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COMPLETE REAL KÄHLER EUCLIDEAN HYPERSURFACES ARE CYLINDERS

by Luis A. FLORIT & Fangyang ZHENG (*)

ABSTRACT. — In this note we show that any complete Kähler (immersed) Euclidean hypersurface $M^{2n} \subset \mathbb{R}^{2n+1}$ must be the product of a surface in $\mathbb{R}^3$ with an Euclidean factor $\mathbb{C}^{n-1} \cong \mathbb{R}^{2n-2}$.

RÉSUMÉ. — Dans cet article nous montrons que toute hypersurface Kählerienne complète immergée dans un espace Euclidien $M^{2n} \subset \mathbb{R}^{2n+1}$ est le produit d'une surface de $\mathbb{R}^3$ et d'un facteur Euclidien $\mathbb{C}^{n-1} \cong \mathbb{R}^{2n-2}$.

1. The statement

The purpose of this paper is to give a proof of the following result, that was proved by Abe ([2]) under the additional assumption that either f is real analytic, or the scalar curvature of M^{2n} is everywhere nonnegative or everywhere negative:

THEOREM 1.1. — *Let $f : M^{2n} \to \mathbb{R}^{2n+1}$ be an isometric immersion of a complete Kähler manifold. Then $M^{2n} = \Sigma^2 \times \mathbb{C}^{n-1}$ and $f = f_1 \times \iota$ split, where $f_1 : \Sigma^2 \to \mathbb{R}^3$ is an isometric immersion and ι is the identity map of $\mathbb{C}^{n-1} \cong \mathbb{R}^{2n-2}$.*

The problem of classifying all real Kähler hypersurfaces in real space forms was raised in [12] and solved in [11], both locally and globally, when the ambient space has non-zero constant curvature. In this case, the examples are essentially unique, even locally, being the product of two space forms. One of the main ingredients used to show this is the well-known fact

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that the *index of relative nullity* ν of the immersion satisfies that $\nu \geqslant 2n-2$ ([12]). Here $\nu(x)$ is the dimension of the nullity of the second fundamental form at $x \in M^{2n}$. The function $r = n - \nu$ is often called the *rank* of the immersion.

The situation when the ambient space is flat is fairly more complicated. The local classification for hypersurfaces without flat points was carried out in [3]. It was shown that they are abundant and can be parametrized by means of a pseudoholomorphic surface in $\mathbb{S}^{2n}$ (the Gauss image) and a smooth function over it. In particular, generically they are locally irreducible.

Since any isometric immersion $f : N^n \to \mathbb{R}^{n+p}$ with $\nu \geqslant n-1$ is flat, the classical Hartman's cylinder theorem ([9]) implies that, when N^n is also complete, $N^n = \Gamma\times\mathbb{R}^{n-1}$ and $f = f_1\times\iota$, where $\Gamma = \mathbb{R}$ or $\mathbb{S}^1$, $f_1 : \Gamma \to \mathbb{R}^{1+p}$ is a smooth curve and ι is the identity map of $\mathbb{R}^{n-1}$. In other words, f is a $(n-1)$-cylinder.

Later on, Abe ([1]) obtained the complex analogue, which states that for any holomorphic isometric immersion $f : M^{2n} \to \mathbb{C}^{n+p}$ of a complete Kähler manifold into the complex Euclidean space, if $\nu \geqslant 2n-2$, it holds that $f = f_1\times\iota$, where $f_1 : \Sigma \to \mathbb{C}^{1+p}$ is a holomorphic curve and ι is the identity map of $\mathbb{C}^{n-1}$. That is, any complete complex submanifold M^{2n} of the complex Euclidean space with (real) rank at most two is a complex $(n-1)$-cylinder.

After these two classical results, it is only natural to consider the cylinder problem in the hybrid case, namely, for complete real Kähler submanifolds with rank at most two. In view of the proof of Theorem 1.1, we will state the question in the form of a conjecture:

Conjecture 1.2. — *Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be an isometric immersion of a complete Kähler manifold into Euclidean space. If $\nu \geqslant 2n-2$, then $M^{2n} = \Sigma^2 \times \mathbb{R}^{2n-2}$ and $f = f_1\times\iota$ split, where $f_1 : \Sigma^2 \to \mathbb{R}^{2+p}$ and ι is the identity map of $\mathbb{R}^{2n-2}$.*

Note that when M^{2n} is flat (or more generally, when M^{2n} has nonnegative scalar curvature), the above is true by Hartman's cylinder theorem. The above conjecture was also proved by Dajczer and Gromoll in [4] under the additional assumption that the set of non-flat points of M^{2n} is dense and connected. When M^{2n} is not everywhere flat, it is easy to see that, if the above conjecture is true, then the decomposition $M^{2n} = \Sigma^2\times\mathbb{R}^{2n-2} \cong \Sigma^2\times\mathbb{C}^{n-1}$ is a Kähler one. For related splitting results of real Kähler Euclidean submanifolds in higher codimensions, see [6], [8] and [7].

2. The proof

Let $f : M^{2n} \to \mathbb{R}^{2n+1}$ be an isometric immersion of a Kähler manifold of (real) dimension $2n$, i.e., a *real Kähler Euclidean hypersurface*. Denote by $\Delta(x)$ the *relative nullity of f at $x \in M^{2n}$*, that is, the kernel of the second fundamental form at x. Its dimension $\nu(x)$ is the index of relative nullity of f at x. It is well-known that, along any open subset where ν is constant, Δ is a smooth distribution with totally geodesic leaves in both M^{2n} and $\mathbb{R}^{2n+1}$. Moreover, on the open subset where ν attains its minimum, the leaves are complete if M^{2n} is complete.

The next observation follows easily from the Gauss equation and the fact that $R(X,Y) \circ J = J \circ R(X,Y)$ for all $X, Y \in TM$, where R is the curvature tensor of M^{2n}.

Lemma 2.1. — ([12]) *For any real Kähler hypersurface $f : M^{2n} \to \mathbb{R}^{2n+1}$, the relative nullity index ν satisfies that $\nu \geqslant 2n-2$, and in the open set of non-flat points*

$$U = \{x \in M^{2n} : \nu(x) = 2n-2\},$$

Δ is a complex distribution, that is, $J\Delta = \Delta$.

Let $f : M^{2n} \to \mathbb{R}^{2n+1}$ be as in Theorem 1.1. If M^{2n} is flat, then by the cylinder theorem of Hartman-Nirenberg ([10]), f is a $(2n-1)$-cylinder over a plane curve $f_1 : \Gamma \to \mathbb{R}^2$. Let T be a unit tangent vector of Γ. Then JT is constant in the Euclidean space since $\tilde{\nabla}_T JT = \nabla_T JT = J\nabla_T T = 0$, where $\tilde{\nabla}$ and ∇ stand for the Levi-Civita connections of $\mathbb{R}^{2n+1}$ and M^{2n}, respectively. Therefore, JT defines a line $\mathbb{R} \subset \mathbb{R}^{2n-1}$ and if we set $\Sigma = \Gamma \times \mathbb{R}$ as the product of Γ with that line, then we have the Kähler decomposition $M^{2n} = \Sigma \times \mathbb{C}^{n-1}$ and Theorem 1.1 follows in this case. Hence, since an arbitrary Euclidean hypersurface is flat if and only if its rank is at most one, in view of Lemma 2.1 from now on we can assume that U is not empty.

By Lemma 2.1 each leaf of Δ in U is a complete complex submanifold of M^{2n} that has to be $\mathbb{C}^{n-1}$ since f maps each leaf of Δ in U onto a linear subvariety of $\mathbb{R}^{2n+1}$, that is, an affine subspace of $\mathbb{R}^{2n+1}$. The aforementioned result in [2] is consequence of the following lemma together with the Cheeger-Gromoll splitting theorem (see also [4]):

Lemma 2.2. — ([2]) *In any connected component of U, the leaves of Δ are mapped by f onto linear subvarieties in $\mathbb{R}^{2n+1}$ that are parallel to each other.*

Remark 2.3. — The same proof holds for any open subset of M^{2n} with a complex totally geodesic distribution $\Delta' \subset \Delta$ with complete leaves of complex codimension 1 in M^{2n}.

Our goal is to extend this foliation in U to the entire M^{2n}. First let us fix some notations. Denote by

$$U' = \{x \in M^{2n} \mid \nu(x) \leqslant 2n-1\}, \quad U_1 = U' \setminus U.$$

Then U' is open, with $M^{2n} \setminus U'$ being the set of totally geodesic points. U_1 is the set of points where f has constant rank 1. For convenience, we will denote by V the set of interior points of U_1, and write $F = U_1 \setminus V$. Clearly, U' is the disjoint union of the open subsets U, V and the closed set F, with $F = U_1 \cap \overline{U}$.

Now let us construct a distribution L in the open set U'. For any rank 2 point $x \in U$, we will simply set $L_x = \Delta_x$. For $x \in V$, define

$$L_x = \Delta_x \cap J\Delta_x,$$

where $\Delta_x \cong \mathbb{R}^{2n-1}$ is the kernel of α and J is the almost complex structure. L_x is again a J-invariant subspace of T_xM of codimension 2. Since Δ is a totally geodesic foliation in V with flat leaves, L is also a totally geodesic foliation in V and its leaves are flat complex submanifolds. Let us denote by $L(y)$ the leaf of L in $U \cup V$ passing through y.

For any $x \in F$, since $F \subseteq \overline{U}$, there exists a sequence of points $\{x_k\}$ in U approaching to x. Then by passing to a subsequence if necessary, we may assume that L_{x_k} converges, so $\lim L(x_k)$ will be a totally geodesic flat $\mathbb{C}^{n-1}$ passing through x that is mapped by f onto a linear subvariety. We will call it a *limit position* at x.

We claim that the limit positions at any given $x \in F$ must be unique. To see this, let us assume there are two sequences $\{x_k\}$ and $\{y_k\}$ in U, both approaching x, such that $P := \lim L(x_k)$ and $Q := \lim L(y_k)$ both exist but are different. Since P and Q are (closed) complex hypersurfaces of M^{2n}, both intersect at x *transversally.* So, nearby leaves will also have to intersect. That is, for sufficiently large k, $L(x_k)$ and $L(y_k)$ will intersect each other, which is only possible when they coincide. This contradicts $P \neq Q$.

Now we have the uniqueness of limit position at any $x \in F$, let us just denote it by $F(x)$, and write L_x for the tangent space of $F(x)$ at x. Since for each k, L_{x_k} is a J-invariant subspace where α vanishes, so L_x is a J-invariant subspace of Δ_x. Thus by the fact that $\nu(x) = 2n-1$, we get $L_x = \Delta_x \cap J\Delta_x$, consistent with our definition of L in V. In addition, since each connected component of U is a cylinder over a surface in $\mathbb{R}^3$ by Lemma

2.2, for any $y \in F(x)$ we have that $\nu(y) = \nu(x) = 2n - 1$. Therefore, $F(x)$ must be contained in F, and then F is the disjoint union of these limit positions.

Now let us fix any $x \in V$, and consider the leaf $L(x)$ of L in V. We claim that this totally geodesic submanifold is complete. Let $\gamma : [0, \infty) \to M$ be a geodesic such that $\gamma(0) = x$, and $\gamma([0, a)) \subseteq L(x)$, where $a > 0$. It is well-known (cf. [5]) that ν does not increase along such a geodesic. So $\gamma(a) \in U_1$. If $y = \gamma(a) \in F$, then since

$$\gamma'(t) \in L_{\gamma(t)} = \Delta_{\gamma(t)} \cap J\Delta_{\gamma(t)}$$

for any $t < a$, we would have $\gamma'(a) \in L_y$. But the limit position $F(y)$ through y is totally geodesic, so γ is contained in $F(y) \subset F$, contradicting the assumption that $x \in V$. So we have proved that $\gamma(a) \in V$. This proves the completeness of $L(x)$.

In summary, we showed that the open subset U' is foliated by L, whose leaves are totally geodesic flat $\mathbb{C}^{n-1}$ that are mapped under f onto linear subvarieties in $\mathbb{R}^{2n+1}$. Also, in view of the flatness of the metric in V, Lemma 2.2 and the continuity of L, in each connected component of U', these linear subvarieties are parallel to each other. (Another proof of the parallelism of the leaves in V can be obtained with the standard splitting tensor argument used in the proof of Lemma 2.2; see Remark 2.3).

Our next goal is to extend the foliation L across the set $W := M \setminus U'$ of totally geodesic points. First, for any $x \in \overline{U'} \setminus U'$, the leaves of L give limit positions at x, which are totally geodesic complex submanifolds (of complex codimension 1) passing through x. Such limit positions at x must be unique for the exact same reason as before (when we extended L from U to F). So $\overline{U'}$ is now foliated by flat, locally parallel $\mathbb{C}^{n-1}$.

Denote by W^0 the set of interior points of W. Suppose W^0 is not empty and let Ω be a connected component of W^0. Fix any $x \in \Omega$, and consider $H = T_{f(x)}M \cong \mathbb{C}^n$ as a linear subvariety of $\mathbb{R}^{2n+1}$. For convenience, we will identify a subset $P \subseteq M$ with its image $f(P)$ if $f|_P$ is an embedding. Denote by $\partial\Omega \subset \overline{U'}$ the set of boundary points of Ω. Since f is totally geodesic in W, we know that the restriction of f on $\overline{\Omega}$ is an embedding, and $\overline{\Omega} \subset H$. Note that $\partial\Omega \neq \emptyset$ since $W \neq M^{2n}$.

For any $y \in \partial\Omega$, y is a boundary point of U'. Hence, there exists a totally geodesic flat $L(y) \cong \mathbb{C}^{n-1}$ passing through y, and f embeds $L(y)$ onto a linear subvariety. Since $T_{f(y)}M = H$, we have $L(y) \subset H$ as a complex affine hyperplane. Thus, if z is another point in $\partial\Omega$, unless $L(y)$ is parallel to $L(z)$ they will intersect. Suppose $w \in L(y) \cap L(z)$. Since $L(y)$ and $L(z)$

are both contained in $\overline{U'}$, w is a point in $\overline{U'}$, and then we have $L(w)$, which must agree with $L(y)$ and $L(z)$ by the uniqueness of the leaves of L on $\overline{U'}$. Therefore, we get $L(y) = L(z)$. That is, for any two points y and z in $\partial\Omega$, the complex linear subvarieties $L(y)$ and $L(z)$ in H are parallel to each other.

We conclude that there is a unique complex linear subvariety of complex codimension 1 in H passing through x, which is parallel to $L(y)$ for all $y \in \partial\Omega$. Call it $L(x)$. Clearly, $L(x) \subset \Omega$ since it cannot touch the boundary part of Ω due to the fact that $L(y) \subset \overline{U'}$ for all $y \in \partial\Omega$.

Now our entire manifold M^{2n} is foliated by totally geodesic flat $\mathbb{C}^{n-1}$, with each leaf mapped by f onto a linear subvariety, and nearby leaves are parallel to each other. The proof of Theorem 1.1 now follows easily.

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