ANNALES

DE

L'INSTITUT FOURIER

Florin Alexandru BELGUN,
Nicolas GINOUX & Hans-Bert RADEMACHER

A Singularity Theorem for Twistor Spinors

Tome 57, nº 4 (2007), p. 1135-1159.

<http://aif.cedram.org/item?id=AIF_2007__57_4_1135_0>

Ann. Inst. Fourier, Grenoble
57, 4 (2007) 1135-1159

A SINGULARITY THEOREM FOR TWISTOR SPINORS

**by Florin Alexandru BELGUN,
Nicolas GINOUX & Hans-Bert RADEMACHER (*)**

Abstract. — We study spin structures on orbifolds. In particular, we show that if the singular set has codimension greater than 2, an orbifold is spin if and only if its smooth part is. On compact orbifolds, we show that any non-trivial twistor spinor admits at most one zero which is singular unless the orbifold is conformally equivalent to a round sphere. We show the sharpness of our results through examples.

Résumé. — Nous étudions les structures spin sur les orbifolds. Nous montrons en particulier que, si la codimension de l'ensemble des singularités est supérieure à 2, alors une orbifold est spin si et seulement si sa partie lisse l'est. Nous prouvons également que, sur une orbifold compacte, tout spineur-twisteur non identiquement nul admet au plus un zéro qui est alors singulier sauf si l'orbifold est conformément équivalente à une sphère ronde. Nous illustrons l'optimalité de nos résultats sur des exemples.

1. Introduction

The twistor operator (also called the Penrose operator) and the Dirac operator on a Riemannian spin manifold are obtained by composing the Levi-Civita covariant derivative with some natural linear maps. They are actually the two *natural* first order linear differential operators on spinors. The solutions of the corresponding P.D.E.'s (i.e., the kernels of these operators) are the *twistor spinors* and, respectively, the *harmonic spinors*, and they are both *conformally covariant*. Moreover, if we consider some appropriate *conformal weights*, they appear to be conformally invariant objects (as sections of some weighted spinor bundle).

Keywords: Orbifolds, twistor-spinors, ALE spaces.
Math. classification: 53C21, 53A30, 32C10.
(*) This work started while the second named author enjoyed the hospitality of the Max-Planck Institute for Mathematics in the Sciences, which he would like to thank for his support.

However, it turns out that the *norm* of a twistor spinor defines a special metric in the conformal class, for which the corresponding spinor is actually parallel, or, more generally, sum of two *Killing spinors*. This metric is then Einstein (of vanishing scalar curvature if and only if the corresponding spinor is parallel).

Therefore, a dichotomy occurs: the study of the twistor spinors without zeros reduces to the study of parallel resp. Killing spinors, and the study of twistor spinors with zeros which seems to be a "purely conformal" problem. In both cases the corresponding manifolds and spinors can be described in full detail if we look for compact, smooth, solutions.

The aim of this paper is to extend the above notions to the *orbifold* case, where quotient-like singularities are allowed, and to concentrate on the twistor equation on compact orbifolds. Although the notions concerning spin geometry on orbifolds may occur frequently as mathematical folklore, there is – at our knowledge – very little written material focusing on the study of spinors on orbifolds (see, however, [6] for a definition of spin orbifolds), and all questions could be legitimate in this setting (in particular which topological conditions characterize a spin structure on an orbifold, see Theorem 1), however this type of quotient-like singularities naturally occur in the setting of twistor equations, as we explain below.

Indeed, using the solution of the Yamabe problem A. Lichnerowicz proved that the only compact manifolds admitting twistor spinors with zeros are the standard spheres [17], therefore conformally flat. Actually, the first example of a (non-compact) non conformally flat Riemannian spin manifold carrying a twistor spinor with one zero was given by W. Kühnel and H.B. Rademacher [14] using a *conformal inversion* of the Eguchi-Hanson gravitational instanton [7]. The resulting metric is invariant under an involution having a fixed point, suggesting thus the possibility of extending the theory to orbifolds.

The main result of this paper, Theorem 3, establishes the following facts:

(1) On a compact orbifold (different from the standard sphere) admitting non-trivial solutions with non-empty vanishing set to the twistor equation, the zero of such a spinor is unique and singular;
(2) The (finite) cardinality of the *singularity group* of such a zero is not smaller than for any other point on the orbifold, and the equality case characterizes the quotients of the standard sphere.

Furthermore, we give a series of examples of compact orbifolds carrying twistor spinors with zeros, cf. Section 6. They rely on a general version of *conformal inversion* that allows us to compactify conformally an asymptotically locally Euclidean (ALE for short) metric on some open manifold (or orbifold, in general) M by adding one singular point at infinity, p_∞. The resulting orbifold $N = M \cup p_\infty$ admits then a twistor spinor with zero at p_∞ if and only if M admits a parallel spinor, and therefore 3 possible holonomy groups occur for M: $SU(n)$ (in which case M is a non-flat Ricci-flat ALE Kähler space, see below), $Sp(n)$ (in which case M should be ALE hyperkähler). Such an example cannot be obtained by the below described construction of Joyce, [11, Theorem 8.2.4]), or when the holonomy group is G_2 (in which case M is 7-dimensional and thus p_∞ is not an isolated singularity, therefore M is not smooth). In the present paper we give examples coming from the conformal compactification of a smooth manifold, therefore those examples belong all to the first class.

Our fundamental examples can be then described as follows: We consider S^{2n} as the conformal gluing of two copies of $\mathbb{C}^n$ (through stereographical projections). The quotient of S^{2n} by the standard action of a finite subgroup of $SU(n)$ is then an orbifold with 2 singular points, each of them being the origin in the corresponding chart. In one of the charts we resolve the singularity (we must assume there is a *crepant* resolution, see [11, 8.2]) and get, in some cases(1) a smooth open manifold M admitting an ALE Kähler metric, see [11]. Joyce's Theorem [11, Theorem 8.2.3] states then the existence of a Ricci-flat ALE Kähler metric on M. Adding one point p_∞ to M or, equivalently, gluing to it the other copy of $\mathbb{C}^n/\Gamma$ gives us an orbifold S^{2n}_Γ. The resulting orbifold is thus obtained by resolving only one of the singular points of S^{2n}/Γ (which obviously admits itself two families of twistor spinors for the standard metric, each of them vanishing in one of the two singularities).

We point out that, although the principle of conformal inversion holds for all orbifolds in general, without restriction to the isolated singularity case, there is no non-trivial example, so far, of Ricci-flat ALE structure on an orbifold with non-isolated singularities. In particular, the existence of odd-dimensional orbifolds admitting a twistor spinor with zero, as well as the existence of compact orbifolds admitting more than two linearly independent such spinors remain open questions.

Aknowledgements The authors would like to thank the anonymous referee whose remarks led to an improvement of the material.

(1) there is no criterium establishing that in general; that can be shown only on some explicit examples, cf. Section 6

2. Spin orbifolds

In this section we will recall the basic properties of an orbifold and define a spin structure on it.

Definition 2.1. — *An n-dimensional* orbifold *M is a Hausdorff topological space, together with an atlas of charts (U, f), where U is an open set in M and $f : U \to \mathbb{R}^n/\Gamma_f$ is a homeomorphism, where Γ_f is a finite subgroup of $GL(n, \mathbb{R})$ (acting on $\mathbb{R}^n$), such that the transition functions $g \circ f^{-1} : f(U \cap V) \to g(U \cap V)$ are differentiable in the following sense:*

For any $x \in U \cap V$ there exists a small neighbourhood $W \subset U \cap V$ of x and a differentiable map (called the lift of the transition function) $F : \pi_f^{-1}(f(W)) \to \pi_g^{-1}(g(W))$ such that $g \circ f^{-1} \circ \pi_f = \pi_g \circ F$, where $\pi_f : \mathbb{R}^n \to \mathbb{R}^n/\Gamma_f$ and $\pi_g : \mathbb{R}^n \to \mathbb{R}^n/\Gamma_g$ are the canonical projections.

Remark. Note that the lift F is not unique unless $\pi_f^{-1}(f(W))$ and $\pi_g^{-1}(g(W))$ are connected, but it is always a local diffeomorphism.

Remark. As any finite subgroup of $GL(n, \mathbb{R})$ is conjugated to one sitting in $O(n)$, we will suppose the groups Γ above are orthogonal. An *orientation* on M is defined, as in the case of a manifold, by the choice of an atlas of charts, such that the Jacobian of the lift of transition functions has positive determinant. This is well-defined if the groups Γ actually lie in $SO(n)$. As we are interested in oriented orbifolds, we will always suppose the groups Γ to lie in $SO(n)$.

The concept of a differentiable map between orbifolds can also be defined as in the definition above. Of course, the charts and their compositions with the canonical projections $\pi_\Gamma : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ are differentiable.

Definition 2.2. — *The (singularity) group of a point $x \in M$ is the conjugacy class of a minimal (with respect to the inclusion) finite subgroup $\Gamma_x \subset SO(n)$ such that there exists a chart (U, f) (which is then called* minimal*) around x with $\Gamma_f = \Gamma_x$. If $\Gamma_x = \{1\}$ we say that x is a smooth point; otherwise it is called singular.*

If the singularity group of a singular point x acts freely on $\mathbb{R}^n \setminus \{0\}$, then the singularity is *isolated*, i.e., it is surrounded by smooth points. This can only happen in even dimensions (in the oriented case).

Remark. A chart (U, f) as in the definition above is called *minimal* because there is no other chart around x having a group Γ' with less elements than Γ_x. Such a chart, composed with the canonical projection from $\mathbb{R}^n$ to $\mathbb{R}^n/\Gamma$, yields a *ramified covering* of a neighbourhood of x by an open set in $\mathbb{R}^n$.

A minimal chart yields, therefore, a C^∞ map $\tilde{f}$ from a neighbourhood of 0 in $\mathbb{R}^n$ onto a neighborhood of x, such that $\tilde{f}^{-1}(x) = \{0\}$.

Example. In dimension 2, the group of a singularity is equal to $\mathbb{Z}_n$, $n \geqslant 2$, and the "total angle" around such a point is $2\pi/n$ (around a smooth point it is 2π). So here the singularities are always isolated (note that we restricted ourselves to oriented orbifolds). A basic neighbourhood of such a "conical" point is actually homeomorphic to a disk, so every 2-dimensional oriented orbifold is homeomorphic to a manifold. In larger dimensions the basic neighbourhood of an isolated singularity is a cone over S^{n-1}/Γ_x, and, for $n > 2$, a quotient of a sphere is never homeomorphic to the sphere itself.

If a singularity x is not isolated, then the group of a neighbouring point y is the isotropy group of a point $z \in \tilde{f}^{-1}(y) \subset \mathbb{R}^n$ under the action of Γ_x, so Γ_y is isomorphic to a subgroup of Γ_x (the minimal chart is obtained by restricting $\tilde{f}$ to a Γ_y-invariant open subset U of z, disjoint from $\gamma(U)$, for any $\gamma \in \Gamma_x \smallsetminus \Gamma_y$). Then Γ_y will fix not only z, but actually a whole vector subspace of $\mathbb{R}^n$, containing z. The set of singularities of M is then a (not necessarily disjoint) union of orbifolds, each of even codimension (because of the orientability condition).

Fundamental remark: Any object on an orbifold M can be seen, in a neighbourhood of a point x, as a Γ_x-invariant object on a local ramified covering by a smooth manifold (obtained from a minimal chart composed with a canonical projection - we call this *the (minimal) smooth covering of M around x*).

We can now consider tensors on orbifolds: Locally they must come from Γ_x-invariant tensors on the minimal smooth covering around x.

Remark. A vector field on an orbifold must vanish on any isolated singularity and, in general, it must be tangent to the singular set, because in these points x it must be Γ_x-invariant. In general, the tensor fields on M must have particular (Γ_x-invariant) values in the singular points. In particular, a *metric* on an orbifold is locally a Γ_x-invariant metric on the minimal smooth covering around x.

We can carry on most of the differential-geometric constructions on orbifolds: for example, we can consider the Levi-Civita connection, differentiate vector fields, take their Lie bracket etc. All those operations may be performed locally in a chart, and there we will work on the smooth covering with Γ-invariant objects.

We are interested now in putting a spin structure on an orbifold. Before doing that, recall that the Spin bundle is a double covering of the total space of the bundle of orthonormal frames, which is non-trivial on each fiber. Note that, locally around x, the frame "bundle" $SO(M)$ is just the quotient of the frame bundle of the smooth covering under the action (by isometries) of Γ_x. This action is always free, so the frame bundle of an orbifold is a smooth manifold (but no longer a fiber bundle).

We will, however, continue to call $SO(M)$ the *bundle* of orthonormal frames on M, and, in general, we will continue to use the term *bundle* for quotients of (locally trivial) fiber bundles on the local smooth coverings, and for objects which are locally of this type.

Definition 2.3. — (see also [6]) *A Spin structure on an orbifold M is given by a two-fold covering of the frame bundle $SO(M)$, which is non-trivial over each fiber SO_x, $\forall x \in M$.*

We can then describe a spin structure on an orbifold as being locally a Γ_x-invariant spin structure on the smooth covering around x, but first we have to be able to lift the action of the group Γ_x of isometries to the Spin bundle of the smooth covering.

Definition 2.4. — *A singularity x is said to be* spin *if there is a lift $G_x \subset Spin(n)$ of $\Gamma_x \subset SO(n)$ which projects isomorphically onto Γ_x via the canonical projection from $Spin(n)$ to $SO(n)$.*

Such a lift does not always exist; for example, in dimension 2, the group $\mathbb{Z}_n$ can be lifted to $Spin(2)$ if and only if n is odd. In that case the lift is unique (as for all cyclic groups of odd order). On the other hand, any cyclic group of even order admits two lifts (if any): for example, $\mathbb{Z}_n \subset SU(2) \subset SO(4)$ admits an obvious lift in the first factor of $SU(2)\times SU(2) \simeq Spin(4)$, generated by an element $(\gamma, 1)$, and a second lift, generated by $(-\gamma, -1)$.

Remark. There is another, deeper, motivation for the definition above: As any spin structure on M restricts to one on the smooth part $M \setminus S$, it should induce spin structures on the quotients of small spheres around any singular point, i.e., on S^{n-1}/Γ. The condition above is necessary and sufficient for S^{n-1}/Γ to be spin [9, p. 47].

Remark. In [9] precise algebraic conditions on Γ are given for the existence of a lift G of Γ in $Spin(n)$. If $n = 4k+2$, $k \in \mathbb{N}^*$, such a lift is always unique, if it exists [9, sec. 2.2].

Example The group $\{\pm 1\} \subset SO(4)$ is the group of a spin singularity: indeed, on $S^3/\{\pm 1\} \simeq \mathbb{R}P^3$ there are exactly 2 (inequivalent) spin structures. On the other hand, there are 2 possible lifts of $\{\pm 1\}$ in $Spin(4) \simeq SU(2) \times SU(2)$, which are $\{\pm 1\} \times \{1\}$ and $\{1\} \times \{\pm 1\}$.

Remark. The action of Γ by isometries on $SO(\mathbb{R}^n)$ commutes with the right action of $SO(n)$ on the fibers of the frame bundle on $\mathbb{R}^n$. The lifted action of G on $Spin(\mathbb{R}^n)$ equally commutes with the right action of $Spin(n)$ on the fibers. It follows that G equally acts on every associated bundle $Spin(\mathbb{R}^n) \times_\rho F$, where $\rho : Spin(n) \times F \to F$ is a C^∞ representation of $Spin(n)$.

In particular, G acts on the spinor bundles $\Sigma^\pm(\mathbb{R}^n)$ of positive, resp. negative Weyl spinors of $\mathbb{R}^n$ (if n is even; we will mainly focus on this case). That leads to the following definition (see also [6]):

Definition 2.5. — *The total spaces of the spinor bundles $\Sigma^\pm(M)$ over an orbifold M are the orbifolds obtained by gluing together the spinor bundles $\Sigma^\pm(M \setminus S)$ with $\Sigma^\pm(U_x) \simeq \Sigma^\pm(\mathbb{R}^n)/G_x$, for all $x \in S$. A spinor field on an orbifold M is a pair of smooth sections in each of these two bundles, such that the lifts in any local smooth covering around $x \in M$ are smooth (G_x-equivariant) spinors.*

Remark. If the quotient $\Sigma^\pm(\mathbb{R}^n)/G$ is not smooth, then the value of any spinor field in 0 must lie in the singular set, more precisely in the set of fixed points of G in $\Sigma^\pm(\mathbb{R}^n)$. We will focus later on twistor spinors; they have the property that if the spinor ϕ and the value of $D\phi$ (where D is the Dirac operator) simultaneously vanish in some point, then the twistor spinor is everywhere zero (in a connected manifold), see next section. As ϕ and $D\phi$ are sections in the 2 different spinor bundles, they cannot both vanish at a singularity x unless ϕ identically vanishes, which means first of all that G_x must have non-zero fixed points at least on $\Sigma^+(\mathbb{R}^n)$ or on $\Sigma^-(\mathbb{R}^n)$. This implies certain constraints in dimension 4:

Proposition 2.6. — *Let ϕ^+ be a positive and ϕ^- a negative Weyl spinor field on a 4-dimensional orbifold. In a singular point x at least one of them vanishes. Moreover, they both vanish unless there is a complex structure on $\widetilde{T}_xM$ such that $\Gamma_x \subset SU(\widetilde{T}_xM) \simeq SU(2)$. In that latter case, if $\Gamma_x \neq \{\pm 1\}$, then, for any other local spin structure around x, every spinor field ψ vanishes at x.*

The proof follows from the identity $Spin(4) \simeq SU(2) \times SU(2)$, so any group $G \subset Spin(4)$ having a fixed point in Σ^+ must be totally contained in the second factor and conversely. If this is the case, then any other lift of the corresponding projection $\Gamma \subset SO(4)$ will act nontrivially on Σ^+ (always), and on Σ^- (unless $G = \Gamma = \{\pm 1\}$).

3. A characterization of spin orbifolds

The following result shows that is is enough to look at the smooth part of an orbifold to see if it is spin and to determine its spin structure.

THEOREM 1. — *Let M be an oriented orbifold, and let S be the set of its singularities. Assume S is of codimension at least 4. Then M is spin if and only if the manifold $M \smallsetminus S$ is spin. Moreover, the Spin structures on M are in 1-1 correspondence with the spin structures on $M \smallsetminus S$.*

The codimension condition means the following: for any non-trivial element of any singularity group, its fixed point set is of codimension at least 4. Note that this is essential: there are non-spin quotients of $\mathbb{R}^2$ by $\Gamma \subset SO(2)$, even if $\mathbb{R}^2 \smallsetminus \{0\}/\Gamma$ is always a cylinder, therefore spin. On the other hand this codimension is always even for oriented orbifolds.

Proof. — The proof follows the following steps:

(1) First we characterize topologically the total spaces of the frame bundles of spin orbifolds (these are always smooth manifolds); we introduce the concept of *orbifold universal covering*. In particular, extending a spin structure becomes equivalent to extending a certain discrete group action on a smooth manifold.
(2) We show that any spin structure on an n-sphere can be uniquely filled in to get one on the $n+1$-dimensional ball (the so-called *local model*).
(3) We conclude using an induction on the dimension of the orbifold.

Recall that the frame bundle $SO(M)$ of an orbifold M, as defined before, is smooth. Moreover, it comes with an action of the Lie algebra $\mathfrak{so}(n)$, that induces a right action of the simply connected group $Spin(n)$ (recall that $n \geqslant 4$). This is actually a right $SO(n)$-action, therefore the (non-trivial) element $-\mathbf{1}$ lying in the kernel of the projection $Spin(n) \to SO(n)$ acts trivially on $SO(M)$.

PROPOSITION 3.1. — *A spin structure on an orbifold M is a double covering of $SO(M)$ on which the element $-\mathbf{1} \in Spin(n)$ acts freely. Moreover, $-\mathbf{1}$ acts either trivially (in which case M is non-spin) or freely on the universal covering $\widetilde{SO(M)}$, the action of $Spin(n)$ being induced by the infinitesimal action of $\mathfrak{so}(n)$.*

Therefore, a spin structure on M is just a quotient of $\widetilde{SO(M)}$ by a subgroup $G \subset \pi_1(SO(M))$, such that

$$\pi_1(SO(M)) = G \ltimes \{\mathbf{1}, -\mathbf{1}\}.$$

Remark. The advantage of this point of view is that it reduces the extension of a spin structure to the extension (by continuity, as we shall see) of the action of a group G from an open dense set of $\widetilde{SO(M)}$ to its whole.

Definition 3.2. — *In both cases, the orbifold* $\tilde{M} := \widetilde{SO(M)}/Spin(n)$ *is called the* orbifold universal covering *of* M.

Remark. As its name suggests, the orbifold universal covering $\widetilde{M}$ can be characterized by a universality property in the category of orbifolds, similar with the one satisfied by the smooth universal covering of a manifold (the coverings are to be understood here as orbifolds maps, so they are not necessarily locally invertible).

Proof of Proposition 3.1

The claim is well-known if M is smooth. For a general spin orbifold, we have defined a spin structure $Spin(M)$ to be a double-covering of $SO(M)$ which is fiberwise non-trivial. This implies that $-\mathbf{1}$ has no fixed points over the smooth part of M. Now, a singularity P of a spin orbifold is spin itself, and the restriction of the Spin-bundle $Spin(M)$ over a chart U around P is a spin structure of an orbifold of the type $\mathbb{R}^n/\Gamma$, with $\Gamma \subset SO(n)$. Let us describe now the spin structures of this basic type of orbifold:

Lemma 1. — *The spin structures on B^n/Γ and on S^{n-1}/Γ are both in 1-1 correspondence with the lifts G of Γ in $Spin(n)$. Moreover, any spin structure on S^{n-1}/Γ can be uniquely filled in on B^n/Γ. Here, B^n is the unit ball in $\mathbb{R}^n$, Γ is the singularity group of* 0 *and the codimension of the fixed point set of* Γ *is at least* 4.

Proof. — Let us first describe the SO-, resp. $Spin$-bundles on the orbifold $\mathbb{R}^n/\Gamma$. The orthogonal and spin frame bundles over $\mathbb{R}^n$, denoted by $SO(\mathbb{R}^n)$ and by $Spin(\mathbb{R}^n)$, respectively, are both Lie groups acting by isometries on $\mathbb{R}^n$ (actually $SO(\mathbb{R}^n)$ is the group of Euclidean transformations on $\mathbb{R}^n$), and the canonical projection from the last to the former is a group homomorphism.

The group $\Gamma \subset SO(n) \subset SO(\mathbb{R}^n)$ is then a subgroup, and so is $G \subset Spin(\mathbb{R}^n)$. As $SO(\mathbb{R}^n/\Gamma) = SO(\mathbb{R}^n)/\Gamma$ (here we notice that Γ acts on $SO(\mathbb{R}^n)$ by left multiplication – hence commutes with the (right) $SO(n)$-action on the fibers), we see that the fundamental group of $M_\Gamma := SO(\mathbb{R}^n/\Gamma)$ is the preimage $\widetilde{\Gamma} \subset Spin(n)$ of Γ under the fundamental projection $p : Spin(n) \to SO(n)$. So we have the following exact sequence:

$$1 \to \{\pm 1\} \to \pi_1(M_\Gamma) \to \Gamma \to 1.$$

On the other hand, if $\widetilde{M_\Gamma}$ is a two-fold covering of M_Γ, it is itself covered by $Spin(\mathbb{R}^n)$ which is the universal covering of both M_Γ and $\widetilde{M_\Gamma}$, so we have the exact sequence

$$1 \to G \to \pi_1(M_\Gamma) \to \{\pm 1\} \to 1.$$

$\widetilde{M_\Gamma}$ is a spin structure on $\mathbb{R}^n/\Gamma$ if, moreover, the two-fold covering $\widetilde{M_\Gamma} \to M_\Gamma$ is non-trivial on each fiber SO_x, $x \in \mathbb{R}^n/\Gamma$. This implies that G cannot contain the non-trivial element in the $\mathbb{Z}_2$ from the first sequence, so the second sequence is a splitting of the first one.

So G must be a lift of Γ. It is worth mentioning that the double covering $\widetilde{M_\Gamma}$ of M_Γ is actually determined by the subgroup G of $\widetilde{\Gamma}$.

If we remove the singularity (and the corresponding fiber from the frame bundles), the fundamental groups of the (smooth!) frame bundles remain the same, because the codimension of the removed object is $n > 2$. We can carry on the same argument to conclude (see also [9, Ch. 2.2] for the case when Γ acts freely on S^{n-1}) that there is a 1-1 correspondence between the lifts G of Γ and the spin structures on $(\mathbb{R}^n \smallsetminus \{0\})/\Gamma$, or, equivalently, the spin structures on S^{n-1}/Γ. Note that the SO, resp. $Spin$ bundle of S^{n-1} can be identified with the Lie group $SO(n)$, resp. $Spin(n)$. □

We first show now that, if $-\mathbf{1}$ has some fixed point on $\widetilde{SO(M)}$, then it must act trivially. We will show that $\widetilde{SO(M)}/\{-\mathbf{1},\mathbf{1}\}$ is actually the frame bundle of $\widetilde{M}$, therefore is smooth. Note that $\widetilde{M}$ has been defined, so far, as the space of orbits of $Spin(n)$ in $\widetilde{SO(M)}$ and we can see, using the previous Lemma, that the preimage W_x in $\widetilde{SO(M)}$ of a small ball U_x around $x \in M$ is a union of copies of connected open sets, each of which is a covering of $SO(U_x)$ and, therefore, diffeomorphic to a quotient of $Spin(B_n)$ by a subgroup H_x of $\widetilde{\Gamma}_x$, the $\mathbb{Z}_2$ extension of the singularity group of x. If $-\mathbf{1} \in H_x$ then it acts trivially on W_x, if not, it has no fixed point there. So the fixed point set of $-\mathbf{1}$ is open and closed, therefore it is either empty or the total space. In the second case, M is clearly non-spin and $\widetilde{SO}(M)$ is the frame bundle of $\widetilde{M}$; in the first case, and again looking at the local model $W_x/Spin(n)$, which projects over U_x, we can identify $\widetilde{SO(M)}/\{-\mathbf{1},\mathbf{1}\}$ with the frame bundle of $\widetilde{M}$, therefore the orbifold universal covering of M is spin.

Let $Spin(M)$ be a spin structure on M, i.e., a double covering of $SO(M)$ which is fiberwise non-trivial. Again, looking at the local model provided by the Lemma 1, we can see that $Spin(M)$ is locally a quotient of $Spin(B_n)$ by a subgroup of $\widetilde{\Gamma}$ of index 2, using the notations from Lemma 1. Again, if

$-\mathbf{1}$ acts non-trivially (as it must be the case at least over the frame space of any smooth point), then it does not belong to this subgroup of index 2 and, therefore, acts freely on the quotient. So $-\mathbf{1}$ has no fixed point on $Spin(M)$. $\square$

LEMMA 2. — *If the codimension of the singular set of M is at least 3, then the universal covering of $M \smallsetminus S$ is open dense in $\widetilde{M}$.*

Proof. — $\widetilde{SO(M)}$ is simply-connected and so is the preimage of $M \smallsetminus S$, because the preimage of S is a union of submanifolds of codimension at least 3, as we can see from the local model, and removal of such submanifolds does not change the fundamental group of the total space.

But the preimage of $M \smallsetminus S$ is either the SO or the $Spin$ bundle of a smooth manifold open and dense in $\widetilde{M}$. This manifold is a covering of $M \smallsetminus S$ and is also simply-connected, because its frame bundle is. This proves the Lemma. $\square$

Remark. Note that S is not necessarily a union of smooth submanifolds, but the set of frames of all points in S form a submanifold in $SO(M)$. In $\widetilde{M}$, some of the singularities may be lifted (e.g., if $M = \mathbb{R}^n/\Gamma$), so in general $\widetilde{M \smallsetminus S}$ is just *contained* in the smooth part of $\widetilde{M}$.

The extension of the spin structure on $M \smallsetminus S$ to M proceeds now as follows: Let $Spin(M \smallsetminus S)$ be a spin structure on $M \smallsetminus S$. From Proposition 3.1 we know $Spin(M \smallsetminus S) = \widetilde{SO(M \smallsetminus S)}/G$, where $G \subset \pi_1(SO(M \smallsetminus S))$ and is isomorphic to $\Gamma = \pi_1(M \smallsetminus S)$. Because of the codimension condition, Lemma 2 states that $(\widetilde{SO(M} \smallsetminus S)$ is open dense in $\widetilde{SO(M)} = Spin(\widetilde{M})$, and actually Γ acts on $\widetilde{M}$ by isometries, freely on the preimage of $(M \smallsetminus S)$ (which is therefore the universal covering of $M \smallsetminus S$).

We have to show that the action of G can be uniquely extended by continuity to $Spin(\widetilde{M})$. We will do this using Lemma 1 and an induction on the dimension of M.

For $n = 4$ the singular set S is a discrete set of points. Using the local characterization of spin structures, as in Lemma 1, on small balls around the points of S, we can extend these actions of Γ, resp. G on the covering orbifold and on its spin bundle. This is just the extension by continuity, but we can see it more precisely as follows: Let x be a singular point, and let U_x be a small ball around it. Then U_x is covered by a union of copies of V_x, each of which is itself diffeomorphic to a quotient of an Euclidean ball by a group Γ_0 of isometries. Then we have

$$\{1\} \to \Gamma_0 \to \Gamma \to \bar{\Gamma} \to \{1\},$$

where $\bar{\Gamma}$ sends isometrically one copy of $V_x \setminus \{x\}$ into another, and Γ_0 preserves the copies of $V_x \setminus \{x\}$. The local model identifies Γ_0 with a subgroup of $SO(n)$ and the extension of its action on whole V_x is canonical. We can do the same for G, and note that, in the local model, G_0 projects onto Γ_0 and does not contain $-\mathbf{1}$, actually $-\mathbf{1}$ commutes with the action of G_0 (before and after the extension), and acts freely on the quotient $Spin(\widetilde{M})/\Gamma$. We get, therefore, a spin structure on M starting from one on $M \setminus S$, as claimed.

We remark that the extension procedure is the one used in Lemma 1, using here the fact that the small geodesic spheres around singular points are smooth and lie, therefore, in $M \setminus S$.

For the induction step we suppose that on any $(n-1)$-dimensional oriented orbifold whose singular set is of codimension at least 4, and such that the smooth part of it is spin, the action of the discrete group G as above can be extended to the singular set.

We will use this induction hypothesis only for the small geodesic spheres around singular points. Let $x \in S$, then we get an extension of the action of G on the preimage in $Spin(\widetilde{M})$ of a small sphere around x, because that sphere quotient is of dimension $(n-1)$ and the action of G is well-defined above the smooth part of it. But then we can fill in the obtained spin structure on this sphere to get one on the corresponding small ball around x, using again Lemma 1. By doing so, we actually extend the action of G, by continuity, over the preimage of x in $Spin(\widetilde{M})$. As this extension is unique, it is independent of the choices made and we obtain a free action of G on $Spin(\widetilde{M})$, such that $-\mathbf{1}$ acts freely on the quotient, and such that the further quotient under $-\mathbf{1}$ is $SO(M)$. Therefore, $Spin(\widetilde{M})/G$ is a spin structure on M.

4. Zeros of twistor spinors and conformal inversion

Let us first recall the twistor equation on a Riemannian spin manifold of dimension n : We call a spinor field ϕ a *twistor spinor* if the following *twistor equation* holds for all tangent vectors X :

$$\nabla_X \phi + \frac{1}{n} X \cdot D\phi = 0\,. \tag{4.1}$$

Here ∇ is the *spin-connection* on the spinor bundle $\Sigma(M)$, the dot "$\cdot$" denotes the Clifford multiplication and D the Dirac operator. This definition extends to the orbifold case: around a singularity x the spinor field ϕ

and tangent vectors X must then have G_x-equivariant liftings. The spin-connection $\nabla_X\phi$ and the Dirac operator $D\phi = \sum_{i=1}^n e_i \cdot \nabla_i \phi$ of a G_x-equivariant spinor field ϕ are again G_x-equivariant.

If ϕ is a twistor spinor then one computes for the derivative of the Dirac operator $D\phi$:

$$\nabla_X D\phi = \frac{n}{2} L(X) \cdot \phi \,. \tag{4.2}$$

Here L is the $(1,1)-$*Schouten tensor* defined by

$$L(X) = \frac{1}{n-2}\left(\frac{s}{2(n-1)}X - \mathrm{Ric}(X)\right)$$

with the *Ricci tensor* Ric and the scalar curvature s. One can view Equation (4.1) and Equation (4.2) as a parallelism condition: We define on the double spinor bundle $E = \Sigma(M) \oplus \Sigma(M)$ the connection

$$\nabla^E_X = \begin{pmatrix} \nabla_X & \frac{1}{n}X\cdot \\ -\frac{n}{2}L(X)\cdot & \nabla_X \end{pmatrix},$$

i.e.,

$$\nabla^E_X(\phi,\psi) = \left(\nabla_X\phi + \frac{1}{n}X\cdot\psi, -\frac{n}{2}L(X)\cdot\phi + \nabla_X\psi\right).$$

Then we obtain

Lemma 3. — [1, ch. 1.4, Thm. 4]. *A twistor spinor ϕ on a Riemannian spin manifold is uniquely determined by a parallel section of the bundle E with connection ∇^E. More precisely, if ϕ is a twistor spinor, then $(\phi, D\phi)$ is a parallel section of (E, ∇^E) and if (ϕ,ψ) is a parallel section of (E,∇^E) then ϕ is a twistor spinor and $\psi = D\phi$.*

Corollary 1. — *Any zero P of a twistor spinor ϕ is isolated, more precisely $\nabla_X\phi(P) \neq 0$ for any non-zero vector $X \in T_PM$.*

Proof. — If both ϕ and $D\phi$ vanish at P, then ϕ is identically zero (as unique parallel section in E with this initial data). Therefore $\nabla_X\psi = -\frac{1}{n}X\cdot D\phi$ is non-zero if $X \neq 0$, as Clifford product of a non-zero vector with a non-zero spinor $D\phi$. □

We use the following notation for the open Euclidean ball of radius R : $B_R := \{x \in \mathbb{R}^n \,|\, \|x\| < R\}$, then $\overline{B}_R := \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant R\}$ is the corresponding closed Euclidean ball.

Definition 4.1. — (a). *Let Γ be a finite subgroup of $SO(n)$ that acts freely on $\mathbb{R}^n \setminus \{0\}$ and let U be an open subset of a Riemannian manifold. Then the open subset $U \subset M$ carries an* Asymptotically Locally Euclidean (ALE) coordinate system $y = (y_1, \ldots, y_n)$ *of order (τ, μ) if the* *following conditions are satisfied: There is for some* $R > 0$ *a diffeomorphism* $y \in \left(\mathbb{R}^n - \overline{B}_R\right)/\Gamma \mapsto \phi(y) \in U$ *such that the metric coefficients* $g_{ij}(y) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$ *with respect to the coordinates* $y = (y_1, \dots, y_n)$ *and its derivatives*

$$\partial_{i_1 \dots i_k} g_{ij} = \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} g_{ij}$$

have the following asymptotic behaviour for $\rho = \|y\| = \sqrt{\sum_1^n y_i^2} \to \infty$:

$$g_{ij} - \delta_{ij} = O(\rho^{-\tau}); \partial_{i_1 \dots i_k} g_{ij} = O\left(\rho^{-\tau-k}\right)$$

for all $k = 1, 2, \dots, \mu$. *If the group* $\Gamma = \{1\}$ *is trivial, then the coordinate system is called* Asymptotically Euclidean.

(b). *We call a non-compact Riemannian manifold* $\overline{M}$ *of dimension* n Asymptotically Locally Euclidean *or short* ALE *of order* (τ, μ) *if there is a compact subset* M_0 *such that the complement* $M - M_0$ *carries an asymptotically Euclidean coordinate system of order* (τ, μ).

Remark. There is a close relationship between manifolds carrying twistor spinors with zeros and non-compact manifolds with parallel spinors with an end carrying an ALE coordinate system, more precisely: Let (M, g) be a Riemannian spin manifold with a twistor spinor ϕ having a zero p. It is isolated and the length $\|\phi\|$ behaves like a distance function in the neighbourhood of p (see Corollary 1). Given normal coordinates $x = (x_1, \dots, x_n) \in B_\epsilon \subset \mathbb{R}^n$ in a neighbourhood U of p we define *inverted normal coordinates* $y = (y_1, \dots, y_n) \in \mathbb{R}^n - B_{\epsilon^{-1}}, y = x/\|x\|^2$. Then the conformally equivalent metric $\left(U - \{p\}, \overline{g} = g/\|\phi\|^4\right)$ carries a parallel spinor and an asymptotically Euclidean coordinate system of order $(3, 2)$. It is locally irreducible unless it is flat (see [16, Theorem 1.2] for details).

On the other hand one can use a metric with parallel spinors having an end with an ALE coordinate system to produce examples of twistor spinors with zeros:

Lemma 4. — *If on the open subset* U *diffeomorphic to* $\left(\mathbb{R}^n \setminus \overline{B}_R\right)/\Gamma$ *of a smooth, i.e.* C^∞ *Riemannian manifold* (M, g) *there is an ALE-coordinate system* y *with radius function* $\rho = \|y\| = \sqrt{\sum_1^n y_i^2}$ *of order* (τ, μ) *with* $\mu \geqslant \tau - 1 \geqslant 2$, *then the conformally equivalent metric* $\overline{g} = \rho^{-4} g$ *extends as a* $C^{\tau-1}$ *metric to the one-point completion* $U \cup p_\infty$ *diffeomorphic to* B_R/Γ.

Proof. — We denote by $y = (y_1, \dots, y_n), \rho > R$ asymptotically Euclidean coordinates and denote

$$g_{ij}(y) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \delta_{ij} + h_{ij}(y).$$

Then $h_{ij} = O(\rho^{-\tau})$, $\frac{\partial^k}{\partial y_{i_1}\cdots y_{i_k}} h_{ij}(y) = O(\rho^{-\tau-k})$ for all $k = 1, 2, \ldots, \mu$. We use the *inversion* $z = \rho^{-2} y$ and obtain with the formula $\frac{\partial}{\partial z_i} = \rho^{-2}\frac{\partial}{\partial y_i} - 2\frac{z_i}{\rho^2}\sum_{k=1}^n z_k \frac{\partial}{\partial y_k}$ for the coefficients $\overline{g}_{ij}$ of the conformally equivalent metric $\overline{g}$ with respect to the inverted coordinates $z = (z_1, \ldots, z_n)$; $z_i = \rho^{-2} y_i$:

$$\begin{aligned} \overline{g}_{ij}(z) &= \hat{g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \delta_{ij} + h_{ij} - \frac{2}{\rho^2}\left(z_i \sum_k z_k h_{kj} + z_j \sum_l z_l h_{il}\right) \\ &\quad + \frac{4}{\rho^4} z_i z_j \sum_{k,l} z_k z_l h_{kl} \,. \end{aligned} \tag{4.3}$$

It follows that $\overline{g}_{ij}(z) = \delta_{ij} + O(r^\tau)$ with $r = \|z\| = \sqrt{\sum_1^n z_i^2} = \rho^{-1}$ and $\frac{\partial^k}{\partial z_{i_1}\cdots\partial z_{i_k}} \overline{g_{ij}(z)} = O\left(r^{\tau-k}\right)$. Hence we obtain that the function $\overline{g}_{ij}(z)$ extends to $z = 0$ as a $(\tau-1)$-times continuously differentiable function. □

The following result establishes a *conformal completion* of a Ricci-flat ALE Kähler metric which will provide us with examples of conformal orbifolds admitting twistor spinors with zero.

Theorem 2. — *Let Γ be a subgroup of $SO(n), n = 2m$ acting freely on $\mathbb{R}^n \setminus \{0\}$ and let (M, g) be an ALE Riemannian spin manifold (M, g) (with a C^∞-metric) of order (τ, τ) (i.e. asymptotically Euclidean to $\mathbb{R}^n/\Gamma$ at infinity) with $\tau \geqslant 2$ and holonomy group* SU(m). *Then there is a one point conformal completion $N = M \cup \{p_\infty\}$ of (M, g) to a compact Riemannian spin orbifold with singular point p_∞ whose singularity group is Γ. The metric $\overline{g}$ on $M = N - \{p_\infty\}$ is C^∞-smooth, conformally equivalent to g on $M = N - \{p_\infty\}$ and it is a $C^{\tau-1}$ metric on N.*
Then there is a spin structure on the orbifold N with a two-dimensional space of twistor spinors, and all nontrivial twistor spinors have exactly one zero point, which is the singularity point p_∞.

Proof. — Since the holonomy group is SU(m) one can conclude that the manifold M is spin and has a preferred spin structure for which the space of parallel spinors is two-dimensional, cf. [11, Corollary 3.6.3]. It follows from Lemma 4 that the conformally equivalent metric $\rho^{-4}g$ can be extended to the orbifold $N = M \cup \{p_\infty\}$. If U is a sufficiently small neighbourhood of p_∞ then there is a Riemannian covering $(V \setminus \{p\}, \overline{g})$ of $(U \setminus \{p\}, \overline{g})$ with Riemannian covering group Γ which is diffeomorphic to $B_1^n \setminus \{0\}$ carrying a twistor spinor ϕ invariant under Γ. It follows from the behaviour of twistor spinors under conformal changes that its norm with respect to the Hermitian metric on the spinor bundle of $(V \setminus \{p\}, \overline{g})$ is given by $\|\phi\| = r$, i.e. the spinor field ϕ can be continuously extended to U by setting $\phi(p_\infty) = 0$. As pointed out above a twistor spinor ϕ is in one-to-one correspondence with a parallel section of the bundle E with connection ∇^E. Then it follows from the next Lemma 5 that it extends to a continuously differentiable section, hence it is a Γ-invariant twistor spinor on $(V, \overline{g})$. Therefore the orbifold $(N, \overline{g})$ carries a twistor spinor with zero in p. □

In the proof of Lemma 4 we used the following general result to extend the twistor spinor into the singularity:

Lemma 5. — *Let $E \to M$ be a $\mathcal{C}^1$ vector bundle equipped with a continuous linear connection (i.e., in a $\mathcal{C}^1$ trivialization map, the coefficients of the connection form are continuous forms on M, with values in the set of linear endomorphisms of the fiber), and let σ be a parallel section over $M \smallsetminus \{p\}$. Then σ can be uniquely extended to a $\mathcal{C}^1$ global section (hence parallel).*

Proof. — By replacing M, if necessary, with a smaller neighbourhood of p, we can suppose the bundle is trivial, so $E = M \times F$, and the connection form is a continuous 1-form ω on M, with values in $End(F)$. Consider some Riemannian metric on M and an Euclidean metric on F. Without loss of generality we can assume that $\omega_p = 0$, and (by restricting it, if necessary, to a smaller neighbourhood around p),

$$|\omega_x(X) \cdot V| \leqslant \varepsilon |X||V|, \tag{4.4}$$

where the dot means matrix multiplication, and the norms of $X \in T_xM$, $V \in F$ are computed using the above chosen Riemannian, resp. Euclidean Metric on M and F.

We consider a parallel lift of a $\mathcal{C}^1$ loop c in M, passing through x, and contained in a compact ball around x. It is a curve $t \mapsto (c(t), \gamma(t))$ in $M \times F$. This lift is characterized by the following linear ODE:

$$\gamma'(t) = -\omega_{c(t)}(c'(t)) \cdot \gamma(t)$$

Suppose $|c'| \equiv 1$, so c is an arc length parameterization. By taking the scalar product in F with $\gamma(t)$, and using (4.4), we obtain

$$|(|\gamma(t)|^2)'| \leqslant 2\varepsilon |\gamma(t)|^2,$$

hence

$$|\gamma(t)| \leqslant e^{\varepsilon t} |\gamma(0)|,$$

and

$$|\gamma'(t)| \leqslant \varepsilon e^{\varepsilon t} |\gamma(0)|. \tag{4.5}$$

This gives us a bound for the length of γ, depending continuously on the length of c, and linearly on the initial data $\gamma(0)$. The first conclusion is that the lift is defined for all times.

Let $\sigma : M \smallsetminus \{p\} \to F$ be a parallel section outside p. Let c be an arbitrary $\mathcal{C}^1$ curve on M, such that $c(0) = p$ and $c'(0) \neq 0$. Starting from $c(t_0) \neq p$, for some small time t_0, we consider the lift $t \mapsto (c(t), \sigma(c(t)))$, which is horizontal, because σ is parallel. But we can define this lift for all times, including 0. We obtain a curve γ, depending on c, and a point $(p, \gamma(0)) = (p, A)$ in the fiber over p.

We want to show that this point does not depend on the choice of the curve c. Suppose that, for another curve $\tilde{c}$, equally arc length parametrized, we obtain a lift $\tilde{\gamma}$, such that $\tilde{\gamma}(0) = B \neq A$.

Using c for the beginning, $-\tilde{c}$ for the end (i.e., we run $\tilde{c}$ in the reverse sense), and connecting them smoothly, we can get, for any $n \in \mathbb{N}^*$, a sequence of $\mathcal{C}^1$ arc length parametrized loops c_n in M, of length $l_n \leqslant 1/n$, such that $c_n(0) = c_n(l_n) = p$.

The corresponding lifts γ_n through σ are well defined over $M \smallsetminus \{p\}$, and because of the argument above, $\gamma_n(0) = \gamma(0) = A$ (the curves c and $c(n)$ coincide around 0), and $\gamma_n(l_n) = \tilde{\gamma}(0)$ (the curves $-\tilde{c}$ and c_n coincide around 0, resp. l_n).

From (4.5), the length of γ_n is smaller than $|A|(e^{\varepsilon/n} - 1)$, therefore the distance between A and B must be smaller that this expression.

So $A = B$.

This allows us to define $\sigma(p) := A$ and get a continuous global section. Because the lifts through σ of $\mathcal{C}^1$ curves through p are parallel lifts through A, we easily obtain the differentiability σ in p, and actually that σ is $\mathcal{C}^1$.

It is also, by construction, parallel. □

5. Main results

THEOREM 3. — *Let M be a compact Riemannian spin orbifold. If M carries a non-trivial twistor spinor with zero at p_0 then this zero is unique and p_0 is a singular point unless the orbifold is conformally equivalent to a round sphere. In addition, for every point $q \neq p_0$ the order $\#\Gamma_q$ of the singularity group satisfies*

$$\#\Gamma_q \leqslant \#\Gamma_{p_0}$$

with equality only if the orbifold is a quotient of a standard sphere, hence conformally flat.

Then conformally flat case is discussed, under some assumptions, in Proposition 5.3

Proof. — Let ϕ be the corresponding twistor spinor and $u := \langle\phi,\phi\rangle$, then the orbifold $\overline{M} = (M - Z_\phi, \overline{g} = u^{-2}g)$ is a Ricci flat Riemannian orbifold carrying a parallel spinor (recall that Z_ϕ is the zero set of ϕ). This follows from the analogue of the Remark in the previous section, applied to orbifolds. If $Z_\phi = \{p_1, \dots, p_m\}$ then $\overline{M}$ has m ends, at any of which the metric $\bar{g}$ is asymptotically locally Euclidean (i.e. asymptotic to $\mathbb{R}^n/\Gamma_{p_j}$) in inverted normal coordinates.

As in the Riemannian case we first show that if $m > 1$ there is a geodesic line in $\overline{M}$. This geodesic line is obtained as a limit of segments connecting points approaching two different ends (see, for example, [18], Ch. 9).

Let us remark that the corresponding segments can be chosen to avoid the singularities. Indeed, first note that the singularity set Σ is of codimension at least 2 (because of the orientability of M), so $\overline{M} \smallsetminus \Sigma$ is connected, so we can choose the endpoints of the segments to be smooth points. On the other hand, no segment (i.e., minimizing geodesic) can touch the singular set without being entirely contained in it, otherwise it wouldn't be minimizing (the singular set is totally geodesic, for symmetry reasons).

So the singularities play no role in the process, which carries over exactly as in the smooth case and we get a geodesic line (which avoids singularities, as well).

On the other hand, there is an orbifold generalization of the splitting Theorem of Cheeger and Gromoll:

Proposition 5.1. — ([3, Theorem 1]) *If $\overline{M}$ is a complete Riemannian orbifold of dimension n with non-negative Ricci curvature carrying a geodesic line, then $\overline{M}$ is isometric to the product $\mathbb{R} \times N$ of the real line with a Riemannian orbifold N of nonnegative Ricci curvature.*

Therefore, if we have more than one end, there should be such a splitting. But $N \times \mathbb{R}$ is ALE, so the curvature must tend to zero for large t along the line $\{(x,t) \mid t \in \mathbb{R}\}$, for any fixed $x \in N$. Therefore the curvature of N must be identically zero. On the other hand, if the ends of that geodesic line are different ends of $\overline{M}$, then N must be compact. So N is a flat Riemannian orbifold.

On the other hand, every end of $\overline{M}$ corresponds to a removed (singular) point of singularity group Γ, therefore $N \simeq S^{n-1}/\Gamma$, which would imply the existence of a flat metric on the sphere, contradiction (here we need $n > 2$).

So there is only one end, hence the zero set of ϕ contains only the point p, as claimed.

The inequality in the conclusion of the Theorem is proved using growth estimates for the volume of balls (more precisely, an extension to orbifolds of Bishop's Theorem):

Proposition 5.2. — ([2, Proposition 20]) *Let $\overline{M}$ be a complete Riemannian orbifold with singular set Σ and non-negative Ricci curvature* $\mathrm{Ric} \geqslant 0$. *Then for every point $p \in \overline{M}$ the function*

$$r \in (0, \infty) \mapsto \psi(r) := \frac{\mathrm{vol}B(p,r)}{\omega_n r^n}$$

is non-increasing. Here $\mathrm{vol}B(p,r)$ is the volume of the geodesic ball $B(p,r)$ of radius r around p and ω_n is the volume of a unit ball in Euclidean space $\mathbb{R}^n$. Furthermore

$$\lim_{r\to 0} \psi(r) = \frac{1}{\#\Gamma_p}$$

where Γ_p is the isotropy subgroup of the point p. Moreover, if, for some $r > 0$, $\psi(r) = \frac{1}{\#\Gamma_p}$, then $B(p,r)$ is isometric to the quotient B^n_r/Γ_p of the ball of radius r in $\mathbb{R}^n$ with Γ_p.

Following the same estimates as in [16], we can show that, under very weak assumptions, the volume of large balls in an ALE spoace approaches the one in the flat model:

Lemma 6. — *Let $(\overline{M}, g)$ be an ALE Riemannian orbifold (with corresponding group Γ) of order (τ, μ), with $\tau > 1, \mu \geqslant 0$. Then the function ψ (representing the relative volume of a ball in $\overline{M}$ w.r.t. the Euclidean space) satisfies:*

$$\lim_{r\to\infty} \psi(r) = \frac{1}{\#\Gamma}.$$

We apply these inequalities for the proof of Theorem 3: now, $\bar{M}$ is an ALE Ricci-flat orbifold, whose end is asymptotic to $\mathbb{R}^n/\Gamma$ (Here, Γ is actually the group Γ_{p_0} corresponding to the zero point p_0 of the twistor spinor ϕ).

Choose a point $p \in \overline{M}$ (if p is a smooth point, then $\Gamma_p = \{1\}$). Since the function $\psi(p;r) = \frac{\mathrm{vol}B(p,r)}{\omega_n r^n}$ is non-increasing it follows from Lemma 6 and Proposition 5.2 that

$$\#\Gamma_p \leqslant \#\Gamma.$$

The equality occurs if and only if $\overline{M}$ is isometric to $\mathbb{R}^n/\Gamma_p$, therefore $\Gamma \simeq \Gamma_p$ and M is conformally equivalent to S^n/Γ (where Γ acts by isometries and has two mutually opposed fixed points). This finishes the proof of the Theorem. □

Remark. We can show that the vanishing locus $\{p_1, \dots, p_k\}$ of a twistor spinor on a compact orbifold contains only singular points without using Proposition 5.1. Indeed, from the volume estimates given in Proposition 5.2 we obtain

$$1 \geqslant \frac{1}{\#\Gamma_{p_1}} + \cdots + \frac{1}{\#\Gamma_{p_k}},$$

so if any p_i is smooth, it must be unique and we get equality in the volume estimate, so the metric is flat. This gives a simpler proof of the Lichnerowicz' Theorem [17].

Note that the above inequality does not exclude the existence of multiple *singular* zeros of a twistor spinor, so the use of Proposition 5.1 is necessary.

Corollary 2. — *Let (M, g) be a compact Riemannian spin orbifold with finite singularity set carrying a twistor spinor with non-empty zero set. Then either M is a quotient of S^n or it is a* non-trivial *Riemannian orbifold.*

Remark. We prefer to use the attribute *non-trivial* to denote an orbifold that can not be obtained as a quotient of a smooth manifold, as opposed to the slightly misleading term *bad orbifold*, sometimes used in the literature to denote the same object.

Proof. — If $M = N/\Gamma$ for a compact Riemannian spin manifold N then also N carries a twistor spinor with non-empty zero set. Hence it is a round sphere. □

Cross-application of the inequality in Theorem 3 leads to:

Corollary 3. — *If a non-trivial Riemannian orbifold has two (non-trivial) twistor spinors with zero, then they have the same zero set.*

We can actually say a little more about the sphere quotients that admit twistor spinors with zeros: they have exactly 2 singular points. Moreover, we can characterize them (under some technical assumption) merely by the flatness of the conformal structure:

Proposition 5.3. — *Any compact, conformally flat orbifold M, with isolated singularity set, carrying a twistor spinor with zero is conformally equivalent to S^n/Γ, where $\Gamma \subset SO(n)$ acts on $\mathbb{R}^n \subset \mathbb{R}^{n+1} \supset S^n$.*

Proof. — We keep the notations as above. $\overline{M}$ is a flat, complete orbifold with singularity set $\bar{\Sigma}$. $\overline{M} \setminus \bar{\Sigma}$ is a manifold, therefore it has a universal covering Q. Now, in a neighbourhood U of a singular point $P \in \bar{\Sigma}$, the map $\pi : Q \to \overline{M} \setminus \bar{\Sigma}$ looks like a disjoint union of connected coverings of $\mathbb{R}^n \smallsetminus \{0\}/\Gamma_P$, where Γ_P is the singularity group of P. Each of these connected components V_i of $\pi^{-1}(U)$ can be completed by adding a point P_i to V_i. Furthermore, $\bar{Q} := Q \cup \{P_i \mid P \in \bar{\Sigma}\}$ can be given the structure of a flat orbifold, and $\pi : \bar{Q} \to \overline{M}$ is then a ramified covering of orbifolds.

Since Q is simply connected and flat, any parallel vector field on Q can be globally extended to a parallel one on $\bar{Q}$. But V_i is isometric to some quotient of a ball, and its tangent space is spanned by parallel vector fields if and only if it is a ball itself. So all V_i's are isometric to balls and $V_i \cup \{P_i\}$ are smooth balls. So $\bar{Q}$ is a simply connected smooth manifold, flat and complete, thus $\bar{Q} \simeq \mathbb{R}^n$. So it has, like $\overline{M}$, one end, therefore the covering π is finite (the end of $\overline{M}$ has finite fundamental group).

So $\overline{M} \simeq \mathbb{R}^n/G$, where $G \subset \mathrm{Isom}(R^n)$ is finite and has only isolated fixed points. As each $g \in G$ may have at most one fixed point, the set Σ of all fixed points is finite, and $\bar{\Sigma} \subset \pi(\Sigma)$.

Let K be the convex envelop of Σ and let $P \in \Sigma$ be one of the vertices of the polyhedron K. Then $\Sigma \cap \bar{H} = \{P\}$ for a certain "exterior" closed halfspace $\bar{H}$. Let $P' \neq P$ be another point in Σ. Then $P'' := \sum_{k=1}^{N_g} g^k(P')$ is another fixed point for g, where $g \in G \smallsetminus \{1\}$ is such that $g(P) = P$ and $g^{N_g} = 1$. But $P'' \in \mathbb{R}^n \smallsetminus \bar{H}$, therefore $P'' \neq P$. In this case g would have two distinct fixed points, so it would leave the whole line connecting them fixed, contradiction with the finiteness of Σ.

So Σ consists of only one point and the conclusion easily follows. □

So the equality case in our Theorem characterizes the conformally flat orbifolds carrying twistor spinors with zero (which actually turn out to be sphere quotients).

6. Examples

At our knowledge, no examples of parallel or Killing spinors (corresponding to non-vanishing solutions of the twistor equation) on compact, non-smooth, orbifolds are known so far. The following examples all have exactly one zero and, as pointed out by the Corollary 2, they are not only non-smooth, but equally non-trivial (they are not covered by a manifold).

Let Γ be a finite subgroup of the group $\mathrm{Sp}(1) = \mathrm{SU}(2)$ acting freely on $\mathbb{H} \smallsetminus \{0\}$, here $\mathbb{H} \cong \mathbb{R}^4$ is the space of quaternions. Then a hyperkähler ALE space (of order $(4, \infty)$) for this group is also called *gravitational instanton* in the physics literature. Outside a compact set the metric is asymptotic to the quotient $\mathbb{H}/\Gamma$ with order $(4, \infty)$ (cf. Definition 4.1) and in addition also the hyperkähler structure is asymptotic to the Euclidean hyperkähler structure, cf. [11, Definition 7.2.1]. A hyperkähler space is in particular Ricci flat, therefore if follows from the rigidity part of the Bishop-Gromov volume comparison theorem [16] that the group is non-trivial unless the metric is flat.

Example 1. — The first examples of a gravitational instanton are the *Eguchi-Hanson spaces* (M_{EH}, g_{EH}), for the subgroup $\Gamma = \mathbb{Z}_2 = \{\pm 1\}$. These can be given explicitly, cf. [7], [11, Example 7.2.2]. The space M_{EH} is the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at 0, it can be identified with the cotangent bundle $T^*\mathbb{CP}^1 \cong T^*S^2$ of the 1-dimensional complex projective space resp. the 2-sphere. Hence this space is simply-connected and spin. It is shown in [14] that one can conformally compactify the Eguchi-Hanson space to a compact orbifold (with C^∞-metric) with one singular point p_∞ whose singularity group is $\mathbb{Z}_2$. The existence of this compactification (at least as a C^3-metric) follows also from Theorem 2.
Hence we obtain a compact 4-dimensional Riemannian spin orbifold (M, g) with one singular point p_∞ whose singularity group is $\mathbb{Z}_2$ carrying two linearly independent twistor spinors ψ_1, ψ_2. The singularity point is the unique zero point of ψ_1, ψ_2.

Example 2. — Gibbons and Hawking generalized the Eguchi-Hanson construction and obtained hyperkähler ALE spaces asymptotic to $\mathbb{H}/\mathbb{Z}_k$ for all $k \geqslant 2$, cf. [10]. Finally Kronheimer described in [13] and [12] the construction and classification of hyperkähler ALE spaces asymptotic to $\mathbb{H}/\Gamma$ for nontrivial finite subgroup $\Gamma \subset \mathrm{SU}(2)$, see also [11, Theorem 7.2.3].

Example 3. — On $\mathbb{C}^m \setminus \{0\}$ the group $\mathbb{Z}_m$ generated by complex multiplication with $\zeta = \exp(2\pi\sqrt{-1}/m)$ acts freely. The blow-up X of $\mathbb{C}^m/\mathbb{Z}_m$ at 0 can be identified with a complex line bundle over the $(m-1)$-dimensional complex projective space $\mathbb{CP}^{m-1}$. One can explicitly write down an ALE Kähler metric with holonomy $\mathrm{SU}(m)$, cf. [4], [8] or [11, Example 8.2.5]. Using Theorem 2 or the explicit form given in [15] one obtains a compact spin orbifold of real dimension $n = 2m$ with one singular point p_∞ and singularity group $\mathbb{Z}_m$.
This orbifold carries a spin structure with a two-dimensional space of twistor spinors whose common zero point is the singular point.

All the examples described so far are defined using a *crepant resolution* of the isolated singularity $0 \in \mathbb{C}^2$. For a definition of a crepant resolution see [11, ch. 6.4]. In particular for a crepant resolution the first Chern class vanishes, hence one can show that a resolution of $\mathbb{C}^m/\Gamma$ carries a Ricci flat ALE Kähler metric only if the resolution is crepant, cf. [11, Proposition

8.2.1]. In complex dimension 2 there is a unique crepant resolution of $\mathbb{C}^m/\Gamma$ (here $\Gamma \subset SU(n)$ and acts freely on the sphere in $\mathbb{C}^n$), and in dimension 3, for each Γ as above, there exist crepant resolutions (not unique, in general), but in higher dimensions this does no longer hold, for example $\mathbb{C}^4/\{\pm 1\}$ does not carry any crepant resolution, cf. [11, Example 6.4.5].

For crepant resolutions M of the quotient singularity $\mathbb{C}^m/\Gamma$, if there is an ALE Kähler metric on M, there is a solution of the ALE Calabi conjecture, as stated and proved by D. Joyce, [11, Theorem 8.2.3], [11, Theorem 8.2.4]: In every Kähler class of ALE Kähler metrics (with order $(2m, \infty)$) there is a unique Ricci-flat Kähler metric with holonomy $\mathrm{SU}(m)$. Then we can conclude from [11, Corollary 3.6.3] that this metric is spin and carries a 2-dimensional space of parallel spinors. The resolution is simply-connected, hence there is only one spin structure. Hence starting from a crepant resolution of the quotient sigularity $\mathbb{C}^m/\Gamma$ with an ALE Kähler metric (of order $(2m, \infty)$) one obtains a compact spin orbifold of dimension $n = 2m$ with a C^{2m-1}-Riemannian metric which carries a 2-dimensional space of twistor spinors. All twistor spinors have exactly one zero in the only singularity point.

In general, the existence of ALE Kähler metrics on crepant resolutions is an open question, however, in dimension 3, ALE Kähler (non-Ricci flat) metrics have been constructed on crepant resolutions of $\mathbb{C}^3/\Gamma$ using symplectic quotient techniques, [19],[5].

Therefore we can add the following class of examples of 6-dimensional orbifolds:

Example 4. — Let $\Gamma \subset SU(3)$ be any finite subgroup that acts freely on $S^5 \subset \mathbb{C}^3$. Then there exists a crepant resolution M of $\mathbb{C}^3/\Gamma$, admitting an ALE Kähler metric of order $(6, \infty)$, [5], and, therefore, an ALE Ricci-flat one as well, cf. [11]. The one-point compactification of M is then spin, has one singular point at ∞ (of singularity group Γ), and admits two linearly independent twistor spinors that vanish at that singular point.

Remark. Theorem 2 shows only that the metric has some finite regularity C^k around the zero of the twistor spinor, depending on the decay rate of the ALE Kähler Ricci-flat metric on the complement. It is unknown whether this can be improved, possibly after changing the metric in the conformal class.

BIBLIOGRAPHY

[1] H. Baum, T. Friedrich, R. Grunewald & I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 124, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991, 180 pages.

[2] J. Borzellino, "Orbifolds of maximal diameter", *Ind. Univ. Math. J.* **42** (1993), p. 37-53.

[3] J. Borzellino & S.-H. Zhu, "The splitting theorem for orbifolds", *Ill. J. Math.* **38** (1994), p. 679-691.

[4] E. Calabi, "Métriques kählériennes et fibrés holomorphes", *Annal. scient. École Norm. Sup.* **12** (1979), p. 269-294.

[5] A. Degeratu, "Geometrical McKay Correspondence for Isolated Singularities", math.DG/0302068, 2003.

[6] C. Dong, K. Liu & X. Ma, "On orbifold elliptic genus", in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemp. Math, vol. 310, Amer. Math. Soc., Providence, 2002, p. 87-105.

[7] T. Eguchi & A. Hanson, "Asymptotically flat solutions to Euclidean gravity", *Phys. Lett.* **74B** (1978), p. 249-251.

[8] D. Freedman & G. Gibbons, "Remarks on supersymmetry and Kähler geometry", in *Superspace and Supergravity* (S.W. Hawking, M. Roček, ed.), Proc. Nuffield workshop, Cambridge 1980, Cambridge Univ. Press, 1981.

[9] T. Friedrich, *Dirac-Operatoren in der riemannschen Geometrie*, Adv. lect. Math., Vieweg Verlag, Braunschweig, 1997.

[10] G. Gibbons & S. Hawking, "Gravitational multi-instantons", *Phys. Lett.* **78 B** (1978), p. 430-432.

[11] D. Joyce, *Compact manifolds with special holonomy*, Adv. lect. Math., Oxford Math. Monographs, Oxford, 2000.

[12] P. Kronheimer, "A Torelli-type theorem for gravitational instantons", *J. Diff. Geom.* **29** (1989), p. 685-697.

[13] ———, "The construction of ALE spaces as hyperkähler quotients", *J. Diff. Geom.* **29** (1989), p. 665-683.

[14] W. Kühnel & H.-B. Rademacher, "Twistor Spinors and Gravitational Instantons", *Lett. Math. Phys.* **38** (1996), p. 411-419.

[15] ———, "Conformal completion of $U(n)$–invariant Ricci flat Kähler metrics at infinity", *Zeitschr. Anal. Anwend.* **16** (1997), p. 113-117.

[16] ———, "Asymptotically Euclidean Manifolds and Twistor Spinors", *Commun. Math. Phys.* **196** (1998), p. 67-76.

[17] A. Lichnerowicz, "Killing spinors, twistor–spinors and Hijazi inequality", *J. Geom. Phys.* **5** (1988), p. 2-18.

[18] P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics, vol. 171, Springer, 1998.

[19] A. Sardo-Infirri, "Partial Resolutions of orbifold singularities via moduli spaces of HYM-type bundles", alg-geom/9610004.

Manuscrit reçu le 7 juin 2006,
accepté le 14 septembre 2006.

Florin Alexandru BELGUN
Universität Leipzig
Mathematisches Institut
Johannisgasse 26
04109 Leipzig (Allemagne)
Florin.Belgun@math.uni-leipzig.de

Nicolas GINOUX
Universität Potsdam
Institut für Mathematik - Geometrie
Am Neuen Palais 10
14469 Potsdam (Allemagne)
ginoux@math.uni-potsdam.de

Hans-Bert RADEMACHER
Universität Leipzig
Mathematisches Institut
Johannisgasse 26
04109 Leipzig (Allemagne)
Hans-Bert.Rademacher@math.uni-leipzig.de