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## ON GAPS IN RÉNYI $\beta$ -EXPANSIONS OF UNITY FOR $\beta > 1$ AN ALGEBRAIC NUMBER

by Jean-Louis VERGER-GAUGRY

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ABSTRACT. — Let  $\beta > 1$  be an algebraic number. We study the strings of zeros (“gaps”) in the Rényi  $\beta$ -expansion  $d_\beta(1)$  of unity which controls the set  $\mathbb{Z}_\beta$  of  $\beta$ -integers. Using a version of Liouville’s inequality which extends Mahler’s and Güting’s approximation theorems, the strings of zeros in  $d_\beta(1)$  are shown to exhibit a “gappiness” asymptotically bounded above by  $\log(M(\beta))/\log(\beta)$ , where  $M(\beta)$  is the Mahler measure of  $\beta$ . The proof of this result provides in a natural way a new classification of algebraic numbers  $> 1$  with classes called  $Q_i^{(j)}$  which we compare to Bertrand-Mathis’s classification with classes  $C_1$  to  $C_5$  (reported in an article by Blanchard). This new classification relies on the maximal asymptotic “quotient of the gap” value of the “gappy” power series associated with  $d_\beta(1)$ . As a corollary, all Salem numbers are in the class  $C_1 \cup Q_0^{(1)} \cup Q_0^{(2)} \cup Q_0^{(3)}$ ; this result is also directly proved using a recent generalization of the Thue-Siegel-Roth Theorem given by Corvaja.

RÉSUMÉ. — Soit  $\beta > 1$  un nombre algébrique. Nous étudions les plages de zéros (“lacunes”) dans le  $\beta$ -développement de Rényi  $d_\beta(1)$  de l’unité qui contrôle l’ensemble  $\mathbb{Z}_\beta$  des  $\beta$ -entiers. En utilisant une version de l’inégalité de Liouville qui étend des théorèmes d’approximation de Mahler et de Güting, on montre que les plages de zéros dans  $d_\beta(1)$  présentent une “lacunarité” asymptotiquement bornée supérieurement par  $\log(M(\beta))/\log(\beta)$ , où  $M(\beta)$  est la mesure de Mahler de  $\beta$ . La preuve de ce résultat fournit de manière naturelle une nouvelle classification des nombres algébriques  $> 1$  en classes appelées  $Q_i^{(j)}$  que nous comparons à la classification de Bertrand-Mathis avec les classes  $C_1$  à  $C_5$  (reportée dans un article de Blanchard). Cette nouvelle classification repose sur la valeur asymptotique maximale du “quotient de lacune” de la série “lacunaire” associée à  $d_\beta(1)$ . Comme corollaire, tous les nombres de Salem sont dans la classe  $C_1 \cup Q_0^{(1)} \cup Q_0^{(2)} \cup Q_0^{(3)}$ ; ce résultat est également obtenu par un théorème récent qui généralise le théorème de Thue-Siegel-Roth donné par Corvaja.

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## 1. Introduction

The exploration of the links between symbolic dynamics and number theory of  $\beta$ -expansions, when  $\beta > 1$  is an algebraic number or more generally a real number, started with Bertrand-Mathis [6] [7]. Bertrand-Mathis, in Blanchard [8], reported a classification of real numbers according to their  $\beta$ -shift, using the properties of the Rényi  $\beta$ -expansion  $d_\beta(1)$  of 1. A lot of questions remain open concerning the distribution of the algebraic numbers  $\beta > 1$  in this classification. The Rényi  $\beta$ -expansion of 1 is important since it controls the  $\beta$ -shift [38] and the discrete and locally finite set  $\mathbb{Z}_\beta \subset \mathbb{R}$  of  $\beta$ -integers [13] [18] [25] [26]. The aim of this note is to give a new Theorem (Theorem 1.1) on the gaps (strings of 0's) in  $d_\beta(1)$  for algebraic numbers  $\beta > 1$ , and investigate how it provides (partial) answers to some questions of [8], in particular for Salem numbers (Corollary 1.2).

Theorem 1.1 provides an upper bound on the asymptotic quotient of the gap of  $d_\beta(1)$  and is obtained by a version of Liouville's inequality extending Mahler's and Gütting's approximation theorems. The proof of Theorem 1.1 turns out to be extremely instructive in itself since it leads to a new classification of the algebraic numbers  $\beta$  as a function of the asymptotics of the gaps in  $d_\beta(1)$  and "intrinsic features", namely the Mahler measure  $M(\beta)$ , of  $\beta$  (the definition of  $M(\beta)$  is recalled in Section 3). The existence of this double parametrization, symbolic and algebraic, was guessed in [8] p 137. This new classification complements Bertrand-Mathis's (Blanchard [8] pp 137–139) and both are recalled below for comparison's sake. The question whether an algebraic number  $\beta > 1$  is contained in one class or another has already been discussed by many authors [5] [6] [7] [8] [9] [10] [11] [12] [17] [22] [32] [33] [38] [39] [41] [42] and depends at least upon the distribution of the conjugates of  $\beta$  in the complex plane. Only the conjugates of  $\beta$  of modulus strictly greater than unity intervene in Theorem 1.1 via the Mahler measure of  $\beta$ . Corollary 1.2 is readily deduced from this remark. We deduce that Salem numbers belong to  $C_1 \cup C_2 \cup Q_0$ , whereas the Pisot numbers are in  $C_1 \cup C_2$  [45].

Another proof of Corollary 1.2 consists of controlling the gaps of  $d_\beta(1)$  by stronger Theorems of Diophantine Geometry which allow suitable collections of places of the number field  $\mathbb{Q}(\beta)$  associated with the conjugates of  $\beta$  and the properties of  $d_\beta(1)$  to be taken into account simultaneously. This alternative proof of Corollary 1.2, just sketched in Section 4, is obtained using the Theorem of Thue-Siegel-Roth given by Corvaja [1] [15].

THEOREM 1.1. — Let  $\beta > 1$  be an algebraic number and  $M(\beta)$  be its Mahler measure. Denote by  $d_\beta(1) := 0.t_1t_2t_3\dots$ , with  $t_i \in A_\beta := \{0, 1, 2, \dots, \lceil \beta - 1 \rceil\}$ , the Rényi  $\beta$ -expansion of 1. Assume that  $d_\beta(1)$  is infinite and gappy in the following sense: there exist two sequences  $\{m_n\}_{n \geq 1}$ ,  $\{s_n\}_{n \geq 0}$  such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \dots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \dots$$

with  $(s_n - m_n) \geq 2$ ,  $t_{m_n} \neq 0, t_{s_n} \neq 0$  and  $t_i = 0$  if  $m_n < i < s_n$  for all  $n \geq 1$ . Then

$$(1.1) \quad \limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} \leq \frac{\log(M(\beta))}{\log(\beta)}.$$

Moreover, if  $\liminf_{n \rightarrow +\infty} (m_{n+1} - m_n) = +\infty$ , then

$$(1.2) \quad \limsup_{n \rightarrow +\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leq \frac{\log(M(\beta))}{\log(\beta)}.$$

As in Ostrowski [37] the quotient  $s_n/m_n \geq 1$  is called *the quotient of the gap*, relative to the  $n$ th-gap (assuming  $t_j \neq 0$  for all  $s_n \leq j \leq m_{n+1}$  to characterize uniquely the gaps). Note that the term “lacunary” has often other meanings in literature and is not used here to describe “gappiness”. Denote by  $\mathcal{L}(S_\beta)$  the language of the  $\beta$ -shift [8] [23] [24] [34]. Bertrand-Mathis’s classification ([8] pp 137–139) is as follows:

- $C_1$  :  $d_\beta(1)$  is finite.
- $C_2$  :  $d_\beta(1)$  is ultimately periodic but not finite.
- $C_3$  :  $d_\beta(1)$  contains bounded strings of 0’s, but is not ultimately periodic.
- $C_4$  :  $d_\beta(1)$  does not contain some words of  $\mathcal{L}(S_\beta)$ , but contains strings of 0’s with unbounded length.
- $C_5$  :  $d_\beta(1)$  contains all words of  $\mathcal{L}(S_\beta)$ .

Present classes of algebraic numbers, with the notations of Theorem 1.1:

- $Q_0^{(1)}$  :  $1 = \lim_{n \rightarrow +\infty} \frac{s_n}{m_n}$  with  $(m_{n+1} - m_n)$  bounded.
- $Q_0^{(2)}$  :  $1 = \lim_{n \rightarrow +\infty} \frac{s_n}{m_n}$  with  $(s_n - m_n)$  bounded and  $\lim_{n \rightarrow +\infty} (m_{n+1} - m_n) = +\infty$ .
- $Q_0^{(3)}$  :  $1 = \lim_{n \rightarrow +\infty} \frac{s_n}{m_n}$  with  $\limsup_{n \rightarrow +\infty} (s_n - m_n) = +\infty$ .
- $Q_1$  :  $1 < \limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} < \frac{\log(M(\beta))}{\log(\beta)}$ .
- $Q_2$  :  $\limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} = \frac{\log(M(\beta))}{\log(\beta)}$ .

What are the relative proportions of each class in the whole set  $\overline{\mathbb{Q}}_{>1}$  of algebraic numbers  $\beta > 1$  ? Comparing  $C_2, C_3$  and  $Q_0^{(1)}$ , what are the relative proportions in  $Q_0^{(1)}$  of those  $\beta$  which give ultimate periodicity in

$d_\beta(1)$  and those for which  $d_\beta(1)$  is not ultimately periodic ? Schmeling ([41] Theorem A) has shown that the class  $C_3$  (of real numbers  $\beta > 1$ ) has Hausdorff dimension one. We have:

- $\overline{Q}_{>1} \cap C_2 \subset Q_0^{(1)}$ ,
- $\overline{Q}_{>1} \cap C_3 \subset Q_0^{(1)} \cup Q_0^{(2)}$ , with  $C_3 \cap Q_0^{(3)} = \emptyset$ ,
- $\overline{Q}_{>1} \cap C_4 \subset Q_0^{(3)} \cup Q_1 \cup Q_2$ .

The Pisot numbers  $\beta$  are contained in  $C_1 \cup Q_0^{(1)}$  since they are such that  $d_\beta(1)$  is finite or ultimately periodic (Parry [38], Bertrand-Mathis [5]). Recall that a Perron number is an algebraic integer  $\beta > 1$  such that all the conjugates  $\beta^{(i)}$  of  $\beta$  satisfy  $|\beta^{(i)}| < \beta$ . Conversely, as shown in Lind [32], Denker, Grillenberger, Sigmund [17] and Bertrand-Mathis [7], if  $\beta > 1$  is such that  $d_\beta(1)$  is ultimately periodic (finite or not), then  $\beta$  is a Perron number. Not all Perron numbers are attained in this way: a Perron number which possesses a real conjugate greater than 1 cannot be such that  $d_\beta(1)$  is ultimately periodic ([8] p 138). Parry numbers belong to  $C_1 \cup C_2$ . Let  $Q_0 = Q_0^{(1)} \cup Q_0^{(2)} \cup Q_0^{(3)}$ .

**COROLLARY 1.2.** — *Let  $\beta > 1$  be a Salem number which does not belong to  $C_1$ . Then  $\beta$  belongs to the class  $Q_0$ .*

The attribution of Salem numbers to  $C_1$ ,  $Q_0^{(1)}$ ,  $Q_0^{(2)}$  and  $Q_0^{(3)}$  is an open problem in general, except in low degree. Boyd [9] [12] has shown that Salem numbers of degree 4 belong to  $C_2$ , hence to  $Q_0^{(1)}$ . It is also the case of some Salem numbers of degree 6 and  $\geq 8$  in the framework of a probabilistic model [11] [12]. In Section 5 we ask the question whether Corollary 1.2 could still be true for Perron numbers.

The definition of the class  $Q_0$  does not make any allusion to  $\beta$ , i.e. to  $M(\beta)$ , to the conjugates of  $\beta$ , to the minimal polynomial of  $\beta$  or to its length, etc, but takes only into account the quotients of the gaps in  $d_\beta(1)$ . Hence this class  $Q_0$  can be applied to real numbers  $\beta > 1$  in full generality instead of only to algebraic numbers  $> 1$ . The question whether there exist transcendental numbers  $\beta > 1$  which belong to the class  $Q_0$  was asked in [8]; what proportion appears in each subclass ? Examples of transcendental numbers (Komornik-Loreti constant [2] [29], Sturmian numbers [14]) in  $Q_0$  are given in Section 5.

In the present note, we deal with the algebraicity of values of "gappy" series, deduced from  $d_\beta(1)$ , at the algebraic point  $\beta^{-1}$ . In a related context, more related to transcendency, Nishioka [36] and Corvaja Zannier [16] have followed different paths and applied the Subspace Theorem [43] to deduce different results.

## 2. Definitions

For  $x \in \mathbb{R}$  the integer part of  $x$  is  $[x]$  and its fractional part  $\{x\} = x - [x]$ . The smallest integer larger than or equal to  $x$  is denoted by  $\lceil x \rceil$ . For  $\beta > 1$  a real number and  $z \in [0, 1]$  we denote by  $T_\beta(z) = \beta z \pmod{1}$  the  $\beta$ -transform on  $[0, 1]$  associated with  $\beta$  [38] [40], and iteratively, for all integers  $j \geq 0$ ,  $T_\beta^{j+1}(z) := T_\beta(T_\beta^j(z))$ , where by convention  $T_\beta^0 = Id$ .

Let  $\beta > 1$  be a real number. A beta-representation (or  $\beta$ -representation, or representation in base  $\beta$ ) of a real number  $x \geq 0$  is given by an infinite sequence  $(x_i)_{i \geq 0}$  and an integer  $k \in \mathbb{Z}$  such that  $x = \sum_{i=0}^{+\infty} x_i \beta^{-i+k}$ , where the digits  $x_i$  belong to a given alphabet  $(\subset \mathbb{N})$  [23] [24] [34]. Among all the beta-representations of a real number  $x \geq 0, x \neq 1$ , there exists a particular one called Rényi  $\beta$ -expansion, which is obtained via the greedy algorithm: in this case,  $k$  satisfies  $\beta^k \leq x < \beta^{k+1}$  and the digits

$$(2.1) \quad x_i := \lfloor \beta T_\beta^i\left(\frac{x}{\beta^{k+1}}\right) \rfloor \quad i = 0, 1, 2, \dots,$$

belong to the finite canonical alphabet  $\mathbb{A}_\beta := \{0, 1, 2, \dots, \lceil \beta - 1 \rceil\}$ . If  $\beta$  is an integer, then  $\mathbb{A}_\beta := \{0, 1, 2, \dots, \beta - 1\}$ ; if  $\beta$  is not an integer, then  $\mathbb{A}_\beta := \{0, 1, 2, \dots, \lfloor \beta \rfloor\}$ . We denote by

$$(2.2) \quad \langle x \rangle_\beta := x_0 x_1 x_2 \dots x_k . x_{k+1} x_{k+2} \dots$$

the couple formed by the string of digits  $x_0 x_1 x_2 \dots x_k x_{k+1} x_{k+2} \dots$  and the position of the dot, which is at the  $k$ -th position (between  $x_k$  and  $x_{k+1}$ ). By definition the integer part (in base  $\beta$ ) of  $x$  is  $\sum_{i=0}^k x_i \beta^{-i+k}$  and its fractional part (in base  $\beta$ ) is  $\sum_{i=k+1}^{+\infty} x_i \beta^{-i+k}$ . If a Rényi  $\beta$ -expansion ends in infinitely many zeros, it is said to be finite and the ending zeros are omitted. If it is periodic after a certain rank, it is said to be eventually periodic (the period is the smallest finite string of digits possible, assumed not to be a string of zeros); for the substitutive approach see [19] [39].

The Rényi  $\beta$ -expansion which plays an important role in the theory is the Rényi  $\beta$ -expansion of 1, denoted by  $d_\beta(1)$  and defined as follows: since  $\beta^0 \leq 1 < \beta$ , the value  $T_\beta(1/\beta)$  is set to 1 by convention. Then using the formulae (2.1)

$$(2.3) \quad t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{ \beta \} \rfloor, t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor, \dots$$

The writing

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots$$

corresponds to

$$1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}.$$

Links between the set  $\mathbb{Z}_\beta$  of beta-integers and  $d_\beta(1)$  are evoked in [18] [21] [27] [26] [46]. A real number  $\beta > 1$  such that  $d_\beta(1)$  is finite or eventually periodic is called a *beta-number* or more recently a *Parry number* (this recent terminology appears in [18]). In particular, it is called a *simple beta-number* or a *simple Parry number* when  $d_\beta(1)$  is finite. Beta-numbers (Parry numbers) are algebraic integers ([38]) and all their conjugates lie within a compact subset which looks like a fractal in the complex plane [44]. The conjugates of beta-numbers are all bounded above in modulus by the golden mean  $\frac{1}{2}(1 + \sqrt{5})$  ([20] [44]).

### 3. Proof of Theorem 1.1

Since algebraic numbers  $\beta > 1$  for which the Rényi  $\beta$ -expansion  $d_\beta(1)$  of 1 is finite are excluded, we may consider that  $\beta$  does not belong to  $\mathbb{N}$ . Indeed, if  $\beta = h \in \mathbb{N}$ , then  $d_h(1) = 0.h$  is finite (Lothaire [34], Chap. 7). If  $\beta \notin \mathbb{N}$ , then  $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$  and the alphabet  $A_\beta$  equals  $\{0, 1, 2, \dots, \lfloor \beta \rfloor\}$ , where  $\lfloor \beta \rfloor$  denotes the greatest integer smaller than or equal to  $\beta$ .

Let  $f(z) := \sum_{i=1}^{+\infty} t_i z^i$  be the “gappy” power series deduced from the representation  $d_\beta(1) = 0.t_1 t_2 t_3 \dots$  associated with the  $\beta$ -shift (*gappy* in the sense of Theorem 1.1). Since  $d_\beta(1)$  is assumed to be infinite, its radius of convergence is 1. By definition, it satisfies

$$(3.1) \quad f(\beta^{-1}) = 1,$$

which means that the function value  $f(\beta^{-1})$  is algebraic, equal to 1, at the real algebraic number  $\beta^{-1}$  in the open disk of convergence  $D(0, 1)$  of  $f(z)$  in the complex plane. This fact is a general intrinsic feature of the Rényi expansion process which leads to the following important consequence by the theory of admissible power series of Mahler [35].

PROPOSITION 3.1. —

$$(3.2) \quad \limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} < +\infty.$$

*Proof.* — This is a consequence of Theorem 1 in [35]. Indeed, if we assume that there exists a sequence of integers  $(n_i)$  which tends to infinity such that  $\lim_{i \rightarrow +\infty} s_{n_i}/m_{n_i} = +\infty$ , then  $f(z)$  would be *admissible* in the sense of [35]. Since  $f(z)$  is a power series with nonnegative coefficients, which is not a polynomial, the function value  $f(\beta^{-1})$  should not be algebraic. But it equals 1. Contradiction.  $\square$

Let us improve Proposition 3.1. Assume that

$$(3.3) \quad \limsup \frac{s_n}{m_n} > \frac{\log(M(\beta))}{\log(\beta)}$$

and show the contradiction with (1.1) and (1.2). Recall that, if

$$P_\beta(X) = \sum_{i=0}^d \alpha_i X^i = \alpha_d \prod_{i=0}^{d-1} (X - \beta^{(i)})$$

with  $d \geq 1$ ,  $\alpha_0 \alpha_d \neq 0$ , denotes the minimal polynomial of  $\beta = \beta^{(0)} > 1$ , having  $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(d-1)}$  as conjugates, the Mahler measure of  $\beta$  is by definition

$$M(\beta) := |\alpha_d| \prod_{i=0}^{d-1} \max\{1, |\beta^{(i)}|\}.$$

Gütting [28] has shown that the approximation of algebraic numbers by algebraic numbers is fairly difficult to realize by polynomials. In the present proof, we use a version of Liouville's inequality which generalizes approximation theorems obtained by Gütting [28], and apply it to the values of the "polynomial tails" of the power series  $f(z)$  at the algebraic number  $\beta^{-1}$ , to obtain the contradiction. Let us write

$$(3.4) \quad f(z) = \sum_{n=0}^{+\infty} Q_n(z)$$

with

$$(3.5) \quad Q_n(z) := \sum_{i=s_n}^{m_{n+1}} t_i z^i, \quad n = 0, 1, 2, \dots$$

By construction the polynomials  $Q_n(z)$ , of degree  $m_{n+1}$ , are not identically zero and  $Q_n(1) > 0$  is an integer for all  $n \geq 0$ .

Denote by  $S_n(z) = -1 + \sum_{i=1}^{m_n} t_i z^i$  the  $m_n$ th-section polynomial of the power series  $f(z) - 1$  for all  $n \geq 1$ . Recall that, for  $R(X) = \sum_{i=0}^v \alpha_i X^i \in \mathbb{Z}[X]$ ,  $L(R) := \sum_{i=0}^v |\alpha_i|$  denotes the length of the polynomial  $R(X)$ . We have:  $L(S_n) = 1 + \sum_{i=1}^{m_n} t_i = 1 + \sum_{j=0}^{n-1} Q_j(1)$ . From Theorem 5 in [28] we deduce that only one of the following cases (G-i) or (G-ii) holds, for all  $n \geq 1$ :

$$(3.6) \quad (G-i) \quad S_n(\beta^{-1}) = 0,$$

$$(3.7) \quad (G-ii) \quad |S_n(\beta^{-1})| \geq \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(L(P_\beta^*)\right)^{m_n}},$$



where  $P_\beta^*(X) = X^d P_\beta(1/X)$  is the reciprocal polynomial of the minimal polynomial of  $\beta$ , for which  $L(P_\beta) = L(P_\beta^*) \in \mathbb{N} \setminus \{0, 1\}$ .

Case (G-i) is impossible for any  $n$ . Indeed, if there exists an integer  $n_0 \geq 1$  such that (G-i) holds, then, since all the digits  $t_i$  are positive and that  $\beta^{-1} > 0$ , we would have  $t_i = 0$  for all  $i \geq s_{n_0}$ . This would mean that the Rényi expansion of 1 in base  $\beta$  is finite, which is excluded by assumption. Contradiction. Therefore, the only possibility is (G-ii), which holds for all integers  $n \geq 1$ . From Lemma 3.10 and Liouville's inequality (Proposition 3.14) in Waldschmidt [47] the inequality (G-ii) can be improved to

$$(3.8) \quad (L - ii) \quad |S_n(\beta^{-1})| \geq \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} (M(\beta))^{m_n}}.$$

This improvement may be important; recall the well-known inequalities:

$$M(\beta) \leq L(P_\beta) \leq 2^{\deg(\beta)} M(\beta)$$

and see [47] p. 113 for comparison with different heights. On the other hand, since  $|S_n(\beta^{-1})| = \sum_{i=s_n}^{+\infty} t_i \beta^{-i}$  for all integers  $n \geq 1$ , we deduce

$$(3.9) \quad |S_n(\beta^{-1})| \leq \frac{\lfloor \beta \rfloor}{1 - \beta^{-1}} \beta^{-s_n} \quad n = 1, 2, \dots$$

Putting together (3.8) and (3.9), we deduce that

$$(3.10) \quad \frac{\beta^{s_n}}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} M(\beta)^{m_n}} \leq \frac{\lfloor \beta \rfloor}{1 - \beta^{-1}}$$

should be satisfied for  $n = 1, 2, 3, \dots$ . Denote

$$u_n := \frac{\beta^{s_n}}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} M(\beta)^{m_n}} \quad \text{for all } n \geq 1.$$

*Proof of (1.1):* from (3.3) assumed to be true there exists a sequence of integers  $(n_i)$  which tends to infinity and an integer  $i_0$  such that

$$\frac{s_{n_i}}{m_{n_i}} > \frac{\log(M(\beta))}{\log(\beta)} \quad \text{for all } i \geq i_0.$$

Now,

$$(3.11) \quad \left( \frac{1}{1 + \lfloor \beta \rfloor m_{n_i}} \right)^{d-1} \left( \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{M(\beta)} \right)^{m_{n_i}} \leq \frac{1}{\left( 1 + \sum_{j=0}^{n_i-1} Q_j(1) \right)^{d-1}} \left( \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{M(\beta)} \right)^{m_{n_i}} \leq u_{n_i}.$$

For  $i \geq i_0$  the inequality

$$(3.12) \quad 1 = \frac{\beta^{\frac{\log(M(\beta))}{\log(\beta)}}}{M(\beta)} < \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{M(\beta)}$$

holds. This implies that the left-hand side member of (3.11) tends exponentially to infinity when  $i$  tends to infinity. By (3.11) this forces  $u_{n_i}$  to tend to infinity. The contradiction now comes from (3.10) since the sequence  $(u_n)$  should be uniformly bounded.

*Proof of (1.2):* for  $n = 1, 2, \dots$ , let us rewrite the  $n$ -th quotient

$$(3.13) \quad \frac{u_{n+1}}{u_n} = \frac{\beta^{s_{n+1}-s_n}}{M(\beta)^{m_{n+1}-m_n}} \frac{\left( 1 + \sum_{j=0}^{n-1} Q_j(1) \right)^{d-1}}{\left( 1 + \sum_{j=0}^n Q_j(1) \right)^{d-1}}$$

as

$$(3.14) \quad \frac{u_{n+1}}{u_n} = \frac{\left( \frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{M(\beta)} \right)^{m_{n+1}-m_n}}{(m_{n+1} - m_n + 1)^{(d-1)}} \left[ \frac{(m_{n+1} - m_n + 1)^{(d-1)} \left( 1 + \sum_{j=0}^{n-1} Q_j(1) \right)^{d-1}}{\left( 1 + \sum_{j=0}^n Q_j(1) \right)^{d-1}} \right]$$

and denote

$$(3.15) \quad U_n := \frac{1}{(m_{n+1} - m_n + 1)^{(d-1)}} \left( \frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{M(\beta)} \right)^{m_{n+1}-m_n}$$

and

$$(3.16) \quad W_n := (m_{n+1} - m_n + 1)^{(d-1)} \left( \frac{1 + \sum_{j=0}^{n-1} Q_j(1)}{1 + \sum_{j=0}^n Q_j(1)} \right)^{d-1}$$

so that  $u_{n+1}/u_n = U_n W_n$ .

LEMMA 3.2. —

$$(3.17) \quad 0 < \liminf_{n \rightarrow +\infty} W_n$$

*Proof.* — Assume the contrary. Then there exists a subsequence  $(n_i)$  of integers which tends to infinity such that  $\lim_{i \rightarrow +\infty} W_{n_i} = 0$ . In other terms, for all  $\epsilon > 0$ , there exists  $i_1$  such that  $i \geq i_1$  implies  $W_{n_i} \leq \epsilon$ , equivalently

$$(3.18) \quad (m_{n_i+1} - m_{n_i} + 1) \left(1 + \sum_{j=0}^{n_i-1} Q_j(1)\right) \leq \epsilon^{\frac{1}{d-1}} \times \left(1 + \sum_{j=0}^{n_i} Q_j(1)\right).$$

Since, by hypothesis,  $t_{s_n} \geq 1$  and  $t_{m_n+1} \geq 1$  for all  $n \geq 1$ , we have:  $n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1)$ . On the other hand,  $Q_{n_i}(1) \leq \lfloor \beta \rfloor (m_{n_i+1} - m_{n_i} + 1)$ . Then, from (3.18) with  $\epsilon$  taken equal to 1, we would have

$$(3.19) \quad n_i \leq 1 + \sum_{j=0}^{n_i-1} Q_j(1) \leq \frac{Q_{n_i}(1)}{(m_{n_i+1} - m_{n_i} + 1) - 1} \leq \lfloor \beta \rfloor \times \frac{m_{n_i+1} - m_{n_i} + 1}{m_{n_i+1} - m_{n_i}} \leq \frac{3}{2} \lfloor \beta \rfloor.$$

But the left-hand side member of (3.19) tends to infinity which is impossible. Contradiction. □

Let us assume that (1.2) does not hold and show the contradiction ; that is, assume that  $\liminf_{n \rightarrow +\infty} (m_{n+1} - m_n) = +\infty$  and  $\limsup_{n \rightarrow +\infty} (s_{n+1} - s_n) / (m_{n+1} - m_n) > \log(M(\beta)) / \log(\beta)$  hold. Then

$$(3.20) \quad 1 = \frac{\beta^{\frac{\log(M(\beta))}{\log(\beta)}}}{M(\beta)} < \frac{\beta^{\frac{s_{n_i+1} - s_{n_i}}{m_{n_i+1} - m_{n_i}}}}{M(\beta)}$$

for some sequence of integers  $(n_i)$  which tends to infinity. This proves that  $\limsup_{n \rightarrow +\infty} U_n = +\infty$  since  $\lim_{i \rightarrow +\infty} U_{n_i} = +\infty$  exponentially, by (3.15) and (3.20).

By Lemma 3.2 there exists  $r > 0$  such that  $W_n \geq r$  for all  $n$  large enough. Therefore,  $u_{n+1}/u_n = U_n W_n \geq r U_n$  for all  $n$  large enough. Since  $\limsup_{n \rightarrow +\infty} U_n = +\infty$  we conclude that  $\limsup_{n \rightarrow +\infty} u_{n+1}/u_n = +\infty$ , hence that  $\limsup_{n \rightarrow +\infty} u_n = +\infty$ . This contradicts (3.10) and proves (1.2).

### 4. A direct proof of Corollary 1.2

Let  $\beta > 1$  be a Salem number such that  $\beta \notin C_1$ . Using the notations of Theorem 1.1 we show that the assumption

$$(4.1) \quad \limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} > 1$$

leads to a contradiction.

Denote by  $\mathbb{K}$  the algebraic number field  $\mathbb{Q}(\beta)$ , considered as a multivalued field with the product formula [15] [43] (see also [30]).

The present proof is merely an adaptation of that of Theorem 1 in [1], though the aims are different, and therefore does not merit publication. We simply point out a few hints for the interested reader.

The main result which is used is Corollary 1 of the Main Theorem in [15], as in [1]. This is a version of the Thue-Siegel-Roth Theorem given by Corvaja which is stronger than Roth Theorem for number fields [31] [43] to the extent it allows us to introduce a *missing proportion of places* of  $\mathbb{K}$  by considering the projective approximation of the point at infinity in  $\mathbb{P}^1(\mathbb{K})$ . Since  $\beta$  is a Salem number, it is a unit [4]. Hence, this missing proportion has just to be chosen among the pairwise distinct Archimedean places of  $\mathbb{K}$ .

## 5. On the class $\mathcal{Q}_0$

### 5.1. Perron numbers

Let us give, after Solomyak ([44], p 483), the example of a Perron number which is not a beta-number and therefore which is not in the class  $\mathcal{C}_2$ , without knowing whether it is in the class  $\mathcal{Q}_0$ . This example allows us to estimate the sharpness of the upper bound  $\log(M(\beta))/\log(\beta)$  in (1.1). Recall that a real number  $\beta > 1$  is a beta-number if the orbit of  $x = 1$  under the transformation  $T_\beta : x \rightarrow \beta x \pmod{1}$  is finite [34] [39]. The set of all conjugates of all beta-numbers is the union of the closed unit disc in the complex plane and the set of reciprocals of zeros of the function class  $\{f(z) = 1 + \sum a_j z^j \mid 0 \leq a_j \leq 1\}$ . The closure of this domain, say  $\Phi$ , is compact and was studied by Flatto, Lagarias and Poonen [20] and Solomyak [44]. After [44], the Perron number  $\beta = \frac{1}{2}(1 + \sqrt{13})$ , dominant root of  $P_\beta(X) = X^2 - X - 3$ , is not a beta-number, though its only conjugate  $\beta' = \frac{1}{2}(1 - \sqrt{13})$  lies in the interior  $\text{int}(\Phi)$ . We have  $M(\beta) = 3$ . By Theorem 1.1 the “quotients of the gaps” are asymptotically bounded above by  $\log(3)/\log(\beta) = 1.3171\dots$ , a much better bound than the degree  $d = 2$  of  $\beta$  (see Lemma 5.1). This does not suffice to conclude that  $\frac{1}{2}(1 + \sqrt{13})$  belongs to  $\mathcal{Q}_0$ .

Do all Perron numbers belong to  $\mathcal{Q}_0$ ? Let  $\beta > 1$  be a Perron number of degree  $d \geq 2$  and denote by  $\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(d-1)}$  the conjugates of  $\beta = \beta^{(0)}$ , roots of the minimal polynomial  $P_\beta(X)$  of  $\beta$ . Let  $K_\beta := \max\{|\beta^{(i)}| \mid i = 1, 2, \dots, d-1\}$ .

LEMMA 5.1. — *Let  $n = n_\beta$  (with  $2 \leq n_\beta \leq d$ ) be the number of conjugates of  $\beta$  of modulus strictly greater than unity (including  $\beta$ ). Then*

$$(5.1) \quad \frac{\log(M(\beta))}{\log(\beta)} \leq n - \frac{n-1}{(d\beta)^{6d^3} \log \beta}.$$

*Proof.* — Obvious since (Lemma 2 in [33]):  $K_\beta < \beta(1 - \frac{1}{(d\beta)^{6d^3}})$ .  $\square$

The upper bound (5.1) does not allow us to give a positive answer to the question and has probably to be improved.

## 5.2. Transcendental numbers

Let us show that the Komornik-Loreti constant [2] [29] belongs to  $\mathbb{Q}_0^{(1)}$ .

THEOREM 5.2. — *There exists a smallest  $q \in (1, 2)$  for which there exists a unique expansion of 1 as  $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$ , with  $\delta_n \in \{0, 1\}$ . Furthermore, for this smallest  $q$ , the coefficient  $\delta_n$  is equal to 0 (respectively, 1) if the sum of the binary digits of  $n$  is even (respectively, odd). This number  $q$  can then be obtained as the unique positive solution of  $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$ . It is equal to 1.787231650...*

This constant  $q$  is named Komornik-Loreti constant. Allouche and Cosnard [2] have shown the following result.

THEOREM 5.3. — *The constant  $q$  is a transcendental number, where the sequence of coefficients  $(\delta_n)_{n \geq 1}$  is the Prouhet-Thue-Morse sequence on the alphabet  $\{0, 1\}$ .*

The uniqueness of the development of 1 in base  $q$  given by Theorem 5.2 allows us to write

$$d_q(1) = 0.\delta_1\delta_2\delta_3\dots,$$

the coefficients  $\delta_n$  being the digits of the Rényi  $q$ -expansion of 1. Since the strings of zeros and 1's in the Prouhet-Thue-Morse sequence are known (Thue, 1906/1912; [3]) and uniformly bounded, the constant  $q$  belongs to the class  $\mathbb{Q}_0^{(1)}$ .

As second example, let us show that Sturmian numbers in the interval  $(1, 2)$  (in the sense of [14]) belong to  $\mathbb{Q}_0^{(1)}$ .

A real number  $\beta > 1$  is called a Sturmian number if  $d_\beta(1)$  is a Sturmian word over a binary alphabet  $\{a, b\}$ , with  $0 \leq a < b = \lfloor \beta \rfloor$ . Chi and Kwon [14] have shown the following theorem.

THEOREM 5.4. — *Every Sturmian number is transcendental.*

Let us consider all the Sturmian numbers  $\beta \in (1, 2)$  for which the two-letter alphabet is  $\{0, 1\}$ . For such numbers gappiness appears in  $d_\beta(1)$  (in the sense of Theorem 1.1). By Theorem 3.3 in [14] strings of zeros, resp. of 1's, cannot be arbitrarily long. This gives the claim.

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