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SMALL DIVISORS AND LARGE MULTIPLIERS

by Boele BRAAKSMA & Laurent STOLOVITCH

Abstract. — We study germs of singular holomorphic vector fields at the origin of $\mathbb{C}^n$ of which the linear part is 1-resonant and which have a polynomial normal form. The formal normalizing diffeomorphism is usually divergent at the origin but there exists holomorphic diffeomorphisms in some "sectorial domains" which transform these vector fields into their normal form. In this article, we study the interplay between the small divisors phenomenon and the Gevrey character of the sectorial normalizing diffeomorphisms. We show that the Gevrey order of the latter is linked to the diophantine type of the small divisors.

Résumé. — Nous étudions des germes de champs de vecteurs holomorphes singuliers à l'origine de $\mathbb{C}^n$ dont la partie linéaire est 1-résonante et qui admettent une forme normale polynomiale. En général, bien que le difféomorphisme formel normalisant soit divergent à l'origine, il existe néanmoins des difféomorphismes holomorphes dans des "domaines sectoriels" qui les transforment en leur forme normale. Dans cet article, nous étudions la relation qui existe entre le phénomène de petits diviseurs et le caractère Gevrey de ces difféomorphismes sectoriels normalisants. Nous montrons que l'ordre Gevrey de ce dernier est relié au type diophantien des petits diviseurs.

1. Introduction

In this article, we are concerned with the study of some germs of holomorphic vector fields in a neighborhood of a fixed point. More precisely, we shall consider holomorphic non-linear perturbations

$$X = \sum_{i=1}^{n} (\lambda_i x_i + f_i(x))\, \partial/\partial x_i$$

of the diagonal linear vector field $s = \sum_{i=1}^{n} \lambda_i x_i \partial/\partial x_i$, where $n \geqslant 2$. Hence, the functions f_i vanish as well as their first derivatives at the origin. Two

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such germs of vector fields X_1, X_2 are holomorphically conjugate (resp. equivalent) if there exists a germ of biholomorphism Φ of $(\mathbb{C}^n, 0)$ which conjugates them (resp. up to the muliplication by a holomorphic unit): $\Phi_* X_1(y) := D\Phi(\Phi^{-1}(y))X_1(\Phi^{-1}(y)) = X_2(y)$. It is well known (see [1] for instance) that such a vector field is formally conjugate (*i.e.* by means of a formal diffeomorphism named *normalizing diffeomorphism*) to a *normal form* $\widehat{X}_{\mathrm{norm}}$, that is a formal vector field which commutes with the linear part s. In coordinates, we have

$$\widehat{X}_{\mathrm{norm}} = \sum_{i=1}^{n} \left(\lambda_i y_i + \sum_{\substack{(Q,\lambda)=\lambda_i \\ Q \in \mathbb{N}_2^n}} a_Q^i y^Q \right) \frac{\partial}{\partial y_i}$$

the sum being taken over the multiindices $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ such that $|Q| := q_1 + \cdots + q_n \geqslant 2$ (we shall write $Q \in \mathbb{N}_2^n$) and which satisfy a *resonance relation* $(Q, \lambda) := q_1\lambda_1 + \cdots + q_n\lambda_n = \lambda_i$. The a_Q^i's are complex numbers.

If there exists a monomial x^r which is a first integral of the linear part s (*i.e.* $s(x^r) = 0$) but which is not a first integral of a normal form then the formal normalizing diffeomorphism is generally a (vector of) divergent power series (see [4, 14]). We are interested in this situation. More precisely, Ichikawa [9] has shown that, if s is 1-*resonant* (*i.e.* the formal non-linear centralizer of s is generated by the sole relation $(r, \lambda) = 0$ for a nonzero $r \in \mathbb{N}^n$), then X has only a finite number of formal invariants if and only if y^r is not a first integral of a normal form (and thus of any normal form); this is Ichikawa's **condition** (I). This means that X has a polynomial normal form.

The analytic classification of these objects in dimension two is due to J. Martinet and J.-P. Ramis in two seminal articles [15, 16]. They showed that the divergent normalizing diffeomorphism is in fact *summable* in some sectorial domain. This means that there exist germs of a holomorphic diffeomorphism in some *large sectorial domain* (with vertex at the origin) having the formal diffeomorphism as asymptotic expansion in the domain and conjugating the vector field to its polynomial normal form. The counterpart for one dimensional diffeomophisms is due to Écalle-Voronin-Malgrange [26, 12]. This study was continued in a more general setting (in particular in higher dimension) by J. Écalle by a completely different manner in the article [7]. The second author has given a unified approach of the two articles of Martinet and Ramis while treating the n-dimensional case [24]. In this situation, he also proved that the formal normalizing diffeomorphism could be realized as the asymptotic expansion of some genuine holomorphic diffeomorphism in some sectorial domain which normalizes the vector field (in the case of one zero eigenvalue, in dimension 3 and without small divisors, the result is due to Canille Martins [5]). Due to the presence of small divisors, the summability property of the formal power series does not hold as we already noticed in [24] (there is no small divisor in the two-dimensional problem). But, we only had a qualitative approach of this phenomenon. In his article [7], J. Écalle gave some statements (see propositions 9.1–9.4, p.136–137) in a more general setting with *"preuves succintes"*. One of them is (we refer to his article for the definitions, $\tilde{g}(z)$ is a normalizing series):

> *"Théorème 9.1 - Les séries formelles $\tilde{g}(z)$ associés à l'objet local X ou f sont toujours de classe Gevrey $1+\delta^{int}+0$ mais pas inférieur en général."*

In this article, we shall **quantify the interplay** between the **Gevrey property** (this is due to the "grands multiplicateurs" that Poincaré mentions in [17, p. 392]) of the normalizing tranformations and the **small divisors phenomenon** (compare with Theorem 4.3).

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2. Normal form of 1-resonant vector fields

Let $s = \sum_{i=1}^{n} \lambda_i x_i \partial/\partial x_i$ be a linear diagonal vector field.

Definition 2.1. — *We shall say that s is 1-resonant if there is a monomial x^r where $r \in \mathbb{N}_1^n$ such that if $i \in \{1, \ldots, n\}$, then for all monomials x^Q such that $[s, x^Q \partial/\partial x_i] = 0$, we have $x^Q = (x^r)^l x_i$ for some nonnegative integer l.*

The resonance monomial x^r generates the ring

$$\widehat{\mathcal{O}}_n^s = \{f \in \mathbb{C}[[x_1, \ldots, x_n]] \mid \mathcal{L}_s(f) = 0\}$$

of first integrals. We shall assume that s is diophantine in the sense that it satisfies the Brjuno small divisors condition

$$(\omega) \qquad -\sum_{k\geqslant 0}\frac{\ln(\omega_{k+1})}{2^k}<+\infty,$$

where

$$\omega_k=\inf\big\{|(Q,\lambda)-\lambda_i| \mid |(Q,\lambda)-\lambda_i|\neq 0,\\ i=1,\dots,n,\ Q\in\mathbb{N}^n,\ 2\leqslant|Q|\leqslant 2^k\big\}.$$

Without any loss of generality, we assume that $r_i\neq 0$ if $1\leqslant i\leqslant p$ and $r_{p+1}=\cdots=r_n=0$ where $r=(r_1,\dots,r_n)$. We shall set $p=n$ if none of the r_i's vanish. Let us define the map $\pi\colon\mathbb{C}^n\to\mathbb{C}$ defined by $\pi(x)=x^r$. Let D_π (resp. $\widehat{D}_\pi$) be the group of germs of biholomorphisms (resp. formal diffeomorphisms) at the origin of $\mathbb{C}^n$, fixing the origin and leaving the map π invariant (*i.e.* $\pi\circ\Phi=\pi$).

Let $k\geqslant 1$ be an integer and let us define the space $\mathcal{E}_k$ (resp. $\widehat{\mathcal{E}}_k$) of germs of 1-resonant holomorphic (resp. 1-resonant formal) vector fields in a neighborhood of the origin $0\in\mathbb{C}^n$ of the form:

$$X=\sum_{i=1}^n x_i(\lambda_i+P_{i,k}(x^r))\frac{\partial}{\partial x_i}+\sum_{i=1}^p x_i\left(x^r\right)^k f_i(x)\frac{\partial}{\partial x_i}\\ +\sum_{i=p+1}^n\left(x^r\right)^{k+1}g_i(x)\frac{\partial}{\partial x_i}$$

where x^r is the resonance monomial and $P_{i,k}$'s are polynomials in the variable u, vanishing at zero and of degree at most k such that $\sum_{i=1}^p r_iP_{i,k}(u)=\beta u^k$ with $\beta\in\mathbb{C}^*$. The f_i's and g_i's are germs of holomorphic functions (resp. formal power series) in a neighborhood of 0 such that $\sum_{i=1}^p r_if_i(x)=0$. We shall say that elements of $\mathcal{E}_k$ (resp. $\widehat{\mathcal{E}}_k$) are *well prepared* vector fields. We recall that a vector field X is holomorphically equivalent to Y if it is holomorphically conjugate to Y up to multiplication by a unit of $\mathcal{O}_n$.

Proposition 2.2 ([24], Proposition 3.2.1). — *Any germ of a 1-resonant vector field satisfying the Ichikawa transversality condition (I) and (ω) is holomorphically equivalent to a well prepared germ. This means that there exists an integer k such that X is equivalent to an element of $\mathcal{E}_k$.*

Proposition 2.3 ([24], Proposition 3.2.2). — *Let $\widehat{X}\in\widehat{\mathcal{E}}_k$ be a well prepared formal vector field of the form*

$$\widehat{X}=\sum_{i=1}^n x_i(\lambda_i+P_{i,k}(x^r))\frac{\partial}{\partial x_i}+\sum_{i=1}^p x_i\left(x^r\right)^k f_i(x)\frac{\partial}{\partial x_i}\\ +\sum_{i=p+1}^n\left(x^r\right)^{k+1}g_i(x)\frac{\partial}{\partial x_i}.$$

Then there exists a unique formal diffeomorphism $\hat{\phi} \in \widehat{D}_\pi$ *tangent to the identity at zero such that*

$$\hat{\phi}_* \widehat{X} = \sum_{i=1}^{n} y_i(\lambda_i + P_{i,k}(y^r)) \frac{\partial}{\partial y_i}.$$

Let $\alpha \in \mathbb{C}^n$ such that $(r, \alpha) \neq 0$. Let $\mathcal{E}_{k,\lambda,\alpha} \subset \mathcal{E}_k$ be the set of germs of well prepared holomorphic vector fields at the origin of the form:

$$\sum_{i=1}^{n} \left(x_i(\lambda_i + \alpha_i(x^r)^k) + (x^r)^k f_i(x)\right) \frac{\partial}{\partial x_i}, \tag{2.1}$$

with $\sum_{i=1}^{p} r_i\alpha_i =: \beta \neq 0$ and $f_i(x) = x_i\tilde{f}_i(x)$, $1 \leqslant i \leqslant p$, where the $\tilde{f}_i$'s are germs of holomorphic functions in a neighborhood of $0 \in \mathbb{C}^n$. Moreover, they satisfy $\sum_{i=1}^{p} r_i\tilde{f}_i(x) = 0$. Let

$$X_{k,\lambda,\alpha} = \sum_{i=1}^{n} x_i \left(\lambda_i + \alpha_i(x^r)^k\right) \frac{\partial}{\partial x_i}$$

be the normal form of such a vector field.

Let us make the following assumptions:

- (H_1') all the eigenvalues λ_i have a nonnegative imaginary part and if zero is not an eigenvalue then there are at least two real eigenvalues.
- (H_2') there exists $\lambda_{i_0} \in \mathbb{R}^*$ such that
$$\min_{i \neq i_0} \mathrm{Re} \left(\frac{\alpha_i}{\beta} - \frac{\lambda_i}{\lambda_{i_0}}\frac{\alpha_{i_0}}{\beta}\right) > 0,$$
where the minimum is taken over all indices $i \neq i_0$ such that $\lambda_i \in \mathbb{R}$.
- (H_3') if λ_i is not real then $f_i = x_i\tilde{f}_i$ where $\tilde{f}_i$ is a germ of a holomorphic function at the origin.

We shall call *sectorial domain* a domain of $\mathbb{C}^n$ of the form:

$$DS_j(\rho, R) = \Big\{y \in \mathbb{C}^n \mid \left|\arg y^r - \frac{1}{k}\pi(j + \frac{1}{2})\right| < \frac{\pi}{k} - \epsilon, \\ 0 < |y^r| < \rho,\ |y_i| < R \text{ for } i = 1, \dots, n\Big\}$$

where ρ, $R > 0$, $0 < \epsilon < \pi/k$ and $0 \leqslant j \leqslant 2k - 1$ is an integer.

Stolovitch has shown the following result

Theorem 2.4 (Sectorial normalization; [24], Théorème 3.3.1). — *Let* $\epsilon < \pi/k$ *be a positive number and let* X *belong to* $\mathcal{E}_{k,\lambda,\alpha}$. *Under assumptions* (H_1'), (H_2') *and* (H_3'), *for any even integer* $0 \leqslant j \leqslant 2k - 1$ *there exists a local change of coordinates* $x_i = y_i + \phi_i^j(y)$, $i = 1, \dots, n$, *tangent* *at the identity, holomorphic in the sectorial domain $DS_j(\rho, R)$ with ρ, R sufficiently small, in which the vector field* (2.1) *can be written as*

$$X = \sum_{i=1}^{n} y_i \left(\lambda_i + \alpha_i (y^r)^k\right) \frac{\partial}{\partial y_i}. \tag{2.2}$$

This change of coordinates preserves the function x^r. Each function ϕ_i^j admits the formal power series $\hat{\phi}_i$ as asymptotic expansion in y^r in the sense of Gérard-Sibuya in the domain $DS_j(\rho, R)$. Here, $x_i = y_i + \hat{\phi}_i(y)$, $i = 1, \ldots, n$, is the unique formal coordinate system in which the vector field X is in its normal form (2.2)*. Moreover, if all the eigenvalues are real then the result holds also for j odd.*

We refer to definition 3.2 in the next section for the notion of asymptotic expansion in the sense of Gérard-Sibuya. The proof of this theorem reduces to the proof of the sectorial linearization of the non-linear system with an irregular singularity at the origin

$$\beta z^{k+1} \frac{dx_i}{dz} = x_i(\lambda_i + \alpha_i z^k) + z^k f_i(x), \; i = 1, \ldots, n. \tag{2.3}$$

By sectorial linearization, we mean that there is a change of coordinates $x_i = y_i + g_i(z, y)$, $i = 1, \ldots, n$, holomorphic in $S \times P$, where S is a sector in $\mathbb{C}$ with vertex at 0 (variable z) and P a polydisc centered at $0 \in \mathbb{C}^n$ (variables y) in which the system can be written as:

$$\beta z^{k+1} \frac{dy_i}{dz} = y_i(\lambda_i + \alpha_i z^k), \; i = 1, \ldots, n. \tag{2.4}$$

In [24]) it is shown that the function $\phi_i(y)$ is nothing but $g_i(y^r, y)$. Moreover, the g_i's have an expansion at the origin of the form

$$g_i(z, y) = \sum_{Q \in \mathbb{N}_2^n} g_{i,Q}(z) y^Q \tag{2.5}$$

where the $g_{i,Q}$'s are holomorphic functions in S.

3. Gevrey functions and summability

Here we recall some definitions of Gevrey asymptotics and summability which will be used further on (for more details see [13, 2, 20, 19]).

Definition 3.1. — *A holomorphic function f in an open bounded sector S with vertex 0 in $\mathbb{C}$ is said to to admit an asymptotic expansion of Gevrey order $s > 0$ if there exists a formal power series $\hat{f} = \sum_{j=0}^{\infty} f_j z^j$*

such that for every compact subsector S' of $S \cup \{0\}$ there exist positive constants A and C such that for all $z \in S'$ and $N \in \mathbb{N}$

$$\Big| f(z) - \sum_{j=0}^{N-1} f_j z^j \Big| \leqslant CA^N \Gamma(1+Ns)|z|^N,$$

where $\Gamma(x)$ is the Gamma-function. Such a function f will be called an s-Gevrey function on S or shortly f is s-Gevrey on S.

Equivalently: A holomorphic function f on S is s-Gevrey on S if all derivatives of f are continuous at 0 and if S' is as above then there exist positive constants A and C such that for all $N \in \mathbb{N}$ and all $z \in S'$:

$$\frac{1}{N!}\Big|\frac{\partial^N f(z)}{\partial z^N}\Big| \leqslant CA^N \Gamma(1+Ns).$$

Definition 3.2 ([15, 8]). — *Let $\hat{f} = \sum_{Q\in\mathbb{N}^n} \hat{f}_Q(z)x^Q \in \mathbb{C}[[z, x_1, \dots, x_n]]$ be a formal power series. We shall say that an analytic function f on $S \setminus \{0\} \times \Delta$ ($S \subset \mathbb{C}$ is an open sector and $\Delta \subset \mathbb{C}^n$ an open polydisc centered at $0 \in \mathbb{C}^n$), $f(z,x) = \sum_{Q\in\mathbb{N}^n} f_Q(z)x^Q$ admits $\hat{f}$ as* **asymptotic expansion in the sense of Gérard-Sibuya** *in $S \setminus \{0\} \times \Delta$, if each function $f_Q(z)$ admits $\hat{f}_Q(z)$ as an asymptotic expansion in the sense of Poincaré, in the sector S and for every compact subsector S' of $S \cup \{0\}$, every compact subset Δ' of Δ and every $N \in \mathbb{N}_1$ there exists a constant K such that*

$$\Big| f(z,x) - \sum_{|Q|<N} f_Q(z)x^Q \Big| \leqslant K|x|^N \text{ for all } (z,x) \in S' \times \Delta'.$$

Definition 3.3. — *If $k > 0$ and f is $1/k$-Gevrey in a sector S with opening $> \pi/k$ then $\hat{f}$ is k-summable in the direction of the bisector of S and its k-sum on S is f. In this case f is uniquely determined by $\hat{f}$ and we say that f is a k-sum on S.*

The notion of summability is due to Borel and generalized by Ramis (*cf.* [18]).

Suppose that f is a holomorphic function on $S \times P_n(0,r)$ (where S is as above and $P_n(0,r)$ is the open polydisc in $\mathbb{C}^n$ with center 0 and radius r). Then $f(z,x)$ *is said to be s-Gevrey in z on S uniformly in x on $P_n(0,r')$* for some $r' \in (0,r)$ if there exists a formal power series $\hat{f}(z,x) = \sum_{j=0}^{\infty} f_j(x)z^j$ where the coefficients $f_j(x)$ are holomorphic on $P_n(0,r')$ such that for every compact subsector S' of $S \cup \{0\}$ there exist positive constants A and C such that for all $z \in S'$, all $x \in P_n(0,r')$ and all $N \in \mathbb{N}$:

$$\left| f(z,x) - \sum_{j=0}^{N-1} f_j(x) z^j \right| \leqslant CA^N \Gamma(1+Ns)|z|^N. \tag{3.1}$$

The latter condition is equivalent to the existence of A, C, r' as above such that for all $z \in S'$, all $x \in P_n(0,r')$ and all $N \in \mathbb{N}$:

$$\frac{1}{N!}\left|\frac{\partial^N f(z,x)}{\partial z^N}\right| \leqslant CA^N \Gamma(1+Ns).$$

If the opening of S is $> \pi s$ then $\hat{f}(z,x)$ is $1/s$-summable in z on S uniformly on $P_n(0,r')$ (*cf.* [22]).

Remark 3.4. — If $f(z,x)$ is s-Gevrey in z on S uniformly in x on $P_n(0,r')$ then $f(z,x)$ is also s-Gevrey in (z,x) on $S \times P_n(0,r'')$ for some $r'' \in (0,r')$ in the sense that for every S' as above there exist positive constants A and C such that for all $N \in \mathbb{N}$ and all $(z,x) \in S' \times P_n(0,r'')$:

$$\left| f(z,x) - f^{(N-1)}(z,x) \right| \leqslant CA^N \Gamma(1+Ns) \|(z,x)\|^N. \tag{3.2}$$

Here $f^{(N-1)}(z,x)$ denotes the $(N-1)$-jet of f in $0 \in \mathbb{C}^{n+1}$ and $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{C}^{n+1}$.

Proof. — We have $f_j(x) = 1/j!\ \partial^j/\partial z^j\ f(0,x)$ and therefore $|f_j(x)| \leqslant CA^j\Gamma(1+js)$ for $x \in P_n(0,r')$. From this and Cauchy's formula it follows that for sufficiently small $\delta > 0$ and all $Q \in \mathbb{N}^n$:

$$\frac{1}{Q!}\left|\frac{\partial^Q}{\partial x^Q} f_j(0)\right| \leqslant CA^j \Gamma(1+js)\delta^{-|Q|}.$$

Let

$$\widetilde{R}_N(z,x) = \sum_{j=0}^{N-1} \sum_{|Q| \geqslant N-j} \frac{1}{Q!}\frac{\partial^Q}{\partial x^Q} f_j(0) z^j x^Q.$$

Then it follows from (3.1) that

$$|f(z,x) - f^{(N-1)}(z,x) - \widetilde{R}_N(z,x)| \leqslant CA^N\Gamma(1+Ns)|z|^N \tag{3.3}$$

and

$$|\widetilde{R}_N(z,x)| \leqslant C \sum_{j=0}^{N-1} \sum_{|Q| \geqslant N-j} A^j \Gamma(1+js)\delta^{-|Q|}|z^j x^Q|.$$

Using

$$\sharp\{Q \in \mathbb{N}^n \colon |Q| = m\} = \binom{n+m-1}{h} \leqslant 2^{n+m-1} \tag{3.4}$$

we obtain for $|x| \leqslant \delta/4$ and $\delta \leqslant 2/A$:

$$|\widetilde{R}_N(z,x)| \leqslant C2^{n-1}\sum_{j=0}^{N-1}(A|z|)^j\Gamma(1+js)\Big(\frac{2|x|}{\delta}\Big)^{N-j}\Big(1-2|x|\delta\Big)^{-1}$$
$$\leqslant C2^{n+N}\delta^{-N}\Gamma(1+(N-1)s)\sum_{j=0}^{N-1}|z|^j|x|^{N-j}.$$

From this and (3.3) the assertion follows. □

Let $r \in \mathbb{N}^n$ a nonzero multiindex and let $\rho\colon \mathbb{C}^n \to \mathbb{C}^{n+1}$ be the map defined to be $\rho(x) = (x^r, x)$.

Definition 3.5 ([16], p. 6). — *Let $f \in \mathbb{C}[[x]]$ be a formal power series in $\mathbb{C}^n$ (resp. smooth function in some domain). We shall say that f is k-summable (resp. a Gevrey function) in the monomial x^r if $f \in \rho^*\mathbb{C}\{x\}\{z\}_k$, that is f is the pull-back of a formal series (resp. smooth function) $g(z,x)$ which is k-summable (resp. a Gevrey function) in the variable z in some sector, uniformly in x on a polydisk.*

Let $k > 0$ and let d denote a direction in the complex plane. We define the **Borel transform of order** k in the direction d as

$$\mathcal{B}_k f(t) := \frac{1}{2i\pi}\int_{\gamma_k} f(z)e^{(t/z)^k}d(z^{-k}).$$

Here we assume that f is holomorphic in a sector $S = \{z \in \mathbb{C}^*\colon |z| < \rho$, $|d - \arg z| < \alpha\}$ where $\rho > 0$ and $\alpha > \pi/(2k)$ and γ_k is the path from 0 along the ray $\arg z = d - \alpha_1$ till $|z| = \rho_1$ then along the circle $|z| = \rho_1$ to the ray $\arg z = d + \alpha_1$ and then back to the origin along this ray. Here $0 < \rho_1 < \rho$ and $\pi/(2k) < \alpha_1 < \alpha$. If $\hat{f} = \sum_{n=1}^{\infty} f_n z^n$ is a formal power series, then the formal Borel transform of order k is defined as the power series

$$\mathcal{B}_k\hat{f}(t) = \sum_{n=1}^{\infty}\frac{f_n}{\Gamma(n/k)}t^{n-k},$$

where $\Gamma(x)$ is the Gamma-function. We define the *Laplace transform of order* k in the direction d (inverse Borel transform) as

$$\mathcal{L}_k\tilde{f}(z) := \int_0^{\infty:d}\tilde{f}(t)e^{-(t/z)^k}d(t^k).$$

An equivalent definition for k-summability in a direction d is:

Definition 3.6 ([18, 20]). — *A formal power series $\hat{f}$ at the origin of the complex plane will be said to be k-summable in the direction d if its formal k-Borel transform defines a holomorphic function in a neighborhood* *of the origin which can be continued holomorphically in some small sector bisected by the direction d and is of exponential growth of order at most k at infinity. In this case, the function $\mathcal{L}_k \circ \mathcal{B}_k \hat{f}$ is holomorphic in a large sector bisected by d and of opening $> \pi/k$. Moreover, this is the unique function which admits $\hat{f}$ as asymptotic expansion in this sector.*

Another equivalent definition has been given by Tougeron as follows: Let η, R be positive numbers and let $k > 1/2$. Let $\mathcal{S}(\eta, R)$ denote the sector

$$\mathcal{S}_{d,\pi/k}(\eta, R) := \left\{ z \in \mathbb{C}^* \mid |\arg z - d| < \frac{\pi}{2k} + \eta, 0 < |z| < R \right\}.$$

Let $\theta > 0$. Let us define the *sectorial neighborhood of the origin* of order q to be

$$\mathcal{S}_{q,d,\pi/k,\theta}(\eta, R) := \left\{ z \in \mathbb{C} \mid |z| < \frac{R}{(q+1)^{\theta}} \right\} \cup \mathcal{S}_{d,\pi/k}(\eta, R).$$

Theorem 3.7 (Tougeron's definition of summability, [25]). — *A function f is a k-sum in the direction d if and only if it has a representation as a sum $\sum_{q=0}^{+\infty} f_q$ of functions f_q, each of which is holomorphic in the sectorial neighborhood $\mathcal{S}_{d,\pi/k,1/k}(\eta, R)$ and satisfies*

$$\|f_q\|_{\mathcal{S}_{q,d,\pi/k,1/k}(\eta,R)} := \sup_{z \in \mathcal{S}_{q,d,\pi/k,1/k}(\eta,R)} |f_q(z)| \leqslant C\rho^q,$$

for some constants η, R, C, ρ independent of q.

4. Main results

Our first main result shows that a series of functions defines a Gevrey function on a sector if the Borel transforms satisfy good estimates in some well chosen domains.

Let k be a positive integer, α and β real numbers with $\alpha < \beta$. Define for $0 \leqslant \epsilon < (\beta - \alpha)/2, \rho > 0$:

$$S_\epsilon(\rho) = \{ z \in \mathbb{C}^* \mid \alpha + \epsilon \leqslant \arg z \leqslant \beta - \epsilon, |z| \leqslant \rho \}. \tag{4.1}$$

Theorem 4.1. — *Let k, α and β be as above with $\beta - \alpha > \pi/k$. Let $\rho > 0$, $R > 0$. Suppose $g(z, y) = \sum_{Q \in \mathbb{N}^n} g_Q(z) y^Q$ is a scalar-valued holomorphic function in $S_0(\rho) \times \overline{P}_n(0, R)$. Moreover, suppose $(\mathcal{B}_k g_Q)(t)$ exists and is holomorphic in $\overline{\Delta}_m := \overline{P}_1(0, cm^{-\gamma})$ and satisfies*

$$|(\mathcal{B}_k g_Q)(t)| \leqslant K^m \text{ in } \overline{\Delta}_m, \text{ if } m = |Q| \geqslant 1, \tag{4.2}$$

where $\gamma \geqslant 0$ and c and K are positive constants.

Then for all $\epsilon \in (0, (\beta-\alpha)/2)$ *the function* $g(z, y)$ *is a Gevrey function of order* $\gamma + 1/k$ *in* z *in* $S_\epsilon(\rho')$ *uniformly in* y *in* $P_n(0, R')$ *for some* $\rho' \in (0, \rho)$ *and* $R' \in (0, R)$ *both depending on* ϵ*. Moreover, if* $0 \leqslant \gamma < (\beta-\alpha)/\pi - 1/k$ *then* $g(z, y)$ *is a* $k/(k\gamma + 1)$*-sum.*

Definition 4.2. — *We shall say that the linear part* $s = \sum_{i=1}^n \lambda_i x_i \partial/\partial x_i$ *is diophantine of the type* $\gamma \geqslant 0$ *if there exists* $c > 0$ *such that, for all* $Q \in \mathbb{N}_2^n$ *and for all* $1 \leqslant i \leqslant n$ *then*

$$|(Q, \lambda) - \lambda_i| > \frac{c}{|Q|^\gamma} \text{ unless } (Q, \lambda) - \lambda_i = 0. \tag{4.3}$$

Our second main result gives the Gevrey property of a sectorial normalizing tranformation of a well prepared vector field.

Theorem 4.3. — *If the linear part of (2.1) is diophantine of type* $\gamma \geqslant 0$ *and the assumptions of Theorem 2.4 are satisfied, then the* **sectorial normalizing biholomorphisms** *defined by Theorem 2.4 are* **Gevrey functions of order** $(1+\gamma)/k$ *in the resonance monomial* x^r*. Moreover, if* $\gamma = 0$ *then the formal normalizing transformation is* k*-summable.*

Remark 4.4 (Important remark). — Using Theorem 4.1, we could show that if $0 \leqslant \gamma < 1$, then the formal normalizing transformation is $k/(\gamma+1)$-summable. Nevertheless, we should emphasize that there is no $\lambda \in \mathbb{C}^n$ which satisfies (4.3) with $0 \neq \gamma < 1$. This was pointed out by Yann Bugeaud who refers to [21]. However, for a fixed non zero $\lambda \in \mathbb{C}^n$, there are infinite sequences of multiindexes $\{Q_m\}$ such that $|(Q_m, \lambda) - \lambda_i| > c/|Q_m|^\gamma$ with $0 \neq \gamma < 1$ unless $(Q_m, \lambda) - \lambda_i = 0$. Hence, if it happens that, in our normalization process, the sole monomials that appear in the Taylor expansions of our objects belong to such a sequence, then we will obtain the claimed summability property.

It is a remarkable fact that we obtain the summability property even when some singularities accumulate at the origin in the Borel plane. This is due to the slow rate ($\gamma < 1$) at which this accumulation occurs. The fact that there are no λ which satisfies (4.3) with $\gamma < 1$ has nothing to do with this phenomenon. It is just an arithmetic property.

In fact, we show that $\phi_i(y) = g_i(y^r, y)$ where $g_i(z, y)$, given by (2.5), is shown to be a Gevrey function in z in some sector at the origin of $\mathbb{C}$, uniformly in y in a polydisk centered at the origin in $\mathbb{C}^n$.

As far as we know, it is the first time that such an interplay between the rate of accumulation of small divisors at zero and the Gevrey character of the normalizing transformation is characterized. In other situations, such an interplay seems to be guessed (see for instance [23, 11, 10]).

We can show that the formal k-Borel transform of $g_{i,Q}(z)$ has no singularity in the disc centered at the origin and of radius $r < \inf_{|P|\leqslant|Q|}\{|(P,\lambda)-\lambda_i| \neq 0\}$. This is the main reason for which the g_i's are k-sums when there are no small divisors (*i.e.* there exists $c > 0$ such that for all $P \in \mathbb{N}_2^n$, $|(P,\lambda)-\lambda_i| > c$ if the number on the left hand side is not zero). In fact, the $\mathcal{B}_k g_{i,Q}$'s are holomorphic on the *same* disc centered at the origin and of radius $c/2$ (it is easy to show that the $g_{i,Q}$ have asymptotic expansions $\hat{g}_{i,Q}$ which are $1/k$-Gevrey power series). In particular, this is the case in dimenson 2 [15, 16]. Our main result will quantify the interplay between the small divisor phenomenon and the Gevrey character of the normalizing transformation.

As in the proof of Theorem 2.4 [24, p. 132–135], the main theorem reduces to the proof of the Gevrey character (in the variable z, uniformly in the variables x) of the linearizing transformation of the associated non-linear system (2.3) with an irregular singularity at the origin. The proof of this fact relies on Theorem 4.1 the proof of which is postponed to the end of the article.

5. Proof of Theorem 4.3

In the same way as the proof of Theorem 2.4 reduces to the proof of the sectorial linearization of the non-linear system (2.3), the proof of Theorem 4.3 reduces to the proof of the Gevrey character of the sectorial linearization of (2.3) (see section 5.4). It is sufficient to consider the case $\beta = 1$ in (2.3). We will consider a little bit more generally:

$$z^{k+1}\frac{dx}{dz} = (\Lambda + z^k A)x + z^k f(z,x), \tag{5.1}$$

where $\Lambda = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\}, A = \mathrm{diag}\{\alpha_1,\dots,\alpha_n\}$ and

$$f(z,x) = \sum_{j\in\mathbb{N}_2^n} f_j(z)x^j,$$

a convergent series for $|x| \leqslant \rho_1, |z| \leqslant \rho_2$ and the coefficients $f_j(z)$ are $\mathbb{C}^n$-valued holomorphic functions for $|z| \leqslant \rho_2$.

We introduce the following hypotheses:

(H_1) the eigenvalues λ_i are not all 0 and all have a nonnegative imaginary part,

(H_2) if there are real eigenvalues λ_i then either these are all 0 and then $\Re\alpha_i > 0$ or there exists $\lambda_{i_0} \in \mathbb{R}^*$ such that

$$\min_{i\neq i_0} \mathrm{Re}\left(\alpha_i - \frac{\lambda_i}{\lambda_{i_0}}\alpha_{i_0}\right) > 0,$$

where the minimum is taken over all indices $i \neq i_0$ such that $\lambda_i \in \mathbb{R}$.

(H_3) if there exists a resonance relation $(Q,\lambda) = \lambda_i$ for some $Q \in \mathbb{N}_2^n$ then $\alpha_i - (Q,\alpha) \notin \mathbb{N}$.

(H_4) if λ_i is not real then f_i/x_i is holomorphic in a neighborhood of the origin.

(H_5) there exists $r = (r_1,\dots,r_n) \in \mathbb{N}_1^n$ such that if $\mathcal{I} = \{1 \leqslant i \leqslant n \mid r_i \neq 0\}$ then

- $\forall i \in \mathcal{I}$, $f_i = x_i\tilde{f}_i$ and $\tilde{f}_i$ is holomorphic in a neighborhood of the origin.
- $\sum_{i\in\mathcal{I}} r_i\tilde{f}_i = 0$.

Let

$$S_j(\epsilon,\rho) = \left\{z \in \mathbb{C}^* \mid \left|\arg z - \frac{\pi}{k}\big(j+\frac{1}{2}\big)\right| \leqslant \frac{\pi}{k} - \epsilon, |z| \leqslant \rho\right\} \tag{5.2}$$

where $\rho > 0$, $j = 0,\dots,2k-1$ and $0 < \epsilon < \pi/k$ fixed.

Then

Theorem 5.1. — *Assume that hypotheses (H_1), (H_2), (H_3) and (H_4) are satisfied and that Λ is diophantine of type $\gamma \geqslant 0$. Then for even j with $0 \leqslant j \leqslant 2k-2$ there exists a unique change of variables $x = y + g^j(z,y)$ with $g^j(z,0) = 0, D_y g^j(z,0) = 0$ which transforms (5.1) into*

$$z^{k+1}\frac{dy}{dz} = (\Lambda + z^k A)y, \tag{5.3}$$

and where $g^j(z,y)$ is a Gevrey function of order $(1+\gamma)/k$ in z in $S_j(\epsilon,\rho)$ uniformly for $y \in \overline{P}_n(0,R)$ for all ϵ in $(0,\pi/(2k))$ and positive numbers ρ and R depending on ϵ.

If $\gamma = 0$ then $g^j(z,y)$ is k-sum of $\hat{g}(z,y)$ in the direction $\pi/k(j+1/2)$ where $\hat{g}(z,y)$ is the asymptotic expansion in z of $g(z,y)$.

If all λ_h are real then these statements also hold for odd j with $1 \leqslant j \leqslant 2k-1$ and direction $\pi/2$ replaced by $-\pi/2$.

If $\gamma = 0$ then $g(z,y)$ is Borel-sum of $\hat{g}(z,y)$ in the

If condition (H_5) is satisfied then $x^r = y^r$.

Except for the Gevrey property this is Theorem 2.7.1 in [24]. However, the Gevrey property follows from Theorem 4.1 once condition (4.2) has been verified and then Theorem 5.1 follows. For simplicity, we shall verify this condition first in the case $k = 1$. We shall indicate at the end of the section the changes in the proof in case $k > 1$. Moreover, it is sufficient to prove the theorem only for the case $j = 0$ since the other cases may be obtained from this by a rotation of the independent variable z.

In the next two subsections we derive properties of and estimates for the Borel transform of the coefficients $g_Q(z)$ in the expansion (2.5) on $S_0 \times \overline{P}_n(0, R)$. Here g_Q is holomorphic on $S_0 := S_0(\epsilon, \rho)$ (*cf.* (5.2)).

5.1. Properties of the Borel transform of g_Q in case $k = 1$

The function g satisfies

$$z^2 \frac{d}{dz} g(z, y(z)) = (\Lambda + zA) g(z, y(z)) + z f(z, y + g(z, y(z))) \tag{5.4}$$

together with

$$z^2 \frac{dy(z)}{dz} = (\Lambda + zA) y(z).$$

From the series expansion for $g(z, y)$ and Cauchy's inequality it follows that there exists a positive constant M such that

$$|g_Q(z)| \leqslant M R^{-|Q|}, \; z \in S_0 = S_0(\epsilon, \rho). \tag{5.5}$$

Let $g_{\mathbf{e}_l} := \mathbf{e}_l$ for $l = 1, \ldots, n$, where $\mathbf{e}_l := (\delta_{i,l})_{1 \leqslant i \leqslant n}$ with $\delta_{i,l} = 0$ if $i \neq l$ and 1 otherwise. Then

$$f(z, y + g(z, y)) = \sum_{j \in \mathbb{N}_2^n} f_j(z) \Big(\sum_{Q \in \mathbb{N}_1^n} g_Q(z) y^Q \Big)^j$$

and so

$$f(z, y + g(z, y)) = \sum_{Q \in \mathbb{N}_2^n} y^Q t_Q(z) \tag{5.6}$$

where

$$t_Q(z) := \sum_{2 \leqslant |j| \leqslant |Q|} f_j(z) \Sigma' \prod_{l=1}^{n} \prod_{q=1}^{j_l} (g_{i_{l,q}}(z))_l, \; z \in S_0. \tag{5.7}$$

Here $(v)_l$ denotes the lth component of a vector $v \in \mathbb{C}^n$, and $\sum'$ denotes that the sum has to be taken over all $i_{l,q} \in \mathbb{N}_1^n$ such that $\sum_{l=1}^{n} \sum_{q=1}^{j_l} i_{l,q} = Q$. So $1 \leqslant |i_{l,q}| \leqslant |Q| - |j| + 1 \leqslant |Q| - 1$ and $t_Q = 0$ if $|Q| \leqslant 1$. From this and (5.4) it follows that for $Q \in \mathbb{N}_2^n$:

$$z^2 g_Q'(z) + (\lambda_Q + z\alpha_Q) g_Q(z) = z t_Q(z), \tag{5.8}$$

where $\lambda_Q := \big(\lambda, Q\big) - \Lambda, \; \alpha_Q := (\alpha, Q) - A, \; \alpha = \big(\alpha_1, \ldots, \alpha_n\big)$.

Let $w_Q := z^{|Q|}g_Q$. Then it follows from (5.8) and (5.7) that

$$z^2 w_Q'(z) + (\lambda_Q + z\beta_Q)w_Q(z) = zu_Q(z), \tag{5.9}$$

where $\beta_Q := \alpha_Q - |Q|\,\mathrm{Id}$ and

$$u_Q(z) := z^{|Q|}t_Q(z) = \sum_{2\leqslant|j|\leqslant|Q|} f_j(z)\Sigma' \prod_{l=1}^{n}\prod_{q=1}^{j_l}(w_{i_{l,q}}(z))_l$$

since in $\sum'$ we have $\sum_{l=1}^{n}\sum_{q=1}^{j_l} i_{l,q} = Q$.

Let $G_Q := \mathcal{B}g_Q$, $W_Q = \mathcal{B}w_Q$ and $U_Q = \mathcal{B}u_Q$. These functions exist in $S' := \{t \in \mathbb{C}^* | 2\epsilon \leqslant \arg t \leqslant \pi - 2\epsilon\}$ and in a neighborhood of the origin since $\hat{g}_Q(z)$ is a series of Gevrey order 1. Moreover, $W_Q(t)$ and $U_Q(t))$ are $O(t^{|Q|-1})$ on S' since $w_Q(z) = O(z^{|Q|})$. Furthermore

$$G_Q(t) = \frac{d^m}{dt^m}W_Q(t) \text{ where } m = |Q|. \tag{5.10}$$

Since $w_{\mathbf{e}_l} = zg_{\mathbf{e}_l} = z\mathbf{e}_l$ and $\mathcal{B}z = 1$ we have $W_{\mathbf{e}_l} = \mathbf{e}_l$ for $l = 1, \dots, n$.

We apply the Borel transform to both sides of (5.9). Since

$$\mathcal{B}\Big(z^2\frac{d}{dz}w_Q(z)\Big)(t) = tW_Q(t),$$

we obtain from (5.9)

$$(t + \lambda_Q)W_Q + \beta_Q * W_Q = 1 * U_Q \tag{5.11}$$

where

$$U_Q = \sum_{2\leqslant|j|\leqslant|Q|}\Big(f_j(0) + (\mathcal{B}f)_j *\Big)\Sigma' \prod_{l=1}^{n} * \prod_{q=1}^{j_l} *(W_{i_{l,q}})_l. \tag{5.12}$$

Here, $\prod *$ denotes the product with respect to the convolution product (see also [6, 3]). Differentiating (5.11) we get

$$(t + \lambda_Q)W_Q' + (1 + \beta_Q)W_Q = U_Q. \tag{5.13}$$

5.2. Proof of condition (4.2) in case $k = 1$

Define for $m \in \mathbb{N}_1$

$$\rho_m := \min\big\{|(P,\lambda) - \lambda_j| \mid P \in \mathbb{N}_1^n,\ |P| \leqslant m,\ j = 1, \dots, n, (P,\lambda) \neq \lambda_j\big\}. \tag{5.14}$$

The diophantine condition (4.3) implies that there exists a positive constant c such that

$$\rho_m \geqslant cm^{-\gamma}. \tag{5.15}$$

First we prove

Lemma 5.2. — *$U_Q(t)$ and $W_Q(t)$ are holomorphic for $|t| < \rho_m$ and have a zero of order at least $m-1$ at 0, where $m = |Q|$.*

Proof. — For $m = 1$ we have $W_{\mathbf{e}_l} = \mathbf{e}_l, U_{\mathbf{e}_l} = 0$. Next suppose that the property holds for some $m \geqslant 1$. Let $P \in \mathbb{N}^n$ with $|P| = m+1$. Since $t^{a-1} * t^{b-1} = B(a,b)t^{a+b-1}$ we may deduce from (5.12) that $U_P(t)$ is holomorphic for $|t| < \rho_m$ and has a zero of order m at 0. If $\lambda_{P,l} \neq 0$ then (5.13) implies that $W_{P,l}$ is holomorphic for $|t| < \rho_{m+1}$ and has a zero of order m at 0, whereas if $\lambda_{P,l} = 0$ then using (H_3) we may construct a formal power series solution of the lth component of (5.13) with a zero of order m at 0 and this series converges for $|t| < \rho_m$ since 0 is a regular singular point of (5.13). □

From hypothesis (H_2) we deduce

Lemma 5.3. — *There exist $m_0 \in \mathbb{N}_1$ and $\delta_0 > 0$ such that for all $l \in \{1, \ldots, n\}$*

$$\Re((\alpha, Q) - \alpha_l) \geqslant \delta_0 \textit{ if } (\lambda, Q) = \lambda_l, |Q| \geqslant m_0. \tag{5.16}$$

Proof. — We may order the λ_j such that $\lambda_j \in \mathbb{R}$ if $j \leqslant q$ and $\Im\lambda_j > 0$ if $j > q$. Let Q' arise from Q by replacing the last $n-q$ components by 0 and let $Q'' := Q - Q'$. From $(Q, \lambda) = \lambda_l$ it follows that $\Im(Q'', \lambda) = \Im\lambda_l$. Since $\min\{\Im\lambda_j | j = q+1, \ldots, n\} > 0$ if $q < n$ we see that there exist positive constants M_1 and M_2 such that

$$|Q''| \leqslant M_1,\ |(Q', \lambda)| = |\lambda_l - (Q'', \lambda)| \leqslant M_2.$$

First suppose that there is an index $i \leqslant q$ such that $\lambda_i \neq 0$. Let δ_1 be the minimum of the lefthand side in hypothesis (H_2). Let $P = Q' - Q_{i_0}\mathbf{e}_{i_0}$, $p_0 = \Re\alpha_{i_0}/\lambda_{i_0}$. From hypothesis (H_2) it follows that $\Re(P, \alpha) \geqslant |P|\delta_1 + (P, \lambda)p_0$ and therefore $\Re(Q', \alpha) \geqslant |P|\delta_1 + (P, \lambda)p_0 + Q_{i_0}\Re\alpha_{i_0} = |P|\delta_1 + (Q', \lambda)p_0$. Hence

$$\Re(Q, \alpha) \geqslant |P|\delta_1 + (Q', \lambda)p_0 + \Re(Q'', \alpha) \geqslant |P|\delta_1 - M_2|p_0| - M_1|\alpha|.$$

Let $M_3 > (M_2|p_0| + (M_1+1)|\alpha|)/\delta_1$. If $|P| < M_3$ then since $Q_{i_0}\lambda_{i_0} = (Q', \lambda) - (P, \lambda)$ we get $|Q_{i_0}| < (M_2 + M_3|\lambda|)/|\lambda_{i_0}| =: M_4$ and so $|Q| < M_3 + M_4 + M_1 =: M_0$. Therefore if $|Q| \geqslant M_0$ then $|P| \geqslant M_3$ and $\Re((Q, \alpha) - \alpha_l) \geqslant M_3\delta_1 - M_2|p_0| - (M_1+1)|\alpha| =: \delta_0 > 0$.

Next suppose that $\lambda_i = 0$ if $i \leqslant q$. Then $\Re\alpha_i > 0$ for $i \leqslant q$. Let $\delta_2 = \min_{i \leqslant q} \Re\alpha_i$. So $\delta_2 > 0$. It follows that $\Re\{(Q', \alpha) - \alpha_l\} \to \infty$ if $|Q'| \to \infty$. Since $|Q''| \leqslant M_1$ we also have $\Re\{(Q, \alpha) - \alpha_l\} \to \infty$ if $|Q| \to \infty$. □

We will give estimates of W_Q in a neighborhood of 0 in terms of

$$R_m(t) := \frac{|t|^{m-1}}{(m-1)!},\ m \in \mathbb{N}_1. \tag{5.17}$$

LEMMA 5.4. — *There exist positive constants K_0 and c_0 with $c_0 < 1$ such that for all $Q \in \mathbb{N}_1^n$*

$$|W_Q(t)| \leqslant K_0^m R_m(t) \text{ if } m = |Q|,\ |t| \leqslant c_0\rho_m. \tag{5.18}$$

Proof. — We choose $c_0 \in (0,1)$ as follows. Let $p_{Q,l} = |Q|^{-1}((Q,\alpha) - \alpha_l)$, $l = 1,\dots,n$. Then there exists a constant $M > 0$ such that $|p_{Q,l}| \leqslant M$ for all $Q \in \mathbb{N}_1^n$ and all $l \in \{1,\dots,n\}$. Now choose c_0 so small that $\Re(pt+1)/(t+1) > 0$ for all $|t| \leqslant c_0$ and all $p \in \mathbb{C}$ such that $|p| \leqslant M$.

From Lemma 5.2 it follows that there exist positive constants μ_Q such that

$$|W_Q(t)| \leqslant \mu_Q R_{|Q|}(t) \text{ if } |t| \leqslant c_0\rho_{|Q|} \tag{5.19}$$

for all $Q \in \mathbb{N}_1^n$. We choose these constants first for all Q with $|Q| < m_0$, where m_0 is given in Lemma 5.3. For $|Q| \geqslant m_0$ we determine suitable μ_Q by means of a recurrence relation. Suppose μ_Q have been determined for $1 \leqslant |Q| < m$ such that (5.19) holds for these Q. Here we assume that $m \geqslant m_0$.

We first estimate $|U_Q|$ for $|Q| = m$. There exists a positive constant K_1 such that

$$|f_j(0)| \leqslant K_1^{|j|}, |(\mathcal{B}f_j)(t)| \leqslant K_1^{|j|}$$

for all $t \in \mathbb{C}$ with $|t| \leqslant \rho_1$ and $j \in \mathbb{N}$. From (5.12) it follows that for all $|t| \leqslant c_0\rho_{m-1}$ we have

$$\begin{aligned}|U_Q(t)| &\leqslant \sum_{2\leqslant|j|\leqslant|Q|} K_1^{|j|}|(1+1*)\Sigma'' \prod_{q=1}^{|j|} *\{\mu_{i_q} R_{|i_q|}(t)\}| \\ &\leqslant (1+\rho_1) \sum_{2\leqslant|j|\leqslant|Q|} K_1^{|j|} R_{|Q|}(t)\Sigma'' \prod_{q=1}^{|j|} \mu_{i_q},\end{aligned} \tag{5.20}$$

where in $\sum''$ we sum over $i_q \in \mathbb{N}_1^n$ with $\sum_{q=1}^{|j|} i_q = Q$. Using (3.4) we obtain for all $|t| \leqslant c_0\rho_{m-1}$ and $|Q| = m \geqslant 2$

$$|U_Q(t)| \leqslant \nu_Q R_m(t),\ \nu_Q := c_1 \sum_{h=2}^{|Q|} (2K_1)^h \Sigma'' \prod_{q=1}^{h} \mu_{i_q}, \tag{5.21}$$

where $c_1 := (1+\rho_1)2^{n-1}$. We will use this estimate in an integral representation of W_Q for $|Q| \geqslant m_0$ which follows from (5.13):

$$(5.22)\quad W_{Q,l}(t) = (t+\lambda_{Q,l})^{-1-\beta_{Q,l}} \int_0^t (s+\lambda_{Q,l})^{\beta_{Q,l}} U_{Q,l}(s)ds,\ l = 1,\dots,n.$$

From now on we will fix $l \in \{1,\dots,n\}$ and delete the index l.

First we suppose $\lambda_Q = 0$. From (5.22) and (5.21) we obtain

$$|W_Q(t)| \leqslant \nu_Q \int_0^t \Big|\Big(\frac{s}{t}\Big)^{\beta_Q+1} \frac{s^{m-2}}{(m-1)!} ds\Big| \leqslant \nu_Q R_m(t)(\Re\alpha_Q)^{-1}$$

if $|t| \leqslant c_0\rho_{m-1}$. With lemma 5.3 we see that (5.19) holds with $\mu_Q \geqslant \delta_0^{-1}\nu_Q$.

Next suppose $\lambda_Q \neq 0$. Let $t = \lambda_Q\tau$ and $v_Q(\tau) = U_Q(t)/R_{|Q|}(t)$. So $|v_Q(\tau)| \leqslant \nu_Q$ if $|\tau| \leqslant |\lambda_Q|^{-1}c_0\rho_{m-1}$. We subsitute $s = \lambda_Q\sigma$ in (5.22) with $|Q| = m$. Since $\beta_Q = m(p_Q-1)$ we obtain for all $|t| \leqslant c_0\rho_{m-1}$

$$(5.23)\quad |W_Q(t)| \leqslant \frac{|t|^{m-1}}{(m-1)!} \int_0^\tau \Big|\Big(\frac{\sigma(1+\sigma)^{p_Q-1}}{\tau(1+\tau)^{p_Q-1}}\Big)^{m-1}\Big(\frac{1+\sigma}{1+\tau}\Big)^{p_Q-1}\frac{v_Q(\sigma)}{1+\tau}d\sigma\Big|.$$

Let $h(\sigma) := \log\{\sigma(1+\sigma)^{p_Q-1}\}$, so that the first factor in the integrand equals $\exp\{(m-1)(h(\sigma)-h(\tau))\}$. Then $h'(\sigma) = (1+p_Q\sigma)/(\sigma(1+\sigma))$. Due to the choice of c_0 made above we have $\Re(1+p_Q\sigma)/(1+\sigma) > 0$ for all $Q \in \mathbb{N}_2^n$ if $|\sigma| \leqslant c_0$. Hence $\Re\{d/d\xi\ h(\xi\tau)\} = \Re(1+p_Q\xi\tau)/(\xi(1+\xi\tau)) > 0$ if $|\tau| \leqslant c_0$, $0 < \xi \leqslant 1$ and consequently $\Re(h(\sigma)-h(\tau)) < 0$ if $\sigma \in [0,\tau)$, $|\tau| \leqslant c_0$ for all $Q \in \mathbb{N}_2^n$. It follows that there is a positive constant c_2 independent of Q and t such that

$$(5.24)\qquad |W_Q(t)| \leqslant R_m(t)\int_0^\tau \Big|\Big(\frac{1+\sigma}{1+\tau}\Big)^{p_Q-1}\frac{d\sigma}{1+\tau}\Big|\nu_Q \leqslant c_2\nu_Q R_m(t)$$

if $|Q| = m$, $|t| \leqslant c_0\min\{|\lambda_Q|, \rho_{m-1}\}$, so in particular for $|t| \leqslant c_0\rho_m$. Enlarging c_2 such that $c_2 \geqslant \delta_0^{-1}$ we define $\mu_Q = c_2\nu_Q$ implying estimate (5.19) for both cases $\lambda_Q = 0$ and $\lambda_Q \neq 0$.

Thus with (5.21) we obtain the recurrence relation

$$(5.25)\qquad \mu_Q = c_1c_2\sum_{h=2}^{|Q|}(2K_1)^h\Sigma''\prod_{q=1}^h \mu_{i_q} \text{ if } |Q| \geqslant m_0$$

with Σ'' as before.

Define formally

$$F(x) := \sum_{Q\in\mathbb{N}_1^n}\mu_Q x^Q,\ x \in \mathbb{C}^n.$$

Let $c_1c_2\sum_{h=2}^\infty(2K_1F(x))^h = \sum_{Q\in\mathbb{N}_1^n}\tilde\mu_Q x^Q$ formally. From (5.25) we may deduce that $\tilde\mu_Q = \mu_Q$ if $|Q| \geqslant m_0$. Hence $F - c_1c_2(2K_1F)^2(1-2K_1F)^{-1} = F_1$ where F_1 is a polynomial in x of degree $< m_0$ and $F_1(0) = 0$. This quadratic equation for F has a unique holomorphic solution $F = F_1 + O(F_1^2)$ as $F_1 \to 0$. So the formal series $F(x)$ converges in a neighborhood of the origin and consequently there exists a constant $K_0 > 0$ such that $\mu_Q \leqslant K_0^{|Q|}$ for all $Q \in \mathbb{N}_1^N$. This implies (5.18). □

Corollary 5.5. — *Let $Q \in \mathbb{N}_1^n$ and $m = |Q|$. There exists a positive constant K such that $\mathcal{B}g_Q = G_Q$ satisfies*

$$|G_Q(t)| \leqslant K^m \text{ if } |t| \leqslant \rho_m/2. \tag{5.26}$$

Proof. — Let t be as above and C the circle with radius $\rho_m/2$ and center t. From Cauchy's Theorem, (5.10) and (5.18) it follows that

$$|G_Q(t)| \leqslant \frac{m!}{2\pi} \int_C \left| \frac{W_Q(t+s)}{s^{m+1}} ds \right| \leqslant m! K_0^m \frac{(\rho_m)^{m-1}}{(m-1)!} (\rho_m/2)^{-m},$$

which implies (5.26). □

Hence (4.2) holds. So the assumptions of Theorem 4.1 with $\alpha = -\pi/2$ and $\beta = 3\pi/2$ are satisfied and Theorem 5.1 follows in case $k = 1$.

5.3. Proof in case $k > 1$

As before we only need to prove the Gevrey property of the normalizing transformation $g(z, y)$. It is sufficient to consider the case $j = 0$ only.

With the coefficients g_Q in (2.5) —which are holomorphic in S_0— we now associate $w_Q(z) = z^{k|Q|} g_Q(z)$ and $u_Q(z) = z^{k|Q|} t_Q(z)$. Now instead of (5.9) we have

$$z^{k+1} w_Q'(z) + (\lambda_Q + z^k \beta_Q) w_Q(z) = z^k u_Q(z), \text{ where } \beta_Q = \alpha_Q - k|Q| \operatorname{Id}.$$

Let the operator σ_k be defined by $(\sigma_k \phi)(z) = \phi(z^{1/k})$ and define $\tilde{w}_Q = \sigma_k w_Q$, $\tilde{g}_Q = \sigma_k g_Q$, $\tilde{u}_Q = \sigma_k u_Q$. Then

$$\tilde{w}_Q = z^{|Q|} \tilde{g}_Q, k z^2 \tilde{w}_Q'(z) + (\lambda_Q + z\beta_Q) \tilde{w}_Q(z) = z \tilde{u}_Q(z). \tag{5.27}$$

Let $\widetilde{G}_Q = \mathcal{B}\tilde{g}_Q$, $\widetilde{W}_Q = \mathcal{B}\tilde{w}_Q$, $\widetilde{U}_Q = \mathcal{B}\tilde{u}_Q$. Then instead of (5.13) we now have

$$(kt + \lambda_Q) \widetilde{W}_Q'(t) + (k + \beta_Q) \widetilde{W}_Q(t) = \widetilde{U}_Q(t),$$

and instead of Lemma 5.2 we now have that $\widetilde{U}_Q$ and $\widetilde{W}_Q$ are holomorphic functions of $t^{1/k}$ for $|t| < \rho_m/k$ and both are $O(t^{m-1})$ as $t \to 0$. In Lemma 5.4 we have to replace W_Q by $\widetilde{W}_Q$ and the condition on $|t|$ becomes $|t| \leqslant c_0 \rho_m/k$.

Corollary 5.5 now becomes:

The k-Borel transform $\mathcal{B}_k g_Q(t) = G_Q(t)$ is holomorphic for $|t| \leqslant c_1\rho_m^{1/k}$ and $|G_Q(t)| \leqslant K^m$ on this set. Here c_1 and K are positive constants and $m = |Q|$.

Proof. — From the definition of the Borel transform it follows that $\widetilde{G}_Q = \sigma_k G_Q$ and therefore $G_Q(t) = \widetilde{G}_Q(t^k)$. Let $\rho_m' := c_0\rho_m/k$. If $|t^k| < \rho_m'$ then

$$\widetilde{W}_Q(t^k) = \frac{1}{2\pi i}\int_C \frac{\widetilde{W}_Q(s^k)}{s-t}ds$$

where C is the positively oriented circle $|s| = (\rho_m')^{1/k}$. From (5.27) we deduce that if $|t| < \rho_m'$ then

$$\widetilde{G}_Q(t) = \frac{d^m}{dt^m}\widetilde{W}_Q(t) = \frac{1}{2\pi i}\int_C \widetilde{W}_Q(s^k)\frac{d^m}{dt^m}(s-t^{1/k})^{-1}ds.$$

There exists $K_1 > 0$ such that

$$\left|\frac{d^m}{dt^m}(s-t^{1/k})^{-1}\right| \leqslant m!K_1^m|s|^{-km-1} \text{ if } |t| \leqslant |s|^k/2$$

for all $m \in \mathbb{N}$ and $s \neq 0$. This follows easily by substituting $t = s^k u$. Using this estimate and the k-version of Lemma 5.4 in the preceding integral we obtain $|\widetilde{G}_Q(t)| \leqslant m(K_0K_1)^m/\rho_m'$ if $|t| \leqslant \rho_m'/2$ and since $G_Q(t) = \widetilde{G}_Q(t^k)$ this implies the corollary. □

Hence the assumptions of Theorem 4.1 are satisfied with $\alpha = -\pi/(2k)$ and $\beta = 3\pi/(2k)$ if we replace γ by γ/k and Theorem 5.1 follows in case $k > 1$.

5.4. Proof of Theorem 4.3 from Theorem 5.1

The argument is the same as in the proof of Theorem 4.3.1 in [24]. For the convenience of the reader we give the sketch of it.

To a well prepared holomorphic vector field $X \in \mathcal{E}_{k,\lambda,\alpha}$ of the form (2.1) with $\beta = 1$, we associated a germ of holomorphic vector field $\widetilde{X}$ in $(\mathbb{C}^{n+1}, 0)$

$$\widetilde{X} = \sum_{i=1}^n \left(x_i(\lambda_i + \alpha_i z^k) + z^k f_i(x)\right)\frac{\partial}{\partial x_i} + z^{k+1}\frac{\partial}{\partial z}.$$

It it tangent to the germ of variety $\Sigma = \{z = x^r\}$ at the origin and its restriction to it is equal to X. To $\widetilde{X}$, we associate a non-linear system with irregular singularity at the origin

$$z^{k+1}\frac{dx_i}{dz} = x_i(\lambda_i + \alpha_i z^k) + z^k f_i(x),\ i = 1, \dots, n. \tag{5.28}$$

If the assumptions $(H_i)_{i=1,\dots,4}$ are satisfied then we can apply Theorem 5.1. Since the original vector field is well prepared, system (5.28) satisfies condition (H_5). Hence, the sectorial linearizing diffeomorphisms $x = y + g^j(z, y)$ preserve the monomial x^r. Therefore, we can restrict the associated vector field of $\mathbb{C}^{n+1}$ as well as the sectorial diffeomorphisms to Σ. We obtain that $x = y + g^j(y^r, y)$ transforms $X = \widetilde{X}_{|\Sigma}$ to

$$\sum_{i=1}^{n} y_i \left(\lambda_i + \alpha_i (y^r)^k\right) \frac{\partial}{\partial y_i}$$

and have the good Gevrey properties. As already noticed in remark 4.3.1 of [24], we can associate different $\widetilde{X}$ to X. Each of them differs from the others by a vector field vanishing on Σ and gives rise to a set of linearizing sectorial diffeomorphisms. The point is that, althought these sets depends on the chosen $\widetilde{X}$, their restriction to Σ don't. Hence, the Gevrey property makes sense.

6. Proof of Theorem 4.1

6.1. Estimates for $G_Q^{(N)}$

Let $\rho_m := cm^{-\gamma}$ for $m \in \mathbb{N}_1$ and $G_Q(t) = \mathcal{B}_k g_Q(t)$. From the properties of $g(z, y)$ we deduce that (5.5) holds on $S_0 := S_0(\rho)$ (cf. (4.1)) and that G_Q is holomorphic on $S_1 := \{t \in \mathbb{C}^* | \alpha + \pi/2k + \epsilon_1 \leqslant \arg t \leqslant \beta - \pi/2k - \epsilon_1\}$. Here we choose ϵ_1 sufficiently small: $0 < \epsilon_1 < \pi/(4k)$, $\epsilon_1 < (\beta - \alpha - \pi/k)/2$. Since $G_Q(t)$ is holomorphic for $|t| \leqslant \rho_m$ and satisfies (4.2), Cauchy's inequality shows that

$$|\frac{G_Q^{(N)}(t)}{N!}| \leqslant K^m \left(\frac{\rho_m}{2}\right)^{-N} \text{ if } |t| \leqslant \frac{\rho_m}{2}. \tag{6.1}$$

Next we estimate $G_Q(t)$ for $t \in S_1$. For these values of t we have

$$G_Q(t) = \frac{1}{2\pi i} \int_\Gamma e^{t^k s} g_Q(s^{-1/k}) ds$$

where Γ is a contour consisting of the arc Γ_0 of the circle $|s| = \rho^{-k}$ in the sector $|\arg(t^k s)| \leqslant \pi/2 + k\epsilon_1$ and the two half lines $\Gamma_\pm$ consisting of the part of the rays $\arg(t^k s) = \pm(\pi/2 + k\epsilon_1)$ outside this circle. Hence

$$\frac{G_Q^{(N)}(t)}{N!} = \frac{1}{2\pi i} \int_\Gamma \frac{1}{N!} \left\{ \frac{d^N}{dt^N} e^{t^k s} \right\} g_Q(s^{-1/k}) ds. \tag{6.2}$$

Here

$$\Big|\frac{1}{N!}\frac{d^N}{dt^N}e^{t^k s}\Big| \leqslant \frac{1}{2\pi}\int_{|\sigma|=\mu}\Big|\frac{\exp\{t^k(1+\sigma)^k s\}}{t^N\sigma^{N+1}}d\sigma\Big|.$$

We choose $\mu\in(0,1)$ such that $|\arg(1+\sigma)|\leqslant \epsilon_1/2$ for $|\sigma|=\mu$. Thus we obtain for $|(\mu t)^N/N!\; d^N/dt^N\; e^{t^k s}|$ the upper bound $\exp(|t|(1+\mu)/\rho)^k$ for $s\in\Gamma_0$ and the upper bound $\exp\{-|t|^k(1-\mu)^k|s|\sin k\epsilon_1/2\}$ for $s\in\Gamma_\pm$ since on $\Gamma_\pm$ we have $(\pi+k\epsilon_1)/2\leqslant|\arg(s(t(1+\sigma))^k)|\leqslant\pi$. Using these estimates and (5.5) in (6.2) we deduce that for $t\in S_1$ we have

$$\Big|\int_{\Gamma_0}\Big| \leqslant M_1R^{-m}(\mu|t|)^{-N}\exp(|t|(1+\mu)/\rho)^k,$$

$$\Big|\int_{\Gamma_\pm}\Big| \leqslant M_2R^{-m}(\mu|t|)^{-N}\int_0^\infty \exp\Big\{-|t|^k(1-\mu)^k|s|\sin\frac{k\epsilon_1}{2}\Big\}d|s|$$

and thus

$$\Big|\frac{G_Q^{(N)}(t)}{N!}\Big| \leqslant M_3R^{-m}(\mu|t|)^{-N}\{|t|^{-k}+\exp\Big(|t|\frac{(1+\mu)}{\rho}\Big)^k\} \tag{6.3}$$

Here $t\in S_1$ and M_1, M_2, M_3 are some positive constants. We use this estimate for $|t|\geqslant\rho_m/2$ and (6.1) for $|t|\leqslant\rho_m/2$. Then we get for $t\in(S_1\cup\overline{\Delta}(\rho_m/2))$

$$|\frac{G_Q^{(N)}(t)}{N!}| \leqslant M_4R_1^mR_2^N\rho_m^{-N}\exp(|t|R_3)^k \tag{6.4}$$

where M_4, R_1, R_2 and R_3 are positive constants.

6.2. Estimates for $g_Q(z)$ in a subsector of S

We use the Laplace representation for g_Q:

$$\begin{aligned} g_Q(z) &= g_Q(0)+\int_0^{\infty:\theta} e^{-(t/z)^k}G_Q(t)dt^k \\ &= g_Q(0)+z^k\int_0^{\infty:k(\theta-\arg z)} e^{-s}G_Q(zs^{1/k})ds \end{aligned}$$

for $\arg z\in[\alpha+2\epsilon_1,\beta-2\epsilon_1]$ and $|z|$ sufficiently small where $\alpha+\pi/2k+\epsilon_1\leqslant\theta\leqslant\beta-\pi/2k-\epsilon_1$, $|\theta-\arg z|\leqslant\pi/2k-\epsilon_1$. From this we deduce

$$\frac{1}{N!}g_Q^{(N)}(z) = \int_0^{\infty:k(\theta-\arg z)} e^{-s}G(z,s)ds \tag{6.5}$$

where

$$G(z,s)=\sum_{l=0}^{k}\binom{k}{l}z^{k-l}\frac{s^{(N-l)/k}}{(N-l)!}G_Q^{(N-l)}(zs^{1/k}).$$

With (6.4) we obtain

$$\frac{1}{N!}|g_Q^{(N)}(z)| \leqslant M_4 R_1^m \sum_{l=0}^{k} \binom{k}{l} |z^{k-l}| R_2^{N-l} \rho_m^{l-N} \cdot \int_0^{\infty : k(\theta - \arg z)} |e^{-s+(|z|R_3)^k |s|} s^{(N-l)/k} ds|.$$

As $|\theta - \arg z| \leqslant \pi/2k - \epsilon_1$ we have $\Re s \geqslant |s| \sin(k\epsilon_1)$ on the path of integration. Let $|z| \leqslant (\sin(k\epsilon_1)/2)^{1/k}/R_3 =: \rho'$. Then $|(zR_3)^k s| - \Re s \leqslant -(|s| \sin(k\epsilon_1))/2$ and therefore

$$\frac{1}{N!}|g_Q^{(N)}(z)| \leqslant M_4 R_1^m \sum_{l=0}^{k} \binom{k}{l} |z^{k-l}| R_2^{N-l} \rho_m^{l-N} \cdot \int_0^{\infty} e^{-(|s|\sin(k\epsilon_1))/2} |s|^{(N-l)/k}) d|s|.$$

From this we deduce

$$\frac{1}{N!}\left|g_Q^{(N)}(z)\right| \leqslant C_1^m \left(\frac{C_2}{\rho_m}\right)^N \Gamma\left(\frac{N}{k}+1\right) \tag{6.6}$$

for $z \in S_{2\epsilon_1}(\rho')$ (*cf.* (4.1)). Here C_1 and C_2 are positive constants.

6.3. End of proof of Theorem 4.1

Since $\partial^N/\partial z^N \; g(z,y) = \sum_{Q \in \mathbb{N}_1^n} g_Q^{(N)}(z) y^Q$ we obtain with the help of (6.6)

$$\frac{1}{N!}\left|\frac{\partial^N}{\partial z^N} g(z,y)\right| \leqslant \sum_{Q \in \mathbb{N}_1^n} \Gamma\left(\frac{N}{k}+1\right) C_1^m \left(\frac{C_2}{\rho_m}\right)^N |y|^m, \tag{6.7}$$

provided the righthand side converges and where $m = |Q|$ and $z \in S_{2\epsilon_1}(\rho')$. Now use (3.4) and the diophantine condition (5.15) in (6.7) and obtain for $z \in S_{2\epsilon_1}(\rho')$

$$\left|\frac{1}{N!}\frac{\partial^N}{\partial z^N} g(z,y)\right| \leqslant \Gamma\left(\frac{N}{k}+1\right) C_2^N c^{-N} 2^{n-1} \sum_{m=1}^{\infty} m^{\gamma N} (2C_1|y|)^m. \tag{6.8}$$

Finally we use that for $0 < \delta < 1$ there exist positive constants C_3 and C_4 such that for $|x| \leqslant 1 - \delta$ and $\mu > 0$

$$\left|\sum_{m=1}^{\infty} m^{\mu} x^m\right| \leqslant C_3 C_4^{\mu} \Gamma(\mu) \tag{6.9}$$

(*cf.* proof below). So the series in the righthand side of (6.7) converges for $|y| \leqslant R' := (1-\delta)/2C_1$. Combining (6.8) with (6.9) and Stirling's formula we see that there exists a positive constant C_5 such that

$$\Big|\frac{1}{N!}\frac{\partial^N}{\partial z^N} g(z,y)\Big| \leqslant C_5^N \Gamma\Big(\Big(\gamma+\frac{1}{k}\Big)N\Big) \tag{6.10}$$

if $|y| \leqslant R',\ z \in S_{2\epsilon_1}(\rho')$. This proves the Gevrey property in Theorem 4.1 for sufficiently small positive ϵ and this suffices.

Proof of (6.9). — Let the lefthand side of (6.9) be denoted by $F(\mu, x)$ where $|x| < 1$. We apply Hankel's formula

$$\frac{m^\mu}{\Gamma(\mu+1)} = \frac{1}{2\pi i}\int_\Gamma e^{ms} s^{-\mu-1} ds$$

to get

$$\begin{aligned} F(\mu,x) &= \Big|\frac{\Gamma(\mu+1)}{2\pi i}\int_\Gamma s^{-\mu-1}\sum_{m=1}^{\infty}(e^s x)^m ds\Big| \\ &= \Big|\frac{\Gamma(\mu+1)}{2\pi i}\int_\Gamma s^{-\mu-1}\frac{e^s x}{1-e^s x} ds\Big|. \end{aligned}$$

Here Γ is a contour from $\infty e^{-\pi i}$ to $\infty e^{\pi i}$ turning once around 0 in the positive sense such that $|e^s x| < 1$ on Γ. If $|x| \leqslant 1-\delta$ we may choose Γ such that it consists of the circle $|s| = -\log(1-\delta)/2$ and the parts of the half lines $\arg s = \pm\pi$ outside this circle. Then it is easy to estimate the integral by $2\pi C_3 C_4^\mu/\mu$ with some positive constants C_3 and C_4 and (6.9) follows. □

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