).$$

Here we have used (3.3) with Y replaced by X. On the other hand

$$\sigma_x(\alpha_\xi(x)_M, X_M(x)) = \sigma^{\mathfrak{t}}(\alpha_\xi(x), X) = 0$$

for every $X \in \mathfrak{t}$ if $\alpha_\xi(x) \in \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$, and (5.2) now follows from Definition 5.3. □

The equation (5.2) improves upon (5.1) if $\mathfrak{l}$ is a proper linear subspace of $\mathfrak{t}$, that is, if $\sigma^{\mathfrak{t}} \neq 0$. If the principal orbits are Lagrangian submanifolds of M, then $\mathfrak{l} = \mathfrak{t}$ and (5.2) is the same as (5.1).

5.2. Special admissible connections

Recall the Hamiltonian torus T_{h}, the unique maximal stabilizer subgroup T_{h} of T as in Remark 3.12 and Lemma 3.6, with Lie algebra $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l}$. In our quest for nice admissible lifts, we will use a decomposition of T into the subtorus T_{h} and a complementary subtorus T_{f}, as in Lemma 2.8 with $U = T_{\mathrm{h}}$. Note that the torus T_{f} acts freely on M, because if $x \in M$, then $T_x \subset T_{\mathrm{h}}$, hence $T_x \cap T_{\mathrm{f}} \subset T_{\mathrm{h}} \cap T_{\mathrm{f}} = \{1\}$. This explains our choice of the subscript f in T_{f}. Note also that the choice of a complementary subtorus T_{f} to T_{h} is far from unique if $\{1\} \neq T_{\mathrm{h}} \neq T$. See Remark 2.9. We will refer to T_{f} as *a freely acting complementary torus to the Hamiltonian torus* T_{h}.

If $\mathfrak{t}_{\mathrm{f}}$ denotes the Lie algebra of T_{f}, then we have a corresponding direct sum decomposition $\mathfrak{t} = \mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$ of Lie algebras. Each linear form on ${\mathfrak{t}_{\mathrm{h}}}^*$ has a unique extension to a linear form on $\mathfrak{l}$ which is equal to zero on $\mathfrak{t}_{\mathrm{f}}$. This leads to an isomorphism of ${\mathfrak{t}_{\mathrm{h}}}^*$ with the linear subspace $(\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^*$ of $\mathfrak{l}^*$. This isomorphism depends on the choice of the complementary freely acting torus T_{f} to the Hamiltonian torus in T. Note that the direct sum decomposition $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$ implies the direct sum decomposition

$$\mathfrak{l}^* = (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \oplus (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*. \tag{5.3}$$

Let

$$\mu : M \to \Delta \subset (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \simeq {\mathfrak{t}_{\mathrm{h}}}^* \tag{5.4}$$

denote the projection $\pi : M \to M/T$, followed by the projection from $M/T \simeq \Delta \times (N/P)$ onto the first factor. Here we use the isomorphism $\Phi_p : \Delta \times (N/P) \to M/T$ of Proposition 3.8, with $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ and $C = (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \simeq {\mathfrak{t}_{\mathrm{h}}}^*$. Note that $\mu : M \to {\mathfrak{t}_{\mathrm{h}}}^*$ is a momentum mapping for the Hamiltonian T_{h}-action on M as in Corollary 3.11.

With these notations, Proposition 5.5 below yields the existence of an admissible connection for which both the Lie brackets and the symplectic products of the L_ξ take an extremely simple form. (We are tempted to call such a connection a "minimal admissible connection", see also Subsection 8.1, but we do not have a proposal for a functional which is minimized exactly by the connections in Proposition 5.5.) In Remark 5.7 and Remark 5.8 we discuss the topological meaning of the antisymmetric bilinear form $c : N \times N \to \mathfrak{l}$. From these remarks it follows that c is unique. The freedom in the choice of the admissible connection in Proposition 5.5 will be described in Lemma 5.9.

Proposition 5.5. — *There exists an admissible connection $\mathfrak{l}^* \ni \xi \mapsto L_\xi$ as in Definition 5.3, and an antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$, with the following properties.*

i) $[L_\eta, L_{\eta'}] = 0$ *for all* $\eta, \eta' \in C$,
ii) $[L_\eta, L_\zeta] = 0$ *for all* $\eta \in C$ *and* $\zeta \in N$,
iii) $[L_\zeta, L_{\zeta'}] = c(\zeta, \zeta')_M$ *for all* $\zeta, \zeta' \in N$,
iv) $\sigma(L_\eta, L_{\eta'}) = 0$ *for all* $\eta, \eta' \in C$,
v) $\sigma(L_\eta, L_\zeta) = 0$ *for all* $\eta \in C$ *and* $\zeta \in N$, *and finally*
vi) $\sigma_x(L_\zeta(x), L_{\zeta'}(x)) = -\mu(x)(c_{\mathrm{h}}(\zeta, \zeta'))$ *for all* $\zeta, \zeta' \in N$ *and* $x \in M$. *Here* $c_{\mathrm{h}}(\zeta, \zeta')$ *denotes the* $\mathfrak{t}_{\mathrm{h}}$*-component of* $c(\zeta, \zeta')$ *in the direct sum decomposition* $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$.

The antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ in part iii) satisfies the relation

$$\zeta(c(\zeta', \zeta'')) + \zeta'(c(\zeta'', \zeta)) + \zeta''(c(\zeta, \zeta')) = 0 \tag{5.5}$$

for every $\zeta, \zeta', \zeta'' \in N$. Note that ζ is a linear form on $\mathfrak{l}$ which vanishes on $\mathfrak{t}_{\mathrm{h}}$, and therefore $\zeta(c(\zeta', \zeta''))$ is a real number which only depends on the projection of $c(\zeta, \zeta')$ to $\mathfrak{l}/\mathfrak{t}_{\mathrm{h}}$.

Proof. — We start with an arbitrary admissible connection $\mathfrak{l}^* \ni \xi \mapsto L_\xi$, which exists according to Lemma 5.4, and first simplify the Lie brackets.

We use the isomorphism $\Phi_p : \Delta \times (N/P) \to M/T$ of Proposition 3.8, in order to identify M/T with $\Delta \times (N/P) = (\Delta \times N)/P \subset \mathfrak{l}^*/P$. In view of Lemma 4.1, the smooth basic k-forms on M satisfy $\omega = \pi^*\nu$ on M_{reg} for uniquely determined smooth k-forms ν on $(\Delta_{\mathrm{reg}} \times N)/P$, such that ν extends to a smooth k-form on $\mathfrak{l}^*/P$. This leads to an identification of the space of all smooth basic k-forms on M with the space of all restrictions to $(\Delta \times N)/P$ of smooth k-forms on $\mathfrak{l}^*/P$. If we view $\xi \in \mathfrak{l}^*$ as a constant vector field on $\mathfrak{l}^*/P$, then the fact that $\mathrm{T}\pi$ maps L_ξ to ξ implies that $(\pi^*\nu)(L_{\xi^1}, \ldots, L_{\xi^k}) = \nu(\xi^1, \ldots, \xi^k)$. Because π intertwines the flow of L_ξ with the flow of the constant vector field ξ, we also have the identity $\mathrm{L}_{L_\xi}(\pi^*\nu) = \pi^*(\mathrm{L}_\xi\, \nu)$ for the Lie derivatives. In particular the differentiation of T-invariant smooth functions on M in the direction of the vector field L_ξ corresponds to the differentiation ∂_ξ of smooth functions on M/T in the direction of the constant vector field ξ.

Let X_i, $1 \leqslant i \leqslant d_{\mathfrak{l}} := \dim \mathfrak{l}$, be a basis of $\mathfrak{l}$. We will write $\alpha = \alpha^i X_i$, in which the real numbers α^i are the coordinates of $\alpha \in \mathfrak{l}$ with respect to this basis, and we use Einstein's summation convention when summing over indices which run from 1 up to $d_{\mathfrak{l}}$.

The local model lifts L_ξ^Φ and $L_{\xi'}^\Phi$, introduced in Definition 5.1, commute because they are constant vector fields on $K \times E_0$ in the local model of Lemma 2.10, where we use the (θ, ρ)-coordinates for z. If $L_\xi = L_\xi^\Phi + \alpha_\xi^i (X_i)_M$ as in Definition 5.3, then the fact that the vector fields L_ξ^Φ and $L_{\xi'}^\Phi$ commute as well as the vector fields $\alpha_\xi^i (X_i)_M$ and $\alpha_{\xi'}^j (X_j)_M$, implies that $[L_\xi, L_{\xi'}] = \beta_{\xi,\xi'}^i (X_i)_M$, in which the uniquely determined T-invariant functions $\beta_{\xi,\xi'}^i$ on $U = \Phi(K \times E_0)$ are given by

$$\beta_{\xi,\xi'}^i = L_\xi \alpha_{\xi'}^i - L_{\xi'} \alpha_\xi^i = \partial_\xi \alpha_{\xi'}^i - \partial_{\xi'} \alpha_\xi^i.$$

That is, $\beta = \mathrm{d}\alpha$ on U, if we define the basic $\mathfrak{l}$-valued one-form α and two-form β by $\alpha(\xi) = \alpha_\xi^i X_i$ and $\beta(\xi, \xi') = \beta_{\xi,\xi'}^i X_i$, respectively. Because β is locally exact, it follows that the globally defined smooth basic two-form β is closed.

Any other connection $\mathfrak{l}^* \ni \xi \to \widetilde{L}_\xi$ is admissible, if and only if

$$\widetilde{L}_\xi = L_\xi + \widetilde{\alpha}^i_\xi (X_i)_M, \tag{5.6}$$

in which $\widetilde{\alpha}(\xi) := \widetilde{\alpha}^i_\xi X_i$ defines a smooth basic $\mathfrak{l}^*$-valued one-form $\widetilde{\alpha}$ on M. With the same reasoning as above we obtain that $[\widetilde{L}_\xi, \widetilde{L}_{\xi'}] = \widetilde{\beta}^i_{\xi,\xi'}(X_i)_M$, in which $\widetilde{\beta}(\xi,\xi') := \widetilde{\beta}^i_{\xi,\xi'} X_i$ defines a smooth basic $\mathfrak{l}^*$-valued two-form $\widetilde{\beta}$, such that $\widetilde{\beta} = \beta + \mathrm{d}\widetilde{\alpha}$.

According to Corollary 3.10, the de Rham cohomology class of β contains a unique $c = \widetilde{\beta}$, such that $c(\xi,\xi') = 0$ when $\xi \in C$ or $\xi' \in C$, and for $\xi, \xi' \in N$ the $\mathfrak{l}$-valued function $c(\xi,\xi')$ is a constant, equal to the average of $\beta(\xi,\xi')$ over any N/P-orbit in M/T. This leads to the desired properties of the Lie brackets, where the uniqueness of c follows from the injectivity of the mapping $\Lambda^2 N^* \to \mathrm{H}^2(N/P, \mathbb{R})$ in Corollary 3.10.

We now turn to the symplectic inner products. Let $\xi, \xi' \in \mathfrak{l}^*$. It follows from Definition 5.1 that $\sigma(L^\Phi_\xi, L^\Phi_{\xi'}) = 0$. In view of Definition 5.3 and formula (5.2) we conclude that $\sigma(L_\xi, L_{\xi'})$ is a smooth function on M, which moreover is T-invariant. Therefore $(\xi,\xi') \mapsto \sigma(L_\xi, L_{\xi'})$ defines a smooth basic two-form s on M. If in (5.6) we take $\widetilde{\alpha}^i_\xi = \partial_\xi \varphi^i$ for $\varphi^i \in \mathrm{C}^\infty(M/T)$, that is, $\widetilde{\alpha} = \mathrm{d}\varphi$, then the Lie brackets do not change, but

$$\widetilde{s}(\xi,\xi') := \sigma(\widetilde{L}_\xi, \widetilde{L}_{\xi'}) = s(\xi,\xi') + \partial_{\xi'}(\varphi(\xi)) - \partial_\xi(\varphi(\xi')),$$

where we have used that $\sigma((X_i)_M, (X_j)_M) = \sigma^{\mathfrak{t}}(X_i, X_j) = 0$, because $X_i, X_j \in \mathfrak{l}$. This means that $\widetilde{s} = s - \mathrm{d}\varphi$.

In order to investigate the exterior derivative of s, we recall the identity

$$\begin{aligned}(\mathrm{d}\omega)(u,v,w) = {} & \partial_u(\omega(v,w)) + \partial_v(\omega(w,u)) + \partial_w(\omega(u,v)) \\ & + \omega(u,[v,w]) + \omega(v,[w,u]) + \omega(w,[u,v]),\end{aligned} \tag{5.7}$$

which holds for any smooth two-form ω and smooth vector fields u, v, w. It follows that

$$\begin{aligned}(\mathrm{d}s)(\xi,\xi',\xi'') &= \partial_\xi s(\xi',\xi'') + \partial_{\xi'} s(\xi'',\xi) + \partial_{\xi''} s(\xi,\xi') \\ &= L_\xi(\sigma(L_{\xi'}, L_{\xi''})) + L_{\xi'}(\sigma(L_{\xi''}, L_\xi)) + L_{\xi''}(\sigma(L_\xi, L_{\xi'})) \\ &= -\sigma(L_\xi, [L_{\xi'}, L_{\xi''}]) - \sigma(L_{\xi'}, [L_{\xi''}, L_\xi]) - \sigma(L_{\xi''}, [L_\xi, L_{\xi'}]) \\ &= -\xi(c(\xi',\xi'')) - \xi'(c(\xi'',\xi)) - \xi''(c(\xi,\xi')),\end{aligned}$$

where in the third identity we have used that $\mathrm{d}\sigma = 0$, and in the last identity we have inserted $[L_\xi, L_{\xi'}] = c(\xi,\xi')^i (X_i)_M$ and (5.1).

This shows that $\mathrm{d}s$ is constant. In the notation of Corollary 3.10, we have that $\mathrm{d}(\iota_p^* s) = \iota_p^*(\mathrm{d}s)$ is constant, and cohomologically equal to zero, which in view of the first part of the last statement in Corollary 3.10 implies that $\iota_p^*(\mathrm{d}s) = 0$. That is, $(\mathrm{d}s)(\xi, \xi', \xi'') = 0$ when $\xi, \xi', \xi'' \in N$, which in turn is equivalent to (5.5). On the other hand it follows from the already proved statements about the Lie brackets that $c(\xi, \xi') = 0$ if $\xi \in C$ or $\xi' \in C$, and hence $(\mathrm{d}s)(\xi, \xi', \xi'') = 0$ unless one of the vectors ξ, ξ', ξ'' belongs to C and the other two belong to N. Moreover, if $\xi \in C$ and $\xi', \xi'' \in N$, then we obtain that $(\mathrm{d}s)(\xi, \xi', \xi'') = -\xi(c(\xi', \xi''))$.

In other words, the smooth basic two-form $S := s + \mu c_{\mathrm{h}}$ is closed. Here μ is viewed as a $\mathfrak{t}_{\mathrm{h}}{}^*$-valued T-invariant function on M, and the pairing with the $\mathfrak{t}_{\mathrm{h}}$-valued antisymmetric bilinear form c_{h} yields a smooth basic two-form μc_{h} on M. According to Corollary 3.10, the smooth basic one-form φ can be now chosen such that if $\widetilde{S} = S - \mathrm{d}\varphi$, then $\widetilde{S}(\xi, \xi') = 0$ when $\xi \in C$ or $\xi' \in C$, and for $\xi, \xi' \in N$ the function $\widetilde{S}(\xi, \xi')$ is constant.

We finally observe that if $\alpha : \xi \mapsto \alpha_\xi$ is a linear mapping from $\mathfrak{l}^*$ to $\mathfrak{l}$, which is viewed as a constant, hence closed one-form on the $\mathfrak{l}^*$-parallel space M/T, then the Lie brackets of the L_ξ's do not change if we replace L_ξ by $L_\xi + (\alpha_\xi)_M$. However, $\sigma(L_\xi, L_{\xi'})$ then gets replaced by $\sigma(L_\xi, L_{\xi'}) + \xi(\alpha_{\xi'}) - \xi'(\alpha_\xi)$. Because any antisymmetric bilinear form on $\mathfrak{l}^*$ is of the form $(\xi, \xi') \mapsto \xi(\alpha_{\xi'}) - \xi'(\alpha_\xi)$, for a suitable linear mapping $\alpha : \mathfrak{l}^* \to \mathfrak{l}$, we can arrange that $\widetilde{S} = 0$, which leads to vi) in Proposition 5.5. □

Remark 5.6. — Because the left hand side of (5.5) is antisymmetric in ζ, ζ', ζ'', it is automatically equal to zero when $\dim N = \dim \mathfrak{l} - \dim \mathfrak{t}_{\mathrm{h}} \leqslant 2$. However, when $\dim N \geqslant 3$, then the equations (5.5) impose nontrivial conditions on the $\mathfrak{l}$-valued two-form c on N.

Remark 5.7. — For every $x \in M_{\mathrm{reg}}$, let H_x denote the linear span in $\mathrm{T}_x M$ of the vectors $L_\xi(x)$, $\xi \in \mathfrak{l}^*$. Then the H_x, $x \in M_{\mathrm{reg}}$, define a T-invariant infinitesimal connection of the principal T-bundle M_{reg} over $(M/T)_{\mathrm{reg}} \simeq \Delta^{\mathrm{int}} \times (N/P)$. Here Δ^{int} denotes the interior of the Delzant polytope Δ. Any connection of this principal T-bundle has a *curvature form* which is a smooth $\mathfrak{t}$-valued two-form on M_{reg}/T. The cohomology class of the curvature form is an element of $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$, which is independent of the choice of the connection. The action of N on M/T leaves $M_{\mathrm{reg}}/T \simeq (M/T)_{\mathrm{reg}}$ invariant, with orbits isomorphic to the torus N/P, and the pull-back to the N-orbits defines an isomorphism from $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ onto $\mathrm{H}^2(N/P, \mathfrak{t})$, which in turn is identified with $(\Lambda^2 N^*) \otimes \mathfrak{t}$ as in Corollary 3.10.

The proof of Proposition 5.5 shows that the element $c \in (\Lambda^2 N^*) \otimes \mathfrak{l} \subset (\Lambda^2 N^*) \otimes \mathfrak{t}$ is equal to the negative of the pull-back to an N-orbit of the cohomology class of the curvature form. This proves in particular that the antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ in Proposition 5.5 is independent of the choice of the freely acting complementary torus T_{f} to the Hamiltonian torus in T. More precisely, if $(\widetilde{M}, \widetilde{\sigma}, T)$ is another symplectic manifold with an effective symplectic T-action with coisotropic principal orbits, then Proposition 5.5 with (M, σ, T) replaced by $(\widetilde{M}, \widetilde{\sigma}, T)$ yields an antisymmetric bilinear mapping $\widetilde{c}$ instead of c. If there exists a T-equivariant symplectomorphism Φ from (M, σ, T) onto $(\widetilde{M}, \widetilde{\sigma}, T)$ then $\widetilde{c} = c$.

Remark 5.8. — The retrivializations of the principal T-bundle $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ define a one-cocycle of smooth $\mathfrak{t}$-valued functions on M_{reg}/T, of which the sheaf (= Čech) cohomology class τ in $\mathrm{H}^1(M_{\mathrm{reg}}/T, \mathrm{C}^\infty(\cdot, T))$ classifies the principal T-bundle $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$. Because the sheaf $\mathrm{C}^\infty(\cdot, \mathfrak{t})$ is fine, the short exact sequence

$$0 \to T_{\mathbb{Z}} \to \mathrm{C}^\infty(\cdot, \mathfrak{t}) \overset{\exp}{\to} \mathrm{C}^\infty(\cdot, T) \to 1$$

induces an isomorphism $\delta : \mathrm{H}^1(M_{\mathrm{reg}}/T, \mathrm{C}^\infty(\cdot, T)) \to \mathrm{H}^2(M_{\mathrm{reg}}/T, T_{\mathbb{Z}})$. Here exp denotes the exponential mapping $\mathfrak{t} \to T$. The cohomology class $\delta(\tau) \in \mathrm{H}^2(M_{\mathrm{reg}}/T, T_{\mathbb{Z}})$ is called the *Chern class of the principal T-bundle* $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$. It is a general fact, see for instance the arguments in [13, Sec. 15.3], that the image of $\delta(\tau)$ in $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ under the coefficient homomorphism $\mathrm{H}^2(M_{\mathrm{reg}}/T, T_{\mathbb{Z}}) \to \mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ is equal to the negative of the cohomology class of the curvature form of any connection in the principal T-bundle. In view of Remark 5.7, we therefore conclude that c represents the Chern class of the principal T-bundle $\pi : M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$.

In view of the canonical isomorphism between sheaf cohomology and singular cohomology, this implies that the integral of c over every two-cycle in $(M/T)_{\mathrm{reg}}$ belongs to $T_{\mathbb{Z}}$. If $\zeta, \zeta' \in P$, then for every $p \in (M/T)_{\mathrm{reg}}$ the mapping

$$\iota_{\zeta,\zeta'} : (t, t') \mapsto p + (t\zeta + t'\zeta') : \mathbb{R}^2/\mathbb{Z}^2 \to (M/T)_{\mathrm{reg}}$$

defines a two-cycle in $(M/T)_{\mathrm{reg}}$, and

$$c(\zeta, \zeta') = \int_{\mathbb{R}^2/\mathbb{Z}^2} (\iota_{\zeta,\zeta'})^* c.$$

It follows that $c(\zeta, \zeta') \in T_{\mathbb{Z}}$ for every $\zeta, \zeta' \in P$.

In Lemma 7.1, this conclusion will be proved by means of a group theoretical consideration. Other topological interpretations of c will be given in Proposition 8.2 and Proposition 8.1.

Let $\mathrm{Lin}(E, F)$ denote the space of all linear mappings from a vector space E to a vector space F. In the following lemma we use that a smooth T-invariant mapping $\alpha : M \to \mathrm{Lin}(\mathfrak{l}^*, \mathfrak{l})$ corresponds to a unique smooth basic $\mathfrak{l}$-valued one-form on M, which we also denote by α, such that $\alpha(x)(\xi) = \alpha_x(v)$ for every $v \in \mathrm{T}_x M$ such that $\widehat{\sigma}_x(v) = \xi$. In view of Lemma 4.1, α can also be viewed as the restriction to M/T of a smooth $\mathfrak{l}$-valued one-form on $\mathfrak{l}^*/P$, if we identify the $\mathfrak{l}^*$-parallel space M/T with a subset of $\mathfrak{l}^*/P$ as in Proposition 3.8.

Lemma 5.9. — *Let $\mathfrak{l}^* \ni \xi \mapsto L_\xi$ be a connection as in Proposition 5.5. Then $\mathfrak{l}^* \ni \xi \mapsto \widetilde{L}_\xi$ is an admissible connection, if and only if there exists a smooth T-invariant mapping $\alpha : x \mapsto (\xi \mapsto \alpha_\xi(x))$ from M to $\mathrm{Lin}(\mathfrak{l}^*, \mathfrak{l})$, such that $\widetilde{L}_\xi(x) = L_\xi(x) + \alpha_\xi(x)_M(x)$ for every $x \in M$ and $\xi \in \mathfrak{l}^*$. Proposition 5.5 holds with L replaced by $\widetilde{L}$, if and only if α is closed when considered as a smooth basic $\mathfrak{l}$-valued one-form on M, and moreover α is symmetric in the sense that*

$$\xi(\alpha_{\xi'}(x)) - \xi'(\alpha_\xi(x)) = 0 \tag{5.8}$$

for all $\xi, \xi' \in \mathfrak{l}^*$ and all $x \in M$.

Proof. — The first statement follows from Definition 5.3, the definition of admissible connections. It follows from the proof of Proposition 5.5 that $[\widetilde{L}_\xi, \widetilde{L}_{\xi'}] \equiv [L_\xi, L_{\xi'}]$ if and only if α is closed. In view of the uniqueness of c, see Remark 5.7, we have iv), v), vi) in Proposition 5.5 with L replaced by $\widetilde{L}$, if and only if $\sigma(\widetilde{L}_\xi, \widetilde{L}_{\xi'}) \equiv \sigma(L_\xi, L_{\xi'})$, which is equivalent to (5.8). □

6. Delzant submanifolds

Let (M, σ) be our compact connected symplectic manifold, together with an effective action of the torus T by means of symplectomorphisms of (M, σ), such that some (all) principal orbits of the T-action are coisotropic submanifolds of (M, σ).

Recall the especially nice admissible connection introduced in Proposition 5.5, the construction of which is based on the identification in Proposition 3.8 of the orbit space M/T with the $\mathfrak{l}^*$-parallel space $\Delta \times (N/P)$. Proposition 6.1 below implies that the vector fields Y_M, $Y \in \mathfrak{t}_{\mathrm{h}}$, and L_η, $\eta \in C$, are tangent to the fibers of a fibration of M by Delzant submanifolds. From this section on, the word *fibration* is short for a locally trivial smooth fiber bundle. The remainder of this section is devoted to the proof and further precision of Proposition 6.1. For any subset Y of a set X, the *inclusion mapping* ι_Y is the identity on Y, viewed as a mapping from Y to X.

Proposition 6.1. — *Let* $\mathfrak{l}^* \ni \xi \mapsto L_\xi$ *be an admissible connection as in Proposition 5.5. Then there is a unique smooth T-invariant distribution D on M such that, for every $x \in M_{\mathrm{reg}}$, D_x is equal to the linear span in $\mathrm{T}_x M$ of the vectors $Y_M(x)$ with $Y \in \mathfrak{t}_{\mathrm{h}}$ and $L_\eta(x)$, $\eta \in C := (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \simeq \mathfrak{t}_{\mathrm{h}}{}^*$.*

The distribution D is integrable. Each integral manifold manifold I of D is invariant under the action of the Hamiltonian torus T_{h}, and $(I, \iota_I{}^*\sigma, T_{\mathrm{h}})$ is a Delzant manifold with the Delzant polytope Δ introduced in Proposition 3.8. Here $\iota_I : I \to M$ is the inclusion mapping from I into M. The integral manifolds of D form a smooth fibration of M into Delzant submanifolds with Delzant polytope Δ.

Proof. — This follows from Lemma 6.3 below, which in turn uses Lemma 6.2. □

Lemma 6.2. — *Let $\pi_{N/P} : M \to N/P$ be the mapping which is equal to $\pi : M \to M/T$, followed by the inverse $M/T \to \Delta \times (N/P)$ of the isomorphism Φ_p in Proposition 3.8, iii), followed by the projection $\Delta \times (N/P) \to N/P$ onto the second factor.*

Then $\pi_{N/P} : M \to N/P$ defines a smooth fibration of M over the torus N/P. Each fiber F of $\pi_{N/P} : M \to N/P$ is a connected compact T-invariant smooth submanifold of M. For each fiber F of $\pi_{N/P} : M \to N/P$, $F \cap M_{\mathrm{reg}}$ is dense in F.

Proof. — Because $\pi : M_{\mathrm{reg}} \to (M/T)_{\mathrm{reg}}$ and the projection from $(M/T)_{\mathrm{reg}} \simeq \Delta_{\mathrm{reg}} \times (N/P)$ onto N/P are smooth fibrations with connected fibers, it follows that the restriction to M_{reg} of $\pi_{N/P}$ is a smooth fibration with connected fibers.

In the local model of Lemma 2.10, the mapping $\pi_{N/P}$ corresponds to the mapping

$$(k, \lambda, z) \mapsto \lambda_{(\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*} + P \in N/P,$$

where we have used the direct sum decomposition (5.3). This shows that $\pi_{N/P}$ is a smooth submersion. Moreover, for each fiber F of $\pi_{N/P} : M \to N/P$, $F \cap M_{\mathrm{reg}}$ is dense in F, because the point (k, λ, z) is regular if and only if $z^j \neq 0$ for every j. Because the fiber $F \cap M_{\mathrm{reg}}$ of the restriction to M_{reg} of $\pi_{N/P}$ is connected, it follows that F is connected. Because M is compact, the submersion $\pi_{N/P}$ is proper, and because every proper submersion is a fibration, it follows that $\pi_{N/P}$ is a fibration. □

As observed in the beginning of Subsection 5.2, the action on M of the complementary torus T_{f} to T_{h} is free. This exhibits each fiber F of $\pi_{N/P}$

as a principal T_{f}-bundle $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$, in which the T_{f}-orbit space F/T_{f} is a compact, connected smooth manifold, on which we still have the action of the Hamiltonian torus T_{h}. The following lemma says that there is a symplectic form $\sigma_{F/T_{\mathrm{f}}}$ on F/T_{f} such that

$$(F/T_{\mathrm{f}}, \sigma_{F/T_{\mathrm{f}}}, T_{\mathrm{h}}) \tag{6.1}$$

is a Delzant manifold defined by the Delzant polytope Δ, and that the fibration $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$ is trivial, exhibiting F as the Cartesian product of the Delzant manifold F/T_{f} with T_{f}.

LEMMA 6.3. — *There is a unique smooth distribution D on M such that, for every $x \in M_{\mathrm{reg}}$, D_x is equal to the linear span in $\mathrm{T}_x M$ of the vectors $Y_M(x)$, $Y \in \mathfrak{t}_{\mathrm{h}}$, and $L_\eta(x)$, $\eta \in C$. The distribution D is integrable and T-invariant.*

For every fiber F of the fibration $\pi_{N/P}$ in Lemma 6.2, we have $D|_F \subset \mathrm{T}F$, which implies that $I \subset F$ or $I \cap F = \emptyset$ for every integral manifold I of D. Let $f_0 \in F$ and let I_0 be the integral manifold of D such that $f_0 \in I_0$. For each $y \in F/T_{\mathrm{f}}$ there is a unique $i(y) \in I_0$ such that $\pi_{F/T_{\mathrm{f}}}(i(y)) = y$. The mapping

$$(y, t_{\mathrm{f}}) \mapsto t_{\mathrm{f}} \cdot i(y) : (F/T_{\mathrm{f}}) \times T_{\mathrm{f}} \to F$$

is the inverse of a trivialization τ of the principal T_{f}-fibration $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$. The trivialization τ is T-equivariant, where $t \in T$ acts on $(F/T_{\mathrm{f}}) \times T_{\mathrm{f}}$ by sending $(\pi_{F/T_{\mathrm{f}}}(f), \widetilde{t_{\mathrm{f}}})$ to $(\pi_{F/T_{\mathrm{f}}}(t_{\mathrm{h}} \cdot f), t_{\mathrm{f}}\widetilde{t_{\mathrm{f}}})$, if $t = t_{\mathrm{h}} t_{\mathrm{f}}$, with $t_{\mathrm{h}} \in T_{\mathrm{h}}$ and $t_{\mathrm{f}} \in T_{\mathrm{f}}$.

Finally, there is a unique symplectic form $\sigma_{F/T_{\mathrm{f}}}$ on F/T_{f} such that, for any integral manifold I of D in F,

$$(\pi_{F/T_{\mathrm{f}}} \circ \iota_I)^* \sigma_{F/T_{\mathrm{f}}} = {\iota_I}^* \sigma, \tag{6.2}$$

if $\iota_I : I \to F$ denotes the inclusion mapping from I into F. With this symplectic form, (6.1) is a Delzant manifold with Delzant polytope Δ. For each integral manifold I of D in F, $(I, {\iota_I}^*\sigma, T_{\mathrm{h}})$ is a Delzant manifold with Delzant polytope Δ, and $\pi_{F/T_{\mathrm{f}}} \circ \iota_I$ is a T_{h}-equivariant symplectomorphism from $(I, {\iota_I}^*\sigma, T_{\mathrm{h}})$ onto the Delzant manifold (6.1).

Proof. — In order to investigate the D_x with $x \in M_{\mathrm{reg}}$ near a singular point x_0, we use a local model as in Lemma 2.10, with the (θ, ρ)-coordinates in $\mathbb{C}^m$ as in (3.2). Here $H = T_{x_0}$ is a subtorus of the Hamiltonian torus T_{h}. Let K_0 be a complementary subtorus to H in T_{h}. We will take $K = K_0 T_{\mathrm{f}}$ as the complementary subtorus to H in T. For the Lie algebras we have the corresponding direct sum decompositions $\mathfrak{t}_{\mathrm{h}} = \mathfrak{t}_x \oplus \mathfrak{k}_0$ and $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{t}_{\mathrm{f}}$. The span of the infinitesimal actions of the elements $Y \in \mathfrak{h}$ is equal to the span of the vector fields $(0,0,\partial/\partial\theta_j)$, $1 \leqslant j \leqslant m$, and the vector fields $(Y,0,0)$ with $\mathfrak{k}_0$. The L_η^Φ, $\eta \in C$, are the linear combinations of the vector fields $(0,0,\partial/\partial\rho_j)$, $1 \leqslant j \leqslant m$, and the vector fields $(0,\delta\lambda,0)$, with constant $\delta\lambda \in (\mathfrak{l}/(\mathfrak{h} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}))^*$. According to Definition 5.3, the definition of admissible lifts, $L_\eta = L_\eta^\Phi + v_\eta$ in which the vector field v_η is smooth on M, of the form $v_\eta(x) = \alpha_\eta(x)_M(x)$ for a smooth T-invariant $\mathfrak{l}$-valued function α on M. We write v_j instead of v_η if η is such that $L_\eta^\Phi = (0,0,\partial/\partial\rho_j)$. The problem is that the vector fields $\partial/\partial\theta^j$ and $\partial/\partial\rho_j$ have a zero and a pole at $z^j = p^j + \mathrm{i}\, q^j = 0$.

Now

$$\frac{\partial}{\partial\rho_j} = (2\rho_j)^{-1}(p^j\frac{\partial}{\partial p^j} + q^j\frac{\partial}{\partial q^j}) \quad \text{and} \quad \frac{\partial}{\partial\theta^j} = -q^j\frac{\partial}{\partial p^j} + p^j\frac{\partial}{\partial q^j}$$

imply that

$$p^j L_{\eta^j} - \frac{q^j}{2\rho_j}(Y_j)_M = \frac{\partial}{\partial p^j} + p^j v_j \quad \text{and} \quad q^j L_{\eta^j} + \frac{p^k}{2\rho_k}(Y_j)_M = \frac{\partial}{\partial q^j} + q^j v_j.$$

These two vector fields are smooth and converge to $\partial/\partial p^j$ and $\partial/\partial q^j$, respectively, as $z^j \to 0$. This proves the first statement in the lemma. We also obtain for every $x \in M$ that $\mathrm{T}_x\, M = D_x \oplus E_x$, if E_x denotes the linear span of the $Z_M(x)$, $Z \in \mathfrak{t}_{\mathrm{f}}$, and the $L_\eta(x)$, $\zeta \in N$.

In view of (5.1), conclusion i) in Proposition 5.5, and the commutativity of the infinitesimal action of $\mathfrak{t}_{\mathrm{h}}$ on M, the vector fields Y_M and L_η all commute with each other. This implies that on M_{reg} the distribution D satisfies the Frobenius integrability condition. Because M_{reg} is dense in M, it follows by continuity that D is integrable on M. Because the vector fields Y_M, and L_η are T-invariant, the restriction to M_{reg} of D is T-invariant, and it follows by continuity that D is T-invariant.

For each $x \in M_{\mathrm{reg}}$, the vectors $X_M(x)$, $X \in \mathfrak{t}$, and $L_\eta(x)$, $\eta \in C := (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^*$, together span $\mathrm{T}_x F = \ker \mathrm{T}_x \pi_{N/P}$, hence $D_x \subset \mathrm{T}_x F$. Because $M_{\mathrm{reg}} \cap F$ is dense in F, see Lemma 6.2, it follows by continuity that $D|_F \subset \mathrm{T}F$. This implies in turn that if I is an integral manifold of D in M and $I \cap F \neq \emptyset$, then $I \subset F$ and I is an integral manifold of $D|_F$.

Because for every $x \in F$ the linear subspaces D_x and $((\mathfrak{t}_{\mathrm{f}})_M)_x$ of $\mathrm{T}_x F$ have zero intersection and their dimensions add up to the dimension of F, we have that D_x is a complementary linear subspace to $((\mathfrak{t}_{\mathrm{f}})_M)_x$ in $\mathrm{T}_x F$, and it follows that $D|_F$ defines a T_{f}-invariant infinitesimal connection for the principal T_{f}-bundle $\pi_{F/T_{\mathrm{f}}} : F \to F/T_{\mathrm{f}}$.

It follows from (5.2) and the conclusion v) in Proposition 5.5, that for every $x \in M_{\mathrm{reg}}$ the complementary linear subspaces D_x and E_x of $\mathrm{T}_x M$ are σ_x-orthogonal, and by continuity the same conclusion follows for every

$x \in M$. This implies that, for every $x \in M$, D_x is a symplectic vector subspace of $\mathrm{T}_x M$, and therefore every integral manifold I of D is a symplectic submanifold of (M, σ).

If I is an integral manifold of D, then the restriction to I of $\pi_{F/T_{\mathrm{f}}}$ is a covering from I onto F/T_{f}. Because σ is invariant under the action of T_{f}, there is a unique two-form $\sigma_{F/T_{\mathrm{f}}}$ on F/T_{f} such that (6.2) holds, and because $\pi_{F/T_{\mathrm{f}}} \circ \iota_I$ is a covering, it follows that $\sigma_{F/T_{\mathrm{f}}}$ is a smooth symplectic form on F/T_{f}.

The mapping from F/T_{f} to Δ induced by (5.4), which we also denote by μ, is a momentum mapping for the T_{h}-action on the symplectic manifold $(F/T_{\mathrm{f}}, \sigma_{F/T_{\mathrm{f}}})$. Because for any $q \in N/P$ the pre-image of $\{q\}$ under the projection from $M/T \simeq \Delta \times (N/P)$ onto the second factor is equal to $\Delta \times \{q\}$, and μ forgets the second factor, we have that $\mu(F) = \Delta$, and therefore $\mu(F/T_{\mathrm{f}}) = \Delta$. Because F is compact and connected, see Lemma 6.2, the image F/T_{f} of F under the continuous projection $F \to F/T_{\mathrm{f}}$ is also compact and connected. The conclusion is that (6.1) is a Delzant manifold defined by the Delzant polytope Δ.

Because F/T_{f} is simply connected in view of Lemma 6.4, $(\pi_{F/T_{\mathrm{f}}})|_I : I \to F/T_{\mathrm{f}}$ is a diffeomorphism. The other statements in the lemma now readily follow. □

LEMMA 6.4. — *Every Delzant manifold is simply connected.*

Proof. — Every Delzant manifold can be provided with the structure of a toric variety defined by a complete fan, *cf.* Delzant [11] and Guillemin [20, App. 1], and Danilov [10, Th. 9.1] observed that such a toric variety is simply connected. The argument is that the toric variety has an open cell which is isomorphic to $\mathbb{C}^n$, of which the complement is a complex subvariety of complex codimension one. Therefore any loop can be deformed into the cell and contracted within the cell to a point. □

Remark 6.5. — The pull-back to each T_{f}-orbit of the symplectic form σ on M is given by

$$\sigma_x(X_M(x), Y_M(x)) = \sigma^{\mathfrak{t}}(X, Y) \quad \text{for all} \quad X, Y \in \mathfrak{t}_{\mathrm{f}}.$$

Because $\mathfrak{t} = \mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$ and $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$, we have that this pull-back is equal to zero if and only $\sigma^{\mathfrak{t}} = 0$, that is, the principal T-orbits are Lagrangian. In this case the tangent spaces of the T_{f}-orbits in F are the kernels of the pull-back to F of σ, and the symplectic form $\sigma_{F/T_{\mathrm{f}}}$ on F/T_{f} is the reduced form of the pull-back to F of σ. In other words, $(F/T_{\mathrm{f}}, \sigma_{F/T_{\mathrm{f}}})$ is a reduced phase space for the "momentum mapping" $\pi_{N/P} : M \to N/P$ for the T_{f}-action, where the word momentum mapping is put between parentheses because the free T_{f}-action is not Hamiltonian.

The T-invariant projection $\pi_{N/P} : M \to N/P$ induces a T_{h}-invariant projection $\pi_{N/P} : M/T_{\mathrm{f}} \to N/P$, of which the fibers are canonically identified with the F/T_{f}, where the F are the fibers of $\pi_{N/P} : M \to N/P$. If $\sigma^{\mathfrak{t}} = 0$, then the symplectic leaves in M/T_{f} of the Poisson structure on $\mathrm{C}^\infty(M/T_{\mathrm{f}}) = \mathrm{C}^\infty(M)^{T_{\mathrm{f}}}$ are equal to the fibers F/T_{f} of $\pi_{N/P} : M/T_{\mathrm{f}} \to N/P$, provided with the symplectic forms $\sigma_{F/T_{\mathrm{f}}}$. It is quite remarkable that the symplectic leaves form a fibration, because in general the symplectic leaves of a Poisson structure are only immersed submanifolds, not necessarily closed.

7. A global model

Let (M, σ) be our compact connected symplectic manifold, together with an effective action of the torus T by means of symplectomorphisms of (M, σ), such that some (all) principal orbits of the T-action are coisotropic submanifolds of (M, σ).

In Subsection 7.1 we will show that the T-action together with the infinitesimal action of the vector fields L_ζ, $\zeta \in N$, introduced in Proposition 5.5, lead to an action on M of a two-step nilpotent Lie group G, where G is explicitly defined in terms of the antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ introduced in Proposition 5.5. Subsection 7.1 is a sequence of definitions, together with some of their immediate consequences.

Recall the fibration of M into Delzant submanifolds introduced in Proposition 6.1. The action of G on M will be used to exhibit this fibration as a G-homogeneous bundle over the homogeneous space G/H with fiber equal to a Delzant manifold defined by the Delzant polytope Δ. Here H is a closed Lie subgroup of G which is explicitly defined in terms of c and the period group P in N, defined in Lemma 10.12 with $Q = M/T$, $V = \mathfrak{l}^*$, and $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$. See Proposition 7.2.

The symplectic form on this bundle of Delzant manifolds is given explicitly by means of the formula (7.14), in terms of the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ introduced in Lemma 2.1, the antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ introduced in Proposition 5.5, and the symplectic form σ_{h} on the Delzant manifold M_{h}. In this way we obtain an explicit global model for our symplectic manifold (M, σ) with symplectic T-action.

7.1. An extension G of N by T acting on M

In the sequel, $\mathcal{X}^\infty(M)$ denotes the Lie algebra of all smooth vector fields on M, provided with the Lie brackets $[u, v]$ of $u,\ v \in \mathcal{X}^\infty(M)$ such that $[u, v]f = u(vf) - v(uf)$ for every $f \in \mathrm{C}^\infty(M)$. We denote the flow after time $t \in \mathbb{R}$ of $v \in \mathcal{X}^\infty(M)$ by e^{tv}. This defines an exponential mapping $v \mapsto \mathrm{e}^v$ from $\mathcal{X}^\infty(M)$ to the group $\mathrm{Diff}^\infty(M)$ of all smooth diffeomorphisms of M, which is analogous to the exponential mapping exp from the Lie algebra of any Lie group to the Lie group.

Let $\mathfrak{l}^* \ni \xi \mapsto L_\xi$ be an admissible lift as in Proposition 5.5. For each $\zeta \in N$, L_ζ is a smooth vector field on M, see Definition 5.3, and because M is compact, its flow $\mathrm{e}^{tL_\zeta} : M \to M$ is defined for all $t \in \mathbb{R}$.

A Lie algebra $\mathfrak{g}$ is called *two-step nilpotent* if $[[X, Y], Z] = 0$ for all $X, Y, Z \in \mathfrak{g}$. Because the vector fields L_ζ, $\zeta \in N$ commute with the X_M, $X \in \mathfrak{t}$, and the X_M, $X \in \mathfrak{t}$, commute with each other, it follows from iii) in Proposition 5.5 that the linear span of the X_M, $X \in \mathfrak{t}$, and the L_ζ, $\zeta \in N$, is a two-step nilpotent Lie subalgebra $\mathfrak{g}_M$ of $\mathcal{X}^\infty(M)$. Moreover, if we provide $\mathfrak{g} := \mathfrak{t} \times N$ with the structure of a two-step nilpotent Lie algebra defined by

$$[(X, \zeta), (X', \zeta')] = -(c(\zeta, \zeta'), 0), \quad (X, \zeta), (X', \zeta') \in \mathfrak{g} = \mathfrak{t} \times N, \tag{7.1}$$

then the mapping $(X, \zeta) \mapsto X_M + L_\zeta$ is an injective anti-homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathcal{X}^\infty(M)$, with image equal to $\mathfrak{g}_M$.

The vector space $\mathfrak{t} \times N$, provided with the product

$$(X, \zeta)(X', \zeta') = (X + X' - c(\zeta, \zeta')/2, \zeta + \zeta'), \quad (X, \zeta), (X', \zeta') \in \mathfrak{t} \times N, \tag{7.2}$$

is a two-step nilpotent Lie group with Lie algebra equal to $\mathfrak{g}$ and the identity as the exponential mapping. It follows that the mapping

$$(X, \zeta) \mapsto \mathrm{e}^{X_M + L_\zeta} = \mathrm{e}^{X_M} \circ \mathrm{e}^{L_\zeta} \tag{7.3}$$

is a (left) action of the group $\mathfrak{t} \times N$ on M, that is a homomorphism from the group $\mathfrak{t} \times N$ to the group $\mathrm{Diff}^\infty(M)$, with infinitesimal action given by $(X, \zeta) \mapsto X_M + L_\zeta$. It follows that

$$\mathrm{e}^{X_M + L_\zeta} \circ \mathrm{e}^{X'_M + L_{\zeta'}} = \mathrm{e}^{(X + X' - c(\zeta, \zeta')/2)_M + L_{\zeta + \zeta'}}. \tag{7.4}$$

The kernel of the homomorphism (7.3) is equal to the discrete normal subgroup $T_\mathbb{Z} \times \{0\}$ of $\mathfrak{t} \times N$, in which $T_\mathbb{Z} = \ker \exp$ is the integral lattice in the Lie algebra $\mathfrak{t}$ of T. It follows that the connected Lie group $G = T \times N \simeq (\mathfrak{t}/T_\mathbb{Z}) \times N$ acts smoothly on M, where in $T \times N$ we have the product

$$(t, \zeta)(t', \zeta') = (tt'\, \mathrm{e}^{-c(\zeta, \zeta')/2}, \zeta + \zeta') \tag{7.5}$$

and the action is given by

$$(t, \zeta) \mapsto t_M \circ \mathrm{e}^{L_\zeta} . \tag{7.6}$$

We have the exact sequence of groups $1 \to T \to G \to N \to 1$, where the homomorphism from G onto N corresponds to passing from the action of G on M to the action of N on the orbit space M/T, on which T acts trivially.

Note that the Lie algebra of G is equal to the previously introduced two-step nilpotent Lie algebra $\mathfrak{g} = \mathfrak{t} \times N$. Also note that the T-orbit map $\pi : M \to M/T$ intertwines the action of G on M with the translational action of N on M/T, in the sense that $\pi((t, \zeta) \cdot x) = \pi(x) + \zeta$ for every $(t, \zeta) \in G = T \times N$.

7.2. The holonomy of the connection

Let $\mathfrak{l}^* \ni \xi \mapsto L_\xi$ be an admissible connection as in Proposition 5.5. For each $\zeta \in P$ and $p \in M/T$, the curve $\gamma_\zeta(t) := p + t\zeta$, $0 \leqslant t \leqslant 1$, is a loop in M/T. If $x \in M$ and $p = \pi(x)$, then the curve $\delta(t) = \mathrm{e}^{tL_\zeta}(x)$, $0 \leqslant t \leqslant 1$, is called the *horizontal lift in M of the loop γ_ζ which starts at x*, because $\delta(0) = x$, $\delta'(t) = L_\zeta(\delta(t))$ is a horizontal tangent vector which is mapped by $\mathrm{T}_{\delta(t)}\pi$ to the constant vector ζ, which implies that $\pi(\delta(t)) = \gamma_\zeta(t)$, $0 \leqslant t \leqslant 1$. The element of T which maps the initial point $\delta(0) = x$ to the end point $\delta(1)$ is called the *holonomy $\tau_\zeta(x)$ of the loop γ_ζ and the intial point x* with respect to the given connection. Because $\delta(1) = \mathrm{e}^{L_\zeta}(x)$, we have $\tau_\zeta(x) \cdot x = \mathrm{e}^{L_\zeta}(x)$. In Lemma 7.1 below we investigate the dependence of the holonomy element $\tau_\zeta(x) \in T$ on the point $x \in M$ and the period $\zeta \in P$.

Lemma 7.1. — *Let $\zeta \in N$. Then the following conditions are equivalent.*

i) *There exists an $x \in M$ and a $t \in T$ such that $\mathrm{e}^{L_\zeta}(x) = t \cdot x$.*

ii) *$\zeta \in P$, where P is the period group in N for the translational action of N on M/T, as defined in Lemma 10.12 with $Q = M/T$, $V = \mathfrak{l}^*$, and $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$.*

iii) *The diffeomorphism e^{L_ζ} leaves all T-orbits in M invariant.*

For each $\zeta \in P$ there is a unique T-invariant smooth mapping $\tau_\zeta : M \to T$ such that $\mathrm{e}^{L_\zeta}(x) = \tau_\zeta(x) \cdot x$ for every $x \in M$. We have

$$\tau_\zeta(t \cdot \mathrm{e}^{L_{\zeta'}}(x)) = \mathrm{e}^{c(\zeta,\zeta')}\, \tau_\zeta(x) \tag{7.7}$$

for every $(t, \zeta') \in T \times N$.

We have $c(\zeta, \zeta') \in T_{\mathbb{Z}}$ whenever $\zeta, \zeta' \in P$, and $T \times P$ is a commutative subgroup of G.

Finally, the mapping $\tau_\zeta : M \to T$ is constant on every fiber of the fibration of M into Delzant submanifolds introduced in Proposition 6.1, and satisfies

$$\tau_{\zeta'}(x)\tau_\zeta(x) = \tau_{\zeta+\zeta'}(x)\,\mathrm{e}^{c(\zeta',\zeta)/2}, \quad x \in M, \quad \zeta, \zeta' \in P. \tag{7.8}$$

Proof. — Because the action of e^{L_ζ} on the T-orbits is equal to the transformation $p \mapsto p + \zeta$ in M/T, the equivalence between i), ii), iii) follows from Lemma 10.12 with $Q = M/T$, $V = \mathfrak{l}^*$, and $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$.

If $\zeta \in P$, then e^{L_ζ} leaves each T-orbit invariant. Because, for every $\zeta \in N$, e^{L_ζ} commutes with the T-action, this implies the existence of the smooth mapping τ_ζ in view of Lemma 4.2.

In order to show that (7.7) holds, we observe that

$$\begin{aligned}
\mathrm{e}^{L_{\zeta'}}(\tau_\zeta(\mathrm{e}^{L_{\zeta'}}(x))\cdot x) &= \tau_\zeta(\mathrm{e}^{L_{\zeta'}}(x))\cdot\mathrm{e}^{L_{\zeta'}}(x) = \mathrm{e}^{L_\zeta}(\mathrm{e}^{L_{\zeta'}}(x))\\
&= (\mathrm{e}^{L_\zeta}\circ\mathrm{e}^{L_{\zeta'}}\circ\mathrm{e}^{-L_\zeta})(\mathrm{e}^{L_\zeta}(x))\\
&= \mathrm{e}^{L_{\zeta'}+[L_\zeta,L_{\zeta'}]}(\tau_\zeta(x)\cdot x) = \mathrm{e}^{L_{\zeta'}}(\mathrm{e}^{c(\zeta,\zeta')}\cdot(\tau_\zeta(x)\cdot x)),
\end{aligned}$$

which implies that $\tau_\zeta(\mathrm{e}^{L_{\zeta'}}(x)) = \mathrm{e}^{c(\zeta,\zeta')}\,\tau_\zeta(x)$. In combination with the T-invariance of τ_ζ this yields (7.7).

If $\zeta' \in P$, then we have for every $x \in M$ that $\mathrm{e}^{L_{\zeta'}}(x) \in T\cdot x$, hence $\tau_\zeta(\mathrm{e}^{L_{\zeta'}}(x)) = \tau_\zeta(x)$, which in view of (7.7) implies that $\mathrm{e}^{c(\zeta,\zeta')} = 1$, hence $c(\zeta,\zeta') \in T_{\mathbb{Z}}$. The fact that $c(\zeta,\zeta') \in T_{\mathbb{Z}}$ for all $\zeta,\zeta' \in P$ implies in view of (7.5) that $T \times P$ is a commutative subgroup of $T \times N$.

Because L_ζ commutes with all L_η, $\eta \in C := (\mathfrak{l}/\mathfrak{l}\cap\mathfrak{t}_{\mathrm{f}})^* \simeq {\mathfrak{t}_{\mathrm{h}}}^*$, see ii) in Proposition 5.5, we have

$$\begin{aligned}
\mathrm{e}^{L_\eta}(\tau_\zeta(\mathrm{e}^{L_\eta}(x))\cdot x) &= \tau_\zeta(\mathrm{e}^{L_\eta}(x))\cdot\mathrm{e}^{L_\eta}(x) = \mathrm{e}^{L_\zeta}(\mathrm{e}^{L_\eta}(x))\\
&= \mathrm{e}^{L_\eta}(\mathrm{e}^{L_\zeta}(x)) = \mathrm{e}^{L_\eta}(\tau_\zeta(x)\cdot x),
\end{aligned}$$

which for regular x implies that $\tau_\zeta(\mathrm{e}^{L_\eta}(x)) = \tau_\zeta(x)$. By continuity this identity extends to all $x \in M$. Because also $\tau_\zeta(t\cdot x) = \tau_\zeta(x)$ for all $t \in T_{\mathrm{h}}$, it follows from the definition in Proposition 6.1 of the fibration of M into Delzant submanifolds, that τ_ζ is constant on its fibers.

If $\zeta,\zeta' \in P$, then we obtain, using (7.4), that

$$\begin{aligned}
\tau_{\zeta'}(x)\cdot(\tau_\zeta(x)\cdot x) &= \tau_{\zeta'}(x)\cdot\mathrm{e}^{L_\zeta}(x) = \mathrm{e}^{L_\zeta}(\tau_{\zeta'}(x)\cdot x) = (\mathrm{e}^{L_\zeta}\circ\mathrm{e}^{L_{\zeta'}})(x)\\
&= \mathrm{e}^{-c(\zeta,\zeta')_M/2}\cdot\mathrm{e}^{L_{\zeta+\zeta'}}(x) = \mathrm{e}^{c(\zeta',\zeta)_M/2}\cdot(\tau_{\zeta+\zeta'}(x)\cdot x),
\end{aligned}$$

which implies (7.8). □

Let ε^l, $1 \leqslant l \leqslant d_N := \dim N$, be a $\mathbb{Z}$-basis of P. For any $\zeta \in P$ we have $\zeta = \sum_l \zeta_l \varepsilon^l$ for unique integral coordinates $\zeta_l \in \mathbb{Z}$. With the notation $c^{ll'} := c(\varepsilon^l, \varepsilon^{l'}) \in \mathfrak{l} \cap T_{\mathbb{Z}}$, the formula (7.8) leads to the formula

$$\tau_\zeta(x) = \mathrm{e}^{\sum_{l<l'} \zeta_l \zeta_{l'} c^{ll'}/2} \prod_{l=1}^{d_N} \tau_{\varepsilon^l}(x)^{\zeta_l} \tag{7.9}$$

for $\tau_\zeta(x)$ in terms of the elements $\tau_{\varepsilon^l}(x) \in T$. In other words, all holonomies at a given point $x \in M$ can be expressed in terms of the holonomies of the basic loops γ_{ε^l}, $1 \leqslant l \leqslant d_N$, by means of the formula (7.9).

7.3. M as a G-homogeneous bundle with the Delzant manifold as fiber

In this subsection, let $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$ be one of the Delzant submanifolds of (M, σ, T) in Proposition 6.1. That is, M_{h} is an integral manifold I of the distribution H, and $\sigma_{\mathrm{h}} = {\iota_I}^* \sigma$, if ι_I denotes the inclusion mapping from I to M. Recall that all Delzant manifolds with the same Delzant polytope are T_{h}-equivariantly symplectomorphic, which means that one may identify $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$ with any favourite explicit model of a Delzant manifold with Delzant polytope Δ. We will construct a model for our symplectic T-manifold (M, σ, T) by means of the mapping $A : G \times M_{\mathrm{h}} \to M$ which is defined by

$$A((t, \zeta), x) = t \cdot \mathrm{e}^{L_\zeta}(x), \quad t \in T,\ \zeta \in N,\ x \in M_{\mathrm{h}}. \tag{7.10}$$

Write τ_ζ for the common value of the $\tau_\zeta(x)$ for all $x \in M_{\mathrm{h}}$, see Lemma 7.1. Define

$$H := \{(t, \zeta) \in G \mid \zeta \in P \text{ and } t\tau_\zeta \in T_{\mathrm{h}}\}. \tag{7.11}$$

Then H is a closed Lie subgroup of G, commutative because $T \times P$ is commutative, see Lemma 7.1. Furthermore,

$$((t, \zeta), x) \mapsto (t\tau_\zeta) \cdot x : H \times M_{\mathrm{h}} \to M_{\mathrm{h}} \tag{7.12}$$

defines a smooth action of H on the Delzant manifold M_{h}.

Proposition 7.2. — *The mapping* (7.10) *induces a diffeomorphism α from $G \times_H M_{\mathrm{h}}$ onto M, where $h \in H$ acts on $G \times M_{\mathrm{h}}$ by sending (g, x) to $(gh^{-1}, h \cdot x)$.*

The diffeomorphism α intertwines the action of G on $G \times_H M_{\mathrm{h}}$, which is induced by the action $(g, (g', x)) \mapsto (gg', x)$ of G on $G \times M_{\mathrm{h}}$, with the action of G on M, and therefore also the action of the normal subgroup

$T\times\{0\}\simeq T$ of G with the T-action on M. The projection $(g,\ x)\mapsto g : G\times M_{\mathrm{h}}\to G$ induces a G-equivariant smooth fibration $\psi : G\times_H M_{\mathrm{h}}\to G/H$, and $\delta=\psi\circ\alpha^{-1} : M\to G/H$ is a G-equivariant smooth fibration of which the Delzant submanifolds of M introduced in Proposition 6.1 are the fibers.

Proof. — Let $x_0\in M_{\mathrm{h}}$ and $y\in M$. For each $\zeta\in N$, the projection $\pi_{N/P} : M\to N/P$ defined in Lemma 6.2 intertwines the diffeomorphism e^{L_ζ} in M with the translation in N/P over the vector ζ. Because these translations act transitively on N/P, there exists a $\zeta\in N$ such that $\pi_{N/P}(y)=\pi_{N/P}(x_0)+\zeta$, which implies that $\mathrm{e}^{-L_\zeta}(y)$ belongs to the same fiber F of $\pi_{N/T}$ as x_0. With such a choice of ζ, it follows from Lemma 6.3 that there exists $t_{\mathrm{f}}\in T_{\mathrm{f}}$ and $x\in M_{\mathrm{h}}$ such that $\mathrm{e}^{-L_\zeta}(y)=t_{\mathrm{f}}\cdot x$, or equivalently $y=t_{\mathrm{f}}\cdot\mathrm{e}^{L_\zeta}(x)$. This shows that already the restriction of A to $(T_{\mathrm{f}}\times N)\times M_{\mathrm{h}}$ is surjective.

Let $g,g'\in G$, $x,x'\in M_{\mathrm{h}}$ and $g\cdot x=g'\cdot x'$. Then $x'=h\cdot x$ in which $h:=(g')^{-1}g$. Write $h=(t,\zeta)$ with $t\in T$ and $\zeta\in N$. Then

$$\pi_{N/P}(x)=\pi_{N/P}(x')=\pi_{N/P}(t\cdot\mathrm{e}^{L_\zeta}(x))=\pi_{N/P}(\mathrm{e}^{L_\zeta}(x))=\pi_{N/P}(x)+\zeta$$

implies that $\zeta\in P$, and it follows from Lemma 7.1 that

$$x'=t\cdot\mathrm{e}^{L_\zeta}(x)=t\cdot\tau_\zeta\cdot x=(t\tau_\zeta)_{\mathrm{f}}\cdot((t\tau_\zeta)_{\mathrm{h}}\cdot x.$$

Because x' and $(t\tau_\zeta)_{\mathrm{h}}\cdot x$ belong to the same integral manifold I of H, it follows from Lemma 6.3 that the element $(t\tau_\zeta)_{\mathrm{f}}$ of T_{f} is equal to the identity element, hence $t\tau_\zeta\in T_{\mathrm{h}}$ and $x'=(t\tau_\zeta)\cdot x$. In other words, $h\in H$, $g'=gh^{-1}$ and $x'=(t\tau_\zeta)\cdot x$. This proves that the mapping A induces a bijective mapping α from $G\times_H M_{\mathrm{h}}$ onto M.

The closedness of H in G implies that the right action of H on G is proper and free, hence the action of H on $G\times M_{\mathrm{h}}$ is proper and free, and the orbit space $G\times_H M_{\mathrm{h}}$ has a unique smooth structure for which the projection $G\times M_{\mathrm{h}}\to G\times_H M_{\mathrm{h}}$ is a principal H-bundle. With respect to this smooth structure on $G\times_H M_{\mathrm{h}}$, the mapping $\alpha : G\times_H M_{\mathrm{h}}\to M$ is smooth. The transversality to $\mathrm{T}M_{\mathrm{h}}$ of the span of Z_M, $Z\in\mathfrak{t}_{\mathrm{f}}$ and the L_ζ, $\zeta\in N$, implies that at every point the tangent mapping of A is surjective. Hence α is a submersion, and because α is bijective, it follows from the inverse mapping theorem that α is a diffeomorphism. The other statements in the proposition are general facts about induced fiber bundles $G\times_H M_{\mathrm{h}}$ over G/H with fiber M_{h}, see for instance [15, Sec. 2.4]. □

Remark 7.3. — On G/H we still have the free action of the torus T/T_{h}, which exhibits G/H as a principal T/T_{h}-bundle over the torus $(G/H)/T\simeq$

N/P. Palais and Stewart [46] showed that every principal torus bundle over a torus is diffeomorphic to a nilmanifold for a two-step nilpotent Lie group. In this remark we will give an explicit nilmanifold description of G/H.

The Hamiltonian torus T_{h}, or rather the identity component $H^o = T_{\mathrm{h}} \times \{0\}$ of H, is a closed normal Lie subgroup of both $G = T \times N$ and H, and the mapping $(gH^o)(H/H^o) \mapsto gH$ is a G-equivariant diffeomorphism from $(G/H^o)/(H/H^o)$ onto G/H. The group structure in $G/H^o = (T/T_{\mathrm{h}}) \times N$ is defined by

$$(t,\zeta)(t',\zeta') = (tt'\,\mathrm{e}^{-c_{\mathfrak{l}/\mathfrak{t}_{\mathrm{h}}}(\zeta,\zeta')/2}, \zeta+\zeta'), \quad t,t' \in T/T_{\mathrm{h}}, \quad \zeta,\zeta' \in N, \tag{7.13}$$

and $c_{\mathfrak{l}/\mathfrak{t}_{\mathrm{h}}} : N \times N \to \mathfrak{l}/\mathfrak{t}_{\mathrm{h}}$ is equal to $c : N \times N \to \mathfrak{l}$, followed by the projection $\mathfrak{l} \to \mathfrak{l}/\mathfrak{t}_{\mathrm{h}}$. This exhibits G/H^o as a two-step nilpotent Lie group with universal covering equal to $(\mathfrak{t}/\mathfrak{t}_{\mathrm{h}}) \times N$ and covering group $(T/T_{\mathrm{h}})_{\mathbb{Z}} \simeq T_{\mathbb{Z}}/(T_{\mathrm{h}})_{\mathbb{Z}}$. Also note that $\iota : \zeta \mapsto ({\tau_\zeta}^{-1}, \zeta)H^o$ is an isomorphism from the period group P onto H/H^o.

In view of (7.11), we conclude that the compact homogeneous G-space G/H is isomorphic to the quotient of the simply connected two-step nilpotent Lie group $(\mathfrak{t}/\mathfrak{t}_{\mathrm{h}}) \times N$ by the discrete subgroup of $(\mathfrak{t}/\mathfrak{t}_{\mathrm{h}}) \times N$ which consists of all $(Z,\zeta) \in (\mathfrak{t}/\mathfrak{t}_{\mathrm{h}}) \times P$ such that $\mathrm{e}^Z\,\tau_\zeta \in T_{\mathrm{h}}$.

7.4. The symplectic form on the global model

In Proposition 7.2 we have described the global model $M_{\mathrm{model}} := G \times_H M_{\mathrm{h}}$ for the T-manifold M, where the multiplication in the Lie group $G = T \times N$ is defined by (7.5). We now describe the symplectic form on M_{model}.

PROPOSITION 7.4. — *Let ω be the pull-back of σ to $G \times M_{\mathrm{h}} = (T \times N) \times M_{\mathrm{h}}$ by means of the mapping A in (7.10). Let $\delta a = ((\delta t, \delta\zeta), \delta x)$ and $\delta' a = ((\delta' t, \delta'\zeta), \delta' x)$ be tangent vectors to $G \times M_{\mathrm{h}}$ at $a = ((t,\zeta),x)$, where we identify each tangent space of the torus T with $\mathfrak{t}$. Write $X = \delta t + c(\delta\zeta,\zeta)/2$ and $X' = \delta' t + c(\delta'\zeta,\zeta)/2$. Then*

$$\begin{aligned}\omega_a(\delta a, \delta' a) = \sigma^{\mathfrak{t}}(\delta t, \delta' t) &+ \delta\zeta({X'}_{\mathfrak{l}}) - \delta'\zeta(X_{\mathfrak{l}}) - \mu(x)(c_{\mathrm{h}}(\delta\zeta,\delta'\zeta)) \\ &+ (\sigma_{\mathrm{h}})_x(\delta x, ({X'}_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) - (\sigma_{\mathrm{h}})_x(\delta' x, (X_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) \\ &+ (\sigma_{\mathrm{h}})_x(\delta x, \delta' x).\end{aligned} \tag{7.14}$$

Here X_{h} denotes the $\mathfrak{t}_{\mathrm{h}}$-component of $X \in \mathfrak{t}$ with respect to the direct sum decomposition $\mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$.

If $\pi_{M_{\mathrm{model}}}$ denotes the canonical projection from $G \times M_{\mathrm{h}}$ onto $M_{\mathrm{model}} := G \times_H M_{\mathrm{h}}$, then the T-invariant symplectic form $\sigma_{\mathrm{model}} := \alpha^*\sigma$ on M_{model} is the unique two-form β on M_{model} such that $\omega = {\pi_{M_{\mathrm{model}}}}^*\beta$.

Proof. — It follows from (7.4) that

$$\mathrm{e}^{L_{\zeta'+\zeta}} = \mathrm{e}^{c(\zeta',\zeta)_M/2} \circ \mathrm{e}^{L_{\zeta'}} \circ \mathrm{e}^{L_{\zeta}}\,.$$

Therefore, if we substitute $\zeta' = \epsilon\delta\zeta$ and differentiate with respect to ϵ at $\epsilon = 0$, we get the vector $c(\delta\zeta, \zeta)_M/2 + L_{\delta\zeta}$ at the image point under the mapping e^{L_ζ}. Because e^{L_ζ} commutes with the T-action, it follows, with the notations $y = A(a)$ and $B = t_M \circ \mathrm{e}^{L_\zeta}$, that

$$\delta y = (\mathrm{T}_a A)(\delta a) = (X_M + L_{\delta\zeta})(y) + (\mathrm{T}_x B)(\delta x),$$

in which $X = \delta t + c(\delta\zeta, \zeta)/2$.

If x is a regular point in M_{h}, then we can write $\delta x = (Y_M + L_\eta)(p)$ for uniquely determined $Y \in \mathfrak{t}_{\mathrm{h}}$ and $\eta \in C = (\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \simeq (\mathfrak{t}_{\mathrm{h}})^*$. The vector fields Y_M, $Y \in \mathfrak{t}_{\mathrm{h}}$, and L_η, $\eta \in C$, commute with the vector fields X_M, $X \in \mathfrak{t}$, and L_ζ, $\zeta \in N$, because of ii) in Proposition 5.5 and the fact that all the vector fields are T-invariant. Therefore $(\mathrm{T}_x B)(\delta x) = (Y_M + L_\eta)(y)$, and we obtain that $(\mathrm{T}_a A)(\delta a)$ is equal to the value at $y = A(a)$ of the vector field $(X + Y)_M + L_{\delta\zeta+\eta}$.

In view of (5.2) and iv), v) in Proposition 5.5, the symplectic product of this vector with the one in which δt, $\delta\zeta$, Y, η are replaced by $\delta' t$, $\delta'\zeta$, Y', η', respectively, is equal to

$$(\delta\zeta + \eta)((X' + Y')_{\mathfrak{l}}) - (\delta'\zeta + \eta')((X + Y)_{\mathfrak{l}}) + \sigma_y(L_{\delta\zeta}(y), L_{\delta'\zeta}(y)),$$

in which $X = \delta t + c(\delta\zeta, \zeta)/2 + Y$ and $X' = \delta' t + c(\delta'\zeta, \zeta)/2 + Y'$. Collecting terms and using the equations $\eta(X'_{\mathfrak{l}}) = \eta(X'_{\mathrm{h}}) = (\sigma_{\mathrm{h}})_x(\delta x, (X'_{\mathrm{h}})_{M_{\mathrm{h}}}(x))$, $\eta'(X_{\mathfrak{l}}) = \eta'(X_{\mathrm{h}}) = (\sigma_{\mathrm{h}})_x(\delta' x, (X_{\mathrm{h}})_{M_{\mathrm{h}}}(x))$, $\eta(Y') - \eta'(Y) = (\sigma_{\mathrm{h}})_x(\delta x, \delta' x)$, and vi) in Proposition 5.5, we arrive at (7.14).

Because $A = \alpha \circ \pi_{M_{\mathrm{model}}}$, we have

$$\omega = A^*\sigma = {\pi_{M_{\mathrm{model}}}}^*(\alpha^*\sigma) = {\pi_{M_{\mathrm{model}}}}^*\sigma_{\mathrm{model}}.$$

The uniqueness in the last statement follows because $\pi_{M_{\mathrm{model}}}$ is a submersion. □

Lemma 7.5. — *Let T_{f} be a complementary torus to the Hamiltonian torus T_{h} in T. Then the following conditions are equivalent.*

a) *(M, σ, T) is T-equivariantly symplectomorphic to the Cartesian product of a symplectic T_{f}- space $(M_{\mathrm{f}}, \sigma_{\mathrm{f}}, T_{\mathrm{f}})$ on which the T_{f}-action is free and a Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$. Here $t \in T$ acts on $M_{\mathrm{f}} \times M_{\mathrm{h}}$ by sending $(x_{\mathrm{f}}, x_{\mathrm{h}})$ to $(t_{\mathrm{f}} \cdot x_{\mathrm{f}}, t_{\mathrm{h}} \cdot x_{\mathrm{h}})$, if $t = t_{\mathrm{f}} t_{\mathrm{h}}$ with $t_{\mathrm{f}} \in T_{\mathrm{f}}$ and $t_{\mathrm{h}} \in T_{\mathrm{h}}$.*

b) *$c(P \times P) \subset \mathfrak{t}_{\mathrm{f}}$.*

c) *$c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$.*

d) *The* $\mathfrak{t}_{\mathrm{h}}$*-component* c_{h} *of* c *in the direct sum decomposition* $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$ *is equal to zero.*

Proof. — The equivalence between b) and c) follows from the fact that P has a $\mathbb{Z}$-basis which is an $\mathbb{R}$-basis of N. The equivalence between c) and d) is obvious.

If (M, σ, T) is equal to the Cartesian product of a Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$ and a symplectic T_{f}-space $(M_{\mathrm{f}}, \sigma_{\mathrm{f}}, T_{\mathrm{f}})$ for which the T_{f}-action on M_{f} is free, then we can choose the L_ζ in the direction of the second component M_{f}. In this case we have for every $\zeta, \zeta' \in N$ that $[L_\zeta, L_{\zeta'}] \in \mathfrak{t}_{\mathrm{f}}$, which means that the antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ in Proposition 5.5 has the property that $c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$, or equivalently $c_{\mathrm{h}} = 0$.

For the converse, assume that $c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$, which implies that $c_{\mathrm{h}} = 0$. Then the Lie group G is equal to the Cartesian product $T_{\mathrm{h}} \times G_{\mathrm{f}}$, in which $G_{\mathrm{f}} = T_{\mathrm{f}} \times N$, where the product in G_{f} is defined as in (7.5) with T replaced by T_{f}. According to Subsection 7.5 we can multiply the elements $\tau_{\varepsilon^l}(x)$, for $x \in M_{\mathrm{h}}$, by any element of $\exp(\mathfrak{l})$. Because $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l}$, it follows that we can arrange that $\tau_{\varepsilon^l}(x) \in T_{\mathrm{f}}$ for every $1 \leqslant l \leqslant d_N$, and then it follows from (7.9) that $\tau_\zeta \in T_{\mathrm{f}}$ for every $\zeta \in P$. The mapping $\iota : \zeta \mapsto ({\tau_\zeta}^{-1}, \zeta)$ is a homomorphism from P onto a discrete cocompact subgroup of G_{f}. Write $M_{\mathrm{f}} := G_{\mathrm{f}}/\iota(P)$. It follows that the mapping

$$A_{\mathrm{f}} : ((t_{\mathrm{f}}, \zeta), x) \mapsto t_{\mathrm{f}} \cdot \mathrm{e}^{L_\zeta}(x) : G_{\mathrm{f}} \times M_{\mathrm{h}} \to M \tag{7.15}$$

induces a diffeomorphism α_{f} from $M_{\mathrm{f}} \times M_{\mathrm{h}}$ onto M. Moreover, it follows from (7.14) that the symplectic form ${\alpha_{\mathrm{f}}}^*\sigma$ on $M_{\mathrm{f}} \times M_{\mathrm{h}}$ is equal to ${\pi_{\mathrm{f}}}^*\sigma_{\mathrm{f}} + {\pi_{\mathrm{h}}}^*\sigma_{\mathrm{h}}$, if π_{f} and π_{h} is the projection form $M_{\mathrm{f}} \times M_{\mathrm{h}}$ onto the first and the second factor, respectively, and the symplectic form σ_{f} on M_{f} is given by

$$(\sigma_{\mathrm{f}})_b(\delta b, \delta' b) = \sigma^{\mathfrak{t}}(\delta t, \delta' t) + \delta\zeta({X'}_{\mathfrak{l}}) - \delta'\zeta(X_{\mathfrak{l}}). \tag{7.16}$$

Here $b = (t, \zeta)\iota(P) \in (G_{\mathrm{f}}/\iota(P))$, the tangent vectors $\delta b = (\delta t, \delta\zeta)$ and $\delta' b = (\delta' t, \delta'\zeta)$ are elements of $\mathfrak{t}_{\mathrm{f}} \times N$, the vectors $X := \delta t + c(\delta\zeta, \zeta)/2$ and $X' := \delta' t + c(\delta'\zeta, \zeta)/2$ are elements of $\mathfrak{t}_{\mathrm{f}}$, and finally $\sigma(L_{\delta\zeta}, L_{\delta'\zeta}) = 0$ because in vi) in Proposition 5.5 we have $c_{\mathrm{h}} = 0$. It follows that (M, σ, T) is T-equivariantly symplectomorphic to $(M_{\mathrm{f}}, \sigma_{\mathrm{f}}, T_{\mathrm{f}}) \times (M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$, in which $(M_{\mathrm{f}}, \sigma_{\mathrm{f}}, T_{\mathrm{f}})$ is a compact connected symplectic manifold with a free symplectic action T_{f}-action. □

Remark 7.6. — In the proof of Lemma 7.5 we have also given a global model for (M, σ, T) in the case that the action of T is free. Note that the mapping $g \mapsto gH^o$ defines an isomorphism from G_{f} onto the group G/H^o in Remark 7.3, and an isomorphism from $\iota(P)$ onto H/H^o, which leads to an identification of $M_{\mathrm{f}} = G_{\mathrm{f}}/\iota(P)$ with the manifold G/H. In Remark 7.3, G/H has been described as a principal T_{f}-bundle over the torus N/P, and as a nilmanifold for a two-step nilpotent Lie group.

The example of Kodaira [30, Th. 19, case 3] is equal to $G_{\mathrm{f}}/\iota(P)$ with $T_{\mathrm{f}} = \mathbb{R}^2/\mathbb{Z}^2$, $N = \mathbb{R}^2$, $P = \mathbb{Z}^2$, $\sigma^{\mathrm{t}} = 0$, and $c(e_1, e_2) = m e_1$ if e_1, e_2 denotes the standard basis in $\mathbb{R}^2$. Furthermore, τ_ζ is given by the formula (7.9), in which $\tau_{e_1} = \tau_{e_2} = 0$ modulo $\mathbb{Z}^2$. We learned this reference from Fernández, Gotay and Gray [16]. For $m = 1$ this is the first example of Thurston [51]. For more examples, see McDuff and Salamon [39, Ex. 3.8 on p.88] and the references therein.

Remark 7.7. — If $\{1\} \neq T_{\mathrm{h}} \neq T$, then the choice of a complementary torus T_{f} to T_{h} in T is far from unique, see Remark 2.9. It can happen that for some choice of T_{f} we have $c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$, whereas for another choice we have not.

However, if $c(N \times N) \subset \mathfrak{t}_{\mathrm{h}}$ and $c \neq 0$, then there is no choice of a complementary torus T_{f} to T_{h} such that $c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$, and therefore (M, σ, T) is in no way T-equivariantly symplectomorphic to a Cartesian product of a symplectic manifold with a free torus action and a Delzant manifold.

Remark 7.8. — If $\dim N \leqslant 1$ then $c(\zeta, \zeta') \equiv 0$ because every antisymmetric bilinear form on a one-dimensional space is equal to zero, and we conclude that (M, σ, T) is a Cartesian product of a Delzant manifold with a two-dimensional homogeneous symplectic torus.

If M is four-dimensional, that is, $n = 2$, then it is proven in [48] that we have only few possibilities.

a) The homogeneous symplectic torus, where T acts freely and transitively on M. Cases where T is replaced by a subtorus with coisotropic orbits are treated as subcases.
b) (M, σ, T) is T-equivariantly symplectomorphic with the Cartesian product of a two-dimensional homogeneous symplectic torus and a sphere, provided with a rotationally invariant area form.
c) (M, σ, T) is a four-dimensional Delzant manifold.
d) The action of T is free with Lagrangian orbits, but not in case a). See the proof of Lemma 7.5 for a more detailed description of this case. The example of Kodaira mentioned in Remark 7.6 seems to be the first one in the literature.

7.5. The holonomy invariant

In view of the surjectivity of the mapping A in Proposition 7.2, Lemma 7.1 contains the description of the dependence of the holonomy $\tau(x) : \zeta \mapsto \tau_\zeta(x) : P \to T$ on all points $x \in M$. The only change which occurs is that if x is replaced by $x' = \mathrm{e}^{L_{\zeta'}}(x)$, $\zeta' \in N$, then $\tau_\zeta(x)$ is replaced by $\tau_\zeta(x)\,\mathrm{e}^{c(\zeta,\zeta')}$, see (7.7). We now investigate the dependence of the holonomy on the choice of the admissible connection as in Proposition 5.5.

It follows from Lemma 5.9 that $\mathfrak{l}^* \ni \xi \mapsto \widetilde{L}_\xi$ is another connection as in Proposition 5.5, if and only if there exists a smooth T-invariant mapping $\alpha : x \mapsto (\xi \mapsto \alpha_\xi(x))$ from M to $\mathrm{Lin}(\mathfrak{l}^*, \mathfrak{l})$, which is *closed* when viewed as an $\mathfrak{l}$-valued one-form on M/T, and *symmetric* in the sense of (5.8), such that $\widetilde{L}_\xi(x) = L_\xi(x) + \alpha_\xi(x)_M(x)$ for every $x \in M$ and $\xi \in \mathfrak{l}^*$. The change from L to $\widetilde{L}$ leads to a change from $\tau_\zeta(x)$ to

$$\widetilde{\tau}_\zeta(x) = \tau_\zeta(x)\,\mathrm{e}^{\int_{\gamma_\zeta}\alpha}. \tag{7.17}$$

Here $\gamma_\zeta(t) := \pi(x)+t\zeta$, $0 \leqslant t \leqslant 1$, is a loop in M/T because $\zeta \in P$. Because α is closed, the integral $\int_{\gamma_\zeta}\alpha$ only depends on the de Rham cohomology class of α, which means that for the effect on the $\tau_\zeta(x)$'s we can restrict ourselves to constant $\mathfrak{l}$-valued one-forms on $\mathfrak{l}^*$, that is, linear mappings from $\mathfrak{l}^*$ to $\mathfrak{l}$, or equivalently, bilinear forms on $\mathfrak{l}^*$. Therefore, at a given point $x \in M$, the allowed changes in $\tau_\zeta(x)$, $\zeta \in P$, consist of the multiplications with $\mathrm{e}^{\alpha_\zeta}$, where α ranges over the space of linear mappings $\xi \mapsto \alpha_\xi$ from $\mathfrak{l}^*$ to $\mathfrak{l}$, which are symmetric in the sense of (5.8).

Definition 7.9. — *Let* $\mathrm{Hom}_c(P,T)$ *denote the space of mappings* $\tau : \zeta \mapsto \tau_\zeta : P \to T$ *such that*

$$\tau_{\zeta'}\tau_\zeta = \tau_{\zeta+\zeta'}\,\mathrm{e}^{c(\zeta',\zeta)/2}, \quad \zeta,\zeta' \in P. \tag{7.18}$$

Because $c(P\times P) \subset T_{\mathbb{Z}}$*, the factor* $\mathrm{e}^{c(\zeta',\zeta)/2}$ *in (7.18) is an element of order two in* T*. Therefore the elements of* $\mathrm{Hom}_c(P,T)$ *are quite close to being homomorphisms from* P *to* T*. They are homomorphisms from* P *to* T *if* $c(P\times P) \subset 2T_{\mathbb{Z}}$*.*

If $h : \zeta \mapsto h_\zeta$ is a homomorphism from P to T, then $h\cdot\tau : \zeta \mapsto \tau_\zeta h_\zeta \in \mathrm{Hom}_c(P,T)$ for every $\tau \in \mathrm{Hom}_c(P,T)$, and $(h,\tau) \mapsto h\cdot\tau$ defines a free, proper, and transitive action of $\mathrm{Hom}(P,T)$ on $\mathrm{Hom}_c(P,T)$. Because $\mathrm{Hom}(P,T)$ is a torus group, a compact, connected, and commutative Lie group with Lie algebra equal to the vector space $\mathrm{Hom}(P,\mathfrak{t})$ of dimension $\dim N\dim T$, it follows that $\mathrm{Hom}_c(P,T)$ is diffeomorphic to a torus of dimension $\dim N\dim T$.

DEFINITION 7.10. — *For each* $\zeta' \in N$, $\zeta \mapsto c(\zeta, \zeta')$ *is a homomorphism from* P *to* $\mathfrak{t}$, *actually* $\mathfrak{l}$*-valued. Write* $c(\cdot, N)$ *for the set of all* $c(\cdot, \zeta') \in \mathrm{Hom}(P, \mathfrak{t})$ *such that* $\zeta' \in N$. $c(\cdot, N)$ *is a linear subspace of the Lie algebra* $\mathrm{Hom}(P, \mathfrak{t})$ *of* $\mathrm{Hom}(P, T)$.

Let Sym *denote the space of all linear mappings* $\alpha : \mathfrak{l}^* \to \mathfrak{l}$ *which are symmetric in the sense of (5.8). For each* $\alpha \in \mathrm{Sym}$, *the restriction* $\alpha|_P$ *of* α *to* P *is a homomorphism from* P *to* $\mathfrak{l} \subset \mathfrak{t}$. *In this way the set* $\mathrm{Sym}\,|_P$ *of all* $\alpha|_P$ *such that* $\alpha \in \mathrm{Sym}$ *is another linear subspace of* $\mathrm{Hom}(P, \mathfrak{t})$. *Write*

$$\mathcal{T} := \mathrm{Hom}_c(P, T)/\exp\mathcal{A}, \quad \mathcal{A} := c(\cdot, N) + \mathrm{Sym}\,|_P \tag{7.19}$$

for the orbit space of the action of the Lie subgroup $\exp\mathcal{A}$ *of* $\mathrm{Hom}(P, T)$ *on* $\mathrm{Hom}_c(P, T)$. *Because* $\exp\mathcal{A}$ *need not be a closed subgroup of* $\mathrm{Hom}(P, T)$, *the quotient topology of* $\mathcal{T}$ *need not be Hausdorff.*

It follows from (7.8) and (7.18), that for every choice of a connection as in Proposition 5.5 and every $x \in M$, *the mapping* $\tau(x) : \zeta \mapsto \tau_\zeta(x)$ *is an element of* $\mathrm{Hom}_c(P, T)$. *It is the point of (7.19), that the right hand side in*

$$\overline{\tau} := (\exp\mathcal{A}) \cdot \tau(x) \in \mathcal{T} \tag{7.20}$$

defines an invariant $\overline{\tau}$ *of our symplectic* T*-space* (M, σ, T), *in the sense that it neither depends on the choice of the point* $x \in M$, *nor on the choice of the connection as in Proposition 5.5.*

Remark 7.11. — In order to obtain some more insight in the vector space $\mathcal{A}$ in (7.19), we use the direct sum decomposition $\mathfrak{t} = \mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$, where $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l}$ is the Lie algebra of the Hamiltonian torus T_{h} and $\mathfrak{t}_{\mathrm{f}}$ is the Lie algebra of a complementary torus T_{f} to T_{h} in T. This leads to an identification of $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ with $(\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^*$ and of its linear complement C in $\mathfrak{l}^*$ with $\mathfrak{t}_{\mathrm{h}}{}^*$.

Let $(\mathrm{Sym}_{\mathrm{f}})|_P$ denote the space of all linear mappings $\alpha : (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^* \to \mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$, which satisfy the symmetry condition (5.8) with $\mathfrak{l}$ replaced by $\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$. The space $\mathrm{Sym}\,|_P$ of all restrictions to $P \subset N = (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^*$ of linear mappings $\alpha : \mathfrak{l}^* \to \mathfrak{l}$ which satisfy the the symmetry condition (5.8) is equal to the direct sum of the space $\mathrm{Hom}(P, \mathfrak{t}_{\mathrm{h}})$ of *all* homomorphisms from P to $\mathfrak{t}_{\mathrm{h}}$, and the space $(\mathrm{Sym}_{\mathrm{f}})|_P$. This means that in the space $\mathcal{T}$ in (7.19) we dispose of the T_{h}- components, and in the computation of $\mathcal{A}$ we can replace c by its $\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$-component c_{f}.

Now suppose that $\zeta' \in N$ and $c_{\mathrm{f}}(\cdot, \zeta') \in (\mathrm{Sym}_{\mathrm{f}})|_P$. This is equivalent to the condition that

$$-\zeta'(c(\zeta, \zeta'')) = \zeta''(c(\zeta, \zeta')) + \zeta(c(\zeta', \zeta'')) = \zeta''(c_{\mathrm{f}}(\zeta, \zeta')) - \zeta(c_{\mathrm{f}}(\zeta'', \zeta')) = 0$$

for all $\zeta, \zeta'' \in P \subset N = (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})^*$. Here we have used (5.5) in the first equality. In the second equality we have used the antisymmetry of c and the fact that the elements $\zeta'', \zeta \in N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ are equal to zero on $\mathfrak{t}_{\mathrm{h}}$. In other words, $c_{\mathrm{f}}(\cdot, \zeta') \in (\mathrm{Sym}_{\mathrm{f}})|_P$ if and only if $\zeta' = 0$ on $c(P \times P)$, or equivalently $\zeta' = 0$ on the linear subspace of $\mathfrak{l}$ which is spanned by $c(N \times N)$. Let c^0 denote the space of all $\zeta' \in \mathfrak{l}^*$ which are equal to zero on the linear span of $c(N \times N)$. In view of (5.5) we have $\ker c \subset c^0$, and it follows that the dimension of $c_{\mathrm{f}}(\cdot, N) \cap (\mathrm{Sym}_{\mathrm{f}})|_P$ is equal to $\dim c^0 - \dim\ker c$, whereas the dimension of $c_{\mathrm{f}}(\cdot, N)$ is equal to $\dim N - \dim\ker c_{\mathrm{f}}$. It follows that the dimension of $c_{\mathrm{f}}(\cdot, N) + (\mathrm{Sym}_{\mathrm{f}})|_P$ is equal to $d_N(d_N+1)/2 - (\dim c^0 - \dim\ker c)$, in which $d_N = \dim N = \dim(\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$. Therefore the codimension of $c_{\mathrm{f}}(\cdot, N) + (\mathrm{Sym}_{\mathrm{f}})|_P$ in the ${d_N}^2$-dimensional space $\mathrm{Hom}(P, \mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$ is equal to $d_N(d_N - 3)/2 + \dim\ker c_{\mathrm{f}} - \dim\ker c + \dim c^0$. Because all elements of $\mathcal{A}$ map to $\mathfrak{l}$, it follows that

$$\dim \mathcal{T} = d_N(\dim T - d_N) + d_N(d_N - 3)/2 + \dim\ker c_{\mathrm{f}} - \dim\ker c + \dim c^0. \tag{7.21}$$

8. Applications of the global model

Let (M, σ) be our compact connected symplectic manifold, together with an effective action of the torus T by means of symplectomorphisms of (M, σ), such that some (all) principal orbits of the T-action are coisotropic submanifolds of (M, σ).

In this section, which is not needed for the classification in Section 9, we give some applications of Proposition 7.2 to minimal coupling, the reduced phase spaces, the topology of the torus action, and to the universal covering of our symplectic T-space M.

8.1. Minimal coupling

The fibration of M by Delzant manifolds is a fibration by symplectic submanifolds with a structure group H which acts on the fiber by means of symplectomorphisms. See Proposition 7.2. Moreover, the distribution spanned by the Z_M, $Z \in \mathfrak{t}_{\mathrm{f}}$, and the L_ζ, $\zeta \in N$, which we used in the construction of the model, is the symplectic orthogonal complement of the fibers. This follows from the fact that at the regular points the tangent space to the Delzant submanifold is spanned by the Y_M, $Y \in \mathfrak{t}_{\mathrm{h}}$, and the L_η, $\eta \in C$, combined with Lemma 2.1 and $\mathfrak{t}_{\mathrm{h}} \subset \mathfrak{l} := \ker \sigma^{\mathfrak{t}}$, the equation (5.2), and v) in Proposition 5.5. Because the Z_M, $Z \in \mathfrak{t}_{\mathrm{f}}$, commute with each other and with the L_ζ, $\zeta \in N$, the only nonzero Lie brackets of horizontal vector fields are the $[L_\zeta, L_\zeta] = c(\zeta, \zeta')_M$, $\zeta, \zeta' \in N$, see iii) in Proposition 5.5. The vertical part of $[L_\zeta, L_\zeta]$ is equal to $c_{\mathrm{h}}(\zeta, \zeta')_M$. Because, for every $Y \in \mathfrak{t}_{\mathrm{h}}$, Y_M is the Hamiltonian vector field defined by the function $x \mapsto \mu(x)(Y)$, the vertical part of $[L_\zeta, L_\zeta]$ is the Hamiltonian vector field defined by the function $x \mapsto \mu(x)(c_{\mathrm{h}}(\zeta, \zeta'))$. It follows from vi) in Proposition 5.5 that the derivative of this function is equal to the negative of the derivative of $\sigma(L_\zeta, L_{\zeta'})$, and therefore the vertical part of $[L_\zeta, L_{\zeta'}]$ is equal to $-\operatorname{Ham}_{\sigma(L_\zeta, L_{\zeta'})}$. This equation, which holds in great generality for the curvature of the symplectically orthogonal connection in a fibration by symplectic manifolds, is known as *minimal coupling*, see Guillemin, Lerman and Sternberg [21, Sec. 1.3]. In this way equation vi) in Proposition 5.5 represents the minimal coupling term in the symplectic form on M. This observation was suggested to us by Yael Karshon.

Recall that the fibration of M by Delzant manifolds was not a priori given. It has been constructed using the special admissible connection introduced in Proposition 5.5, and it is not unique if $\{1\} \neq T_{\mathrm{h}} \neq T$.

8.2. The reduced phase spaces

On the symplectic manifold (M, σ) we have the Hamiltonian action of the torus T_{h}, with momentum mapping $\mu : M \to {\mathfrak{t}_{\mathrm{h}}}^*$, where $\mu(M) \simeq \Delta$. Let $q \in \mu(M)$. Then, restricting the discussion to the orbit type stratum which contains $\mu^{-1}(\{q\})$, we obtain that $\mu^{-1}(\{q\})$ is a compact and connected smooth submanifold of M, on which T_{h}/H acts freely, where H denotes the common stabilizer subgroup of the elements in $\mu^{-1}(\{q\})$. It follows that the orbit space $M^q := \mu^{-1}(\{q\})/T_{\mathrm{h}}$ has a unique structure of a compact connected smooth manifold, such that the projection $\pi^q : \mu^{-1}(\{q\}) \to M^q$ is a principal T_{h}/H-fibration.

At each point of $\mu^{-1}(\{q\})$, the kernel of the pull-back to $\mu^{-1}(\{q\})$ of σ is equal to the tangent space of the T_{h}-orbit through that point, and it follows that there is a unique symplectic form σ^q on M^q such that $(\pi^q)^*\sigma^q = (\iota^q)^*\sigma$, if ι^q denotes the inclusion mapping from $\mu^{-1}(\{q\})$ to M. The symplectic manifold (M^q, σ^q) is called the *reduced phase space at the μ-value q* for the Hamiltonian action of T_{h} on (M, σ).

On M^q we still have the action of the torus T/T_{h}, which is free, leaves the symplectic form σ^q invariant, and has coisotropic orbits. The vector fields L_ζ, $\zeta \in N$, are tangent to $\mu^{-1}(\{q\})$, and are intertwined by π^q with unique smooth vector fields L^q_ζ on M^q. In combination with $(\pi^q)^*\sigma^q = (\iota^q)^*\sigma$, this implies that $(\pi^q)^*(\sigma^q(L^q_\zeta, L^q_{\zeta'})) = (\iota^q)^*(\sigma(L_\zeta, L_{\zeta'}))$, as an identity between constant functions on $\mu^{-1}(\{q\})$. It therefore follows from vi) in Proposition 5.5 that

$$\sigma^q(L^q_\zeta, L^q_{\zeta'}) = -q(c_{\mathrm{h}}(\zeta, \zeta')), \quad \zeta, \zeta' \in N. \tag{8.1}$$

We now show that each of the reduced phase spaces $M^q = \mu^{-1}(\{q\})/T_{\mathrm{h}}$ can be identified with the G-homogeneous space $G/H \simeq ((T/T_{\mathrm{h}}) \times N)/\iota(P)$ discussed in Remark 7.3. Moreover, if $c(N \times N) \subset \mathfrak{t}_{\mathrm{h}}$, then (8.1) corresponds to the description of the variation of the cohomology class of the symplectic form of the reduced phase spaces in Duistermaat and Heckman [14].

Let $x \in \mu^{-1}(\{q\})$, and write

$$H_x = \{(t, \zeta) \in T \times P \mid t\tau_\zeta \in T_x\}.$$

Because T_x is a closed Lie subgroup of T_{h}, H_x is a closed Lie subgroup of H, see (7.11), and G/H_x is a compact G-homogeneous space. The mapping $A_x : (t, \zeta) \mapsto t \cdot \mathrm{e}^{L_\zeta}(x) : G \to M$ induces an embedding α_x from G/H_x into M, with image equal to $\mu^{-1}(\{q\})$. This exhibits $\mu^{-1}(\{q\})$ as a compact and connected smooth submanifold of M, and actually as a G-homogeneous space. The pull-back to G/H_x of the symplectic form σ is given by the formula (7.14), in which $\delta x = \delta' x = 0$.

Because $T_{\mathrm{h}}/T_x \simeq H/H_x$, the mapping α_x induces a T/T_{h}-equivariant diffeomorphism β_x from $G/H = (G/H_x)/(H/H_x)$ onto the reduced phase space $M^q = \mu^{-1}(\{q\})/(T_{\mathrm{h}}/T_x)$. Because the dimension of $\mu^{-1}(\{q\})$ jumps down if $q \in \Delta$ moves into a lower-dimensional orbit type stratum, it is quite remarkable that nevertheless the reduced phase spaces M^q for all $q \in \Delta$ are isomorphic to the same space G/H in a natural way. In this model the principal T_{h}/T_x-fibration $\mu^{-1}(\{q\}) \to M^q$ corresponds to the principal H/H_x-fibration $G/H_x \to G/H$, in which $T_{\mathrm{h}}/T_x \simeq H/H_x$ is a torus.

If $c(N \times N) \subset \mathfrak{t}_{\mathrm{h}}$, then the Chern class of the principal H/H_x-fibration $\pi_x : G/H_x \to G/H$, which is an element of $\mathrm{H}^2(G/H, (H/H_x)_{\mathbb{Z}})$ is equal to $\psi^* c$, in which $c \in \mathrm{H}^2(N/P, T_{\mathbb{Z}})$ is the cohomology class corresponding to the antisymmetric bilinear form c introduced in Proposition 5.5, and ψ is the projection from G/H onto $(G/H)/(T/T_{\mathrm{h}}) \simeq N/P$. Therefore, in the case that $c(N \times N) \subset \mathfrak{t}_{\mathrm{h}}$, formula (8.1) shows that the variation of the cohomology class of the symplectic form of the reduced phase spaces is equal to the cohomology class $-c$ of the curvature form.

8.3. The T_{h}-fixed point set modulo T_{f}

The action of the Hamiltonian torus T_{h} on M has fixed points, which are the $x \in M$ such that $\mu(x)$ is equal to a vertex v of the Delzant polytope Δ, if $\mu : M \to \Delta \subset {\mathfrak{t}_{\mathrm{h}}}^*$ denotes the momentum mapping of the Hamiltonian T_{h}-action as in (5.4). Let v be a vertex of Δ. Because $T_x = T_{\mathrm{h}}$ for every $x \in \mu^{-1}(\{v\})$, the reduced phase space $\mu^{-1}(\{v\})/T_{\mathrm{h}}$ at the level v, introduced in Subsection 8.2, is equal to $\mu^{-1}(\{v\})$. Because the reduced phase spaces are connected, the $\mu^{-1}(\{v\})$, where v ranges over the vertices of Δ, are the connected components $\mathcal{F}$ of the fixed point set $M^{T_{\mathrm{h}}}$ of the T_{h}-action in M. Because in this subsection we want to find invariants of the T-action, disregarding the symplectic structure, we use the notation $\mathcal{F}$ for the connected components of $M^{T_{\mathrm{h}}}$, instead of $\mu^{-1}(\{v\})$.

Note that each $\mathcal{F}$ is a global section of the fibration $\delta : M \to G/H \simeq ((T/T_{\mathrm{h}}) \times N)/\iota(P)$ of M by Delzant submanifolds. Using Morse theory with the Hamiltonian functions of infinitesimal T_{h}-actions as Bott-Morse functions, this may lead to useful information about the topology of M in terms of the connected components $\mathcal{F}$ of $M^{T_{\mathrm{h}}}$.

Let T_{f} be a complementary torus to T_{h} in T. Because the action of T_{f} is free, we have the principal T_{f}-fibration $M \to M/T_{\mathrm{f}}$, and because the actions of T_{f} and T_{h} commute, we have an induced action of T_{h} on M/T_{f}. The manifolds $\mathcal{F}/T_{\mathrm{h}}$ are the connected components of the fixed point set $(M/T_{\mathrm{f}})^{T_{\mathrm{h}}}$ of the T_{f}-action in M/T_{f}. The fibration $\delta : M \to G/H \simeq ((T/T_{\mathrm{h}}) \times N)/\iota(P)$ induces a fibration

$$\overline{M} := M/T_{\mathrm{f}} \to (G/H)/T_{\mathrm{f}} \simeq (T_{\mathrm{f}} \times N)/\iota(P))/T_{\mathrm{f}} \simeq N/P$$

by Delzant manifolds, of which each connected component $\overline{\mathcal{F}} := \mathcal{F}/T_{\mathrm{f}}$ of the T_{h}-fixed point set is a global section, diffeomorphic to N/P.

Let $x \in \mathcal{F}$, and write $y = T_{\mathrm{f}} \cdot x \in \overline{\mathcal{F}}$. The tangent action of T_{h} on the normal space $N_y := \mathrm{T}_y \overline{M} / \mathrm{T}_y \overline{\mathcal{F}}$ to $\overline{\mathcal{F}}$ can be identified with the tangent action of T_{h} on the tangent space of the Delzant manifold through x. It follows from the local model in Lemma 2.10, with $H = T_x = T_{\mathrm{h}}$, $\mathfrak{h} = \mathfrak{t}_{\mathrm{h}}$, and $m = d_{\mathrm{h}} := \dim T_{\mathrm{h}}$, that N_y has a direct sum decomposition into T_{h}-invariant two-dimensional linear subspaces E_y^j, $1 \leqslant j \leqslant d_{\mathrm{h}}$, a complex structure on each E_y^j, and a corresponding $\mathbb{Z}$-basis Y_j, $1 \leqslant j \leqslant d_{\mathrm{h}}$ of the integral lattice $(T_{\mathrm{h}})_{\mathbb{Z}}$ in $\mathfrak{t}_{\mathrm{h}}$, such that the tangent action of e^Y, $Y \in \mathfrak{t}_{\mathrm{h}}$ on N_y corresponds to the multiplication with $\mathrm{e}^{2\pi\,\mathrm{i}\,Y^j}$ in E_y^j, if $Y = \sum_{j=1}^{n_1} Y^j Y_j$. Although for their existence we referred to the local model in Lemma 2.10 for our symplectic T-space, all these ingredients are uniquely determined in terms of the linearized action of T_{h} on the normal bundle N of $\overline{\mathcal{F}} := \mathcal{F}/T_{\mathrm{f}}$

in $\overline{M} := M/T_{\mathrm{f}}$, up to a permutation of the indices j. That is, disregarding the symplectic structure.

For each j, the E^j_y, $y \in \overline{\mathcal{F}}$, form a complex line bundle E^j over $\overline{\mathcal{F}} \simeq N/P$, and the normal bundle N of $\overline{\mathcal{F}}$ in $\overline{M}$ is the direct sum of the complex line bundles E^j, $1 \leqslant j \leqslant d_{\mathrm{h}}$.

Any smooth complex line bundle L over a smooth manifold B has a Chern class, which is defined as follows. Let $\mathbb{C}^\times$ denote the multiplicative group of the nonzero complex numbers. The transition functions of local trivializations define a 1-cocycle of germs of smooth $\mathbb{C}^\times$-valued functions, and the bundle L is classified by the sheaf (= Čech) cohomology class $\gamma \in \mathrm{H}^1(B, \mathrm{C}^\infty(\cdot, \mathbb{C}^\times))$ of the 1-cocycle of the transition functions. Because the sheaf $\mathrm{C}^\infty(\cdot, \mathbb{C})$ is fine, the short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{C}^\infty(\cdot, \mathbb{C}) \overset{\mathrm{e}^{2\pi\,\mathrm{i}}}{\to} \mathrm{C}^\infty(\cdot, \mathbb{C}^\times) \to 1$$

induces an isomorphism $\delta : \mathrm{H}^1(B, \mathrm{C}^\infty(\cdot, \mathbb{C}^\times) \to \mathrm{H}^2(B, \mathbb{Z})$, and the *Chern class of the complex line bundle L over B* is defined as the cohomology class $\mathrm{c}(L) := \delta(\gamma) \in \mathrm{H}^2(B, \mathbb{Z})$. With these definitions, we have the following conclusions.

Proposition 8.1. — *Let the $c^j_{\mathrm{h}} \in \Lambda^2 N^*$, $1 \leqslant j \leqslant d_{\mathrm{h}}$, be defined by*

$$c_{\mathrm{h}}(\zeta, \zeta') = \sum_{j=1}^{d_{\mathrm{h}}} c^j_{\mathrm{h}}(\zeta, \zeta') Y_j, \quad \zeta, \zeta' \in N.$$

Let $1 \leqslant j \leqslant d_{\mathrm{h}}$. Viewing c^j_{h} as an element of $\mathrm{H}^2(N/P, \mathbb{R}) \simeq \mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$ as in Corollary 3.10, we have that c^j_{h} is equal to the image in $\mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$ under the coefficient homomorphism $\mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{Z}) \to \mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$, of the Chern class $\mathrm{c}(E^j)$ of the complex line bundle E^j over $\overline{\mathcal{F}} \simeq N/P$.

If M is T-equivariantly diffeomorphic to $M_{\mathrm{f}} \times M_{\mathrm{h}}$, in which T_{h} acts only on M_{h} with isolated fixed points, and T_{f} acts only on M_{f} and freely, then $c_{\mathrm{h}} = 0$, and we have the conclusions c) and a) in Lemma 7.5.

Proof. — Because the vector fields L_ζ, $\zeta \in N$, are invariant under the action of T, hence under the action of T_{h} and T_{f}, they are intertwined by the projection $M \to \overline{M} := M/T_{\mathrm{f}}$ to uniquely determined T_{h}-invariant smooth vector fields on $\overline{M}$, which we also denote by L_ζ. The identity iii) in Proposition 5.5 leads to the identity $[L_\zeta, L_{\zeta'}] = c_{\mathrm{h}}(\zeta, \zeta')_{\overline{M}}$ for vector fields on $\overline{M}$. Because the L_ζ are T_{h}-invariant, their flows leave each connected component $\overline{\mathcal{F}}$ of the T_{h}-fixed point set $\overline{M}^{T_{\mathrm{h}}}$ invariant, and their linearizations define automorphisms of the normal bundle N of $\overline{\mathcal{F}}$ in $\overline{M}$ which commute with the linearized action of T_{h} on N. Therefore these automorphisms leave each of the complex line bundles E^j invariant, and the corresponding infinitesimal automorphisms define vector fields on E_j which we again denote by L_ζ. Because the L_ζ are lifts of the constant vector fields ζ on N/P, we conclude that $N \ni \zeta \mapsto L_\zeta$ is a $\mathbb{T}$-invariant connection in E^j, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. Because the cohomology class in $\mathrm{H}^2(N/P, \mathbb{R}) \simeq \mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$ of the negative of the curvature form is equal to the image of $\mathrm{c}(E^j)$ in $\mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$ under the coefficient homomorphism $\mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{Z}) \to \mathrm{H}^2(\overline{\mathcal{F}}, \mathbb{R})$, the first statement in the proposition follows from the combination of the above discussions with the identifications in Remark 5.7 and Remark 5.8, and the general facts about Chern classes of complex line bundles as for instance in Bott and Tu [8, pp. 270, 267, 72, 73].

For the second statement assume that $M = M_{\mathrm{f}} \times M_{\mathrm{h}}$, in which T_{h} acts only on M_{h} and has isolated fixed points, and T_{f} acts freely on M_{f}. Then $\overline{M} := (M_{\mathrm{f}} \times M_{\mathrm{h}})/T_{\mathrm{f}} = (M_{\mathrm{f}}/T_{\mathrm{f}}) \times M_{\mathrm{h}}$, in which T_{h} only acts on the second component. It follows that the connected components of $\overline{M}^{T_{\mathrm{h}}}$ are of the form $\overline{\mathcal{F}} = (M_{\mathrm{f}}/T_{\mathrm{f}}) \times \{x\}$, in which x ranges over the isolated fixed points of the T_{h}-action on M_{h}, and the normal bundle N of $\overline{\mathcal{F}}$ in $\overline{M}$ is T_{h}-equivariantly isomorphic to $(M_{\mathrm{f}}/T_{\mathrm{f}}) \times \mathrm{T}_x M_{\mathrm{h}}$. This shows that each of the complex line bundles E^j is trivial, which implies that $\mathrm{c}(E^j) = 0$ and therefore $c_{\mathrm{h}}^j = 0$ in view of the first statement in the proposition. Because this holds for every $1 \leqslant j \leqslant d_{\mathrm{h}}$, it follows that $c_{\mathrm{h}} = 0$. □

8.4. A universal covering of M

In Proposition 8.2 below, we describe an explicit universal covering of the manifold M by a Cartesian product $\widetilde{M}$ of a vector space and the Delzant manifold M_{h}, which leads to an explicit description of the fundamental group of M. In Remark 8.4 we recover Corollaire 6.16 of Benoist [6], which states that the universal cover of a compact connected symplectic T-manifold with coisotropic principal orbits is $(\mathfrak{t}_{\mathrm{f}} \times T_{\mathrm{h}})$-equivariantly symplectomorphic to the Cartesian product of a symplectic vector space and a Delzant manifold.

Let ε^l, $1 \leqslant l \leqslant d_N := \dim N$, be a $\mathbb{Z}$-basis of the period group P in N. If $\zeta, \zeta' \in N$ have coordinates ζ_l, ζ'_l with respect to this basis, then we write

$$b(\zeta, \zeta') := \sum_{l<l'} \zeta_l \zeta'_{l'} c^{ll'}, \quad c^{ll'} := c(\varepsilon^l, \varepsilon^{l'}). \tag{8.2}$$

This defines a bilinear mapping $b : N \times N \to \mathfrak{l}$ such that $c(\zeta, \zeta') = b(\zeta, \zeta') - b(\zeta', \zeta)$. We have $\zeta, \zeta' \in P$ if and only if $\zeta_l, \zeta'_l \in \mathbb{Z}$ for all l. Therefore $c(P \times P) \subset T_{\mathbb{Z}}$, see Lemma 7.1, implies that $b(P \times P) \subset T_{\mathbb{Z}}$.

Let $x \in M_{\mathrm{h}}$. For each $1 \leqslant l \leqslant d_N$ we choose $X^l \in \mathfrak{t}$ such that $\tau_{\varepsilon^l}(x) = \mathrm{e}^{X^l}$. Then (7.9) implies that, for each $\zeta \in P$,

$$\tau_\zeta := \tau_\zeta(x) = \mathrm{e}^{b(\zeta,\zeta)+\zeta_l X^l}, \tag{8.3}$$

where in the second term in the exponent we use Einstein's summation convention.

Let T_{f} be the complementary torus to the Hamiltonian torus T_{h} in T which has been used in Proposition 5.5. Finally, let Z_j, $1 \leqslant j \leqslant d_{\mathrm{f}} := \dim T_{\mathrm{f}}$, be a $\mathbb{Z}$-basis of the integral lattice $(T_{\mathrm{f}})_{\mathbb{Z}}$ in the Lie algebra $\mathfrak{t}_{\mathrm{f}}$ of T_{f}. For any $X \in \mathfrak{t}$ we denote by X_{h} and X_{f} the $\mathfrak{t}_{\mathrm{h}}$-component and the $\mathfrak{t}_{\mathrm{f}}$-component of X, respectively. With these notations, we have the following conclusions.

Proposition 8.2. — *The lattice $\Gamma := (T_{\mathrm{f}})_{\mathbb{Z}} \times P$ is a group with respect to the multiplication defined by*

$$(B', \beta')(B, \beta) = (B + B' - b_{\mathrm{f}}(\beta, \beta'), \beta + \beta') \quad (B, \beta), (B', \beta') \in \Gamma. \tag{8.4}$$

Let $\widetilde{M} := (\mathfrak{t}_{\mathrm{f}} \times N) \times M_{\mathrm{h}}$. Let $(B, \beta) \in \Gamma$ act on $\widetilde{M}$ by sending $((Z, \zeta), x)$ to $((Z', \zeta'), x')$, where

$$\begin{aligned} Z' &= Z + B - \beta_l {X^l}_{\mathrm{f}} + b_{\mathrm{f}}(\beta, \beta)/2 + c_{\mathrm{f}}(\beta, \zeta)/2, \quad \zeta' = \zeta + \beta, \\ x' &= (\mathrm{e}^{c_{\mathrm{h}}(\beta,\zeta)/2}(\tau_{-\beta})_{\mathrm{h}}) \cdot x. \end{aligned} \tag{8.5}$$

This defines a proper and free action of Γ on $\widetilde{M}$, and the mapping

$$\widetilde{A} : ((Z, \zeta), x) \mapsto \mathrm{e}^Z \cdot \mathrm{e}^{L_\zeta}(x) : \widetilde{M} \to M. \tag{8.6}$$

is a universal covering of M with the action of Γ on $\widetilde{M}$ as the covering group.

Let $x \in M_{\mathrm{h}}$ and let $\pi_1(M, x)$ be the fundamental group of M with base point x. For any homotopy class $[\gamma]$ of a closed loop γ based at x, let $\iota_x([\gamma])$ be the element of Γ of which the action on $\widetilde{M}$ is equal to the covering transformation defined by γ. Let γ_j be the closed loop $\mathrm{e}^{tZ_j} \cdot x$, $0 \leqslant t \leqslant 1$, and let δ^l be the closed loop based at x which consists of $\mathrm{e}^{tL_{\varepsilon^l}}(x)$, $0 \leqslant t \leqslant 1$, followed by $\mathrm{e}^{(1-t)X^l} \cdot x$, $0 \leqslant t \leqslant 1$. Then the isomorphism $\iota_x : \pi_1(M, x) \xrightarrow{\sim} \Gamma$ is uniquely determined by the condition that $\iota_x([\gamma_j]) = (Z_j, 0)$, $1 \leqslant j \leqslant d_{\mathrm{f}}$, and $\iota_x([\delta^l]) = (0, \varepsilon^l)$, $1 \leqslant l \leqslant d_N$.

Proof. — Let $y \in M$. The surjectivity of the mapping A in (7.10), see Proposition 7.2, implies that there exist $t \in T$, $\zeta \in N$, $x \in M_{\mathrm{h}}$, such that $y = t \cdot \mathrm{e}^{L_\zeta}(x)$. We have $t = t_{\mathrm{h}} t_{\mathrm{f}}$ with $t_{\mathrm{h}} \in T_{\mathrm{h}}$ and $t_{\mathrm{f}} \in T_{\mathrm{f}}$, and subsequently there exists $Z \in \mathfrak{t}_{\mathrm{f}}$ such that $t_{\mathrm{f}} = \mathrm{e}^Z$. Because $(t_{\mathrm{h}})_M$ commutes with $(t_{\mathrm{f}})_M \circ \mathrm{e}^{L_\zeta}$ and leaves M_{h} invariant, it follows that $y = \widetilde{A}((Z, \zeta), t_{\mathrm{h}} \cdot x)$, which proves that the mapping $\widetilde{A}$ is surjective.

Let $\widetilde{A}((Z, \zeta), x) = \widetilde{A}((Z', \zeta'), x')$. The injectivity of the mapping α in Proposition 7.2 implies that there exist $(s, -\beta) \in H$ such that $(\mathrm{e}^{Z'}, \zeta') = (\mathrm{e}^Z, \zeta)(s, -\beta)^{-1}$ and $x' = (s\tau_{-\beta}) \cdot x$. In view of (7.5) and (7.11), this implies that $\beta \in P$, $s\tau_{-\beta} \in T_{\mathrm{h}}$, $\mathrm{e}^{Z'} = \mathrm{e}^Z\, s^{-1}\, \mathrm{e}^{c(\beta,\zeta)/2}$, $\zeta' = \zeta + \beta$, and $x' = s \cdot \tau_{-\beta} \cdot x$. In view of ${s_{\mathrm{f}}}^{-1} = (\tau_{-\beta})_{\mathrm{f}}$ and (8.3) with $\zeta = -\beta$, the T_{f}-part and the T_{h}-part of the equation for Z' mean that

$$Z' \in Z - \beta_l {X^l}_{\mathrm{f}} + b_{\mathrm{f}}(\beta, \beta)/2 + c_{\mathrm{f}}(\beta, \zeta)/2 + (T_{\mathrm{f}})_{\mathbb{Z}}$$

and $s_{\mathrm{h}} = \mathrm{e}^{c_{\mathrm{h}}(\beta,\zeta)/2}$, respectively.

It follows that the fibers of $\widetilde{A}$ are the Γ-orbits, if we let $(B, \beta) \in \Gamma$ act on $\widetilde{M}$ as in (8.5). Note that $\zeta' = \zeta$ implies that $\beta = 0$, and then $Z' = Z$ implies that $B = 0$. Therefore the action of Γ on $\widetilde{M}$ is free, which implies that it is effective, in the sense that the mapping from Γ to the set of diffeomorphisms of $\widetilde{M}$ is injective.

There is a group structure on Γ for which the action of Γ is a group action, a homomorphism from Γ to the group of diffeomorphisms of $\widetilde{M}$, if and only if the composition of the actions of two elements of Γ is an action of an element of Γ. The effectiveness of the action implies that if this is the case, then the group structure on Γ for which this holds is unique.

If we let (B', β') act on (8.5), then we arrive at $((Z'', \zeta''), x'')$, in which

$$\begin{aligned} Z'' &= Z + B - \beta_l {X^l}_{\mathrm{f}} + b_{\mathrm{f}}(\beta, \beta)/2 + c_{\mathrm{f}}(\beta, \zeta)/2 \\ &\quad + B' - \beta'_l {X^l}_{\mathrm{f}} + b_{\mathrm{f}}(\beta', \beta')/2 + c_{\mathrm{f}}(\beta', \zeta + \beta)/2 \\ &= Z + B + B' - b_{\mathrm{f}}(\beta, \beta') \\ &\quad - (\beta + \beta')_l {X^l}_{\mathrm{f}} + b_{\mathrm{f}}(\beta + \beta', \beta + \beta')/2 + c_{\mathrm{f}}(\beta + \beta', \zeta)/2, \end{aligned}$$

$\zeta'' = \zeta + (\beta + \beta')$, and

$$x'' = \mathrm{e}^{c_{\mathrm{h}}(\beta', \zeta+\beta)/2} \cdot (\tau_{-\beta'})_{\mathrm{h}} \cdot \mathrm{e}^{c_{\mathrm{h}}(\beta,\zeta)/2} \cdot (\tau_{-\beta})_{\mathrm{h}} \cdot x = \mathrm{e}^{c_{\mathrm{h}}(\beta+\beta',\zeta)/2} \cdot (\tau_{-(\beta+\beta')})_{\mathrm{h}} \cdot x.$$

Here we have used that $c(\beta', \beta) = b(\beta', \beta) - b(\beta, \beta')$ in the equation for Z''. Furthermore, in the equation for x'' we have used (7.8) and the fact that if $\beta, \beta' \in P$, then $c(\beta', \beta) \in T_{\mathbb{Z}}$, hence $c_{\mathrm{h}}(\beta', \beta) \in (T_{\mathrm{h}})_{\mathbb{Z}}$, and therefore $\mathrm{e}^{c_{\mathrm{h}}(\beta',\beta)} = 1$. This proves that Γ is a group with respect to the multiplication defined by (8.4), and that (8.5) defines a group action of Γ on $\widetilde{M}$.

Because the action of Γ on $\widetilde{M}$ is obviously proper and free, we conclude that $\widetilde{A}$ is a covering with covering group equal to the action of Γ. Because $\widetilde{M}$ is simply connected as the Cartesian product of a vector space and the simply connected Delzant manifold M_{h}, see Lemma 6.4, $\widetilde{M}$ is a universal covering of M.

It follows from general facts about universal coverings, see for instance Greenberg [19, Sec. 5], that ι_x is an isomorphism from $\pi_1(M, x)$ onto Γ. Finally $\widetilde{A}$ maps the curve $((tZ_j, 0), x)$, $0 \leqslant t \leqslant 1$, which runs from $((0,0), x)$ to $((Z_j, 0), x)$, to γ_j. Furthermore $\widetilde{A}$ maps the curve $((0, t\varepsilon^l), x)$, $0 \leqslant t \leqslant 1$, followed by the curve $((-t{X^l}_{\mathrm{f}}, 0), \mathrm{e}^{-t{X^l}_{\mathrm{h}}} \cdot x)$, $0 \leqslant t \leqslant 1$, which runs from $((0,0), x)$ to

$$((-{X^l}_{\mathrm{f}}, \varepsilon^l), \mathrm{e}^{-{X^l}_{\mathrm{h}}} \cdot x) = (0, \varepsilon^l) \cdot ((0,0), x),$$

to δ^l. This shows that $\iota_x([\gamma_j]) = (Z_j, 0)$ and $\iota_x([\delta^l]) = (0, \varepsilon^l)$. Because the elements $(Z_j, 0)$, $1 \leqslant j \leqslant d_{\mathrm{f}}$, and $(0, \varepsilon^l)$, $1 \leqslant l \leqslant d_N$, together generate Γ, this proves the last statement in the proposition. □

Viewing $\mathfrak{t}_{\mathrm{f}}$ as an additive group, the connected commutative Lie group $U := \mathfrak{t}_{\mathrm{f}} \times T_{\mathrm{h}}$ acts on $\widetilde{M}$, where $(Z', t) \in U$ sends $((Z, \zeta), x)$ to $((Z + Z', \zeta), t \cdot x)$. The covering map $\widetilde{A} : \widetilde{M} \to M$ intertwines the U-action on $\widetilde{M}$ with the T-action on M via the covering homomorphism $\epsilon : (Z', t) \mapsto \mathrm{e}^{Z'}\, t : U \to T$, in the sense that $\widetilde{A}(u \cdot p) = \epsilon(u) \cdot \widetilde{A}(p)$ for every $p \in \widetilde{M}$ and $u \in U$.

Corollary 8.3. — *The fundamental group of M is isomorphic to the set $(T_{\mathrm{f}})_{\mathbb{Z}} \times P$, with the group structure defined by (8.4). This group is commutative if and only if $c(P \times P) \subset (T_{\mathrm{h}})_{\mathbb{Z}}$.*

The first homology group $\mathrm{H}_1(M, \mathbb{Z})$ of M with coefficients in $\mathbb{Z}$ is isomorphic to $((T_{\mathrm{f}})_{\mathbb{Z}}/\Theta) \times P$, in which Θ denotes the additive subgroup of $(T_{\mathrm{f}})_{\mathbb{Z}}$ which is generated by the elements $c_{\mathrm{f}}(\beta, \beta')$, such that $\beta, \beta' \in P$. The first Betti number $\dim \mathrm{H}_1(M, \mathbb{R})$ is equal to $\dim M - 2 \dim T_{\mathrm{h}} - \operatorname{rank} \Theta$.

Proof. — A straightforward computation shows that $(B, \beta)^{-1} = (-B - b(\beta, \beta)_{\mathrm{f}}, -\beta)$, that

$$(B, \beta)^{-1} (B', \beta') (B, \beta) = (B' + c_{\mathrm{f}}(\beta', \beta), \beta'),$$

and that the commutator $(B', \beta')^{-1} (B, \beta)^{-1} (B', \beta') (B, \beta)$ is equal to $(c_{\mathrm{f}}(\beta', \beta), 0)$. Therefore the subgroup of $\Gamma = (T_{\mathrm{f}})_{\mathbb{Z}} \times P$ generated by the commutators is equal to $\Theta \times \{0\}$, and Γ is commutative if and only if $\Theta = \{0\}$ if and only if $c(P \times P) \subset T_{\mathbb{Z}} \cap \mathfrak{t}_{\mathrm{h}} = (T_{\mathrm{h}})_{\mathbb{Z}}$.

The canonical homomorphism $\pi_1(M, x) \to \mathrm{H}_1(M, \mathbb{Z})$ is surjective with kernel equal to the subgroup of $\pi_1(M, x)$ generated by the commutators, see Greenberg [19, Th. 12.1]. This induces an isomorphism from $((T_{\mathrm{f}})_{\mathbb{Z}} \times P)/(\Theta \times \{0\}) = ((T_{\mathrm{f}})_{\mathbb{Z}}/\Theta) \times P$ onto $\mathrm{H}_1(M, \mathbb{Z})$. Finally the universal coefficient theorem, *cf.* Greenberg [19, Th. 29.12] implies that for any principal ideal domain R, in particular for $R = \mathbb{R}$, $\mathrm{H}_1(M, R)$ is isomorphic to

$\mathrm{H}_1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Therefore

$$\dim \mathrm{H}_1(M, \mathbb{R}) = d_{\mathrm{f}} - \operatorname{rank} \Theta + d_N = \dim T - \dim T_{\mathrm{h}} - \operatorname{rank} \Theta + \dim \mathfrak{l} - \dim T_{\mathrm{h}},$$

which is equal to $\dim M - 2 \dim T_{\mathrm{h}} - \operatorname{rank} \Theta$ in view of Lemma 2.3. □

It follows that the rank of Θ is a purely topological feature of M, disregarding both the T-action and the symplectic structure on M. Note that the generators of Γ mentioned in Proposition 8.2 were defined in terms of the action of T_{f} and the L_{ε^l}, where the latter were defined in terms of both the T-action and the symplectic form on M.

Remark 8.4. — The symplectic form $\widetilde{A}^* \sigma$ on the universal covering $\widetilde{M} = \mathfrak{t}_{\mathrm{f}} \times N \times M_{\mathrm{h}}$ is given by (7.14), in which a, δa, $\delta' a$ are replaced by $((Z, \zeta), x)$, $((\delta Z, \delta \zeta), \delta x)$, $((\delta' Z, \delta' \zeta), \delta' x)$, respectively, with $Z, \delta Z, \delta' Z \in \mathfrak{t}_{\mathrm{f}}$. We view the linear form

$$\mu(x) c_{\mathrm{h}}(\cdot, \zeta) : \delta \zeta \mapsto \mu(x)(c_{\mathrm{h}}(\delta \zeta, \zeta))$$

on $N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ as an element of $((\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*)^* \simeq \mathfrak{l}/\mathfrak{t}_{\mathrm{h}} \simeq \mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$. Let $\Psi : \widetilde{N} \to \widetilde{N}$ be defined by

$$\Psi : ((Z, \zeta), x) = ((Z + \mu(x) c_{\mathrm{h}}(\cdot, \zeta)/2, \zeta), x), \ ((Z, \zeta), x) \in \widetilde{M} = (\mathfrak{t}_{\mathrm{f}} \times N) \times M_{\mathrm{h}}.$$

Then Ψ is a diffeomorphism from $\widetilde{M}$ onto $\widetilde{M}$, and the symplectic form $\nu := \Psi^*(\widetilde{A}^* \sigma)$ on $\widetilde{N}$ is given by

$$\nu_a(\delta a, \delta' a) = \sigma^{\mathfrak{t}}(\delta Z, \delta' Z) + \delta \zeta(\delta' Z_{\mathfrak{l}}) - \delta' \zeta(\delta Z_{\mathfrak{l}}) + (\sigma_{\mathrm{h}})_x(\delta x, \delta' x).$$

That is, $(\widetilde{M}, \nu)$ is equal to the Cartesian product of a symplectic vector space $(\mathfrak{t}_{\mathrm{f}} \times N, \sigma^{\mathfrak{t}_{\mathrm{f}} \times N})$ and the Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}})$. Here

$$\sigma^{\mathfrak{t}_{\mathrm{f}} \times N}((\delta Z, \delta \zeta), (\delta' Z, \delta' \zeta)) = \sigma^{\mathfrak{t}}(\delta Z, \delta' Z) + \delta \zeta(\delta' Z_{\mathfrak{l}}) - \delta' \zeta(\delta Z_{\mathfrak{l}}).$$

Because Ψ is $(\mathfrak{t}_{\mathrm{f}} \times T_{\mathrm{h}})$-equivariant, we have recovered Benoist [6, Cor. 6.16], in which the "cocycle c" is equal to our $\sigma^{\mathfrak{t}}$.

9. The classification

9.1. Invariants

The model in Proposition 7.2, of a compact connected symplectic manifold (M, σ) with an effective symplectic action of a torus T of which the principal orbits are coisotropic submanifolds of (M, σ), has been described in terms of the following ingredients.

Definition 9.1. — *Let T be a given torus. A* list of ingredients for T *consists of:*

1) *An antisymmetric bilinear form* $\sigma^{\mathfrak{t}}$ *on the Lie algebra* $\mathfrak{t}$ *of* T*.*
2) *A subtorus* T_{h} *of* T*, of which the Lie algebra* $\mathfrak{t}_{\mathrm{h}}$ *is contained in* $\mathfrak{l} := \ker \sigma^{\mathfrak{t}}$.
3) *A Delzant polytope* Δ *in* ${\mathfrak{t}_{\mathrm{h}}}^*$ *with center of mass at the origin.*
4) *A discrete cocompact subgroup* P *of the additive subgroup* $N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ *of* $\mathfrak{l}^*$*.*
5) *An antisymmetric bilinear mapping* $c : N \times N \to \mathfrak{l}$ *with the following properties.*
   5a) *For every* $\zeta, \zeta' \in P$*, the element* $c(\zeta, \zeta') \in \mathfrak{l} \subset \mathfrak{t}$ *belongs to the integral lattice* $T_{\mathbb{Z}}$ *in* $\mathfrak{t}$*, the kernel of the exponential mapping* $\exp : \mathfrak{t} \to T$*.*
   5b) *For every* $\zeta, \zeta', \zeta'' \in N$ *we have that*
$$\zeta(c(\zeta', \zeta'')) + \zeta'(c(\zeta'', \zeta)) + \zeta''(c(\zeta, \zeta')) = 0.$$
6) *An element* $\overline{\tau}$ *of the space* $\mathcal{T}$ *which has been defined in (7.19).*

Remark 9.2. — Regarding the Delzant polytope Δ in 3) in Definition 9.1, we have a corresponding Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}})$, which is a $2 \dim T_{\mathrm{h}}$-dimensional compact connected symplectic manifold, equipped with an effective Hamiltonian T_{h}-action on $(M_{\mathrm{h}}, \sigma_{\mathrm{h}})$, for which Δ is equal to the image of the momentum map.

In 5b), $\zeta \in N$ is viewed as a linear form on $\mathfrak{l}$ which vanishes on $\mathfrak{t}_{\mathrm{h}}$, so $\zeta(c(\zeta', \zeta''))$ is a real number.

The ingredient 6), the holonomy invariant, has been introduced in Subsection 7.5. As explained there, the space $\mathcal{T}$ to which it belongs can have a non-Hausdorff quotient topology.

Definition 9.3. — *Let* M *be a compact and connected smooth manifold,* σ *a symplectic form on* M*, and* T *a torus acting effectively on* (M, σ) *by means of symplectomorphisms and with coisotropic principal orbits. The* list of ingredients *of* (M, σ, T)*, as in Definition 9.1, consists of:*

i) $\sigma^{\mathfrak{t}}(M, \sigma, T)$ *is the antisymmetric bilinear form* $\sigma^{\mathfrak{t}}$ *on* $\mathfrak{t}$ *as defined in Lemma 2.1.*
ii) $T_{\mathrm{h}}(M, \sigma, T)$ *is the Hamiltonian torus* T_{h}*, the unique maximal stabilizer subgroup* T_{h} *for the* T*-action on* M*, see Remark 3.12 and Lemma 3.6.*
iii) $\Delta(M, \sigma, T)$ *is the image* $\Delta = \mu(M)$ *of the momentum map* $\mu : M \to {\mathfrak{t}_{\mathrm{h}}}^*$ *of the* T_{h}*-action on* (M, σ)*, which is Hamiltonian, cf. Corollary 3.11, where we eliminated the translational ambiguity by putting the center of mass of* Δ *at the origin.* Δ *is a translate of the Delzant polytope* Δ_p *in Proposition 3.8.*

iv) *$P(M,\sigma,T)$ is the period group P defined in Lemma 10.12 with $Q = M/T$, $V = \mathfrak{l}^*$, and $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, which according to Proposition 3.8 is a discrete cocompact additive subgroup of N.*

v) *$c(M,\sigma,T)$ is the antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$ defined in Proposition 5.5.*

vi) *$\overline{\tau}(M,\sigma,T)$ is the holonomy invariant of (M,σ,T), the element $\overline{\tau}$ of $\mathcal{T}$ defined in (7.20).*

Note that all the ingredients in Definition 9.1 are defined only in terms of the torus T.

Theorem 9.4. — *Let T be a torus. The list of ingredients of (M,σ,T) is a complete set of invariants for the compact connected symplectic manifold (M,σ) with effective symplectic T-action with coisotropic principal orbits, in the following sense. If (M',σ') is another compact connected symplectic manifold with effective symplectic T-action with coisotropic principal orbits, then there exists a T-equivariant symplectomorphism Φ from (M,σ,T) onto (M',σ',T) if and only if the list of ingredients of (M,σ,T) is equal to the list of ingredients of (M',σ',T).*

Proof. — The property 5a) in Definition 9.1 follows from Remark 5.8, and also from Lemma 7.1. Equation 5b) in Definition 9.1 is the equation (5.5).

Suppose that Φ is a T-equivariant symplectomorphism from (M,σ,T) onto (M',σ',T). We will check that the ingredients of (M,σ,T) and (M',σ',T) are the same. In other words, the ingredients are invariants of the symplectic T-spaces.

If $X, Y \in \mathfrak{t}$, then the T-equivariance of Φ implies that $\Phi^* X_{M'} = X_M$ and $\Phi^* Y_{M'} = Y_M$. In combination with $\sigma = \Phi^*\sigma'$, this implies in view of Lemma 2.1 that

$$\begin{aligned}\sigma^{\mathfrak{t}}(M,\sigma,T)(X,Y) &= \sigma(X_M,Y_M) = (\Phi^*\sigma')(\Phi^*X_{M'},\Phi^*Y_{M'})\\ &= \Phi^*(\sigma'(X_{M'},Y_{M'})) = \Phi^*(\sigma^{\mathfrak{t}}(M',\sigma',T)(X,Y))\\ &= \sigma^{\mathfrak{t}}(M',\sigma',T)(X,Y),\end{aligned}$$

where we have used in the last equation that $\sigma^{\mathfrak{t}}(M',\sigma',T)(X,Y)$ is a constant on M'. This proves that $\sigma^{\mathfrak{t}}(M,\sigma,T) = \sigma^{\mathfrak{t}}(M',\sigma',T)$. The T-equivariance of Φ implies that $T_{\Phi(x)} = T_x$ for every $x \in M$, and therefore $T_{\mathrm{h}}(M',\sigma',T) = T_{\mathrm{h}}(M,\sigma,T)$.

In combination with $\Phi^*\sigma' = \sigma$, the T-equivariance of Φ implies that the $\mathfrak{l}^*$-valued closed basic one-form $\widehat{\sigma}$ defined in Lemma 3.1 is equal to $\Phi^*\widehat{\sigma'}$.

It follows that Φ induces an isomorphism of locally convex polyhedral $\mathfrak{l}^*$-parallel spaces from M/T onto M'/T. In view of Proposition 3.8 this implies that $P(M', \sigma', T) = P(M, \sigma, T)$ and that $\Delta(M', \sigma', T)$ is a translate of $\Delta(M, \sigma, T)$ in $\mathfrak{t}_{\mathrm{h}}{}^*$. Because both $\Delta(M', \sigma', T)$ and $\Delta(M, \sigma, T)$ have their center of mass at the origin, it follows that $\Delta(M', \sigma', T) = \Delta(M, \sigma, T)$.

The T-equivariant symplectomorphism Φ maps an admissible connection as in Proposition 5.5 to an admissible connection as in Proposition 5.5 with (M, σ, T) replaced by (M', σ', T). It follows that $c(M', \sigma', T) = c(M, \sigma, T)$ in view of Remark 5.7 and $\overline{\tau}(M', \sigma', T) = \overline{\tau}(M, \sigma, T)$ in view of Subsection 7.5. This proves the "only if" part of the theorem.

For the "if" part, the completeness of the invariants, we observe that the manifold $M_{\mathrm{model}} := G \times_H M_{\mathrm{h}}$ and the T-invariant symplectic form σ_{model} on M_{model}, see Proposition 7.2 and Proposition 7.4, are defined in terms of the ingredients 1) – 6) in Definition 9.1, and the elements τ_ζ, $\zeta \in P$. Let $x \in M$ and choose an admissible connection for (M, σ, T) as in Proposition 5.5. Then $\overline{\tau}(M', \sigma', T) = \overline{\tau}(M, \sigma, T)$ implies in view of Subsection 7.5 that there exist $x' \in M'$ and a choice of an admissible connection for (M', σ', T) as in Proposition 5.5, such that the holonomy $\tau'_\zeta(x')$, $\zeta \in P$, defined by this connection and with the initial point x', is equal to $\tau_\zeta = \tau_\zeta(x)$, $\zeta \in P$. Therefore the model for (M', σ', T) in Proposition 7.2, with (M, σ, T) replaced by (M', σ', T), can be chosen to be equal to the model for (M, σ, T) in Proposition 7.2. This implies the existence of a T-equivariant symplectomorphism α' from $(M_{\mathrm{model}}, \sigma_{\mathrm{model}}, T)$ onto (M', σ', T), and it follows that $\Phi := \alpha' \circ \alpha^{-1}$ is a T-equivariant symplectomorphism from (M, σ, T) onto (M', σ', T). □

Remark 9.5. — Because $\Delta(M', \sigma', T) = \Delta(M, \sigma, T)$, the point $x' \in M'$ in the last paragraph of the proof of Theorem 9.4 can be chosen such that $\mu'(x') = \mu(x)$, where μ and μ' denote the momentum maps of the Hamiltonian T_{h}-actions on (M, σ) and (M', σ'), respectively. This implies that there is a T_{h}-equivariant symplectomorphism Φ_{h} from the Delzant submanifold of (M, σ) through x onto the Delzant submanifold of (M', σ') through x', which maps x to x'. Using Φ_{h} in order to identify both Delzant manifolds with $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$, under which identifications x and x' are mapped to the same point of M_{h}, we conclude that $\Phi(x) = x'$, if Φ is the T-equivariant symplectomorphism from (M, σ, T) to (M', σ', T), described in the last paragraph of the proof of Theorem 9.4.

Let $\mathrm{Aut}(M, \sigma, T)$ denote the automorphism group of (M, σ, T), the set of all T-equivariant symplectomorphisms from (M, σ, T) to (M, σ, T). Each

$\Phi \in \mathrm{Aut}(M, \sigma, T)$ induces a transformation $\Phi_{M/T}$ of M/T, which is an isomorphism of $\mathfrak{l}^*$-parallel spaces, and therefore of the form $p \mapsto p + \nu(\Phi)$ for a unique element $\nu(\Phi) \in N/P$. The mapping $\nu : \Phi \mapsto \nu(\Phi)$ is a homomorphism from the group $\mathrm{Aut}(M, \sigma, T)$ to the torus N/P. Using the previous paragraph with $M' = M$ and $\sigma' = \sigma$ and using Subsection 7.5, it can be proved that $\nu(\mathrm{Aut}(M, \sigma, T))$ is equal to the set of $\zeta' + P \in N/P$, for which there exists an $\alpha \in \mathrm{Sym}$ such that $\mathrm{e}^{c(\zeta, \zeta')} = \mathrm{e}^{\alpha_\zeta}$ for all $\zeta \in P$, where it is sufficient to satisfy these equations for all ζ in a $\mathbb{Z}$-basis of P.

Using this one can prove that $\nu(\mathrm{Aut}(M, \sigma, T))$ is a Lie subgroup of N/P with Lie algebra equal to c^0, the space of all elements of N which are equal to zero on the span of $c(N \times N)$. Actually, L_ζ is an inifinitesimal symplectomorphism if and only if $\zeta \in c^0$. In general $\nu(\mathrm{Aut}(M, \sigma, T))$ need not be a closed subgroup of the torus N/P, and it neither needs to be connected, but it has countably many connected components.

The kernel of the homomorphism ν from $\mathrm{Aut}(M, \sigma, T)$ to N/P consists of the group
$\mathrm{Aut}_T(M, \sigma, T)$ of all T-equivariant symplectomorphisms $\Phi : (M, \sigma, T) \to (M, \sigma, T)$ which preserve all the T-orbits. This group can be analyzed starting from Remark 4.3.

9.2. Existence

The following existence theorem completes the classification.

THEOREM 9.6. — *Every list of ingredients as in Definition 9.1 is equal to the list of invariants of a compact connected symplectic manifold* (M, σ) *with effective symplectic T-action with coisotropic principal orbits as in Theorem 9.4.*

Proof. — A straightforward verification shows that, for any antisymmetric bilinear mapping $c : N \times N \to \mathfrak{l}$, (7.1) turns $\mathfrak{g} := \mathfrak{t} \times N$ into a two-step nilpotent Lie algebra, and that (7.5) defines a product in $G := T \times N$ for which G is a Lie group with $\mathfrak{g}$ as its Lie algebra.

Choose an element $\tau \in \mathrm{Hom}_c(P, T)$ such that $\overline{\tau} = (\exp \mathcal{A}) \cdot \tau$, see (7.19). Because the τ_ζ, $\zeta \in P$, satisfy (7.18), it follows that (7.11) defines a closed Lie subgroup H of G, and that (7.12) defines a smooth action of H on the Delzant manifold M_{h}. Here we have used a choice of a complementary torus T_{f} to T_{h}, which will be kept fixed in the remainder of the proof.

Because H is a closed Lie subgroup of G, its right action on G is proper and free, and therefore the action of H on $G \times M_{\mathrm{h}}$, for which $h \in H$ sends

(g,x) to $(gh^{-1}, h\cdot x)$, is proper and free. The orbit space $M := G\times_H M_{\mathrm{h}}$ has a unique structure of a smooth manifold such that the canonical projection $\pi_M : G\times M_{\mathrm{h}} \to M$ is a principal H-bundle. Because G and M_{h} are connected, M is connected as the image of the connected set $G\times M_{\mathrm{h}}$ under the continuous mapping π_M. The projection $(g,x)\mapsto g$ induces a G-equivariant smooth fibration $\psi : M \to G/H$ with fiber M_{h}, the fiber bundle induced from the principal fiber bundle $G\to G/H$ by means of the action of H on M_{h}. See [15, Sec. 2.4]. Because P is cocompact in N, the base space G/H is compact, and because the fiber M_{h} is a compact Delzant manifold, it follows that M is compact.

On $G\times M_{\mathrm{h}}$ we define the smooth two-form ω by (7.14). (Note that we cannot use the equation $\omega = A^*\sigma$ here, because we do not have the symplectic form σ on the manifold M yet, we are in the process of defining it.) In (7.14) we have used a choice of a linear projection $X\mapsto X_{\mathfrak{l}}$ from $\mathfrak{t}$ onto $\mathfrak{l}$, which will be kept fixed in the remainder of the proof.

We first verify that ω is closed. The part

$$\sigma^{\mathfrak{t}}(\delta t, \delta' t) + \delta\zeta((\delta' t)_{\mathrm{l}}) - \delta'\zeta((\delta t)_{\mathrm{l}})$$

of (7.14) is closed, because it is defined by a constant two-form on $T\times N$.

For the part

$$\varphi_\zeta(\delta\zeta, \delta'\zeta) := \delta\zeta(c(\delta'\zeta,\zeta)/2) - \delta'\zeta(c(\delta\zeta,\zeta)/2)$$

of (7.14), it follows from (5.7) that $(\mathrm{d}\varphi)(\delta\zeta,\delta'\zeta,\delta''\zeta)$ is equal to the cyclic sum over $\delta\zeta,\delta'\zeta,\delta''\zeta$ of $\delta'\zeta(c(\delta''\zeta,\delta\zeta)/2) - \delta''\zeta(c(\delta'\zeta,\delta\zeta)/2)$, which is equal to zero because of 5b) in Definition 9.1.

If $A_{\mathrm{h}} : (t,x)\mapsto t\cdot x : T_{\mathrm{h}}\times M_{\mathrm{h}} \to M_{\mathrm{h}}$ denotes the action of T_{h} on M_{h}, then the part

$$(\sigma_{\mathrm{h}})_x(\delta x, (\delta' t_{\mathrm{h}})_M(x)) - (\sigma_{\mathrm{h}})_x(\delta' x, (\delta' t_{\mathrm{h}})_M(x)) + (\sigma_{\mathrm{h}})_x(\delta x, \delta' x)$$

of (7.14) is equal to the pull-back of ${A_{\mathrm{h}}}^*\sigma_{\mathrm{h}}$ by means of the mapping

$$p : ((t,\zeta),x)\mapsto (t_{\mathrm{h}}, x) : (T\times N)\times M_{\mathrm{h}} \to T_{\mathrm{h}}\times M_{\mathrm{h}}.$$

This part of (7.14) is closed, because $\mathrm{d}(p^*({A_{\mathrm{h}}}^*\sigma_{\mathrm{h}})) = p^*(\mathrm{d}({A_{\mathrm{h}}}^*\sigma_{\mathrm{h}})) = p^*({A_{\mathrm{h}}}^*(\mathrm{d}\sigma_{\mathrm{h}})) = 0$.

The remaining part of (7.14) is

$$-\mu(x)(c_{\mathrm{h}}(\delta\zeta,\delta'\zeta)) + (\sigma_{\mathrm{h}})_x(\delta x, c_{\mathrm{h}}(\delta'\zeta,\zeta)_{M_{\mathrm{h}}}(x))/2 \\ - (\sigma_{\mathrm{h}})_x(\delta' x, c_{\mathrm{h}}(\delta\zeta,\zeta)_{M_{\mathrm{h}}}(x))/2.$$

Because the action of T_{h} on M_{h} is Hamiltonian with momentum mapping μ, we have for every $Y\in\mathfrak{t}_{\mathrm{h}}$ that $(\sigma_{\mathrm{h}})_x(\delta x, Y_{M_{\mathrm{h}}}(x))$ is equal to the derivative of $x \mapsto \mu(x)(Y)$ in the direction of δx. If we apply this to $Y = c_{\mathrm{h}}(\delta'\zeta, \zeta)/2$, then we obtain that the remaining part of (7.14) is equal to $\mathrm{d}\gamma$, in which the one-form γ is defined by

$$\gamma_{((t,\zeta),x)}((\delta\gamma, \delta\zeta), \delta x) = \mu(x)(c_{\mathrm{h}}(\delta\zeta, \zeta))/2.$$

Because $\mathrm{d}(\mathrm{d}\gamma) = 0$, this completes the proof that $\mathrm{d}\omega = 0$.

The element $(b, \beta) \in H$ sends $((t, \zeta), x)$ to $((\widetilde{t}, \widetilde{\zeta}), \widetilde{x})$ with $\widetilde{t} = tb^{-1}\, \mathrm{e}^{c(\zeta,\beta)/2}$, $\widetilde{\zeta} = \zeta - \beta$, and $\widetilde{x} = (b\tau_\beta)_{\mathrm{h}} \cdot x$. Therefore the tangent map of the action of (b, β) sends $((\delta t, \delta\zeta), \delta x)$ to $((\widetilde{\delta t}, \widetilde{\delta\zeta}), \widetilde{\delta x})$ with $\widetilde{\delta t} = \delta t + c(\delta\zeta, \beta)/2$, $\widetilde{\delta\zeta} = \delta\zeta$, and $\widetilde{\delta x} = \mathrm{T}_x((b\tau_\beta)_{\mathrm{h}})_{M_{\mathrm{h}}}\delta x$. Because $\delta t + c(\delta\zeta, \zeta)/2 = \widetilde{\delta t} + c(\widetilde{\delta\zeta}, \widetilde{\zeta})/2$, and because $((b\tau_\beta)_{\mathrm{h}})_{M_{\mathrm{h}}}$ is a symplectomorphism on M_{h} which leaves μ and infinitesimal T_{h}-actions invariant, it follows that the two-form ω defined by (7.14) is H-invariant.

The condition that $\omega_a(\delta a, \delta' a) = 0$ for every $\delta' a \in \mathrm{T}_a(G \times M_{\mathrm{h}})$ is equivalent to $\delta\zeta = 0$ (take $\delta'\zeta = 0$, $\delta' x = 0$, and let $\delta' t$ range over $\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$), $\delta t \in \mathfrak{l}$ (take $\delta'\zeta = 0$, $\delta' x = 0$, and let $\delta' t$ range over $\mathfrak{t}_{\mathrm{f}}$, where we use that we already have $\delta\zeta = 0$), $\delta x + (\delta t_{\mathrm{h}})_{M_{\mathrm{h}}}(x) = 0$ (take in the remaining equation (7.14) $\delta'\zeta = 0$, $\delta' t = 0$ and let $\delta' x$ range over $\mathrm{T}_x M_{\mathrm{h}}$), and finally $\delta t \in \mathfrak{t}_{\mathrm{h}}$, because the fact that the T_{h}-orbits in M_{h} are isotropic now implies that $-\delta'\zeta(\delta t) = 0$ for all $\delta'\zeta \in (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$. It follows that the kernel of ω_a is equal to $\mathrm{T}_a(H \cdot a) = \ker(\mathrm{T}_a \pi_M)$.

The conclusion is that ω is a basic two-form for the action of H on $G \times M_{\mathrm{h}}$, which implies that there is a unique smooth two-form σ on $M = G \times_H M_{\mathrm{h}}$ such that $\omega = {\pi_M}^* \sigma$. Because ${\pi_M}^*(\mathrm{d}\sigma) = \mathrm{d}({\pi_M}^*\sigma) = \mathrm{d}\omega = 0$ and at every point the tangent mapping of π_M is surjective, we have that $\mathrm{d}\sigma = 0$. Furthermore σ is nondegenerate at every point, because the kernel of ω is equal to the kernel of the tangent mapping of π_M at every point. Therefore σ is a symplectic form on M.

On $G \times M_{\mathrm{h}}$ we have the action of $s \in T$ which sends $((t, \zeta), x)$ to $((st, \zeta), x)$. This action clearly leaves ω invariant, and it follows that the induced action of T on $M := G \times_H M_{\mathrm{h}}$ leaves σ invariant. The tangent vectors to the orbits in $G \times M_{\mathrm{h}}$ are the $((\delta s, 0), 0)$, $\delta s \in \mathfrak{t}$, and if we substitute these as $\delta' a$ in (7.14) then we obtain

$$\sigma^{\mathfrak{t}}(\delta t, \delta s) + \delta\zeta((\delta s)_{\mathfrak{l}}) + (\sigma_{\mathrm{h}})_x(\delta x, (\delta s_{\mathrm{h}})_{M_{\mathrm{h}}}(x)).$$

Requiring that this is equal to zero for all $\delta s \in \mathfrak{t}$ is equivalent to $\delta\zeta = 0$ (let δs range over $\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}}$), $\delta t \in \mathfrak{l}$ (let δs range over $\mathfrak{t}_{\mathrm{f}}$ and use that $\delta\zeta = 0$) and δx is symplectically orthogonal to $\mathrm{T}_x(T_{\mathrm{h}} \cdot x)$. If $x \in (M_{\mathrm{h}})_{\mathrm{reg}}$, then the last condition implies that $\delta x = Y_{M_{\mathrm{h}}}(x)$ for a unique $Y \in \mathfrak{t}_{\mathrm{h}}$. This shows that the principal orbits of the T-action are coisotropic submanifolds of (M, σ).

We now verify that the invariants of the compact connected symplectic manifold (M,σ) with symplectic T-action with coisotropic principal orbits are the ingredients in Definition 9.1 we started out with.

If we substitute $\delta\zeta = \delta'\zeta = 0$ and $\delta x = \delta' x = 0$ in (7.14), then we get $\sigma^{\mathfrak{t}}(\delta t, \delta' t)$, which shows that the pull-back of σ to the T-orbits is given by $\sigma^{\mathfrak{t}}$.

If $s \in T$ and $((t,\zeta),x)$ are such that

$$((st,\zeta),x) = (b,\beta)\cdot((t,\zeta),x) = ((tb^{-1}\,\mathrm{e}^{c(\zeta,\beta)/2}, \zeta-\beta), (b\tau_\beta)_{\mathrm{h}}\cdot x)$$

for some $(b,\beta) \in H$, then $\beta = 0$, $b = s^{-1}$, and $x = {s_{\mathrm{h}}}^{-1}\cdot x$. Because $(b,0) \in H$ implies that $(s^{-1})_{\mathrm{f}} = b_{\mathrm{f}} = 1$, it follows that $s \in T_{\mathrm{h}}$ and $s\cdot x = x$. This shows that $T_{\mathrm{h}}(M,\sigma,T)$, the maximal stabilizer subgroup of the T-action on M, see Remark 3.12, is equal to T_{h}.

The action of the subtorus T_{h} of T on $M = G\times_H M_{\mathrm{h}}$ is induced by the action of T_{h} on the second factor M_{h} of $G\times M_{\mathrm{h}}$. It follows that the action of T_{h} on M is Hamiltonian with image of the momentum mapping equal to a translate of Δ. This proves that $\Delta(M,\sigma,T)$ is equal to a translate of Δ, and therefore equal to Δ if we add a suitable constant to the momentum mapping.

Because $M/T = ((T\times N)\times_H M_{\mathrm{h}})/T \simeq (N/P)\times(M_{\mathrm{h}}/T_{\mathrm{h}}) \simeq (N/P)\times\Delta$, we have that $P(M,\sigma,T) = P$.

For each $\zeta \in N$, the infinitesimal action of $(0,\zeta) \in \mathfrak{g}$ on M defines a smooth vector field L_ζ on M. If the vector fields $L_{\mathrm{h},\eta}$ on M_{h} are lifts of $\eta \in C \simeq {\mathfrak{t}_{\mathrm{h}}}^* \simeq (\mathfrak{l}/\mathfrak{l}\cap\mathfrak{t}_{\mathrm{f}})^*$ as in Proposition 5.5 with (M,σ,T) replaced by $(M_{\mathrm{h}},\sigma_{\mathrm{h}},T_{\mathrm{h}})$, then the vector field $((0,0),L_{\mathrm{h},\eta})$ is intertwined by π_M with a unique vector field L_η on M, and the L_η, L_ζ together form a collection of lifts of η, ζ as in Proposition 5.5, with c replaced by $c(M,\sigma,T)$ in Proposition 5.5, iii). It now follows from (7.1) that $c(M,\sigma,T) = c$.

Finally, if $\zeta \in P$, then $({\tau_\zeta}^{-1},\zeta) \in H$, and therefore

$$\begin{aligned}(0,\zeta)\cdot H\cdot((0,0),x) &= H\cdot((0,\zeta),x) = H\cdot((0,\zeta)(({\tau_\zeta}^{-1},\zeta)^{-1},x)\\ &= H\cdot((\tau_\zeta,0),x) = \tau_\zeta\cdot H\cdot((0,0,x),\end{aligned}$$

which implies that $\overline{\tau}(M,\sigma,T) = \overline{\tau}$. □

According to Theorem 9.4, the symplectic manifold (M,σ) is unique up to T-equivariant symplectomorphisms. In particular the dimension of M is determined in terms of the ingredients in Definition 9.1. Lemma 2.3 implies that $\dim M = \dim T + \dim\mathfrak{l}$.

In the language of categories, see MacLane [35], Theorem 9.4 and Theorem 9.6 can be summarized as follows.

COROLLARY 9.7. — *Let T be a torus. Let $\mathcal{M}$ denote the category of which the objects are the compact connected symplectic manifolds (M, σ) together with an effective symplectic T-action on (M, σ) with coisotropic principal orbits, and the morphisms are the T-equivariant symplectomorphisms. Let $\mathcal{I}$ denote the set of all lists of invariants as in Definition 9.1, viewed as a category with only the identities as morphisms.*

Then the assignment ι in Definition 9.3 defines a full functor of categories from $\mathcal{M}$ onto $\mathcal{I}$. Furthermore, the proper class $\mathcal{M}/\sim$ of isomorphism classes in $\mathcal{M}$ is a set, and the functor $\iota : \mathcal{M} \to \mathcal{I}$ induces a bijective mapping $\iota/\sim$ from $\mathcal{M}/\sim$ onto $\mathcal{I}$.

Proof. — It follows from Theorem 9.4 that $\iota : \mathcal{M} \to \mathcal{I}$ is a functor, which moreover induces an injective mapping from $\mathcal{M}/\sim$ to $\mathcal{I}$. The surjectivity of ι, hence of $\iota/\sim$, follows from Theorem 9.6. □

Remark 9.8. — Let T_{f} be a complementary torus to T_{h} in T. If M is T-equivariantly diffeomorphic to $M_{\mathrm{f}} \times M_{\mathrm{h}}$, in which T_{h} acts only on M_{h} with isolated fixed points, and T_{f} acts freely on M_{f}, then $c(N \times N) \subset \mathfrak{t}_{\mathrm{f}}$. See Proposition 8.1. Conversely, if this condition is satisfied, then Lemma 7.5 implies the stronger statement that (M, σ, T) is T-equivariantly symplectomorphic to the Cartesian product of a symplectic manifold $(M_{\mathrm{f}}, \sigma_{\mathrm{f}}, T_{\mathrm{f}})$ with a free symplectic T_{f}-action and a Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$.

Let c in 5) in Definition 9.1 be such that $c(\zeta, \zeta') \notin \mathfrak{t}_{\mathrm{f}}$ for some $\zeta, \zeta' \in N$. If (M, σ, T) is as in Theorem 9.6, then M is not T-equivariantly diffeomorphic to $M_{\mathrm{f}} \times M_{\mathrm{h}}$, in which T_{f} acts freely on M_{f} and T_{h} acts only on M_{h} and with isolated fixed points. Therefore such (M, σ, T) are counterexamples to Benoist [6, Th. 6.6], if in [6, Th. 6.6] the word "isomorphic" implies "equivariantly diffeomorphic".

There exists c in 5) in Definition 9.1, such that for every choice of a complementary torus T_{f} to T_{h} in T we have $c(\zeta, \zeta') \notin \mathfrak{t}_{\mathrm{f}}$ for some $\zeta, \zeta' \in N$. For instance, if $\dim N \geqslant 2$ and $T_{\mathrm{h}} \neq \{1\}$, then there exists a nonzero antisymmetric bilinear mapping c from $N \times N$ to $\mathfrak{t}_{\mathrm{h}}$, which maps $P \times P$ into the integral lattice $(T_{\mathrm{h}})_{\mathbb{Z}}$ in the Lie algebra $\mathfrak{t}_{\mathrm{h}}$ of T_{h}. Such a c satisfies 5b) because every $\zeta \in N$ is a linear form on $\mathfrak{l}$ which vanishes on $\mathfrak{t}_{\mathrm{h}}$, and it satisfies 5a) by assumption. On the other hand $c(\zeta, \zeta') \notin \mathfrak{t}_{\mathrm{f}}$ as soon as $c(\zeta, \zeta') \neq 0$.

Remark 9.9. — Let $T_{\mathrm{h}} = \{1\}$ and $\sigma^{\mathfrak{t}} = 0$, that is, the action of T is free, $\dim M = 2 \dim T$, and the orbits are Lagrangian submanifolds of (M, σ). In this case the admissible connections are just the smooth T-invariant infinitesimal connections for the principal T-bundle $\pi : M \to M/T \simeq$

$\mathfrak{t}^*/P$ over the torus $\mathfrak{t}^*/P$, as in Remark 5.7. The first step in the proof of Proposition 5.5 consists of the construction of an infinitesimal connection for the principal T-bundle M over the torus $\mathfrak{t}^*/P$, of which the curvature form is a constant two-form on the torus $\mathfrak{t}^*/P$. In this construction the symplectic form did not enter, and the principal T-bundle M over $\mathfrak{t}^*/P$ can be constructed from the ingredients 4), 5), 6), in which condition 5a) is kept, but condition 5b) is dropped.

However, if one has a T-invariant symplectic form σ on M for which the T-orbits are Lagrangian, then (5.5), that is 5b), holds. In combination with Theorem 9.6, we conclude that this principal T-bundle M over $\mathfrak{t}^*/P$ admits a T-invariant symplectic form for which the T-orbits are Lagrangian, if and only if 5b) holds. This interpretation of condition 5b) was suggested to us by Yael Karshon.

If $\dim N \geqslant 3$, then there exist antisymmetric bilinear mappings $c : \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$ for which 5b) does not hold, and it follows that the principal T-bundle over $\mathfrak{t}^*/P$ defined by g does not admit a T-invariant symplectic form for which the T-orbits are Lagrangian.

Remark 9.10. — A slightly different approach to the classification would be to allow morphisms $(\Phi, \iota) : (M, \sigma, T) \to (M', \sigma', T')$, in which Φ is a symplectomorphism from (M, T) onto (M', σ'), ι is an isomorphism of Lie groups from T onto T', and Φ intertwines the T-action on M with the T'-action on M' in the sense that $\Phi(t \cdot x) = \iota(t) \cdot \Phi(x)$ for every $x \in M$ and $t \in T$.

The isomorphisms between tori are classified by the choices of $\mathbb{Z}$-bases in the integral lattices. For instance if we fix a $\mathbb{Z}$-basis e'_i, $1 \leqslant i \leqslant d := \dim T' = \dim T$, of $T'_{\mathbb{Z}}$, then the mapping which assigns to a $\mathbb{Z}$-basis e_i, $1 \leqslant i \leqslant d$, of $T_{\mathbb{Z}}$ the isomorphism $\iota : T \to T'$ such that the tangent mapping of ι at the identity element maps e_i to e'_i, is a bijective mapping from the set of $\mathbb{Z}$-bases of $T_{\mathbb{Z}}$ onto the set of isomorphisms from T onto T'. In turn the set of $\mathbb{Z}$-bases in $T_{\mathbb{Z}}$ is in bijective correspondence with the group $\mathrm{GL}(d, \mathbb{Z})$ of all $d \times d$-matrices with integral coefficients and determinant equal to ± 1.

This can be applied in particular to $T' = \mathbb{R}^d/\mathbb{Z}^d$, with e'_i, $1 \leqslant i \leqslant d$, equal to the standard basis of $\mathbb{R}^d$. If we also choose a $\mathbb{Z}$-basis ε^l, $1 \leqslant l \leqslant \dim N$, in P, then the ingredients 1) – 6) in Definition 9.1 are determined by their coefficients with respect to these bases. Furthermore the groups G and H are identified with $\mathbb{R}^d/\mathbb{Z}^d \times \mathbb{R}^{d_N}$ and $\mathbb{R}^{d_{\mathrm{h}}}/\mathbb{Z}^{d_{\mathrm{h}}} \times \mathbb{Z}^{d_N}$, respectively. This would lead to a presentation of the model in coordinates, except for the Delzant manifold $(M_{\mathrm{h}}, \sigma_{\mathrm{h}}, T_{\mathrm{h}})$. Such a model looks even more explicit than the one in Proposition 7.2.

The disadvantage of this approach is that the invariants are given by coefficient matrices, which are determined uniquely only up to the action on these matrices of the changes of $\mathbb{Z}$-bases. Also the notations become quite a bit heavier if we write out our objects in coordinates.

10. V-parallel spaces

In this section we define the notion of a V-parallel space, and prove that every straight line complete, connected and locally convex V-parallel space is isomorphic to the Cartesian product of a closed convex subset of a vector space and a torus.

DEFINITION 10.1. — *Let V be an n-dimensional vector space. A* V-parallel space *is a Hausdorff topological space Q, together with an open covering Q_α, $\alpha \in A$, of Q and homeomorphisms φ_α from Q_α onto subsets V_α of V such that, for every $\alpha, \beta \in A$ for which $Q_\alpha \cap Q_\beta \neq \emptyset$, the mapping*

$$x \mapsto \varphi_\alpha(x) - \varphi_\beta(x) : Q_\alpha \cap Q_\beta \to V \tag{10.1}$$

is locally constant. A subset U of Q is a V-parallel space with the φ_α replaced by their restrictions to $U \cap Q_\alpha$. If W is another vector space and R is another W-parallel space, then $Q \times R$ is a $V \times W$-parallel space in an obvious way.

The V-parallel space Q is called locally convex *if the V_α are convex subsets of V. The locally convex V-parallel space Q is called* locally convex polyhedral *if for every $\alpha \in A$ there is a convex open subset V'_α of V and there are finitely many linear forms $v^*_{\alpha,i}$ $1 \leqslant i \leqslant m$, on V, such that*

$$V_\alpha = \{v \in V'_\alpha \mid v^*_{\alpha,j}(v) \geqslant 0 \text{ for every } 1 \leqslant j \leqslant m\}. \tag{10.2}$$

Remark 10.2. — If we have (10.2) with linearly independent linear forms $v^*_{\alpha,j}$, and (10.1) is replaced to the weaker condition that for every $\alpha, \beta \in A$ the mapping $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism, then Q is a "manifold with corners" as defined for instance in Mather [37, §1]. If the V_α are open subsets of V and (10.1) is relaxed to the condition that for every $\alpha, \beta \in A$ the mapping $\varphi_\alpha \circ \varphi_\beta^{-1}$ is locally an affine linear transformation, the composition of a linear mapping and a translation, then Q is a "manifold with affine covering" as in Auslander and Markus [4, p. 141] or a "locally affine manifold" as in Auslander [3].

If on the other hand we have (10.1) and (10.2) with $V_\alpha = V$ for all $\alpha \in A$, then Q is called an "affine space modelled over V" in geometry. In order to avoid confusion about the interpretation of the term “affine”, we have replaced the term “affine” by “parallel”, where the latter word reminds of a parallelism, which is defined as a global frame in the tangent bundle, or equivalently a trivialization of the tangent bundle.

Definition 10.3. — *Let V and W be finite-dimensional vector spaces, Q and R a locally convex V-parallel and W-parallel spaces with charts φ_α, $\alpha \in A$ and ψ_β, $\beta \in B$, respectively. If L is a linear mapping from W to V, then an L-*map from R to Q *is a continuous map $f : R \to Q$ such that for each $\beta \in B$ and $\alpha \in A$ and each subset U of $R_\beta \cap f^{-1}(Q_\alpha)$ such that $\psi_\beta(U)$ is a convex subset of W, we have that*

$$\varphi_\alpha(f(p)) - \varphi_\alpha(f(q)) = L(\psi_\beta(p) - \psi_\beta(q)) \quad \text{for all} \quad p, q \in U. \tag{10.3}$$

With such maps as morphisms, the locally convex parallel spaces form a category. In particular two locally convex parallel spaces are called isomorphic *if there exists an L-map from the first one to the second one, which is a homeomorphism and for which the linear mapping L is bijective.*

If $R = I$ is an interval in $\mathbb{R}$, and $L : \mathbb{R} \to V$ a linear mapping determined by the vector $L(1) = v$, then an L-map $\gamma : I \to Q$ is called a motion in Q with constant velocity v. *In this situation the equation (10.3) takes the form*

$$\varphi_\alpha(\gamma(t)) = \varphi_\alpha(\gamma(s)) + (t - s)v \tag{10.4}$$

for all s, t in any interval $J \subset I$ such that $\varphi_\alpha(J) \subset Q_\alpha$.

Definition 10.4. — *Let Q be a locally convex V-parallel space. Q will be called* straight line complete *if, for any motion $\gamma : I \to Q$ in Q with constant velocity v defined on a non-empty interval I with $b := \sup I < \infty$, there exists a point $q \in Q$ such that $\gamma(t)$ converges to q in Q as $t \uparrow b$.*

Note that if C is a convex subset of a finite-dimensional vector space V, then the V-parallel space C is straight line complete if and only if C is a closed subset of V. Proof: if C is not empty, then the relative interior relint(C) of C is not empty, and for every $c \in$ relint(C) and d in the closure of C in V we have that $c + t(d - c) \in$ relint(C) for every $0 \leqslant t < 1$. See [49, Th. 6.2 and Th. 6.1].

Our goal is to prove that every straight line complete, connected and locally convex V-parallel space is isomorphic to the product of a closed convex subset of a vector space and a torus. See Theorem 10.13 below for the precise statement. We begin the proof with a sequence of lemmas.

Lemma 10.5. — *Let Q be a locally convex V-parallel space, as in Definition 10.1, and let $v \in V$. If $\gamma : I \to Q$ and $\delta : J \to Q$ are motions* *in Q with constant velocity v, and $\gamma(s) = \delta(s)$ for some $s \in I \cap J$, then $\gamma(t) = \delta(t)$ for all $t \in I \cap J$.*

Proof. — Let $K = \{t \in I \cap J \mid \gamma(t) = \delta(t)\}$. Then K is a closed subset of $I \cap J$, because γ and δ are continuous.

Suppose $s \in K$. Because the Q_α, $\alpha \in A$, form a covering of Q, there exists an $\alpha \in A$ such that $\gamma(s) = \delta(s) \in Q_\alpha$. Because Q_α is open in Q and γ and δ are continuous, $H := \gamma^{-1}(Q_\alpha) \cap \delta^{-1}(Q_\alpha)$ is an open neighborhood of s in $I \cap J$. For every $t \in H$ we have

$$\varphi_\alpha(\gamma(t)) = \varphi_\alpha(\gamma(s)) + (t-s)v = \varphi_\alpha(\delta(s)) + (t-s)v = \varphi_\alpha(\delta(t)),$$

hence $\gamma(t) = \delta(t)$ because φ_α is injective. Therefore $H \subset K$, and we have proved that K is also an open subset of $I \cap J$. Because $I \cap J$ is connected, it follows that $K = \emptyset$ or $K = I \cap J$. □

COROLLARY 10.6. — *Let Q be a locally convex V-parallel space, $q \in Q$, and let $\gamma : I \to Q$ be a motion in Q with constant velocity v, such that $b := \sup I < \infty$ and there exists a sequence t_j such that $t_j \uparrow b$ and $\gamma(t_j) \to q$ as $j \to \infty$. Then $\gamma(t) \to q$ as $t \uparrow b$.*

In particular it follows that if Q is compact, then Q is straight line complete.

Proof. — There exists $\alpha \in A$ such that $q \in Q_\alpha$, and an index i such that $\gamma(t_j) \in Q_\alpha$ for all $j \geqslant i$. It then follows from (10.4) that

$$\varphi_\alpha(\gamma(t_j)) = \varphi_\alpha(\gamma(t_i)) + (t_j - t_i)v.$$

Taking the limit for $j \to \infty$, we obtain that $\varphi_\alpha(q) = \varphi_\alpha(\gamma(t_i)) + (b - t_i)v$. In view of the convexity of V_α we have that $\varepsilon(t) := \varphi_\alpha(\gamma(t)) + (t - t_i)v \in V_\alpha$ for all $t \in [t_i, b]$. The curve $\delta := {\varphi_\alpha}^{-1} \circ \varepsilon$ is a motion with constant velocity v in Q such that $\delta(t_i) = \gamma(t_i)$, and it follows from Lemma 10.5 that $\delta = \gamma$ on $[t_i, b[$. Because δ is continuous on $[t_i, b]$ and $\delta(b) = q$, the conclusion is that $\gamma(t) \to q$ as $t \uparrow b$. □

Let $s \in \mathbb{R}$ and $p \in Q$. Let $\Gamma^v_{s,p}$ denote the set of all motions γ in Q with constant velocity v, which are defined on an interval I_γ in $\mathbb{R}$ such that $s \in I_\gamma$ and $\gamma(s) = p$. Then it follows from Lemma 10.5 that the $\gamma \in \Gamma^v_{s,p}$ have a common extension $\gamma^v_{s,p}$ to the union $I^v_{s,p}$ of all the intervals I_γ, $\gamma \in \Gamma^v_{s,p}$. $I^v_{s,p}$ is an interval in $\mathbb{R}$ and $\gamma^v_{s,p} : I^v_{s,p} \to Q$ is a motion in Q with constant velocity v, the unique maximal motion $\gamma : I \to Q$ with constant velocity v such that $s \in I$ and $\gamma(s) = p$.

DEFINITION 10.7. — *D is the set of all $(v, p) \in V \times Q$ such that $1 \in I^v_{0,p}$. We write $p + v = \gamma^v_{0,p}(1)$ when $(v, p) \in D$. Note that $p + v \in Q$ when $p \in Q$*

and $v \in V$, whereas $v + w \in V$ when $v, w \in V$. For every $p \in Q$ we write $D_p = \{v \in V \mid (v, p) \in D\}$.

Lemma 10.8. — *Assume that the locally convex V-parallel space Q is straight line complete, see Definition 10.1 and Definition 10.4. Then, insofar as defined, the mapping $(v, p) \mapsto p + v$ is an action of the additive group $(V, +)$ on Q, in the following sense.*

i) *For every $p \in Q$ we have $(0, p) \in D$ and $p + 0 = p$.*
ii) *If $(v, p) \in D$ then $(-v, p + v) \in D$ and $(p + v) + (-v) = p$.*
iii) *If $(v_1, p) \in D$ and $(v_2, p + v_1) \in D$, then $(v_1 + v_2, p) \in D$ and $(p + v_1) + v_2 = p + (v_1 + v_2)$.*

Proof. — The statements i) and ii) follow immediately from the definitions. Our proof of iii) is surprisingly long.

In order to prove iii), assume that $(v_1, p) \in D$ and $(v_2, p + v_1) \in D$.

Let L be the linear mapping from $\mathbb{R}^2$ to V which sends $e_1 = (1, 0)$ to v_1 and $e_2 = (0, 1)$ to v_2. For any $s_1, s_2 \in \mathbb{R}$, let $C(s_1, s_2)$ denote the convex hull of $(s_1, 0)$, $(1, 0)$, and $(1, s_2)$ in $\mathbb{R}^2$, which is a solid triangle.

Let I denote the set of all $s_1 \in [0, 1]$ for which there exists $0 < s_2 \leqslant 1$ and an L-map $f : C(s_1, s_2) \to Q$ such that $f(s_1, 0) = p + s_1 v_1$. Note that if $s_1 \in I$, then $[s_1, 1] \subset I$. Note also that for every $(t_1, t_2), (t'_1, t'_2) \in C(s_1, s_2)$,

$$[0, 1] \ni u \mapsto f(t_1 + u(t'_1 - t_1), t_2 + u(t'_2 - t_2))$$

is a motion in Q with constant velocity $L(t'_1 - t_1, t'_2 - t_2) = (t'_1 - t_1)v_1 + (t'_2 - t_2)v_2$, hence $((t'_1 - t_1)v_1 + (t'_2 - t_2)v_2, f(t_1, t_2)) \in D$ and

$$f(t'_1, t'_2) = f(t_1, t_2) + ((t'_1 - t_1)v_1 + (t'_2 - t_2)v_2).$$

If we apply this with $(t_1, t_2) = (s_1, 0)$, then we see that f is uniquely determined by the formula

$$f(t_1, t_2) = (p + s_1 v_1) + ((t_1 - s_1)v_1 + t_2 v_2), \tag{10.5}$$

where $((t_1 - s_1)v_1 + t_2 v_2, p + s_1 v) \in D$ for every for every $(t_1, t_2) \in C(s_1, s_2)$, which in turn implies that

$$\begin{aligned}(p + s_1 v_1) + ((t'_1 - s_1)v_1 + t'_2 v_2) = ((p + s_1 v_1) &+ ((t_1 - s_1)v_1 + t_2 v_2)) \\ &+ ((t'_1 - t_1)v_1 + (t'_2 - t_2)v_2),\end{aligned} \tag{10.6}$$

where $((t_1 - s_1)v_1 + t_2 v_2, p + s_1 v)$, $((t'_1 - s_1)v_1 + t'_2 v_2, p + s_1 v)$ and $((t'_1 - t_1)v_1 + (t'_2 - t_2)v_2, (p + s_1 v_1) + ((t_1 - s_1)v_1 + t_2 v_2)$ all belong to D for every $(t_1, t_2), (t'_1, t'_2) \in C(s_1, s_2)$.

Let i_1 denote the infimum of I, which implies that $]i_1, 1] \subset I$. We will show that this implies that $i_1 \in I$, which means that $I = [i_1, 1]$. Because trivially $1 \in I$, we may assume that $0 \leqslant i_1 < 1$. Because $(i_1 v_1,\ p) \in D$, there exists an $\alpha \in A$ such that $p + i_1 v_1 \in Q_\alpha$. Because the mapping $t \mapsto p + tv$ is continuous from $I^{v_1}_{0,p}$ to Q, there exists an $s_1 \in\ ]i_1, 1] \subset I$ such that $p + s_1 v_1 \in Q_\alpha$. Because $s_1 \in I$, there exists $0 < s_2 \leqslant 1$ and an L-map $f : C(s_1, s_2) \to Q$ such that $f(s_1, 0) = p + s_1 v_1$. Note that for each $u \in [0, 1]$ we have

$$c(u) := (s_1, 0) + u((1, s_2) - (s_1, 0)) \in C(s_1, s_2).$$

Because f is continuous, there exists $0 < u \leqslant 1$ such that $f(c) \in Q_\alpha$, if we write $c := c(u)$. Because $p + i_1 v_1$, $p + s_1 v_1$, and $f(c)$ all belong to Q_α, the points $\varphi_\alpha(p + i_1 v_1)$, $\varphi_\alpha(p + s_1 v_1)$, and $\varphi_\alpha(f(c))$ all belong to the convex subset V_α of V, which implies that their convex hull B_α in V is contained in V_α. Let B be the convex hull of $(i_1, 0)$, $(s_1, 0)$, and c in $\mathbb{R}^2$. The L-map from B onto B_α which sends $(s_1, 0)$ to $\varphi_\alpha(p + sv_1)$, followed by ${\varphi_\alpha}^{-1}$, defines an L-map e from B to Q such that $e(s_1, 0) = p + s_1 v_1$.

Because of the uniqueness of L-maps which map $(s_1, 0)$ to $p + s_1 v_1$, see (10.5), we have that $e = f$ on $B \cap C(s_1, s_2)$, and therefore e and f have a common extension $g : B \cup C(s_1, s_2) \to Q$. In order to prove that g is an L-map, we observe that the property of being an L-map is local, and because e and f are L-maps on the open subsets $B \setminus C(s_1, s_2)$ and $C(s_1, s_2) \setminus B$ of B and $C(s_1, s_2)$, respectively, we have that g is an L-map on $(B \setminus C(s_1, s_2)) \cup (C(s_1, s_2) \setminus B) = (B \cup C(s_1, s_2)) \setminus (B \cap C(s_1, s_2))$. On the other hand, if $r \in B \cap C(s_1, s_2)$, then there are neighborhoods B_0 and C_0 of r in B and $C(s_1, s_2)$, respectively, such that $\varphi_\alpha(e(p)) - \varphi_\alpha(e(r)) = L(p - r)$ when $p \in B_0 \cap e^{-1}(Q_\alpha)$ and $\varphi_\alpha(f(q)) - \varphi_\alpha(f(r)) = L(p - r)$ when $q \in C_0 \cap f^{-1}(Q_\alpha)$. It follows that

$$\begin{aligned} \varphi_\alpha(g(p)) - \varphi_\alpha(g(q)) &= (\varphi_\alpha(g(p)) - \varphi_\alpha(g(r))) + (\varphi_\alpha(g(r)) - \varphi_\alpha(g(q))) \\ &= (\varphi_\alpha(e(p)) - \varphi_\alpha(e(r))) + (\varphi_\alpha(f(r)) - \varphi_\alpha(f(q))) \\ &= L(p - r) + L(r - q) = L(p - q), \end{aligned}$$

which implies (10.3) with f replaced by g.

Let d be the intersection point of the straight line through $(1, 0)$ and $(1, s_2)$, and the straight line through $(i_1, 0)$ and c. A straightforward calculation shows that $d = (1, s_2')$, with

$$s_2' := \frac{u(s_1 - i_1) + u(1 - s_1)}{s_1 - i_1 + u(1 - s_1)} s_2 \leqslant s_2.$$

Because c is lying on the straight line between $(i_1, 0)$ and d, it follows that the convex hull $C(i_1, s_2')$ of $(i_1, 0)$, $(1, 0)$ and d is equal to to the union of the convex hull B of $(i_1, 0)$, $(s_1, 0)$, c, and the convex hull F of the points

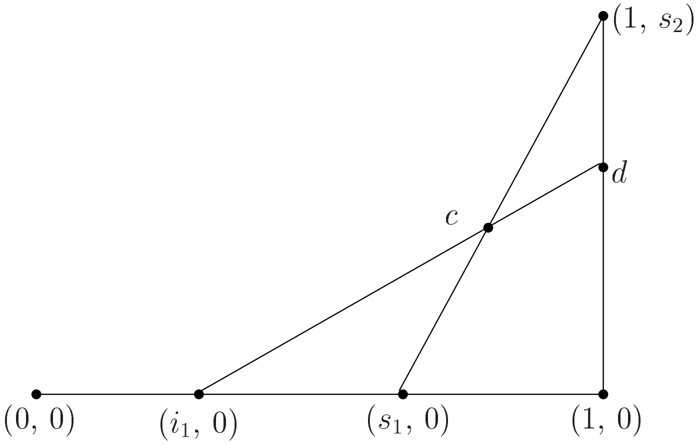

Figure 10.1. The union of $C(s_1, s_2)$ and $C(i_1, s_2')$.

$(i_1, 0)$, c, $(1, 0)$, and d. Because the latter four points all lie in $B \cup C(s_1, s_2)$, it follows that $C(i_1, s_2') \subset B \cup C(s_1, s_2)$. Because the restriction of g to $S(i_1, s_2')$ is an L-map such that $g(s_1, 0) = p + s_1 v_1$, hence

$$g(i_1, 0) = (p + s_1 v_1) + (i_1 - s_1) v_1 = p + i_1 v_1$$

in view of (10.5) with f replaced by g, it follows that $i_1 \in I$.

Now suppose that $i_1 > 0$. With $\alpha \in A$ such that $p + i_1 v_1 \in Q_\alpha$, there exists $0 \leqslant s_1 < i_1$ such that $p + s_1 v_1 \in Q_\alpha$. The same reasoning as above with i_1 and s_1 interchanged, where we use that $i_1 \in I$, leads to the conclusion that $s_1 \in I$, in contradiction with the definition $i_1 = \inf I$ of i_1. We conclude that $i_1 = 0$, or $0 \in I$, which means that there exists $0 < s_2 \leqslant 1$ and an L-map f from $C(0, s_2)$ to Q such that $f(0, 0) = p$.

Define J as the set of all $s_2 \in [0, 1]$ for which there exists an L-map f from $C(0, s_2)$ to Q such that $f(0, 0) = p$, and write $s := \sup J$. The uniqueness of the L-maps f from $C(0, s_2)$ to Q such that $f(0, 0) = p$, see (10.5), where s_2 ranges over J, implies that these L-maps have a common extension to the union C over all $s_2 \in J$ of the triangles $C(0, s_2)$. We denote this common extension to C also by f.

The closure of C in $\mathbb{R}^2$ is equal to $C(0, s)$, and the (relative) interior $\mathrm{relint}(C(0, s))$ of $C(0, s)$ is non-empty and contained in C. Choose $c_0 \in \mathrm{relint}(C(0, s)) \subset C$. For any $c \in C(0, s)$ and $0 \leqslant t < 1$ we have that $c_0 + t(c - c_0) \in \mathrm{relint}(C(0, s)) \subset C$, and the straight line completeness of Q implies that there exists a point $q \in Q$ such that $f(c_0 + t(c - c_0)) \to q$ as

$t \uparrow 1$, see Definition 10.4. Because $q = f(c)$ if $c \in C$, we may write $q = f(c)$ for any $c \in C(0,s)$.

For any $c \in C(0,s)$ there is an $\alpha \in A$ such that $q = f(c) \in Q_\alpha$. There exists $0 \leqslant T < 1$ such that $f(c_0 + t(c - c_0)) \in Q_\alpha$ for every $T \leqslant t < 1$, hence

$$\varphi_\alpha(f(c_0 + t(c - c_0))) = \varphi_\alpha(f(c_0 + T(c - c_0))) + (t - T)L(c - c_0)$$

for every $T \leqslant t < 1$. Because f is continuous on C, these conclusions will still hold if c is replaced by $c' \in C(0,s)$ sufficiently close to c, and it follows that $f(c') \in Q_\alpha$ and

$$\varphi_\alpha(f(c')) = \varphi_\alpha(f(c_0 + T(c' - c_0))) + (1 - T)L(c' - c_0)$$

if $c' \in C(0,s)$ is sufficiently close to c. Again using that f is continuous on C, this formula shows that the extension $f : C(0,s) \to Q$ of the L-map $f : C \to Q$ to the closure $C(0,s)$ of C is continuous, and therefore it is an L-map. It follows that $s \in J$, or equivalently $J = [0,s]$.

If $s < 1$ then the previous argument leading to $i_1 = 0$, with v_1 and v_2 replaced by $v_1 + sv_2$ and $(1 - s)v_2$, respectively, shows that there exists $s < s' \leqslant 1$ and an L-map f' from the convex hull C' of $(0,0)$, $(1,s)$, and $(1,s')$ to Q, such that $f'(0,0) = p$. The uniqueness of L-maps which send $(0,0)$ to p, see (10.5), yields that $f' = f$ on $C' \cap C(0,s)$, which implies that f and f' have a common extension f'' to $C(0,s) \cup C' = C$. As in the argument leading to $i_1 = 0$, we have that f'' is an L-map, and because $f''(0,0) = p$ it follows that $s' \in J$, in contradiction with the definition $s = \sup J$ of s.

We arrive at $s = 1$, or equivalently $1 \in J$, which means that there is an L-map $f : C(0,1) \to Q$ such that $f(0,0) = p$. Now (10.6) with $(s_1, s_2) = (0,1)$, $(t_1, t_2) = (1,0)$ and $(t_1', t_2') = (1,1)$ implies that $(v_1 + v_2, p) \in D$ and $p + (v_1 + v_2) = (p + v_1) + v_2$. This completes the proof of the statement iii) in the lemma. □

Lemma 10.9. — *Let Q be a straight line complete and locally convex V-parallel space. Let $p \in Q$, and let D_p be defined as in Definition 10.7. Then D_p is a closed and convex subset of V, and the mapping $v \mapsto p + v$ is a local homeomorphism from D_p onto an open subset of Q.*

Proof. — Let $v, w \in D_p$, that is, $(v,p) \in D$ and $(w,p) \in D$. It follows from ii) in Lemma 10.8 that $(-v, p + v) \in D$, and then from iii) in Lemma 10.8 with v_1, p and v_2 replaced by $-v$, $p + v$ and w, respectively, that $(w - v, p + v) \in D$ and $(p + v) + (w - v) = p + (v + (w - v)) = p + w$. This implies in view of Definition 10.7 that for each $t \in [0,1]$ we have

$(t(w-v), p+v) \in D$. Using Lemma 10.8 with v_1 and v_2 replaced by v and $t(w-v)$, respectively, we obtain that $(v+t(w-v), p) \in D$, that is, $v+t(w-v) \in D_p$ for every $t \in [0,1]$. This shows that D_p is a convex subset of V.

Let $v \in D_p$, $w \in V$, and $v+tw \in D_p$ for all $0 \leqslant t < 1$. As in the previous paragraph we obtain that $(tw, p+v) \in D$ and $\gamma : t \mapsto p+(v+tw) = (p+v)+tw$ is a motion with constant velocity w in Q. Because Q is straight line complete, it follows that there exists $q \in Q$ such that $(p+v)+tw$ converges to q as $t \uparrow 1$. In other words γ has a continuous extension to a curve $\gamma : [0,1] \to Q$, which again is a motion with contant velocity w and satisfies $\gamma(1) = q$. According to Definition 10.7 this means that $(w, p+v) \in D$, which in combination with $(v,p) \in D$ and iii) in Lemma 10.8 with v_1 and v_2 replaced by v and w, respectively, implies that $(v+w, p) \in D$, that is $v+w \in D_p$. This shows that the convex subset D_p of V is a closed subset of V, see Definition 10.4.

We will now prove that the mapping $v \mapsto p+v$ is a homeomorphism from a neighborhood of 0 in D_p onto a neighborhood of p in Q. There exists $\alpha \in A$ such that Q_α is a neighborhood of p in Q. We will first prove that the translate $V_\alpha - \varphi_\alpha(p)$ of V_α over the vector $-\varphi_\alpha(p)$ is contained in D_p. Indeed, if $v \in V$, $\varphi_\alpha(p)+v \in V_\alpha$, then the convexity of V_α implies that $\varphi_\alpha(p)+tv \in V_\alpha$ for every $t \in [0,1]$. The continuity of ${\varphi_\alpha}^{-1}$ implies that $t \mapsto {\varphi_\alpha}^{-1}(\varphi_\alpha(p)+tv)$ is a motion in Q which has constant velocity v and is equal to p at $t=0$, which shows that $(v,p) \in D$. Note that in passing we have proved that $\varphi_\alpha(p+tv) = \varphi_\alpha(p)+tv$ for all $0 \leqslant t \leqslant 1$, and in particular $\varphi_\alpha(p+v) = \varphi_\alpha(p)+v$

We next claim that $V_\alpha - \varphi_\alpha(p)$ is a neighborhood of 0 in D_p. If this would not be the case, then there is a sequence v_j in $D_p \setminus (V_\alpha - \varphi_\alpha(p))$ which converges to 0 in V as $j \to \infty$. Because Q is a Hausdorff space, there exists an open neighborhood Q_p of p in Q such that the closure $\overline{Q_p}$ of Q_p in Q is contained in Q_α. Let I_j be the set of all $t \in I_{0,p}^{v_j}$ such that $t \geqslant 0$ and $p+tv_j \notin Q_p$, and write $t_j := \inf I_j$. Note that $t_j \geqslant 0$. Because $t \mapsto p+tv_j$ is continuous from $I_{0,p}^{v_j}$ to Q, and $Q \setminus Q_p$ is closed in Q, we have that I_j is a closed subset of $I_{0,p}^{v_j}$, hence $t_j \in I_j$, which implies that $p+t_jv_j \notin Q_p$. On the other hand we have for every $0 \leqslant t < t_j$ that $p+tv_j \in Q_p \subset Q_\alpha$, hence $\varphi_\alpha(p+tv_j) = \varphi_\alpha(p)+tv_j \in V_\alpha$. Because $v_j \notin D_p \setminus (V_\alpha - \varphi_\alpha(p))$, this cannot happen for $t=1$, which proves that $t_j \leqslant 1$. Because $t \mapsto p+tv_j = \gamma_{0,p}^{v_j}$ is continuous and $p+tv_j \in Q_p$ for every $t < t_j$, we have that $p+t_jv_j \in \overline{Q_p} \subset Q_\alpha$, hence $\varphi_\alpha(p+t_jv_j) = \varphi_\alpha(p)+t_jv_j \to \varphi_\alpha(p)$ in V, hence in V_α.

Because φ_{α}^{-1} is continuous, we conclude that $p + t_j v_j \to p$ in Q_{α}, hence in Q, in contradiction with the fact that $p + t_j v_j \notin Q_p$ for every j.

Because $\varphi_{\alpha}(p + v) = \varphi_{\alpha}(p) + v$ for every $v \in V_{\alpha} - \varphi_{\alpha}(p)$, the restriction to the neighborhood $V_{\alpha} - \varphi_{\alpha}(p)$ of 0 in D_p of the mapping $v \mapsto p + v$ is equal to the continuous mapping $v \mapsto {\varphi_{\alpha}}^{-1}(\varphi_{\alpha}(p) + v)$, from $V_{\alpha} - \varphi_{\alpha}(p)$ onto the neighborhood Q_{α} of p in Q, with inverse equal to the continuous mapping $q \mapsto \varphi_{\alpha}(q) - \varphi_{\alpha}(p)$ from Q_{α} onto $V_{\alpha} - \varphi_{\alpha}(p)$.

Let $v \in D_p$. The first paragraph of the proof yielded that the translation $w \mapsto w - v$ maps D_p onto D_{p+v}, and that $p + w = (p + v) + (w - v)$ for every $w \in D_p$. Because the mapping $z \mapsto (p + v) + z$ is a homeomorphism from a neighborhood of 0 in D_{p+v} onto a neighborhood of $p + v$ in Q, and the translation $w \mapsto w - v$ is a homeomorphism from D_p onto D_{p+v}, it follows that the map $w \mapsto p + w$ is a homeomorphism from a neighborhood of v in D_p onto a neighborhood of $p + v$ in Q. □

LEMMA 10.10. — *Let Q be a straight line complete and locally convex V-parallel space which in addition is connected. Then the action of V on Q is transitive, in the sense that for any $p, q \in Q$ there exists $v \in V$ such that $(v, p) \in D$ and $p + v = q$.*

Proof. — We write $p \sim q$ if there exists a $v \in V$ such that $(v, p) \in D$ and $p + v = q$. It follows from Lemma 10.8 that $\sim$ is an equivalence relation in Q. It follows from the last statement in Lemma 10.9 that the equivalence classes are open subsets of Q. Because Q is connected, the equivalence relation $\sim$ has only one equivalence class Q, which proves the transitivity of the action. □

LEMMA 10.11. — *Let Q be a straight line complete, connected and locally convex V-parallel space. For any $v \in V$, the following conditions are equivalent.*

a) *There exists a $p \in Q$ such that $I_{0,p}^{v} = \mathbb{R}$.*

b) *For every $p \in Q$ we have $I_{0,p}^{v} = \mathbb{R}$.*

Let N denote the set of all $v \in V$ such that a) or b) holds. Then N is a linear subspace of V, $(N \times Q) \subset D$, and the restriction to $N \times Q$ of the mapping $(v, p) \mapsto p + v$ defines an action of the additive group N on Q.

Proof. — Assume that a) holds, which implies that $tv \in D_p$ for every $t \in \mathbb{R}$. Let $q \in Q$. Because of the transitivity in Lemma 10.10, there exists a $w \in V$ such that $w \in D_q$ and $q + w = p$. Because of iii) in Lemma 10.8 we have for every $t \in \mathbb{R}$ that $w + tv \in D_q$, which in view of Definition 10.7 implies that $s(w + tv) \in D_q$ for every $0 \leqslant s \leqslant 1$. If for any $t \in \mathbb{R}$ and $0 < s \leqslant 1$ we replace t by t/s and take the limit for $s \downarrow 0$, then we obtain in view of the closedness of D_q that $tv \in D_q$, which implies that $t \in I^v_{0,q}$. Because this holds for every $t \in \mathbb{R}$, it follows that $I^v_{0,q} = \mathbb{R}$.

If $v, w \in N$ then we have for every $r, s \in \mathbb{R}$ that $rv, sw \in N$. Moreover, we have for every $p \in Q$ that $(rv, p) \in D$, $(sw, p+rv) \in D$, hence $(rv+sw, p) \in D$ in view of iii) in Lemma 10.8. If for any $t \in \mathbb{R}$ we replace r and s by tr and ts, respectively, we obtain that $I^{rv+sw}_{0,p} = \mathbb{R}$, hence $rv + sw \in N$, and we conclude that N is a linear subspace of V. Furthermore b) implies that $(v, p) \in D$ for every $v \in N$ and $p \in Q$, and it follows from Lemma 10.8 that the mapping $(v, p) \mapsto p + v$ defines an action of N on Q. □

Lemma 10.12. — *Let Q be a straight line complete and connected locally convex V-parallel space. For any $v \in V$, the following conditions are equivalent.*

i) *There exists $p \in Q$ such that $(v, p) \in D$ and $p + v = p$.*
ii) *For all $p \in Q$ we have $(v, p) \in D$ and $p + v = p$.*

Let P denote the set of all $v \in V$ such that i) or ii) holds. Then P is a discrete additive subgroup of the linear subspace N of V which is defined in Lemma 10.11.

Proof. — Assume that i) holds. Lemma 10.10 implies that for any $q \in Q$ there exists $u \in V$ such that $q = p + u$, and therefore

$$q + v = (p + u) + v = p + (u + v) = p + (v + u) = (p + v) + u = p + u = q,$$

which proves ii).

If $p + v = p$, then it follows by induction on k that $(kv, p) \in D$ and $p + kv = p$ for every positive integer k, and using i), ii) in Lemma 10.8 we obtain the same conclusions for all $k \in \mathbb{Z}$. This implies that $\mathbb{Z} \subset I^v_{0,p}$, which in turn implies that $I^v_{0,p} = \mathbb{R}$, because $I^v_{0,p}$ is an interval in $\mathbb{R}$. In view of Lemma 10.11, we conclude that $v \in N$.

It follows that P is equal to the set of $v \in N$ such that the action of v on Q is trivial, and therefore P is an additive subgroup of N. The last statement in Lemma 10.10 implies that P is a discrete subset of D_p, hence of the closed subset N of D_p. □

Recall that if Q is a compact locally convex V-parallel space, then Q is straight line complete, see Corollary 10.6.

Theorem 10.13. — *Let Q be a straight line complete, connected and locally convex V-parallel space. Let P be the period group in V defined in Lemma 10.12. Denote by $\mathbb{R}P$ the $\mathbb{R}$-linear span of P in V, where we note that $(\mathbb{R}P)/P$ is a torus. Let C be a linear complement in V of the linear span $\mathbb{R}P$ of P in V. Let $p \in Q$. Then there is a convex closed subset Δ_p of C such* *that $D_p = \Delta_p + \mathbb{R}P$. Furthermore the mapping $\Phi_p : (v,\ w) \mapsto p + (v + w)$ defines an isomorphism of V-parallel spaces from $\Delta_p \times ((\mathbb{R}P)/P)$ onto Q. The projection from Q onto the $\mathbb{R}P$-orbit space $Q/(\mathbb{R}P)$ induces an isomorphism from Δ_p, viewed as a C-parallel space, onto the $V/(\mathbb{R}P)$-parallel space $Q/(\mathbb{R}P)$.*

There is a collection of linear forms v_i^* on V and real numbers c_i, where i runs over some index set I, such that D_p is equal to the set of all $v \in V$ such that $v_i^*(v) \geqslant c_i$ for all $i \in I$. For every such collection λ_i, c_i, the linear subspace N of V defined in Lemma 10.11 is equal to the common kernel of the linear forms v_i^*, $i \in I$, on V.

Q is compact if and only if Δ_p is compact, which implies that $N = \mathbb{R}P$, and P is a cocompact discrete subgroup of the additive group N. If Q is a compact connected locally convex polyhedral V-parallel space, then $\Delta_p \simeq Q/(\mathbb{R}P)$ is a convex polytope in $C \simeq V/(\mathbb{R}P)$.

Proof. — Let $(v,p) \in D$, $(v',p) \in D$, and $p + v = p + v'$. It follows from Lemma 10.8 that $(-v', p + v') = (-v', p + v) \in D$, $(v - v', p) = (v + (-v'), p) \in D$ and $p + (v - v') = (p + v) + (-v') = (p + v') + (-v') = p$, which in view of Lemma 10.12 implies that $v - v' \in P \subset N$.

Define $\Delta_p := D_p \cap C$. In view of Lemma 10.9, Δ_p is a closed and convex subset of C. The fact that the additive group N acts on Q, *cf.* Lemma 10.12, and hence its subgroup $\mathbb{R}P$ acts on Q, implies that $D_p = \Delta_p + \mathbb{R}P$.

Because $v, v' \in D_p$ and $p + v = p + v'$ imply that $v - v' \in P$, and C is complementary to $\mathbb{R}P$, the mapping $\Phi_p : \Delta_p \times ((\mathbb{R}P)/P) \to Q$ is injective. On the other hand Lemma 10.10 implies that Φ_p is a surjective local homeomorphism. Because Φ_p is an L-map with $L : C \times (\mathbb{R}P) \to V : (w, z) \mapsto w + z$, it follows that Φ_p is an isomorphism from $\Delta_p \times ((\mathbb{R}P)/P)$ onto Q.

The statement about N, the v_i^* and the c_i follows because $D_p = \Delta_p + (\mathbb{R}P)$ is a closed convex subset of V, and N is equal to the lineality of D_p, the set of direction vectors of lines which are contained in D_p, *cf.* Rockafellar [49, p. 65].

Finally, Q is compact if and only if $\Delta_p \times ((\mathbb{R}P)/P)$ is compact if and only if Δ_p is compact. In view of $N \cap C \subset \Delta_p$, the latter implies that $N \cap C = 0$, hence $N = \mathbb{R}P$, and P is a cocompact discrete subgroup of the additive topological group N. A convex compact subset of a vector space C which is a locally convex polyhedral C-parallel space is a convex polytope in C. □

Remark 10.14. — Theorem 10.13 is a generalization of the theorem of Tietze [52] and Nakajima [41] that any closed, connected, and locally convex subset of a finite-dimensional vector space is convex. Our proof of

Lemma 10.8 is close to the proof of Klee [29, (5.2)] of the generalization of the Tietze-Nakajima theorem to subsets of arbitrary topological vector spaces.

11. The symplectic tube theorem

In this section we describe the local model of Benoist [6, Prop. 1.9] and Ortega and Ratiu [44] for a general proper symplectic Lie group action. See also Ortega and Ratiu [45, Sec. 7.2–7.4] for a detailed proof. For Hamiltonian actions, such local models had been obtained before by Marle [36] and Guillemin and Sternberg [24, Sec. 41].

Let (M, σ) be a smooth symplectic manifold and G a Lie group which acts smoothly on (M, σ) by means of symplectomorphisms. Furthermore assume that the action is *proper*, which means that for any compact subset K of M the set of all $(g, m) \in G \times M$ such that $(m, g \cdot m) \in K$ is compact in $G \times M$. The action of G is certainly proper if G is compact.

For every $g \in G$ we will write $g_M : x \mapsto g \cdot x$ for the action of g on M. For every element X in the Lie algebra $\mathfrak{g}$ of G, the infinitesimal action on M will be denoted by X_M. It is a smooth vector field X_M on M, the flow of which leaves σ invariant.

It follows from the properness of the action, that for every $x \in M$ the stabilizer subgroup $H := G_x := \{g \in G \mid g \cdot x = x\}$ of x in G is a compact, Lie subgroup of G, and the mapping $A_x : g \mapsto g \cdot x : G \to M$ induces a G-equivariant smooth embedding

$$\alpha_x : gH \mapsto g \cdot x : G/H \to M \tag{11.1}$$

from G/H into M, with closed image, equal to the orbit $G \cdot x$ of G through the point x. Here $g \in G$ acts on G/H by sending $g'H$ to $(gg')H$. The Lie algebra $\mathfrak{h} := \mathfrak{g}_x$ of $H := G_x$ is equal to the set of $X \in \mathfrak{g}$ such that $X_M(x) = 0$. The linear mapping $\mathrm{T}_1 A_x : \mathfrak{g} \to \mathrm{T}_x M$ induces induces a linear isomorphism from $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}/\mathfrak{g}_x$ onto $\mathfrak{g}_M(x) := \mathrm{T}_x(G \cdot x)$.

For the description of the symplectic form in the local model, we begin with the closed two-form

$$\sigma^{G/H} := (\alpha_x)^* \sigma \tag{11.2}$$

on G/H, which represents the "restriction" of σ to the orbit $G \cdot x \simeq G/H$ through the point x. Here $\alpha_x : G/H \to M$ is defined in (11.1). The G-invariance of σ and the G-equivariance of α_x imply that $\sigma^{G/H}$ is G-invariant.

If we identify $\mathrm{T}_{1H}(G/H)$ with $\mathfrak{g}/\mathfrak{h}$, and $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the projection $X \mapsto X + \mathfrak{h}$, then $\sigma_{1H}^{G/H}$, and therefore the G-invariant two-form $\sigma^{G/H}$ on G/H, is determined by the antisymmetric bilinear form

$$\sigma^{\mathfrak{g}} := (\mathrm{T}_1 A_x)^* \sigma_x = p^* \sigma_{1H}^{G/H} \tag{11.3}$$

on $\mathfrak{g}$, which is invariant under the adjoint action of H on $\mathfrak{g}$. It follows that the kernel

$$\mathfrak{l} := \ker \sigma^{\mathfrak{g}} = \ker((\mathrm{T}_1 A_x)^* \sigma_x) \tag{11.4}$$

is an $\mathrm{Ad}\, H$-invariant linear subspace of $\mathfrak{g}$. We have $\mathfrak{h} \subset \mathfrak{l}$, because of (11.3) and the fact that $p^* \sigma_{1H}^{G/H}$ vanishes on the kernel $\mathfrak{h}$ of p.

If L is a linear subspace of a symplectic vector space (V, σ), then the symplectic orthogonal complement L^{σ} of L in V is defined as the set of all $v \in V$ such that $\sigma(l, v) = 0$ for every $l \in L$. The restriction to $\mathfrak{g}_M(x)^{\sigma_x}$ of σ_x defines a symplectic form σ^W on the vector space

$$W := \mathfrak{g}_M(x)^{\sigma_x} / (\mathfrak{g}_M(x)^{\sigma_x} \cap \mathfrak{g}_M(x)), \tag{11.5}$$

and the mapping $X + \mathfrak{h} \mapsto X_M(x)$ defines a linear isomorphism from $\mathfrak{l}/\mathfrak{h}$ onto $\mathfrak{g}_M(x)^{\sigma_x} \cap \mathfrak{g}_M(x)$. The linearized action $H \ni h \mapsto \mathrm{T}_x h_M$ of H on $\mathrm{T}_x M$ is symplectic and leaves $\mathfrak{g}_M(x) \simeq \mathfrak{g}/\mathfrak{h}$ invariant, acting on it via the adjoint representation. That is, $h \cdot X_M(x) = (\mathrm{Ad}\, h)(X)_M(x)$, if $X \in \mathfrak{g}$ and $h \in H$. It therefore also leaves $\mathfrak{g}_M(x)^{\sigma_x}$ invariant and induces an action of $H = G_x$ on the symplectic vector space (W, σ^W) by means of symplectic linear transformations.

With $\mathfrak{l}$ as in (11.4), we "enlarge" the vector space W to the vector space

$$E := (\mathfrak{l}/\mathfrak{h})^* \times W, \tag{11.6}$$

on which $h \in H$ acts by sending (λ, w) to $(((\mathrm{Ad}\, h)^*)^{-1}(\lambda), h \cdot w)$.

For any action by linear transformations of a compact Lie group K on a vector space V, any K-invariant linear subspace L of V has an K-invariant linear complement L' in V. For instance, if β is an inner product on V, then the average $\overline{\beta}$ of β over K is a K-invariant inner product on V, and the $\overline{\beta}$-orthogonal complement L' of L in V has the desired properties. Choose $\mathrm{Ad}\, H$-invariant linear complements $\mathfrak{k}$ and $\mathfrak{c}$ of $\mathfrak{h}$ and $\mathfrak{l}$ in $\mathfrak{g}$, respectively. Let $X \mapsto X_{\mathfrak{l}} : \mathfrak{g} \to \mathfrak{l}$ and $X \mapsto X_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ denote the linear projection from $\mathfrak{g}$ onto $\mathfrak{l}$ and $\mathfrak{h}$ with kernel equal to $\mathfrak{c}$ and $\mathfrak{k}$, respectively. Then these linear projections are $\mathrm{Ad}\, H$-equivariant.

If $g \in G$, then we denote by $\mathrm{L}_g : g' \mapsto gg' : G \to G$ the multiplication from the left by means of g. Define the smooth one-form $\eta^{\#}$ on $G \times E$ by

$$\eta^{\#}_{(g,(\lambda,w))}((\mathrm{T}_1\, \mathrm{L}_g)(X), (\delta\lambda, \delta w)) := \lambda(X_{\mathfrak{l}}) + \sigma^W(w, \delta w + X_{\mathfrak{h}} \cdot w)/2 \tag{11.7}$$

for all $g \in G$, $\lambda \in (\mathfrak{l}/\mathfrak{h})^*$, $w \in W$, and all $X \in \mathfrak{g}$, $\delta\lambda \in (\mathfrak{l}/\mathfrak{h})^*$, $\delta w \in W$. Here we identify the tangent spaces of a vector space with the vector space itself and $(\mathfrak{l}/\mathfrak{h})^*$ with the space of linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{h}$.

Let E be defined as in (11.6), with $\mathfrak{l}$ and W as in (11.4) and (11.5), respectively. Let $G \times_H E$ denote the orbit space of $G \times E$ for the proper and free action of H on $G \times E$, where $h \in H$ acts on $G \times E$ by sending (g, e) to $(gh^{-1}, h \cdot e)$. The action of G on $G \times_H E$ is induced by the action $(g, (g', e)) \mapsto (gg', e)$ of G on $G \times E$. Let $\pi : G \times_H E \to G/H$ denote the mapping which is induced by the projection $(g, e) \mapsto g : G \times E \to G$ onto the first component. Because H acts on E by means of linear transformations, this projection exhibits $G \times_H E$ as a G-homogeneous vector bundle over the homogeneous space G/H, which fiber E and structure group H. With these notations, we have the following local normal form for the symplectic G-space (M, σ, G).

Theorem 11.1. — *If $\pi_H : G \times E \to G \times_H E$ denotes H-orbit mapping, then there is a unique smooth one-form η on $G \times_H E$, such that $\eta^{\#} = {\pi_H}^*\eta$. Here $\eta^{\#}$ is defined in* (11.7).

Furthermore, there exists an open H-invariant neighborhood E_0 of the origin in E and a G-equivariant diffeomorphism Φ from $G \times_H E_0$ onto an open G-invariant neighborhood U of x in M, such that $\Phi(H \cdot (1, 0)) = x$ and

$$\Phi^*\sigma = \pi^*\sigma^{G/H} + \mathrm{d}\eta. \tag{11.8}$$

Here the mapping $\pi : G \times_H E_0 \to G/H$ is induced by the projection onto the first component, and $\sigma^{G/H}$ is defined in (11.2).

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