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Embedding subsets of tori Properly into $\mathbb{C}^2$

EMBEDDING SUBSETS
OF TORI PROPERLY INTO $\mathbb{C}^2$

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ABSTRACT. — Let $\mathbb{T}$ be a complex one-dimensional torus. We prove that all subsets of $\mathbb{T}$ with finitely many boundary components (none of them being points) embed properly into $\mathbb{C}^2$. We also show that the algebras of analytic functions on certain countably connected subsets of closed Riemann surfaces are doubly generated.

RÉSUMÉ. — Nous avons fait des progrès sur le problème du plongement des surfaces de Riemann ouvertes dans $\mathbb{C}^2$. Il est connu que pour tout entier naturel $d \geqslant 2$, le nombre $N_d := \left[\frac{3d}{2}\right] + 1$ est le plus petit entier naturel pour lequel il existe un plongement propre de toute variété de Stein de dimension d dans $\mathbb{C}^{N_d}$. Le problème du plongement propre des variétés de Stein de dimension 1 dans $\mathbb{C}^2$ reste ouvert (il existe du plongement propre dans $\mathbb{C}^3$). Dans ce texte nous prouvons le résultat suivant : soit $\mathbb{T}$ un tore complexe de dimension 1 ; alors il existe un plongement propre de toute partie de $\mathbb{T}$, dont la frontière a un nombre fini de composantes (aucune d'elle n'étant un point), dans $\mathbb{C}^2$. Nous prouvons aussi que les algèbres de fonctions analytiques sur certaines surfaces de Riemann sont doublement générées.

1. Introduction, main results and notation

Our main concern is the problem of embedding bordered Riemann surfaces properly into $\mathbb{C}^2$. A (finite) bordered Riemann surface is obtained by removing a finite set of closed disjoint connected components $D_1, ..., D_k$ from a compact surface $\mathcal{R}$, *i.e.*, the bordered surface is $\tilde{\mathcal{R}} := \mathcal{R} \setminus \cup_{i=1}^k D_i$.

For a positive integer $d \geqslant 2$ it is known that there is a lowest possible integer $N_d = [\frac{3d}{2}] + 1$ such that all Stein manifolds of dimension d embed properly into $\mathbb{C}^{N_d}$ [4],[5],[17] (for more details, see for instance the survey [8]). It is also known that all open Riemann surfaces embed properly into

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$\mathbb{C}^3$, but it remains an open question whether the dimension of the target domain in this case always can be pushed down to 2.

For (positive) results when the genus of $\mathcal{R}$ is 0 we refer to the articles [14],[2],[15],[10],[20], and in the case of genus $\geqslant 1$ to [7],[19].

We prove the following theorem:

THEOREM 1.1. — *Let $\mathbb{T}$ be a torus, and let $U \subset \mathbb{T}$ be a domain such that $\mathbb{T} \setminus U$ consists of a finite number of connected components, none of them being points. Then U embeds properly into $\mathbb{C}^2$.*

In [19] we proved that under the assumption that U can be embedded onto a Runge surface in $\mathbb{C}^2$, one can embed arbitrarily small perturbations of U properly into $\mathbb{C}^2$. Our task then is to

(i) Embed U onto a Runge surface,
(ii) Pass from small perturbations to U itself.

(We say that a surface U is Runge if holomorphic functions on U may be approximated uniformly on compacts in U by polynomials).

To achieve (i) we recall from [19] that for any one complementary component D_1, we have that $\mathbb{T} \setminus D_1$ embeds into $\mathbb{C}^2$ by some map ϕ, and that the image is Runge. To embed the smaller domain U onto a Runge surface, we will perturb the image of U by constructing a map that could be described as a local (near some neighborhood of $\phi(U)$) singular shear acting transversally to $\phi(U)$ – the singularities being placed inside each component of $\phi(\mathbb{T} \setminus U)$. This construction is the content of Section 3.

To achieve (ii) we will apply a technique from [10] used by Globevnik and Stensønes to embed planar domains into $\mathbb{C}^2$. He and Schramm have shown that any subset of $\mathbb{T}$ is biholomorphic to a circular subset U' of another torus $\mathbb{T}'$ [13]. This allows us to identify U with a point in $\mathbb{R}^N$ in such a way that the point corresponds to the complex structure on $\mathbb{T}$ and the centers and the radii of the boundary components of U. Now small perturbations of U' embeds properly into $\mathbb{C}^2$, and the perturbation corresponds to some circled subset of some torus, *i.e.*, some (other) point in $\mathbb{R}^N$. So if we identify all subsets of tori close to U with points in a ball B in $\mathbb{R}^N$, we may in this manner construct a map $\psi : B \to \mathbb{R}^N$, such that all circled domains corresponding to points in the image $\psi(B)$ embed properly into $\mathbb{C}^2$. Our goal is to construct the map ψ in such a way that it is continuous and close to the identity. In that case, by Brouwer's fixed point theorem, the point corresponding to U will be contained in the image $\psi(B)$, and the result follows.

Continuity in the setting of uniformization of subsets of tori is treated in Section 2, while continuity regarding the identification of circled subsets with properly embeddable subsets is dealt with in Section 4.

As was pointed out in [7], the question about the embeddability of an open Riemann surface Ω is related to a question about the function algebra $\mathcal{O}(\Omega)$ of all analytic functions on Ω. For an integer $m \in \mathbb{N}$ we say that the algebra $\mathcal{O}(\Omega)$ is m-generated if there exist functions $f_i \in \mathcal{O}(\Omega)$, $i = 1, ..., m$ such that $\mathbb{C}[f_1, ..., f_m]$ is dense in $\mathcal{O}(\Omega)$. Since any Ω embeds properly into $\mathbb{C}^3$ we have that $\mathcal{O}(\Omega)$ is 3-generated, but it is unknown whether or not 2 generators might be sufficient. By the perturbation results in Section 3 we get the following:

THEOREM 1.2. — Let $\mathbb{T}$ be a torus, and let $U \subset \mathbb{T}$ be domain such that each connected component of $\mathbb{T} \setminus U$ has non-empty interior. Then the function algebra $\mathcal{O}(U)$ is 2-generated.

Theorem 1.2 is a special case of the following theorem:

THEOREM 1.3. — Let $\mathcal{R}$ be a closed Riemann surface, let $U \subset \mathcal{R}$ be a domain such that ∂U is a collection of smooth Jordan curves, and let $\phi : U \to \mathbb{C}^2$ be a holomorphic embedding that extends across ∂U. Assume that $\phi(\overline{U})$ is polynomially convex. If $V \subset U$ is a connected open set obtained from U by removing at most countably many disks, then $\mathcal{O}(V)$ is 2-generated.

The proof of the last two theorems will be given in Section 3.

As usual we will denote an ϵ-ball centered at a point p in $\mathbb{R}^n$ or $\mathbb{C}^n$ by $B_\epsilon(p)$ (or simply B_ϵ if the center is the origin), and the corresponding $\epsilon-$disk in $\mathbb{C}$ will be denoted $\triangle_\epsilon(p)$. By a disk in a Riemann surface $\mathcal{R}$ we will mean a subset homeomorphic to $\overline{\triangle}$.

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2. Circled subsets of tori and uniformization

Let $\tau \in \mathbb{C}$ be contained in the upper half plane H^+. If we define the lattice
$$L_\tau := \{m \cdot \tau + n \in \mathbb{C}; m, n \in \mathbb{Z}\},$$

we obtain a torus by considering the quotient $\mathbb{C}/\sim_\tau$, where $z \sim_\tau w \Leftrightarrow z - w \in L_\tau$. It is known that all tori are obtained in this way. For a given τ we let $\mathcal{R}(\Omega(\tau))$ denote the quotient, *i.e.*, the torus, and we let $\Omega(\tau)$ denote $\mathbb{C}$ regarded as its universal cover. We may choose τ with $0 < \mathrm{Re}(\tau) \leqslant 1$.

We are concerned with subsets of tori with finitely many boundary components. Let $\mathbb{T}$ be a torus, let $\tilde{K}_1, \ldots, \tilde{K}_m$ be compact connected disjoint subsets of $\mathbb{T}$, such that $\tilde{\mathbb{T}} := \mathbb{T} \setminus (\cup_{i=1}^m \tilde{K}_i)$ is connected. Then $\mathbb{T}$ may be identified with its cover $\Omega(\tau)$ for some τ, and $\tilde{\mathbb{T}}$ with some subset U of $\Omega(\tau)$. It is clear that U is completely determined by τ and some choice of complementary components $K_1, \ldots, K_m$ of U that intersect the parallelogram with vertices $0, 1, \tau, \tau+1$, and we let $\Omega(\tau, K_1, \ldots, K_m)$ denote such a U. We call such a set an m-domain. We let $\mathcal{R}(\Omega(\tau, \ldots))$ denote the corresponding subset of $\mathcal{R}(\Omega(\tau))$.

Fix an m-domain $\Omega(\lambda, K_1, \ldots, K_m)$, and assume that $\lambda \notin K_i$ for $i = 1, \ldots, m$. We want to consider a space of m-domains "close" to $\Omega(\lambda, K_1, \ldots, K_m)$. For this purpose we recall the definition of the Hausdorff metric: Let X be a metric space with distance function $m : X \times X \to \mathbb{R}^+$. For two closed subsets S_1, S_2 of X one defines first

$$d(S_1, S_2) = \sup_{x \in S_1} \inf\{m(x, y); y \in S_2\}.$$

Then the Hausdorff distance between the sets S_1 and S_2 is defined by

$$d_H(S_1, S_2) = d(S_1, S_2) + d(S_2, S_1).$$

Let $\delta > 0$, let U_0 denote the δ-disk centered at λ, and for $i = 1, ..., m$ let U_i denote the δ-disk centered at the closed connected sets K_i with respect to the Hausdorff metric:

$$U_i = \{S \subset \mathbb{C}; S \text{ is closed}, d_H(S, K_i) < \delta\}.$$

If δ is small enough then if $\lambda' \in U_0$ and if C_i is a connected set $C_i \in U_i$ with $\mathbb{C} \setminus C_i$ connected for $i = 1, ..., m$, then the set $\Omega(\lambda', C_1, \ldots, C_m)$ is an m-domain. (We will also choose δ small enough such that $C_i \in U_i, C_j \in U_j, i \neq j \Rightarrow C_i \cap C_j = \emptyset$, and such that no element $C_i \in U_i$ can intersect the disk U_0). We call the set of these m-domains $X_\delta^m(\Omega(\lambda, K_1, \ldots, K_m))$. Let $\Omega_1 = \Omega(\tau, K_1, \ldots, K_m), \Omega_2 = \Omega(\lambda, C_1, \ldots, C_m) \in X_\delta^m(\Omega(\lambda, K_1, ..., K_m))$, and let $S_1 = \{\tau\} \cup K_1 \cup \ldots \cup K_m$, $S_2 = \{\lambda\} \cup C_1 \cup \ldots \cup C_m$ be the corresponding subsets of $\mathbb{C}$. We then define

$$d_1(\Omega_1, \Omega_2) := d_H(S_1, S_2).$$

As a subset of the set of all m-domains we have all m-domains whose boundary components are all circles. We will let these m-domains be denoted $\Omega(\tau, z_1, r_1, \ldots, z_m, r_m)$, where (z_i, r_i) corresponds to the center and

the radius of the ith boundary component (for some choice of ordering of these components). We will use boldface letters, such as $\mathbf{x}$, to denote a $2m$-tuple $\mathbf{x} = (z_1, r_1, \ldots, z_m, r_m)$ to simplify notation to $\Omega(\tau, \mathbf{x})$. We call such domains circled m-domains, and we denote the set of all such domains T^m.

Let $\Omega(\tau, \mathbf{x})$ be a circled m-domain, and let $X_\delta^m(\Omega(\tau, \mathbf{x}))$ be a space as defined above. For all circled m-domains contained in $X_\delta^m(\Omega(\tau, \mathbf{x}))$ we have a natural ordering of all the boundary components, and we may identify all such domains $\Omega(\lambda, \mathbf{y})$ with points $(\lambda, \mathbf{y}) \in \mathbb{R}^{2+3m}$. So if ϵ is small enough, the points in the ball $B_\epsilon(\tau, \mathbf{x}) \subset \mathbb{R}^{2+3m}$ are in unique correspondence with circled m-domains in $X_\delta^m(\Omega(\tau, \mathbf{x}))$. We may thus give another metric to this (local) space of circled m-domains, henceforth denoted $T_\epsilon^m(\tau, \mathbf{x})$, by defining

$$d_2(\Omega(\tau, \mathbf{x}), \Omega(\lambda, \mathbf{y})) := \|(\tau, \mathbf{x}) - (\lambda, \mathbf{y})\|,$$

where $\| \cdot \|$ is the euclidian distance on $\mathbb{R}^{2+3m}$.

We will now give a lemma regarding conformal mappings of arbitrary m-domains onto circular m-domains. The contents of the lemma are in essence results proved by He and Schramm [13]. Stating the results for the special case of tori, they showed the following: Let $\mathbb{T} \setminus \cup_{i=1}^m K_i$ be an m-connected subdomain of some torus $\mathbb{T}$. Then there exists some torus $\mathbb{T}'$ and a domain $\Omega \subset \mathbb{T}'$ such that the following holds:

(1) Ω is circled, meaning that if we lift Ω to the universal cover of $\mathbb{T}'$ then the complement consists of exact disks (these disks may also be points),

(2) Ω is conformally equivalent to $\mathbb{T} \setminus \cup_{i=1}^m K_i$.

Furthermore they proved that:

(3) A circled domain in the Riemann sphere is unique up to Möbius transformations, i.e., if $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between circled domains, then f is the restriction to Ω_1 of a Möbius transformation.

Formulating (1) and (2) for m-domains as defined above we have the following:

(a) For any $\Omega = \Omega(\lambda, K_1, \ldots, K_m)$ there exists a conformal mapping f that maps Ω onto some $\Omega(\lambda', \mathbf{x}) \in T^m$,

(b) The map f respects the relation $\sim_\lambda$, meaning that $f(z + m + n\lambda) = f(z) + m + nf(\lambda)$ for all $m, n \in \mathbb{Z}$.

In (b) we have normalized so that f fixes the points 0 and 1. By (3) we have then that f is unique.

Now fix a domain $\Omega(\lambda, K_1, \ldots, K_m)$, and consider a space $X_\delta^m(\Omega(\lambda, K_1, \ldots, K_m))$ of nearby m-domains as defined above. For each domain $\Omega' = \Omega(\lambda', C_1, ..., C_m) \in X_\delta^m(\Omega(\lambda, K_1, \ldots, K_m))$ there is a unique map f that

maps Ω' onto a circular m-domain as above, fixing the points 0 and 1, and we may define a map $\varphi : X_\delta^m(\Omega(\lambda, K_1, \ldots, K_m)) \to T^m$ by

$$\varphi(\Omega') = (f(\lambda'), z_1, r_1, \ldots, z_m, r_m),$$

where z_i and r_1 are the center and radius of the boundary component corresponding to C_i. Note that by uniqueness, if $\Omega' = \Omega(\lambda', C_1, \ldots, C_m)$ is a circled m-domain so that Ω' has the representation $\Omega(\lambda', z_1, r_1, \ldots, z_m, r_m)$ where (z_i, r_i) is the center and the radius of C_i, then $\varphi(\Omega') = (\lambda', z_1, r_1, \ldots, z_m, r_m)$. In this respect we may say that $\varphi \mid_{T^m \cap X_\delta^m(\Omega(\lambda, K_1, \ldots, K_m))} = $ id.

We will sum these things up in a lemma, and we want to establish that the map φ is continuous. To prove this we will need the following definitions and theorem from [11]:

Let $\{B_n\}$, for $n = 1, 2, ..$, denote a sequence of domains in the Riemann sphere that include the point $z = \infty$. We define the kernel of this sequence as the largest domain B including $z = \infty$ every closed subset of which is contained in each B_n from some n on. We shall say that the sequence $\{B_n\}$ converges to its kernel B if an arbitrary subsequence has the same kernel B.

THEOREM 2.1. — ([11], page 228). *Let $\{A_n\}$ denote a sequence of domains $A_n, n = 1, 2, ...,$ in the Riemann sphere that include the point $z = \infty$. Suppose that this sequence converges to a kernel A. Let $\{f_n(z)\}$ denote a sequence of functions $\zeta = f_n(z)$ such that for each $n = 1, 2, ...,$ the function $f_n(z)$ maps the domain A_n onto a domain B_n including the point $\zeta = \infty$ in such a way that $f_n(\infty) = \infty$ and $f'_n(\infty) = 1$. Then for the sequence $\{f_n(z)\}$ to converge uniformly in the interior of the domain A to a univalent function $f(z)$ it is necessary and sufficient that the sequence $\{B_n\}$ have a kernel and converge to it, in which case the function $\zeta = f(z)$ maps A univalently onto B.*

We want to apply this theorem for sequences of m-domains. Let A_n be a sequence of m-domains including the origin and converging to an m-domain A. Let A'_n and A' denote the domains in $\mathbb{C}$ including ∞ given by the correspondence $z \mapsto \frac{1}{z}$. Then A'_n is a sequence as above, and A' is its kernel. Let $\{f_n\}$ be a sequence of univalent functions mapping A_n onto a domain B_n including the origin and $f_n(0) = 0$, $f'_n(0) = 1$. For each n define the function $F_n(z) = \frac{1}{f_n(\frac{1}{z})}$ mapping the domain A'_n onto B'_n, where B'_n's relation with B_n is given by the correspondence $z \mapsto \frac{1}{z}$. Then the sequences A'_n and F_n satisfy the conditions in the above theorem. If the sequence $f_n(z)$ converges to a univalent function f on A, the sequence F_n converges to a univalent function F on A'. By the theorem the sequence B'_n

has a kernel B' and converges to it, and F maps A' onto B'. This implies that the sequence B_n has a kernel B and converges to it, and f maps A onto B. On the other hand, if the sequence B_n has a kernel B and converges to it, then the sequence B'_n has a kernel B' and converges to it, and by the theorem F_n converges to a univalent function F on A', mapping A' onto the kernel B'. So the sequence f_n converges to a univalent function f on A mapping A onto the kernel B.

LEMMA 2.2. — *Let* $X_\delta^m(\Omega(\tau, K_1, \ldots, K_m))$ *be a space of m-domains as defined above. There is a map* $\varphi : X_\delta^m(\Omega(\tau, K_1, ..., K_m)) \to T^m$ *such that the following holds:*

(i) $\mathcal{R}(\varphi(\Omega'))$ *is conformally equivalent to* $\mathcal{R}(\Omega')$ *for all* $\Omega' \in T_\delta^m(\Omega(\tau, K_1, \ldots, K_m))$,

(ii) $\varphi \mid_{T^m \cap X_\delta^m(\Omega(\tau, K_1, ..., K_m))} = \mathrm{id}$,

(iii) φ *is continuous with respect to* d_1 *and* d_2.

Proof. — We have already defined φ and established (i) and (ii). To prove continuity we first choose a different normalization of the uniformizing maps. For each map $f : \Omega' \to \mathbb{C}$ as above, we compose with a linear map and assume that $f(0) = 0, f'(0) = 1$.

Let $\Omega(\lambda, Y_1, ..., Y_m) \in X_\delta^m(\Omega(\tau, K_1, ..., K_m))$ and let $f : \Omega(\lambda, Y_1, ..., Y_m) \to \mathbb{C}$ be the corresponding map. Let $\{\Omega(\lambda_j, Y_1^j, .., Y_m^j\} \subset X_\delta^m(\Omega(\tau, K_1, ..., K_m))$ such that $\Omega(\lambda_j, Y_1^j, ..., Y_m^j) \to \Omega(\lambda, Y_1, ..., Y_m)$ and let $f_j : \Omega(\lambda_j, Y_1^j, ..., Y_m^j) \to \mathbb{C}$ be the corresponding maps for those domains. By abuse of notation we will let $f(Y_i)$ and $f_j(Y_i^j)$ denote complementary components of the images. Note that the sequence of domains $\Omega(\lambda_j, Y_1^j, ..., Y_m^j)$ has the domain $\Omega(\lambda, Y_1, ..., Y_m)$ as its kernel and converges to it. We claim that $f_j \to f$ uniformly on compacts in $\Omega(\lambda, Y_1, ..., Y_m)$, and that $f_j(Y_i^j) \to f(Y_i)$. This will prove the continuity of the map φ defined above. That we chose a different normalization does not matter since we will then also have that $\frac{f_j}{f_j(1)} \to \frac{f}{f(1)}$.

To show that $f_j \to f$ it suffices to show that every subsequence of f_j admits a subsequence converging to f. By assumption on the family $X_\delta^m(\Omega(\tau, K_1, \ldots, K_m))$ there exists a $t > 0$ such that $\overline{\triangle}_t = \{\zeta \in \mathbb{C}; |\zeta| \leqslant t\} \subset \Omega(\lambda_j, Y_1^j, ..., Y_m^j)$ for all j. Now let $t_0 < t$ and consider the functions $h_j(z) = \frac{1}{f_j(z)}$ on $W_{t_0}^j = \Omega(\lambda_j, Y_1^j, ..., Y_m^j) \setminus \overline{\triangle}_{t_0}$. By Koebe's $\frac{1}{4}$-Theorem we have that $h_j(W_{t_0}^j) \subset \triangle_{\frac{4}{t_0}}$ for all j, so the sequence h_j is a normal family on $W_{t_0} = \Omega(\lambda, Y_1, ..., Y_m) \setminus \overline{\triangle}_{t_0}$. Passing to a subsequence we assume that $h_j \to h$. Now h cannot be constantly zero, for this would mean that $f_j = \frac{1}{h_j} \to \infty$ uniformly on compacts. This would contradict the fact that

$f_j'(0) = 1$ for all j. But this means that that the sequence f_j converges to some function g on W_{t_0}, hence we may assume that f_j converges to g on $\Omega(\lambda, Y_1, ..., Y_m)$. Since $g'(0) = 1$ we have that g cannot be constant, and we conclude that g maps $\Omega(\lambda, Y_1, ..., Y_m)$ univalently onto some subset of $\mathbb{C}$.

Since f_j converges to g we have now that the for each i, the set $f_j(Y_i^j)$ is a bounded sequence of disks $\triangle_{r_i^j}(z_i^j)$ (some of these disks could be points). So by passing to a subsequence we may assume that each of the sequence of pairs (z_i^j, r_i^j) converges to some pair (z_i, r_i). We have that

$$(B) \qquad f_j(z + m + n\lambda_j) = f_j(z) + mf_j(1) + nf_j(\lambda_j)$$

for all j and for all $m, n \in \mathbb{Z}$. So if we let Q_j be the set of disks in $\mathbb{C}$ generated by the set of disks $\triangle_{r_i^j}(z_i^j)$ and the lattice determined by $f_j(1)$ and $f_j(\lambda_j)$, we get that $f_j(\Omega(\lambda_j, Y_1^j, ..., Y_m^j)) = \mathbb{C} \setminus Q_j$.

From (B) we now get that

$$(C) \qquad g(z + m + n\lambda) = g(z) + mg(1) + ng(\lambda)$$

for all $m, n \in \mathbb{Z}$.

We must have that $g(1)$ and $g(\lambda)$ are linearly independent over $\mathbb{R}$. To see this let V be some open set in $\Omega(\lambda, K_1, ..., K_m)$ containing the point λ. Then $g(V)$ contains an open set around $g(\lambda)$. Now for each $m, n \in \mathbb{Z}$ let $V_{m,n}$ denote the translated sets $V + m + n\lambda$. Then $g(V_{m,n}) = g(V) + mg(1) + ng(\lambda)$, and if $g(1)$ and $g(\lambda)$ are linearly dependent over $\mathbb{R}$ then $g(V_{m,n})$ would intersect the straight line segment between 0 and $g(\lambda)$ for infinitely many choices of $m, n \in \mathbb{Z}$. This would contradict the fact that g is univalent.

Let Q now denote the circled subset of $\mathbb{C}$ generated by the disks $\triangle_{r_i}(z_i)$ and the lattice determined by $g(1)$ and $g(\lambda)$. Now $\mathbb{C} \setminus Q$ is the kernel for sequence $\mathbb{C} \setminus Q_j$, and it follows from Theorem 2.1 that $g(\Omega(\lambda, Y_1, ..., Y_m)) = \mathbb{C} \setminus Q$. But then g is the unique function satisfying $g(0) = 0, g'(0) = 1$ that maps $\Omega(\lambda, Y_1, ..., Y_m)$ onto a circled subset of $\mathbb{C}$ having a cluster point at infinity, and this contradicts (A). We conclude then that $f_j \to f$.

Now from Theorem 2.1 we have that $f(\Omega(\lambda, Y_1, ..., Y_m))$ is the kernel for the sequence $f_j(\Omega(\lambda, Y_1^j, ..., Y_m^j))$ to which it converges. Since an arbitrary subsequence has the same kernel we have that each sequence of disks $f_j(Y_i^j)$ must converge to $f(Y_i)$, and this completes the proof. $\qquad \square$

Now let $\Omega(\tau, \mathbf{x}) \in T^m$ so that no boundary component intersects the point τ, let $X_\delta^m(\Omega(\tau, \mathbf{x}))$ be a space as defined above, and choose $\epsilon > 0$ such that $T_\epsilon^m(\tau, \mathbf{x}) \subset X_\delta^m(\Omega(\tau, \mathbf{x}))$. Let $\varphi : X_\delta^m(\Omega(\tau, \mathbf{x})) \to T^m$ be the map from Lemma 2.2. We then have the following:

LEMMA 2.3. — *For every $\mu > 0$ there exists a $\widehat{\delta} > 0$ such that, if*

$$\psi : T_\epsilon^m(\tau, \mathbf{x}) \to X_\delta^m(\Omega(\tau, \mathbf{x}))$$

is a map with $d_1(\psi(\Omega(\lambda, \mathbf{y})), \Omega(\lambda, \mathbf{y})) < \widehat{\delta}$ for all $\Omega(\lambda, \mathbf{y}) \in T_\epsilon^m(\tau, \mathbf{x})$, then

$$d_2(\varphi \circ \psi(\Omega(\lambda, \mathbf{y})), \Omega(\lambda, \mathbf{y})) < \mu$$

for all $\Omega(\lambda, \mathbf{y}) \in T_\epsilon^m(\tau, \mathbf{x})$.

Proof. — This follows from the facts that $\varphi|_{T^m \cap X_\delta^m(\Omega(\tau, \mathbf{x}))} = \mathrm{id}$, φ is continuous, and $\overline{T_\epsilon^m(\tau, \mathbf{x})}$ is complete. $\qquad\square$

Theorem 1.1 will follow from the previous lemmas and the following proposition. The proof of the proposition will be given in sections 3 and 4.

PROPOSITION 2.4. — *Let $\Omega(\tau, \mathbf{x}) \in T^m$ such that no complementary component of $\Omega(\tau, \mathbf{x}) \in T^m$ intersect the point τ, and such that no boundary component is a single point. Let $X_\delta^m(\Omega(\tau, \mathbf{x}))$ be a space as above. If $\epsilon > 0$ is small enough, then for all $\widehat{\delta} > 0$ there exists a map $\psi : T_\epsilon^m(\Omega(\tau, \mathbf{x})) \to X_\delta^m(\Omega(\tau, \mathbf{x}))$ such that the following holds:*

 (i) *ψ is continuous with respect to d_1 and d_2,*
 (ii) *$d_1(\Omega(\lambda, \mathbf{y}), \psi(\Omega(\lambda, \mathbf{y}))) < \widehat{\delta}$ for all $\Omega(\lambda, \mathbf{y}) \in T_\epsilon^m(\Omega(\tau, \mathbf{x}))$,*
 (iii) *All $\mathcal{R}(\psi(\Omega(\lambda, \mathbf{y})))$ embed properly into $\mathbb{C}^2$.*

Proof of Theorem 1.1. — Lift U to the universal cover of $\mathbb{T}$ and write this lifting as an m-domain $\Omega(\lambda, K_1, ..., K_m)$. By Lemma 2.2, $\Omega(\lambda, K_1, ..., K_m)$ is biholomophic to some circled m-domain $\Omega(\tau, \mathbf{x}) \in T^m$ (see (1),(2),(a) and (b) on page 3), so it is enough to proof the result for $\mathcal{R}(\Omega(\tau, \mathbf{x}))$. By a linear translation we may assume that no boundary component of $\Omega(\tau, \mathbf{x})$ intersect the point τ, and we cannot have that any boundary component of $\Omega(\tau, \mathbf{x})$ is a point, since no K_i is a point. Let $\epsilon > 0$ be in accordance with Proposition 2.4. There exists a $\mu > 0$ such that if $F : B_\epsilon(\tau, \mathbf{x}) \to \mathbb{R}^{2+3m}$ is a continuous map satisfying

$$(*) \qquad\qquad \|F - \mathrm{id}\|_{B_\epsilon(\tau, \mathbf{x})} < \mu,$$

then

$$(**) \qquad\qquad (\tau, \mathbf{x}) \in F(B_\epsilon(\tau, \mathbf{x})).$$

Choose $\widehat{\delta} > 0$ depending on μ as in Lemma 2.3, choose ψ as in Proposition 2.4 depending on $\widehat{\delta}$, and consider the composition

$$F = \varphi \circ \psi$$

(regarded as a map from $B_\epsilon(\tau, \mathbf{x})$ into $\mathbb{R}^{2+3m}$). Then F is a map satisfying $(*)$ so we have $(**)$. We have that all circled m-domains corresponding

to points in $F(B_\epsilon(\tau, \mathbf{x}))$ embed properly into $\mathbb{C}^2$, so $\mathcal{R}(\Omega(\tau, \mathbf{x}))$ embeds properly into $\mathbb{C}^2$. □

3. Perturbing surfaces in $\mathbb{C}^2$ and consequences for function algebras

Let $\mathcal{R}$ be an open Riemann surface, and let U be an open subset of $\mathcal{R}$. We say that U is Runge in $\mathcal{R}$ if every holomorphic function $f \in \mathcal{O}(U)$ can be approximated uniformly on compacts in U by functions that are holomorphic on $\mathcal{R}$. If $\phi(\mathcal{R})$ is an embedded surface in $\mathbb{C}^2$ we will say that $\phi(\mathcal{R})$ is Runge (in $\mathbb{C}^2$) if all functions $f \in \mathcal{O}(\phi(\mathcal{R}))$ can be approximated uniformly on compacts in $\phi(\mathcal{R})$ by polynomials. Now let M be a complex manifold and let $K \subset M$ be a compact subset of M. Recall the definition of the holomorphically convex hull of K with respect to M:

$$\widehat{K}_M = \{x \in M; |f(x)| \leqslant \|f\|_K, \forall f \in \mathcal{O}(M)\}.$$

If $M = \mathbb{C}^n$ we simplify to $\widehat{K} = \widehat{K}_{\mathbb{C}^n}$, and we call $\widehat{K}$ the polynomially convex hull of K. If $K = \widehat{K}$ we say that K is polynomially convex.

For an open Riemann surface $\mathcal{R}$, and a compact set $K \subset \mathcal{R}$, we have that:

(1) $\widehat{K}_\mathcal{R}$ is the union of K and all the relatively compact components of $\mathcal{R} \setminus K$,

(2) An open subset U of $\mathcal{R}$ is Runge if and only if $\widehat{K}_\mathcal{R} \subset U$ for all compact $K \subset U$.

These results can be found in [3],[16].

We will need the following standard result:

LEMMA 3.1. — *Let $U \subset \mathbb{C}^k$ be Runge and Stein, and let $X \subset U$ be an analytic set. For $M \subset\subset X$ we have that*

$$\widehat{M} = \widehat{M}_{\mathcal{O}(U)} = \widehat{M}_{\mathcal{O}(X)}.$$

PROPOSITION 3.2. — *Let $\mathcal{R}$ be a closed Riemann surface, let $V \subset \mathcal{R}$ be a domain such that ∂V is a collection of smooth Jordan curves, and let*

$$\phi : V \to \mathbb{C}^2$$

be an embedding, holomorphic across the boundary. Assume that $\phi(\overline{V})$ is polynomially convex. Then for any finite set of distinct points $\{p_i\}_{i=1}^m \subset V$, there exist arbitrarily small open disks $D_i \subset V$ with $p_i \in D_i$, and a neighborhood Ω of $\phi(\overline{V} \setminus \cup_{i=1}^m D_i)$, such that for all $\epsilon > 0$ there exists an injective holomorphic map

$$\xi : \Omega \to \mathbb{C}^2$$

such that the following holds:

  (i) $\|\xi - \mathrm{id}\|_{\phi(\overline{V} \setminus \cup_{i=1}^m D_i)} < \epsilon$

  (ii) $\xi \circ \phi(\overline{V} \setminus \cup_{i=1}^m D_i)$ *is polynomially convex.*

Proof. — Let $V \subset\subset W$ such that $\phi|_W$ is an embedding. Since $\phi(\overline{V})$ is polynomially convex there is a Runge and Stein neighborhood basis U_j of $\phi(\overline{V})$ in $\mathbb{C}^2$. We may assume that $W_j := \phi(W) \cap U_j$ is a closed submanifold of U_j for all $j \in \mathbb{N}$, and that $\phi(V)$ is Runge in W_j. Let x_i denote $\phi(p_i)$ for $i = 1, ..., m$, and let $Q = \{x_1, ..., x_m\}$.

Now let $\mathcal{N}$ denote the normal bundle of W_1. Since $\mathcal{N}$ is a line bundle and W_1 is a Riemann surface, we have that $\mathcal{N} \cong W_1 \times \mathbb{C}$ (see for instance [6], p.229). For some large enough $j \in \mathbb{N}$ we have that U_j embeds into $\mathcal{N}$ with W_j as the zero section, *i.e.*, there is an injective holomorphic map

$$(*) \qquad\qquad F : U_j \to W_j \times \mathbb{C}$$

such that $F(x) = (x, 0)$ for all $x \in W_j$. We might as well assume that this is true for $j = 1$ (for a reference to these claims see [12] pages 255-258 and Remark 3.3 below).

Let $f \in \mathcal{O}(\phi(W))$ with $f(x) = 0$ for $x \in Q$, and $f(x) \neq 0$ for $x \notin Q$ (see for instance [6]). For any $\delta > 0$ we let

$$\psi_\delta : (W_1 \setminus Q) \times \mathbb{C} \to (W_1 \setminus Q) \times \mathbb{C}$$

be the biholomorphic map defined by $\psi_\delta(x, \lambda) = (x, \lambda + \frac{\delta}{f(x)})$. Then $\psi_\delta(F(W_1 \setminus Q))$ is a closed submanifold of $W_1 \times \mathbb{C}$ for all choices of δ, and we get that $W_1^\delta := F^{-1}(\psi_\delta(F(W_1 \setminus Q)))$ is a closed submanifold of U_1.

Let Ω_j be a neighborhood basis of $\phi(\overline{V} \setminus (\cup_{i=1}^m D_i))$ in $\mathbb{C}^2$. If j is large enough and δ is small enough we have that

$$G_\delta := F^{-1} \circ \psi_\delta \circ F : \Omega_j \to U_1$$

is an injective holomorphic map. Moreover we have that $G_\delta(\phi(\overline{V} \setminus (\cup_{i=1}^m D_i)))$ is holomorphically convex in W_1^δ. Put $\Omega := \Omega_j$, $\xi := G_\delta$, and the result follows by Lemma 3.1. $\qquad\square$

Remark 3.3. — We outline a simple proof of the existence of the map $(*)$ in our setting: Let $g \in \mathcal{O}(U_1)$ be a defining function for W_1, and let $\bigtriangledown g(x)$ denote the gradient of g. Such a function exists since the second Cousin problem has a solution in this setting. Define a map

$$H : W_1 \times \mathbb{C} \to \mathbb{C}^2$$

by $H(x, \lambda) = x + \lambda \cdot \bigtriangledown g(x)$. It is seen that H is injective near $W_1 \times \{0\}$, and we may let $F = H^{-1}$ on U_j if j is big enough.

Proof of Theorem 1.3. — Let $\{K_j\}$ be a holomorphically convex exhaustion of V such that $U \setminus K_j$ has finitely many complementary components for each $j \in \mathbb{N}$. We will repeatedly use Proposition 3.2 to construct an embedding ϕ of V into $\mathbb{C}^2$ such that each $\phi(K_i)$ is polynomially convex, and this will prove the theorem.

We will prove the theorem by induction, and the following will be our induction hypothesis:

We have found a domain $V_i \subset \mathcal{R}$ such that $V \subset\subset V_i$, with K_i holomorphically convex in V_i, and an embedding $\phi_i : V_i \to \mathbb{C}^2$ such that the conditions in Proposition 3.2 are satisfied for the pair (V_i, ϕ_i). In particular we have that $\phi_i(K_i)$ is polynomially convex.

We will show that we can use Proposition 3.2 carry out the inductive step.

Let $T_1, ..., T_k$ denote the connected components of $V_i \setminus K_{i+1}$. If no T_j is relatively compact in V_i we have that K_{i+1} is holomorphically convex in V_i and we define $V_{i+1} := V_i, \phi_{i+1} := \phi_i$. Assume on the other hand that $T_{i_1}, ..., T_{i_s}$ are relatively compact in V_i. By assumption and since K_{i+1} is holomorphically convex in V, we may find points $p_j \in T_{i_j}$ such that $p_j \in (U \setminus V)^\circ$. And so there are disks $D_j \subset V_i \setminus V$ such that $p_j \in D_j$. Define $V_{i+1} = V_i \setminus \cup_{j=1}^s D_j$ and Proposition 3.2 furnishes the map ϕ_{i+1}.

We may now use this procedure to construct an appropriate embedding of V into $\mathbb{C}^2$. Let V_1 be a smoothly bounded domain in $\mathcal{R}$, homeomorphic to U with $U \subset\subset V_1$, and such that ϕ is defined on V_1. Assume that K_1 is a point and define $\phi_1 := \phi$. Notice that for each inductive step we may choose any $\delta_i > 0$ and make sure that $\|\phi_{i+1} - \phi_i\|_{K_{i+1}} < \delta_i$. Therefore we may choose a sequence $\{\phi_i\}$ such that

$$\phi := \lim_{i \to \infty} \phi_i$$

exists on V and is an embedding. Moreover, since $\phi_i(K_i)$ is polynomially convex for each $i \in \mathbb{N}$, and since $\phi(K_{i+1})$ can be made an arbitrarily small perturbation of $\phi_i(K_{i+1})$, we may assume that each $\phi(K_i)$ is polynomially convex. The result follows.   □

Proof of Theorem 1.2. — Let T be a connected component of $\mathbb{T} \setminus V$, and let $p \in T$ be an interior point. Then $\mathbb{T} \setminus \{p\}$ embeds as a closed submanifold of $\mathbb{C}^2$ by some map ϕ. Let D be a smoothly bounded disk such that $D \subset\subset T$, and define $U = \mathbb{T} \setminus D$. The collection (U, ϕ, V) satisfies the conditions in Theorem 1.3.   □

4. Continuous perturbation of families of Riemann surfaces - proof of Proposition 2.4

Briefly the idea behind the proof of Proposition 2.4 is the following: Start with the space $T_\epsilon^m(\Omega(\tau, \mathbf{x}))$ and consider Theorem 4.1 below. In effect we showed in [19] that for each fixed $\Omega(\lambda, \mathbf{y}) \in T_\epsilon^m(\Omega(\tau, \mathbf{x}))$ there exists an arbitrarily small perturbation $U_{(\lambda,\mathbf{y})}$ of $\Omega(\lambda, \mathbf{y})$ such that $U_{(\lambda,\mathbf{y})}$ embeds onto a surface in $\mathbb{C}^2$ satisfying the conditions in Theorem 4.1. *I.e.* $U_{(\lambda,\mathbf{y})}$ embeds properly into $\mathbb{C}^2$. Suppose that we could make sure that the perturbed m-domains vary continuously with the parameter $(\lambda, \mathbf{y})$ (with respect to the metric defined in Section 2). Then the correspondence $\Omega(\lambda, \mathbf{y}) \mapsto U_{(\lambda,\mathbf{y})}$ defines a continuous map $\psi : T_\epsilon^m(\Omega(\tau, \mathbf{x})) \to X^m(\tau, \mathbf{x})$, and all the image domains embed properly into $\mathbb{C}^2$. If ψ could be made arbitrarily close to the identity then Proposition 2.4 would follow from Lemma 2.3. This is indeed what we will prove.

The following theorem is approximately the same as Theorem 1 in [19]. The difference is that Theorem 1 was formulated for surfaces with smooth boundaries, whereas the following is formulated for surfaces with piecewise smooth boundaries. The difference in the proof however is not significant.

THEOREM 4.1. — *Let $M \subset \mathbb{C}^2$ be a Riemann surface whose boundary components are piecewise smooth Jordan curves $\partial_1, ..., \partial_m$. Assume that there are points $p_i \in \partial_i$ such that*

$$\pi_1^{-1}(\pi_1(p_i)) \cap \overline{M} = p_i.$$

Assume that each boundary component ∂_i is smooth near p_i, and that all points p_i are regular points of the restricted projection $\pi_1|_{\overline{M}}$. Then M can be properly holomorphically embedded into $\mathbb{C}^2$.

As outlined above we want to embed families of m-domains onto surfaces satisfying the conditions in this theorem. It seems worth it however to formulate a result for closed Riemann surfaces in general: Fix an integer $g \geqslant 0$. Let B_ϵ denote a ball of radius $r = \epsilon$ in some $\mathbb{R}^N$ and let X be a smooth manifold with a projection $\pi : X \to B_\epsilon$ such that $X_y := \pi^{-1}(y)$ is a closed Riemann surface of genus g for each $y \in B_\epsilon$ — the complex structure on each fibre Y_y being specified by the parameter y. Let $m : X \times X \to \mathbb{R}^+$ be a smooth metric on X that induces the topology.

For $i = 1, ..., m$ let $f_i : B_\epsilon \times \overline{\triangle}$ be a smooth embedding such that $f_i(\{y\} \times \overline{\triangle}) \subset X_y$ for each $y \in B_\epsilon$, and such that the images $f_i(B_\epsilon \times \overline{\triangle})$ are pairwise disjoint. Let $Y := X \setminus \cup_{j=1}^m f_j(B_\epsilon \times \overline{\triangle})$. Then Y is a submanifold of X and each fiber $Y_y \subset X_y$ is an open Riemann surface (specifically a closed

Riemann surface of genus g with m disks removed). For $0 < \delta < 1$ let Y^δ denote $X \setminus \cup_{j=1}^m f_i(B_\epsilon \times \overline{\triangle}_{1-\delta})$.

PROPOSITION 4.2. — Let $F : Y^\delta \to B_\epsilon \times \mathbb{C}^2$ be a smooth map such that $F(y, \cdot) : Y_y^\delta \to \{y\} \times \mathbb{C}^2$ is a holomorphic embedding for each $y \in B_\epsilon$. Assume that $F(y, \overline{Y_y})$ is polynomially convex in each fiber $\{y\} \times \mathbb{C}^2$.

Then, by possibly having to decrease ϵ, for all $\widehat{\delta} > 0$ there exist a family of domains $U_y \subset X_y, y \in B_\epsilon$, and a smooth map $G : \cup_{y \in B_\epsilon}\{y\} \times \overline{U}_y \to B_\epsilon \times \mathbb{C}^2$ such that the following hold for all $y \in B_\epsilon$:

   (i) U_y is homeomorphic to Y_y,
   (ii) $Y_y \subset U_y \subset Y_y^{\widehat{\delta}}$,
   (iii) $d_H(U_{y_j}, U_y) \to 0$ for all $y_j \to y$, $y_j \in B_\epsilon$,
   (iv) $G(y, \cdot)$ is a holomorphic embedding of U_y into $\{y\} \times \mathbb{C}^2$,
   (v) $G(y, \overline{U}_y)$ satisfies the conditions in Theorem 4.1 when regarded as an embedded Riemann surface in the fiber $\{y\} \times \mathbb{C}^2$.

Proof. — We will prove the result in the case that each fiber Y_y is a closed Riemann surface with a single component removed. We will make some comments along the way as regards the general case, which is essentially the same.

We may assume that $F(\overline{Y^\delta}) \subset B_\epsilon \times \triangle \times \mathbb{C}$. For any $0 < r < \widehat{\delta}$ let $s_r \subset \overline{\triangle}$ denote the curve $s_r := \{z \in \mathbb{C}; \mathrm{Im}(z) = 0, -1 \leqslant \mathrm{Re}(z) \leqslant -1 + r\}$, and let $S_r \subset B_\epsilon \times \overline{\triangle}$ denote the manifold $S_r := \cup_{y \in B_\epsilon}\{y\} \times s_r$. Then $f_1(S_r) \subset X$ is a smooth manifold attached to the boundary of Y with $f_1(S_r) \subset Y^\delta \setminus Y$. In each fiber Y_y^δ we have that $c_y := f_1(S_r) \cap Y_y^\delta$ is a smooth curve attached to the Riemann surface Y_y.

Let H denote the composition $F \circ f_1$, and let E_r denote $H(S_r)$. Then E_r is a submanifold of $B_\epsilon \times \mathbb{C}^2$, and each fiber slice $\gamma_y := E_r \cap (\{y\} \times \mathbb{C}^2)$ is a smooth curve attached to the embedded Riemann surface $F(Y_y)$.

Let us first concentrate on some fiber over $y \in B_\epsilon$ and explain how we can modify $F|_{Y_y^\delta}$ to get all claims in the theorem, except of course (iii), for that particular fiber. The idea is the following: We find a neighborhood W_y of $F(Y_y) \cup \gamma_y$ in $\{y\} \times \mathbb{C}^2$ and an injective holomorphic map $\psi_y : W_y \to \{y\} \times \mathbb{C}^2$ such that ψ is close to the identity on $F(Y_y)$ and such that ψ_y stretches the curve γ_y so that $\psi_y(\gamma_y)$ intersects the cylinder $\{y\} \times \partial\triangle \times \mathbb{C}^2$ transversally and at a single point. For a small $\mu > 0$ let V_y^μ denote the μ-neighborhood

$$(*) \qquad\qquad V_y^\mu := \{x \in Y_y^\delta; d(x, Y_y \cup c_y) < \mu\}$$

of $Y_y \cup c_y$ in Y_y^δ. We find a pair (G_y, U_y) as in the proposition by defining $G_y := \psi_y \circ F$ and then

$$(**) \qquad U_y := G_y^{-1}(G_y(V_y^\mu) \cap (\{y\} \times \triangle \times \mathbb{C})).$$

(Meaning that U_y is the connected component of the pullback that contains Y_y). In the general case we attach disjoint curves in a similar manner, one for each boundary component, and stretch each curve.

More detailed we carry out the construction (still focusing on a particular fiber) as follows: Let m_y be a smoothly embedded curve $m_y : [0,1] \to \{y\} \times \mathbb{C}^2$ such that

(i) $m_y \cap F(Y_y^\delta \setminus Y_y) \supset \gamma_y$,
(ii) $(m_y \setminus \gamma_y) \cap F(\overline{Y}_y) = \emptyset$,
(iii) The intersection $\gamma_y \cap (\{y\} \times \partial\triangle \times \mathbb{C})$ consists of a single point (which is not the end point), and the intersection is transversal.

Let $x_0 \in (0,1)$ and let $g : [0,\infty) \times [0,1] \to [0,1]$ be an isotopy of diffeomorphisms such that

(a) $g(t,x) = x$ for all $x \in [0, x_0], t \in [0,\infty)$,
(b) $\lim_{t\to\infty} g(t,x) = 1$ for all $x > x_0$.

Define an isotopy $\phi_y : [0,1] \times m_y \to m_y$ by $\phi_y(t,x) := m_y \circ g(t, m_y^{-1}(x))$. If N_y is a small neighborhood of $F(Y_y)$ in $\{y\} \times \mathbb{C}^2$ we may define an isotopy of diffeomorphisms $\xi_y : [0,1] \times N_y \cup \gamma_y \to N_y \cup \gamma_y$ by

$$\xi_y|_{N_y} := \mathrm{Id}, \ \xi_y(t,x) := \phi_y(t,x) \text{ for } x \in m_y.$$

We will argue in a moment that for arbitrarily small x_0 and arbitrarily large t_0 there is a neighborhood W_y of $F(Y_y) \cup m_y$ in $\{y\} \times \mathbb{C}^2$ such we can approximate the map $\xi_y(t_0, \cdot)$ good in $\mathcal{C}^1$-norm on $F(Y_y) \cup m_y$ by an injective holomorphic map

$$\psi_y : W_y \to \{y\} \times \mathbb{C}^2.$$

Granted the existence of this approximation this proves, by the construction $(*)$ and $(**)$ above, the result (except (iii)) for any particular fiber Y_y.

To get (iii) we carry out this construction simultaneously for all fibers. By possibly having to decrease ϵ we see that we can find a smooth submanifold M of $B_\epsilon \times \mathbb{C}^2$ such that in each fiber we have that $m_y := M \cap \{y\} \times \mathbb{C}^2$ is a smooth curve satisfying (i)-(iii) above. Let $D : B_\epsilon \times [0,1] \to M$ be a diffeomorphism. In the general case we attach several disjoint smooth manifolds, one for each boundary component. For dimension reasons this does not raise a problem.

Let $\varphi : [0,\infty) \times B_\epsilon \times [0,1] \to B_\epsilon \times [0,1]$ be the isotopy $\varphi(t,y,x) = (y, g(t,x))$, and let $\phi : [0,\infty) \times M \to M$ be the isotopy $\phi = D \circ \varphi \circ D^{-1}$.

Now regard $B_\epsilon(\tau, \mathbf{x})$ as the real ϵ-ball contained in $\mathbb{C}^N$, and let $\mathcal{N}$ be a small neighborhood of $F(Y)$ in $\mathbb{C}^N \times \mathbb{C}^2$. Define $\xi : [0, \infty) \times (\mathcal{N} \cup M) \to B_\epsilon(\tau, \mathbf{x}) \times (\mathcal{N} \cup M)$ by

$$\xi(t, x) := x \text{ for } x \in \mathcal{N}, \xi(t, x) := \phi(t, x) \text{ for } x \in M.$$

Since each $F(Y_y)$ is polynomially convex in the fiber over $\{y\}$ it follows by [18] that each $F(Y_y) \cup m_y$ is polynomially convex in the fiber. And so since $B_\epsilon(\tau, \mathbf{x}) \subset \mathbb{C}^N$ is real it follows that $F(Y) \cup M$ is polynomially convex in $\mathbb{C}^N$.

By [9] we have then that for any fixed t_0 and x_0 there is a neighborhood W of $F(Y) \cup M$ such that $\xi(t_0, \cdot)$ can by approximated arbitrarily good by an injective holomorphic map $\psi : W \to B_\epsilon \times \mathbb{C}^2$ preserving fibers, and the approximation is good in $\mathcal{C}^1$-norm.

Define $G := \psi \circ F$, chose a small $\mu > 0$ and define domains U_y as in $(*)$ and $(**)$ above. If μ is small enough then (iii) follows by transversality. $\quad\square$

To prove Proposition 2.4 then, we have to construct manifolds X, Y and Y^δ as above with subsets of tori as fibers, construct a suitable map F, and then apply Proposition 4.2.

Recall the Weierstrass p-function (depending on λ):

$$\varrho_\lambda(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(z - (m + n \cdot \lambda))^2} - \frac{1}{(m + n \cdot \lambda)^2}.$$

This a meromorphic function in z respecting the relation $\sim_\lambda$. Fix a 1-domain $\Omega(\tau, 0, r)$, an $\epsilon > 0$, and let $W_\epsilon := \cup_{\lambda \in \triangle_\epsilon(\tau)} \{\lambda\} \times \Omega(\lambda, 0, r)$. If $\epsilon > 0$ is small enough and p is close to the origin we may define a map

$$\widehat{\phi}_p(\lambda, z) = (\varrho_\lambda(z - p), \varrho_\lambda(z)),$$

from W_ϵ into $\mathbb{C}^2$.

LEMMA 4.3. — *For sufficiently small ϵ and p we have that $\widehat{\phi}_p$ is holomorphic in the variables (λ, z). For each fixed λ we have that $\widehat{\phi}_p(\lambda, \cdot)$ embeds $\mathcal{R}(\lambda, \Omega(\lambda, 0, r))$ into $\mathbb{C}^2$.*

Proof. — If ϵ and p is chosen small enough we have that $\widehat{\phi}_p(\lambda, z)$ is holomorphic in the z-variable for all fixed $\lambda \in \triangle_\epsilon(\tau)$. To prove that $\widehat{\phi}$ is holomorphic in both variables we inspect the standard proof of the fact that $\varrho_\lambda(z)$ converges as a function in the z-variable.

Following Ahlfors [1] we have for $2|z| \leqslant |m + n\tau|$, that

$$\left| \frac{1}{(z - (m + n\tau))^2} - \frac{1}{(m + n\tau)^2} \right| \leqslant \frac{10|z|}{|m + n\tau|^3}.$$

So to prove that $\varrho_\tau(z)$ converges it is enough to prove that

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{|m+n\tau|^3}$$

converges. This in turn is proved by observing that there exists a positive constant K such that

$$|m+n\tau| \geqslant K(|m|+|n|)$$

for all $m, n \in \mathbb{N}$, and then getting the estimate

$$(*) \qquad \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{|m+n\tau|^3} \leqslant 4K^{-3} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

But K may be chosen such that

$$|m+n\lambda| \geqslant K(|m|+|n|)$$

for all λ close to τ, so the inequality $(*)$ holds as we vary τ. This shows that the sum $\varrho_\lambda(z)$ converges uniformly on compacts in W_ϵ in the variables (λ, z). And if the shift determined by p is small enough we have that $\widehat{\phi}_p$ is holomorphic on W_ϵ.

In [19] we demonstrated that the map $z \mapsto (\varrho_\lambda(z-p), \varrho_\lambda(z))$ is an embedding provided that $2p$ is not contained in the lattice determined by λ. So all $\phi_p(\lambda, \cdot)$ are fiberwise embeddings as long as ϵ is small, and p is chosen close to the origin. $\qquad\square$

Let us now construct manifolds X, Y and Y^δ as above. Fix an m-domain $\Omega(\tau, \mathbf{x})$ and let $\epsilon > 0$. We define $X := \cup_{(\lambda,\mathbf{y})\in B_\epsilon(\tau,\mathbf{x})}\{(\lambda, \mathbf{y})\} \times \mathcal{R}(\Omega(\lambda))$ and we let $\pi : X \to B_\epsilon(\tau, \mathbf{x})$ be the obvious projection. Let $q : B_\epsilon(\tau, \mathbf{x}) \times \mathbb{C} \to X$ be the map defined by the standard quotient map on each fiber - $q(\lambda, \mathbf{y}, \zeta) = (\lambda, \mathbf{y}, [\zeta])$ where $[\zeta]$ denotes the equivalence class of ζ in $\mathbb{C}/\sim_\lambda$. Then q induces a differentiable structure on X such that each fiber $X_{(\lambda,\mathbf{y})}$ is a closed Riemann surface which we equip with the complex structure corresponding to λ. Let $m : X \times X \to \mathbb{R}^+$ be a smooth metric that induces the topology.

Next let $V_\epsilon = \cup_{(\lambda,\mathbf{y})\in B_\epsilon(\tau,\mathbf{x})}\{(\lambda, \mathbf{y})\} \times \Omega(\lambda, \mathbf{y})$. Then $Y := q(V_\epsilon) \subset X$ is a submanifold Y of X as above. This is seen by defining $g_i : B_\epsilon(\tau, \mathbf{x}) \times \overline{\triangle} \to B_\epsilon(\tau, \mathbf{x}) \times \mathbb{C}$ by $g_i(\lambda, \mathbf{y}, t) = (\lambda, \mathbf{y}, z_i + t \cdot r_i)$ and $f_i = q \circ g_i$.

To construct the map $F : Y^\delta \to B_\epsilon \times \mathbb{C}^2$ we first let V_ϵ^δ denote the set $q^{-1}(Y^\delta)$, and define a map

$$\phi_p : V_\epsilon^\delta \to B_\epsilon(\tau, \mathbf{x}) \times \mathbb{C}^2$$

by $\phi_p(\lambda, \mathbf{y}, \zeta) = (\lambda, \mathbf{y}, \widehat{\phi}_p(\lambda, \zeta - z_1))$ (here z_1 is a component of the fixed point $(\tau, \mathbf{x})$ and not a variable). This is a well defined mapping if ϵ and p

are small enough. Now define a map

$$\Phi : Y^\delta \to B_\epsilon(\tau, \mathbf{x}) \times \mathbb{C}^2$$

by $\Phi(x) = \phi_p(q^{-1}(x))$ for $x \in Y^\delta$. This is well defined because ϕ_p respects the relation $\sim_\lambda$ on fibers, and it follows from Lemma 4.3 that Φ is a smooth mapping such that $\Phi|_{X_y}$ is an embedding for each fiber X_y. In the following proof of Proposition 2.4 we use Φ to construct F:

Proof of Proposition 2.4. — Let X, Y, Y^δ and Φ be as just defined. By Proposition 3.2 there is an open set $U \subset \mathbb{C}^2$ and an injective holomorphic map $\xi : U \to \mathbb{C}^2$ such that $\Phi(Y^\delta_{(\tau,\mathbf{x})}) \subset \{(\tau, \mathbf{x})\} \times U$, and such that $\xi \circ \Phi(Y_{(\tau,\mathbf{x})})$ is polynomially convex in the fiber $(\tau, \mathbf{x}) \times \mathbb{C}^2$. Define

$$\Psi : B_\epsilon(\tau, \mathbf{x}) \times U \to B_\epsilon(\tau, \mathbf{x}) \times \mathbb{C}^2$$

by $\Psi(\lambda, \mathbf{y}, w_1, w_2) = (\lambda, \mathbf{y}, \xi(w_1, w_2))$.

If ϵ is small enough we have that $\Psi \circ \Phi(Y_{(\lambda,\mathbf{y})})$ is polynomially convex in the fiber $(\lambda, \mathbf{y}) \times \mathbb{C}^2$ for all $(\lambda, \mathbf{y})$. To see this choose a Runge and Stein domain $N \subset \mathbb{C}^2$ such that $\Psi \circ \Phi(Y_{(\tau,\mathbf{x})}) \subset \{(\tau, \mathbf{x})\} \times N$ and $\Psi \circ \Phi(Y^\delta_{(\tau,\mathbf{x})}) \cap \{(\tau, \mathbf{x})\} \times N \subset\subset \Psi \circ \Phi(Y^\delta_{(\tau,\mathbf{x})})$. If ϵ is small then $\Psi \circ \Phi(Y^\delta_{(\lambda,\mathbf{y})}) \cap \{(\lambda, \mathbf{y})\} \times N \subset\subset \Psi \circ \Phi(Y^\delta_{(\lambda,\mathbf{y})})$ for all $(\lambda, \mathbf{y}) \in B_\epsilon(\tau, \mathbf{x})$, i.e., $\Psi \circ \Phi(Y^\delta_{(\lambda,\mathbf{y})}) \cap \{(\lambda, \mathbf{y})\} \times N$ is a closed submanifold of $\{(\lambda, \mathbf{y})\} \times N$. So if ϵ is small the claim follows from Lemma 3.1.

Define $F = \Psi \circ \Phi$ and the pair (Y^δ, F) satisfies the conditions in Proposition 4.2. Let G be as in Proposition 4.2 and define $\psi(\Omega(\lambda, \mathbf{y}))$ to be the m-domain corresponding to $U_{(\lambda,\mathbf{y})}$. Now $(i) - (v)$ guaranties that the conclusions of Proposition 2.4 are satisfied.   □

It is clear that we have proved the following formulation of Theorem 1.1, which we formulate for easier reference in applications to embeddings with interpolation:

THEOREM 1.1'. — *Let $\mathbb{T}$ be a torus, and let $U \subset \mathbb{T}$ be a domain such that $\mathbb{T} \setminus U$ consists of a finite number of connected components, none of them being points. Then U embeds onto a bounded surface in $\mathbb{C}^2$ satisfying the conditions in Theorem 1.1 in* [19].

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