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TILINGS ASSOCIATED WITH NON-PISOT MATRICES

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Abstract. — Suppose $A \in \text{Gl}_d(\mathbb{Z})$ has a 2-dimensional expanding subspace $E_u$, satisfies a regularity condition, called “good star”, and has $A^* \succeq 0$, where $A^*$ is an oriented compound of $A$. A morphism $\theta$ of the free group on $\{1, 2, \ldots, d\}$ is called a non-abelianization of $A$ if it has structure matrix $A$. We show that there is a tiling substitution $\Theta$ whose “boundary substitution” $\theta = \partial \Theta$ is a non-abelianization of $A$. Such a tiling substitution $\Theta$ leads to a self-affine tiling of $E_u \sim \mathbb{R}^2$ with $A_u := A|_{E_u} \in \text{GL}_2(\mathbb{R})$ as its expansion. In the last section we find conditions on $A$ so that $A^*$ has no negative entries.

Résumé. — Supposons que $A \in \text{Gl}_d(\mathbb{Z})$ ait un sous-espace d’extension bidimensionnel $E_u$, satisfaisant une condition de régularité, appelée “bonne étoile”, et telle que $A^* \succeq 0$, où $A^*$ est un composé orienté. Un morphisme $\theta$ du groupe libre sur $\{1, 2, \ldots, d\}$ est une non-abélianisation de $A$ si sa matrice de structure est $A$. Nous prouvons qu’il existe une substitution de pavage $\Theta$ dont la substitution de frontière $\theta = \partial \Theta$ est une non-abélianisation de $A$. Une telle substitution de pavage $\theta$ donne un pavage “auto-affine” de $E_u \sim \mathbb{R}^2$ avec pour expansion $A_u := A|_{E_u} \in \text{GL}_2(\mathbb{R})$. Dans la dernière section nous trouvons des conditions sur $A$ de sorte que $A^*$ n’ait pas de coefficients négatifs.

1. Introduction

A tiling substitution $\Theta$ is a mapping from tiles in $\mathbb{R}^2$ to finite tiling patches that has enough regularity to be extended to a mapping from tiling patches to tiling patches. This permits the iteration of $\Theta$, starting with a single tile, or a small tiling patch, to obtain larger tiling patches, and ultimately tilings of the plane. A tiling substitution $\Theta$ induces a mapping $\theta = \partial \Theta$ on tile boundaries. Assuming there are finitely many polygonal prototiles, having $d$ different boundary segments, this boundary map $\theta$ can

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be viewed as a 1-dimensional substitution on \( B := \{1, 2, \ldots, d\} \). The structure matrix of \( \theta \) is the matrix \( A \) whose \( i,j \)th entry counts the number of times the edge \( j \) occurs in the substitution \( \theta(i) \). Frequently, this matrix \( A \) is hyperbolic, and has a 2-dimensional expanding subspace. This corresponds to how \( \Theta \) “expands” \( \mathbb{R}^2 \). Iterating \( \Theta \) yields an uncountable collection \( X_{\Theta} \) of tilings of \( \mathbb{R}^2 \), which can be regarded as a compact metric space (see e.g., [16]), and on which \( \Theta \) is a “hyperbolic” self homeomorphism. In the best case, the action of \( \Theta \) on \( X_{\Theta} \) is semi-conjugate to the action of \( A \) on \( \mathbb{T}^d \) (as a toral automorphism). In fact, the tilings \( z \in X_{\Theta} \) are closely related to a Markov partition for \( A \) on \( \mathbb{T}^d \) (see e.g., [12], [10]).

In this paper we seek to reverse the process described above. Starting with a hyperbolic matrix \( A \in GL_d(\mathbb{Z}) \) having a 2-dimensional expanding subspace, we want: (1) to find a 1-dimensional substitution \( \theta \), or more generally a free group endomorphism \( \theta \), that has \( A \) as its structure matrix, and (2) find a tiling substitution \( \Theta \) with \( \partial \Theta = \theta \). As it turns out, not every \( A \) works, at least not for the method that we describe here. But we find a sufficient condition for \( A \) to work, namely that a certain “oriented compound” of \( A \), denoted \( A^* \), satisfies \( A^* \geq 0 \) (we also give a sufficient condition on \( A \) for \( A^* \geq 0 \)). Then, given a matrix \( A \) with \( A^* \geq 0 \), we construct what we call a properly ordered endomorphism \( \theta \), which has \( A \) as its structure matrix. Finally, we show how to use such a \( \theta \) to construct a tiling substitution \( \Theta \) with \( \partial \Theta = \theta \).

The idea of starting with a matrix, constructing a corresponding substitution, and using it to define geometric objects first appears in G. Rauzy [15], where the “Rauzy fractal” is introduced (a nice update on this theory appears in [3], highlighting its connections to number theory, dynamics and fractal geometry). The idea of using \( A \) and a 1-dimensional substitution to construct a tiling substitution appears many works by P. Arnoux, S. Ito and their co-workers (see e.g., [12], [4], [6], and [7]). An approach related to the one described here appears in [10].

It should be remarked, however, that all of the work mentioned above assumes some form of the Pisot condition. Recall that a Pisot number (or PV number) is real algebraic integer \( \lambda > 1 \) whose conjugates \( \lambda' \) satisfy \( |\lambda'| < 1 \). The companion matrix \( B \) for a Pisot number \( \lambda \) is a \( d \times d \) hyperbolic integer matrix with a 1-dimensional expanding subspace and contracting subspace of dimension \( d - 1 \). It is this codimension-1 hyperbolicity that is the crux of the Pisot condition. The importance of this condition for most approaches to tiling substitutions cannot be overstated. However, in this paper we do not need the Pisot condition. We also eliminate another
common assumption: that $A \geq 0$. A program similar to ours, (in particular, also without the Pisot assumption) appears in the dissertation [13] of R. Kenyon, but with few details. For other approaches to the non-Pisot case, see [11], [10] [8] and [9].

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2. Linear theory

2.1. The non-Pisot property

Let $A \in \text{Gl}_d(\mathbb{Z})$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ that satisfy
\begin{equation}
|\lambda_1| \geq |\lambda_2| > 1 > |\lambda_3| \geq \cdots \geq |\lambda_d| > 0.
\end{equation}
In particular, $A$ is nonsingular and hyperbolic and has a 2-dimensional expanding subspace, which we denote $E^u$. Let $E^s$ be the contracting subspace, which has dimension $d - 2$, and assume $d - 2 \geq 1$.

We say $A$ satisfies a Pisot condition if $d - 2 = 1$ (i.e., $E^u$ has codimension-1). A matrix $A$ satisfying (2.1) is called non-Pisot of order $n$ if $n = d - 2 > 1$. The main innovation in this paper is that we do not need to assume the Pisot condition. In particular, our examples typically have $d = 4$ and $n = d - 2 = 2$. However, we do not need to exclude the Pisot condition either.

2.2. The good star property

Let $v_1, v_2$ be an ordered basis for $E^u$ (for example, we could take $v_1$ and $v_2$ to be eigenvectors). We call a matrix $P \in \mathbb{R}^{2 \times d}$ a “projection” to $E^u$ if
\begin{equation}
Pv_1 = e_1 \text{ and } Pv_2 = e_2.
\end{equation}

It will often be convenient to identify $E^u$ with its $P$-image, which we think of as the plane $\mathbb{R}^2$. If $P'$ is another projection, corresponding to a different basis for $E^u$, then
\begin{equation}
P' = MP \text{ for } M \in \text{Gl}_2(\mathbb{R}).
\end{equation}
Note that a projection $P$ is onto $\mathbb{R}^2 \sim E^u$ and transverse to $E^s$. By appropriate choice of basis it may be made orthogonal to $E^u$ or parallel to $E^s$.\hfill \square
Let us write $P = (p^1, p^2, \ldots, p^d)$ in terms of its columns. Then $Pe_j = p^j$. In other words, the columns $p^j$ are projections of $e_j$ to $\mathbb{R}^2$. When we draw these vectors in $\mathbb{R}^2$, we refer to the picture as a star of vectors (see Figure 2.1).

**Definition 2.1.** — We call $P = (p^1, p^2, \ldots, p^d)$ a good star of vectors if

$$p^i = \omega p^j$$

for some real $\omega \neq 0$ implies $i = j$.

In other words, the vectors $p^i$ are pairwise non-parallel. We say a matrix $A$ satisfying (2.1) has the good star property if some projection $P = (p^1, p^2, \ldots, p^d)$ to $E^u$ is a good star of vectors.

By (2.3), if (2.4) holds for some projection $P$, then it holds for any projection. Hence the good star property depends only on $A$.

**Definition 2.2.** — Let $A$ satisfy (2.1) and (2.4), and let $P$ be a projection to $E^u$. Define $A_u \in \mathbb{R}^{2 \times 2}$ be the matrix conjugate to $A|_{E^u}$ via $P$. That is,

$$A_u := PA(v_1, v_2).$$

It is easy to see that $A_u$ has eigenvalues $\lambda_1, \lambda_2$.

**2.3. The compound**

Let $B \in \mathbb{C}^{p \times q}$ (the set of $p \times q$ complex matrices), with $p, q \geq n$. Define the $n$th compound of $B$ to be the matrix $C_n(B) \in \mathbb{C}^{(n) \times (n)}$ whose entries are the $n \times n$ minors of $B$ (see [2]). In this paper, $n = 2$. We index $C_2(B)$ by pairs $i \wedge j, k \wedge \ell$, where $i < j$ and $k < \ell$, in lexicographic order.

**Theorem 2.3** (The Binet-Cauchy Theorem, see [2]). — If $B = B_1B_2$ then $C_n(B) = C_n(B_1)C_n(B_2)$. If $B$ is non-singular then $C_n(B^{-1}) = C_n(B)^{-1}$.

Let $P \in \mathbb{R}^{2 \times d}$ be the projection for a matrix $A$ satisfying (2.1) and (2.4). The compound $C_2(P) \in \mathbb{R}^{1 \times \binom{d}{2}}$ has entries

$$p_{i \wedge j} := \det(p^i, p^j) = \sin(\angle p^i p^j) \neq 0,$$

where the last inequality follows $\angle p^i p^j \in (-\pi, 0) \cup (0, \pi)$ by (2.4).
For \( p \neq 0 \), let \( \text{sgn}(p) = p/|p| \). For a vector \( \mathbf{a} \in \mathbb{C}^n \), let \( \text{diag}(\mathbf{a}) \) denote the \( n \times n \) matrix with the entries of \( \mathbf{a} \) along the diagonal, and zeros everywhere else. Define

\[
S(A) = \text{diag}((\text{sgn}(p_1 \land 2), \text{sgn}(p_1 \land 3), \ldots, \text{sgn}(p_{(d-1) \land d}))) \in \{-1, 0, 1\}^{(d^2) \times (d^2)}.
\]

The Binet-Cauchy Theorem and (2.3) imply that \( S(A) \) is well defined up to a change of sign. Define

\[
A^* = S(A)C_2(A)S(A).
\]

This is the matrix with entries \( a_{i \land j, k \land \ell} \text{sgn}(p_{i \land j})\text{sgn}(p_{k \land \ell}) \), where \( a_{i \land j, k \land \ell} \) are the entries of \( C_2(A) \). We call \( A^* \) the oriented compound of \( A \).

From now on, in addition to (2.1) and (2.4), we will assume:

\[
A^* \succeq 0,
\]

i.e., \( A^* \) has no negative entries. It is easy to see that this does not depend on the choice of the basis for \( E^u \).

2.4. Example: The Ammann matrix

Consider the matrix

\[
A = \begin{pmatrix}
-1 & 1 & 0 & -1 \\
0 & -1 & -1 & 0 \\
-1 & 0 & -1 & -1 \\
1 & -1 & -1 & 0
\end{pmatrix},
\]

which is related to the tiling sometimes called the Ammann-Beenker tiling (see [17], [11]). Note that \( A \) is symmetric and has characteristic polynomial \( p(x) = (x^2 + 2x - 1)^2 \). The eigenvalues are \( \lambda_1 = \lambda_2 = -1 - \sqrt{2} \) and \( \lambda_3 = \lambda_4 = \sqrt{2} - 1 \), so \( A \) satisfies (2.1).

Let

\[
Q = (q_1, q_2, q_3, q_4) = \begin{pmatrix}
\sqrt{2} & -1 & -\sqrt{2} & -1 \\
-1 & \sqrt{2} & -1 & -\sqrt{2} \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix},
\]

where the columns \( q_j \) are eigenvectors for \( \lambda_j \). Define the projection \( P \) to be the the first two rows of \( Q^{-1} \):

\[
P = \frac{\sqrt{2}}{4} \begin{pmatrix}
1 & 0 & 1 & \sqrt{2} \\
0 & 1 & \sqrt{2} & 1
\end{pmatrix} = (p^1, p^2, p^3, p^4),
\]
Figure 2.1. The good star of vectors corresponding to the Ammann matrix $A$ in (2.10).

The columns of $P$ are plotted in Figure 2.1. Clearly $A$ satisfies the good star property (2.4).

We have $A_u = \lambda_1 \text{Id} = -(1 + \sqrt{2}) \text{Id}$, which we interpret as a composition of an expansion of $\mathbb{R}^2$ by $1 + \sqrt{2}$, and a rotation by $\pi$. Applying (2.7) we find

\begin{equation}
S(A) = \text{diag}(1, 1, 1, -1, -1, -1),
\end{equation}

so that by (2.8)

\begin{equation}
A^* = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix} \succeq 0.
\end{equation}

Thus $A$ satisfies (2.9). We shall return to this example below.

3. Proto-objects and objects

In the next few sections we are going to consider geometric objects $g$ in $\mathbb{R}^2$, including curves, closed curves, tiles, tiling patches and tilings. Each object $g$ will have a particular location in $\mathbb{R}^2$, and we can move an object by a translation. In particular, $g + w$ denotes the translation of $g$ by the vector $w \in \mathbb{R}^2$. We call two objects $g_1$ and $g_2$ translationally equivalent if $g_1 = g_2 + w$ for some $w \in \mathbb{R}^2$. We denote the translational equivalence class of the object $g$ by $G$, and refer to $G$ as a proto-object. Thus, we will refer to proto-curves, prototiles, etc..
It will be convenient to represent each proto-object $G$ by an arbitrary fixed choice of a geometric object from its translational equivalence class. We think of this object as being “located at the origin” in $\mathbb{R}^2$. Let $\mathcal{G}$ denote the set of all proto-objects $G$. Each object $g$ is then a translation $g = G + w := (w, G)$ of the corresponding proto-object $G \in \mathcal{G}$ by a unique vector $w \in \mathbb{R}^2$.

### 3.1. Words and substitutions

Let $B = \{1, 2, \ldots, d\}$. Let $B^*$ denote the free semi-group of nonempty finite words in $B$. Let $B^\pm = \{i^\pm : i \in B\}$, and let $\mathcal{F}(B)$ denote the free group on $B$, which we think of as the set of reduced words in $(B^\pm)^*$, together with the empty word $\epsilon$.

A substitution is a mapping $\theta : B \to B^*$. Since $B$ is a basis for both $B^*$ and $\mathcal{F}(B)$, a substitution uniquely defines both a semigroup and a group endomorphism. More generally, any mapping $\theta : B \to \mathcal{F}(B)$ defines an endomorphism of $\mathcal{F}(B)$. We always assume an endomorphism is non-erasing, which means $\theta(i) \neq \epsilon$. We think of a non-erasing endomorphism a sort of “generalized substitution”, but refrain from using this language to avoid confusion with its other uses.

The abelianization homomorphism $f : \mathcal{F}(B) \to \mathbb{Z}^d$ is defined on $B$ by $f(i) = e_i$, and extended to $\mathcal{F}(B)$. A endomorphism of $\mathbb{Z}^d$ is given by a matrix $M \in \mathbb{Z}^{d \times d}$. For any endomorphism $\theta$ of $\mathcal{F}(B)$, the abelianization $f$ induces an (abelian group) endomorphism of $\mathbb{Z}^d$, denoted by $L_\theta$, satisfying $L_\theta \circ f = f \circ \theta$. In particular,

$$L_\theta = (f(\theta(1)), f(\theta(2)), \ldots, f(\theta(d))).$$

This matrix is also called the structure matrix of $\theta$, or more formally the abelianization of $\theta$. We note that if each $\theta(i)$ is an efficient word (see Definition 5.3 below), then entry $\ell_{i,j}$ of $L_\theta$ equals the signed number times that $i$ appears in $\theta(j)$ (in particular, this always holds if $\theta$ is a substitution).

For a given matrix $A$, we call any endomorphism $\theta$ that satisfies $L_\theta = A$ a non-abelianization of $A$.

### 3.2. Curves

A curve is continuous, piecewise $C^1$ mapping $w : [a, b] \to \mathbb{R}^2$. Two curves are considered the same if they differ by an orientation preserving piecewise
A change of parameterization. The reverse of a curve $w$, denoted $-w$, is given by $-w(t) = w(-t + a + b)$.

For $x_0, x_1 \in \mathbb{R}^2$ the linear curve from $x_0$ to $x_1$ is defined to be $w(t) = (1-t)x_0 + tx_1$, for $t \in [0, 1]$. Similarly, for a sequence of points $x_0, x_2, \ldots, x_\ell$ we define the piecewise linear curve (or "broken line") connecting the points by $w : [0, \ell] \rightarrow \mathbb{R}^2$ where

$$w(t) = (1 - (t - j))x_j + (t - j)x_{j+1} \text{ for } t \in [j, j+1].$$

We will usually restrict our attention to the following case. Let $P = (p_1, p_2, \ldots, p_d)$ be a good star of vectors. For each $k = 1, \ldots, \ell$, suppose there exists $i_k \in \{1, \ldots, d\}$ and a nonzero $a_k \in \mathbb{R}$ so that

$$(3.2) \quad x_k - x_{k-1} = a_k p^{i_k}.$$ 

In effect, we consider piecewise linear curves whose segments are parallel to the vectors $p_1, \ldots, p_d$.

Now suppose $W \in (B^\pm)^*$ and $x \in \mathbb{R}^2$. In particular, $W = i_1^{a_1} i_2^{a_2} \ldots i_\ell^{a_\ell}$, where $i_k \in B$ and $a_k \in \mathbb{Z}$, $a_k \neq 0$. Define a finite sequence of points $x_0, x_1, \ldots, x_\ell$ inductively, by putting $x_0 = x$, and applying (3.2) for $k = 1, \ldots, \ell$. We define the curve $w := (x, W)$ to be the piecewise linear curve connecting the points $x_0, x_1, \ldots, x_\ell$. In this way a word is a proto-curve.

A curve $w : [a, b] \rightarrow \mathbb{R}^2$ is closed if $w(a) = w(b)$ (we sometimes think of it as mapping $w : S^1 \rightarrow \mathbb{R}^2$). A simple closed curve is a closed curve that is injective on $S^1$. We say $W \in (B^\pm)^*$ is a cyclic if the corresponding curve $w = (x_0, W)$ is closed. This is equivalent to $x_\ell = x_0$. Since

$$(3.3) \quad x_\ell - x_0 = f(W),$$

the curve $w$ is closed, or equivalently the word $W$ is a cyclic, if and only if $f(W) = 0$. A cyclic word is a proto-closed curve. We define the commutator of two words $W_1, W_2$ by

$$(3.4) \quad W = [W_1, W_2] := W_1 W_2 W_1^{-1} W_2^{-1}.$$ 

Note that $f([W_1, W_2]) = f(W_1 W_2 W_1^{-1} W_2^{-1}) = 0$, so that a commutator is always cyclic.

3.3. A topological proposition

Unfortunately, it is not easy to tell if a cycle $W$ corresponds to a simple close curve. However, in this section we obtain a partial result in the case $W = [W_1, W_2]$.  

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The Jordan curve theorem (see e.g., [1]) says that if \( w : S^1 \to \mathbb{R}^2 \) is a simple closed curve then \( \mathbb{R}^2 \setminus w(S^1) \) consists of two open sets: one bounded, called the inside of \( w \), and one unbounded, called the outside. More generally, if \( w \) is piecewise linear closed curve that is not necessarily simple, then \( \mathbb{R}^2 \setminus w(S^1) \) consists of a finite collection of open sets: the outside, which is unbounded, and a finite number of bounded open sets, called the components of the inside.

A chain is a finite sum of the form

\[
(3.5) \quad w = n_1 w_1 + n_2 w_2 + \cdots + n_k w_k,
\]

where each \( w_k \) is a different (not necessarily closed) curve in the plane, and \( n_k \in \mathbb{Z} \) (see [1]). There is a fairly obvious equivalence relation on chains, which in addition to orientation preserving re-parameterization, allows the combination of curves by following a succession of them. Line integrals of real or complex functions \( F \) in the plane are defined over chains. It can be shown that two chains are equivalent if and only if they yield the same line integral for every function \( F \), (see [1]). Equivalent chains are considered to be equal.

A chain is called a cycle if (up to equivalence) each \( w_i \) is a closed curve. We assume in addition that each \( w_i \) is piecewise linear satisfying (3.2). A cycle is a generalization of a closed curve, and like in that case, \( \mathbb{R}^2 \setminus \bigcup_{i=1}^k w_k(S^1) \) consists of a finite collection of open sets: an unbounded outside, and a finite number of bounded inside components. The union of the components is called the inside of the curve or the cycle. The set \( \bigcup_{i=1}^k w_k(S^1) \) is called the trace.

Given a cycle \( w \), and \( x \in \mathbb{R}^2 \) not in the trace, the winding number is defined

\[
(3.6) \quad n_w(x) = \frac{1}{2\pi i} \int_w \frac{dz}{z-x}.
\]

Here we think of \( x \) and \( z \) as a complex numbers. The integral is carried out separately over each closed curve in (3.5), and the results are added.

The basic property of the winding number function \( n_w \) is that it is integer valued, constant on each component of the inside of \( w \), and zero outside (see [1]). It can be shown that two cycles are equivalent if and only if they assign the same winding numbers to all non-trace points (this follows from the generalized Cauchy integral formula, [1]). The change of variables formula for integrals shows that \( n_{w+y}(x+y) = n_w(x) \), and \( n_{-w}(x) = -n_w(x) \). It is easy to see that \( n_{w_1+w_2}(x) = n_{w_1}(x) + n_{w_2}(x) \) provided \( x \) is not in the trace of \( w_1 \) or \( w_2 \).
Let \( w \) be a simple closed curve. Then \( n_w(x) = \pm 1 \) for \( x \) inside \( w \) (see [1]). If \( n_w(x) = 1 \) for \( x \) inside \( w \), we say \( w \) is positive (i.e., it is positively oriented). More generally, we say a (not necessarily simple) closed curve \( w \), or even a cycle \( w \), is positive if \( n_w(x) \geq 0 \) for all \( x \in \mathbb{R}^2 \) not in the trace.

We call a closed curve or cycle \( w \) positive semi-simple if \( n_w(x) \in \{0, 1\} \) for all \( x \in \mathbb{R}^2 \) not in the trace.

**Proposition 3.1.** — Let \( P = (p^1, p^2, \ldots, p^d) \) be a good star of vectors and let \( f : \mathcal{F}(\mathcal{B}) \to \mathbb{Z}^d \) be the abelianization homomorphism. Suppose \( W = [W_1, W_2] \) is the commutator of \( W_1, W_2 \in \mathcal{F}(\mathcal{B}) \), and let \( w = (0, W) \) be the corresponding closed curve. Assume \( w \) is positive and that

\[
(3.7) \quad \det \left( [ Pf(W_1), Pf(W_2) ] \right) > 0.
\]

Then \( w \) is positive semi-simple.

**Proof.** — For simplicity we identify \( W = w \). Let \( v_1 = Pf(W_1) \) and \( v_2 = Pf(W_2) \), which are linearly independent vectors by (3.7). Let \( R \) be the piecewise linear curve in \( \mathbb{R}^2 \) connecting the points \( 0, v_1, v_1 + v_2, v_2, 0 \), and let \( R_1, R_2, R_3, \) and \( R_4 \) denote the four linear segments that make up \( R \). It follows from (3.7) that \( R \) is a positive simple closed curve, and thus \( n_R(x) = 1 \) for \( x \) inside \( R \) and \( n_R(x) = 0 \) outside. We also divide \( W \) into four segments: \( W_1, W_2, W_1^{-1}, \) and \( W_2^{-1} \), corresponding to the four factors of \( W \) with the same names. Note that for each \( i \), the curves \( W_i \) and \( R_i \) start and end at the same place.

Starting at 0, follow \( W_1 \). For a while it may follow \( R_1 \), but assuming, \( W_1 \neq R_1 \), the two curves eventually part. Continue following \( W_1 \) until its first return to \( R_1 \). Call this point \( z_1 \). Then follow \( R_1 \) back to \( 0 \). Call the

![Figure 3.1. The curve \( W = [W_1, W_2] \) and its “linearization” \( R \).](image)
resulting closed curve \( Z_1 \). Let \( Z_1' = -Z_1 + v_2 \) be the reflection of \( Z_1 \) across \( R \).

Next, starting at \( z_1 \), repeat the previous construction. Follow \( W_1 \) until it leaves \( R_1 \) and then returns for the first time at \( z_2 \). Then follow \( R_1 \) back to \( z_1 \). Call the resulting closed curve \( Z_2 \), and define its reflection \( Z_2' = -Z_2 + v_2 \). Continuing in this fashion we get a sequence \( Z_1, Z_2, \ldots, Z_{\ell_1} \) of closed curves. We stop when \( z_{\ell_1} = v_1 \). We also obtain their reflections \( Z_1', Z_2', \ldots, Z_{\ell_1}' \).

Repeat construction with \( W_2 \) and \( R_2 \) to obtain simple closed curves \( Z_{\ell_1+1}, \ldots, Z_{\ell_2} \), and their reflections \( Z_{\ell_1+1}', \ldots, Z_{\ell_2}' \), where \( Z_k' = -Z_k - v_1 \) for \( k = \ell_1 + 1, \ldots, \ell_2 \).

Let \( \Lambda = \{nv_1 + mv_2 : n, m \in \mathbb{Z} \} \) be the lattice in \( \mathbb{R}^2 \) generated by \( v_1 \) and \( v_2 \). Consider the tessellation \( \tilde{R} \) of \( \mathbb{R}^2 \) by the parallelograms \( \Lambda + R \). We can assume without loss of generality that each closed curve \( Z_i \) lies inside one of the parallelograms in \( \tilde{R} \). If not, we can subdivide \( Z_i \) into a sum of closed curves (i.e., a cycle) that follow \( Z_i \) and the lines in \( \tilde{R} \). Simultaneously, we do the same thing to each \( Z_i' \) in reverse.

Now for convenience, we renumber all the reflections

\[ Z_{\ell_1}' , Z_{\ell_1+1}' , \ldots , Z_{\ell_2}' , Z_{\ell_1+\ell_2} \]

as

\[ Z_{\ell_1+\ell_2+1} , Z_{\ell_1+\ell_2+2} , \ldots , Z_{2(\ell_1+\ell_2)} \].

Thus the closed curves \( Z_1, \ldots, Z_{2(\ell_1+\ell_2)} \) come in pairs of reflections (with opposite orientations), and each lies in just one parallelogram of \( \tilde{R} \).

Define the chain

\[
Z := \sum_{j=1}^{2(\ell_1+\ell_2)} Z_j.
\]

Because of the way the curves \( Z_i \) were constructed, \( W = R + Z \). It follows that

\[
n_W(x) = n_R(x) + n_Z(x),
\]

and by (3.8),

\[
n_Z(x) = \sum_{j=1}^{2(\ell_1+\ell_2)} n_{Z_j}(x),
\]

for (non-trace) \( x \in \mathbb{R}^2 \).

Let \( x_0 \in \mathbb{R}^2 \) be such that no \( x \in \Lambda + x_0 \) is in the trace of \( W, Z, R \) or \( \tilde{R} \). Let

\[ I_{x_0} = \{ j : \exists x \in \Lambda + x_0 \text{ with } n_{Z_j}(x) \neq 0 \}. \]
For each $j \in I_{x_0}$ there is a unique $x_j \in \Lambda + x_0$ so that $n_{Z_j}(x_i) \neq 0$. This is because each $Z_j$ lies in a single parallelogram form $\tilde{R}$. Using (3.10) we have

$$\sum_{x \in \Lambda + x_0} n_{Z}(x) = \sum_{x \in \Lambda + x_0} \sum_{j=1}^{2(\ell_1 + \ell_2)} n_{Z_j}(x) = \sum_{j=1}^{2(\ell_1 + \ell_2)} \sum_{x \in \Lambda + x_0} n_{Z_j}(x) = \sum_{j \in I_{x_0}} n_{Z_j}(x_j).$$

(3.11)

For each $j \in I_{x_0}$ there exists $j'$ so that $Z_{j'}$ is the reflection of $Z_j$. Since $Z_{j'} = -Z_j + (x_{j'} - x_j)$, it follows that $n_{Z_{j'}}(x_{j'}) = -n_{Z_j}(x_j)$. This implies the last sum in (3.11) is zero, so that

$$\sum_{x \in \Lambda + x_0} n_{Z}(x) = 0.$$ 

(3.12)

Suppose that $n_W(x_0) > 1$, and consider two cases: (i) $x_0$ is inside $R$ and (ii) $x_0$ is not inside $R$.

In case (i), (3.9) and the fact that $R$ is a positive simple closed curve implies $n_Z(x_0) \geq 1$. It also follows that $n_W(x) = n_Z(x)$ for $x \in \Lambda + x_0$, $x \neq x_0$. By (3.12) there exists such an $x$ so that $n_W(x) = n_Z(x) \leq 1 < 0$. This contradicts the fact that $W$ is positive.

In case (ii), $n_Z(x_0) = n_W(x_0) \geq 2$. It follows that either: (a) there exists $x \in \Lambda + x_0$, $x \neq x_0$, so that $n_Z(x) \leq 2$, or (b) there exist two distinct $x_1, x_2 \in \Lambda + x_0$, not equal to $x_0$, so that $n_Z(x_1) \leq 1$ and $n_Z(x_2) \leq 1$. In case (a), (3.9) implies $n_W(x) < 1$ since $n_R(x) \leq 1$. In case (b) we have $n_R(x_i) = 1$ for $i = 1$ or $i = 2$ only if $x_i$ is inside $R$. This can happen for at most one of the two, since they differ by $\Lambda$. Otherwise $n_R(x_i) = 0$. Then (3.9) implies that $n_W(x_i) < 0$ for at least one of the $x_i$. In both cases we again contradict the fact that $W$ is positive. \hfill \Box

3.4. Tiles, tilings and tiling patches

Let $B^2$ denote the set of all pairs $i \wedge j$ for $i, j \in B$. Define

$$\text{sgn}(i \wedge j) := \text{sgn}(p_{i \wedge j}) = \text{sgn} \left( \sin \left( \angle p^i p^j \right) \right).$$

(3.13)

By (2.6), $\text{sgn}(i \wedge j) \in \{-1, 1\}$ if $i \neq j$ and $\text{sgn}(i \wedge i) = 0$. For $i < j$, (3.13) shows that the entries of the matrix $S(A)$ in (2.7) are $\text{sgn}(i \wedge j)$. When
sgn\((i \wedge j) \neq 0\) we write \(j \wedge i = -(i \wedge j)\). More generally, for \(a, b \in \{-1, 1\}\), we simplify \(i^a \wedge j^b = ab(i \wedge j)\). We call \(i \wedge j\) a positive prototile if \(\text{sgn}(i \wedge j) = 1\), and a negative prototile if \(\text{sgn}(i \wedge j) = -1\). In the latter case, \(j \wedge i = -(i \wedge j)\) is positive. The set of all positive prototiles is denoted by \(B_2^+\). Prototiles \(i \wedge i\) are neither positive or negative, and are called trivial. We simplify \(i^a \wedge i^b = i \wedge i\). Thus \(B^2\) denotes the set of all prototiles.

Figure 3.2. The positive prototiles for the matrix \(A\) in (2.10): \(1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 3 \wedge 2, 4 \wedge 2, 4 \wedge 3\).

Definition 3.2. — Let \(W \in F(B)\) be a positive semi-simple word (i.e., any \((x, W)\) is a positive semi-simple curve). For \(x \in \mathbb{R}^2\), define the tile \(t = (x, W)\) to be the union of the inside of the curve \((x, W)\) with its trace. We write \(\partial(x, W) = (x, W)\) for the boundary of the tile.

Now let \(i \wedge j \in B^2\). To define the geometric realization of the tiles corresponding to \(i \wedge j\), we first define the boundary of \(i \wedge j\) by

\[
\partial(i \wedge j) = [i, j] = iji^{-1}j^{-1}.
\]

We define the tile

\[
t = (x, i \wedge j) := (x, \overline{[i, j]}).
\]

We define the boundary \(\partial(t) = \partial(x, i \wedge j) := (x, iji^{-1}j^{-1})\), which is a piecewise linear simple closed curve (i.e., a parallelogram). The four oriented segments of this curve are called the edges of the tile, and are denoted \(E((x, i \wedge j))\). Two tiles \(t_1, t_2\) are said to be adjacent if \(-E(t_1) \cap E(t_2) \neq \emptyset\), i.e., the two tiles share an oppositely oriented edge.

Now consider a finite sum\(^{(1)}\)

\[
y = n_1 t_1 + n_2 t_2 + \cdots + n_\ell t_\ell
\]

of different tiles \(t_k = (x_k, i_k \wedge j_k)\). Assume that if any two tiles in (3.14) intersect at more than a vertex, they are adjacent. Define \(\partial(y) = \sum n_k \partial(t_k)\), which is a cycle. If the cycle \(\partial(y)\) is positive semi-simple, then we say \(y\) is a (positive) tiling patch. Note that this implies that \(n_1 = n_2 = \cdots = n_\ell = 1\). Tiling patches, modulo translation are, called tiling proto-patches, and are denoted by \(B_2^+\).

\(^{(1)}\)This is in the same spirit as a cycle as a sum of closed curves.
3.5. Positive tiling substitutions

Our goal in this section is to give a definition of a positive tiling substitution analogous to the definition $\theta : \mathcal{B} \to \mathcal{B}^*$ of an ordinary substitution. Basically it will be a mapping $\Theta : \mathcal{B}_+^2 \to (\mathcal{B}_+^2)^*$. But, whereas in the case of a substitution, essentially any mapping works, we have two additional requirements for a tiling substitution: (a) $\Theta$ needs to be able to apply to tiles (i.e., translations of prototiles), and (b) $\Theta$ needs to apply consistently to adjacent tiles in a tiling patch, and still output a tiling patch. We will achieve these two goals using the next definition.

**Definition 3.3.** — A tiling substitution is a mapping $\Theta : \mathcal{B}_+^2 \to (\mathcal{B}_+^2)^*$ such that there is a morphism $\theta : \mathcal{B} \to \mathcal{F}(\mathcal{B})$ with the property that

$$\partial(\Theta(i \land j)) = \theta([i, j]) \text{ for each } i \land j \in \mathcal{B}_+^2.$$  \hfill (3.15)

We abbreviate (3.15) $\theta = \partial \Theta$. In effect, this says that $\Theta(i \land j)$ should be a tiling of the inside of the curve $\theta([i, j])$. Since $\Theta(i \land j)$ is a positive tiling patch, it follows that its boundary $\theta([i, j])$ is a positive semi-simple curve. More generally, we define $\Theta(x, i \land j) := (A_ux, \Theta(i \land j))$.

**Lemma 3.4.** — Let $\Theta$ be a positive tiling substitution. Suppose $y = t_1 + t_2 + \cdots + t_\ell \in (\mathcal{B}_+^2)^*$, i.e., $y$ is a positive tiling patch. Define $\Theta(y) = \Theta(t_1) + \Theta(t_2) + \cdots + \Theta(t_\ell)$. Then $\Theta(y) \in (\mathcal{B}_+^2)^*$. i.e., $\Theta(y)$ is a positive tiling patch.

**Corollary 3.5.** — The iterates $\Theta^n(i \land j)$ for $i \land j \in \mathcal{B}_+^2$, and $\Theta^n(y)$, for $y \in (\mathcal{B}_+^2)^*$, are well defined for all $n \geq 0$.

**Proof of Lemma 3.4.** — For $i \land j$ define $A_u(i \land j)$ to be the tile whose boundary is the curve connecting the points $0, p^i, p^i + p^j, p^j, 0$. For $t_k = (x_k, i_k \land j_k)$ in $y$, let $A_ut_k = (A_ux_k, A_u(i_k \land j_k))$, and let $A_uy = A_u(t_1 + \cdots + A_ut_\ell)$. Then $A_uy$ is a tiling patch.

Similarly, define $\theta(t_k) = \theta(x_k, i_k \land j_k) := (A_ux_k, [\theta(i_k), \theta(j_k)])$ using Definition 3.2, and put $\theta(y) = \theta(t_1) + \cdots + \theta(t_\ell)$. Each of these tiles has four “edges”, which along the boundary are labeled $\theta(i_k), \theta(j_k), \theta(i_k^{-1})$ and $\theta(j_k^{-1})$.

We have

$$\partial(\theta(t_k)) = (A_ux_k, [\theta(i_k), \theta(j_k)])$$

$$= (A_ux_k, \theta([i_k, j_k]))$$

$$= \partial(\Theta(x_k, i_k \land j_k)).$$  \hfill (3.16)
Since $\Theta$ is a positive tiling substitution, its boundary is a positive semi-simple closed curve. Thus by (3.16), $\partial(\theta(t_k))$ is a positive semi-simple closed curve, and $\theta(t_k)$ is a tile.

If $t_h$ and $t_k$ are adjacent in $y$ then $A_u t_h$ and $A_u t_k$ are adjacent in $A_u y$. Moreover, both endpoints of each edge of $A_u t_k$ are endpoints of the edges of $\theta(t_k)$. Thus $\theta(t_h)$ and $\theta(t_k)$ are adjacent in $\theta(y)$. It follows that $\theta(y)$ is a “positive tiling patch” by the tiles $\theta(t_k)$.

Each term $\Theta(t_k)$ in $\Theta(y)$ is a tiling patch with boundary $\partial(\theta(x_k, i_k \wedge j_k))$, which by (3.16) is $\partial(\theta(t_k))$. Since they only overlap on their edges (with opposite orientations), their sum is positive semi-simple closed curve, and it follows that $\Theta(y)$ is a positive tiling patch. □

4. de Bruijn diagrams

4.1. The definition of a de Bruijn diagram

Starting with a cyclic word $W \in (B^\pm)^*$, we will construct an object $Y$, called a de Bruijn diagram, which depends (in a non-unique way) on $W$. We will think of such a diagram as a combinatorial representation of a tiling patch with $W$ as its boundary. However, we will need to drop the restriction that a patch be a tiling by positive tiles, and allow negative and trivial tiles as well.

Suppose $W = i_1 a_1 i_2 a_2 \ldots i_n a_n$. Let $c : [0, 1] \to \mathbb{R}^2$ be a positive simple closed curve. Starting at $c(0)$ we follow $c$ and attach $n$ arrows normal to $c$, labeled $i_1, i_2, \ldots, i_n$. We make the $k$th arrow point in if $a_k = 1$ and point out if $a_k = -1$ (see Figure 4.1 (a)). This is called the frame of the (still to be defined) de Bruijn diagram $Y$. We denote the frame by $\partial(Y)$. Up to a diffeomorphism of $\mathbb{R}^2$, $\partial(Y)$ is uniquely defined by $W$.

Next we describe how to get from the frame $\partial(Y)$ to the diagram $Y$. Since $W$ is cyclic, $f(W) = 0$, and thus for each $i \in B$ there are the same number of $i$ labeled in- and out-arrows. We choose some arbitrary matching of these and connect each matched pair by a non-self intersecting curve through the inside of $c$, called a pseudo-line. Assume that at most two pseudo-lines cross at any point, and any two pseudo-lines cross transversally. The resulting picture is a de Bruijn diagram $Y$ (see Figure 4.1).

Each pseudo-line in $Y$ is labeled by some $i \in B$, and is oriented by its arrows. Two diagrams that differ by an orientation preserving diffeomorphism of $\mathbb{R}^2$ are considered the same. The frame $\partial(Y)$ of a de Bruijn
Figure 4.1. (a) The frame for $W = [W_1, W_2] = 1223^{-1}2^{-1}1^{-1}32^{-1}$ where $c$ is a square. Two corresponding de Bruijn diagrams: (b) the product $W_1 \wedge W_2$, and (c) the only other possibility in this case.

diagram $Y$ determines the cyclic word $W$ up to a cyclic permutation. We write $\partial(Y) = W$.

The vertices of a de Bruijn diagram $Y$ are defined to be the crossings of its pseudo-lines. A diagram $Y$ with no vertices is called trivial. The pseudo-lines in $Y$ divide the inside of $c$ into finitely many faces $F$, the edges $E$ of which are either pseudo-line segments or segments of $c$. A face with no $c$ segment edges is called an internal face. A vertex that is an intersection of two pseudo-lines is called an internal vertex.

Given a de Bruijn diagram $Y$, we can draw a neighborhood of any internal vertex as shown in Figure 4.2, using a (possibly orientation reversing) change of coordinates. In particular, we have a crossing of a straight

\[ \gamma(F_1) \quad \gamma(F_2) \quad \gamma(F_3) \quad \gamma(F_4) \]

\[ F_1 \quad F_2 \quad F_3 \quad F_4 \]

Figure 4.2. (a) A vertex $v$ of type $i \wedge j$ in canonical coordinates, showing the four faces $F_1, F_2, F_3, F_4$. (b) The $\gamma$ image of $F_1, F_2, F_3, F_4$ in the case $i \wedge j$ is positive. In particular, this illustrates the case $i \wedge j = 1 \wedge 3$ in the Ammann matrix example.

pseudo-line segment labeled $i$, pointing up, and a straight pseudo-line segment labeled $j$, pointing left. We say this vertex is of type $i \wedge j$. 
Definition 4.1. — A vertex of type $i \wedge j$ in a de Bruijn diagram $Y$ is called positive if $\text{sgn}(i \wedge j) > 0$. A de Bruijn diagram $Y$ is positive if all its vertices are positive.

4.2. The tiling patch corresponding to a diagram

Here we describe how to go from a nonsingular de Bruijn diagrams $Y$ to the corresponding tiling patch $y$. The idea—to use planar graph duality—comes from de Bruijn’s algebraic theory of Penrose tilings [5]. In particular, an internal type $i \wedge j$ vertex in $Y$ corresponds to a tile $t$ of type $i \wedge j$. Similarly, each face $F$ in $Y$ corresponds, under duality, to a tile vertex in $y$. We will start with this correspondence.

Let $F$ denote the set of all faces in $Y$. Given a vector $x_0$ and some $F_0 \in F$, we define $\gamma : F \to \mathbb{R}^2$ as follows. Put $\gamma(F_0) = x_0$, and proceed by induction. Suppose $\gamma$ has been defined on a set $F_k \subseteq F$ of $k < \#(F)$ faces. Let $F \in F \setminus F_k$ be adjacent to some $F' \in F_k$ across a pseudo-line segment labeled $i$. Put $a = 1$ or $a = -1$ depending on whether segment $i$ is oriented to the left or to the right in crossing from $F'$ to $F$. Define $\gamma(F) = \gamma(F') + ap^i$, and put $F_{k+1} = F_k \cup \{F\}$.

Lemma 4.2. — Suppose $Y$ is a nonsingular de Bruijn diagram. Let $F_1$, $F_2$, $F_3$ and $F_4$ be faces surrounding a type $i \wedge j$ vertex in $Y$, as shown in Figure 4.2 (a). Then $\gamma(F_1)$, $\gamma(F_2)$, $\gamma(F_3)$ and $\gamma(F_4)$ are the vertices of the tile $(\gamma(F_1), i \wedge j)$ (Figure 4.2 (b)), and the boundary of this tile is the piecewise linear curve corresponding to the sequence of points $\gamma(F_1), \gamma(F_2), \gamma(F_3), \gamma(F_4), \gamma(F_1)$.

Proof. — We can assume without loss of generality that $\gamma(F_1) = 0$. Then $\gamma(F_2) = p^i, \gamma(F_4) = p^j, \gamma(F_3) = p^i + p^j$ (see Figure 4.2 (b)). □

Lemma 4.2 says that a positive de Bruijn diagram describes a sort of “locally correct” tiling patch. It gives each tile a precise location, and tiles corresponding to adjacent vertices meet across complete edges. However, there is no a priori guarantee that different parts of the tiling do not overlap. The next result shows this cannot happen if the boundary choice is correct.

Proposition 4.3. — Let $P = (p^1, p^2, \ldots, p^d)$ be a good star of vectors and let $f : F(\mathcal{B}) \to \mathbb{Z}^d$ be the abelianization homomorphism. Suppose $W = [W_1, W_2]$, with $W_1, W_2 \in F(\mathcal{B})$, satisfies

$$\det ([Pf(W_1), Pf(W_2)]) > 0.$$
If $Y$ is a positive de Bruijn diagram with $\partial(Y) = W$, then $Y$ corresponds to a positive tiling patch $y$ with $\partial(y) = (x, W)$ for some $x \in \mathbb{R}^2$.

**Proof.** — By Lemma 4.2, the vertices of $Y$ correspond to positive tiles $t_1, t_2, \ldots, t_\ell$. Let $T_i = \partial(t_i)$. Then $T_i$ is a simple closed curve, which is positive since $Y$ is positive. Thus $n_{T_i}(x) = 1$ if $x$ is inside $T_i$, and $0$ otherwise.

Also

$$\sum_{i=1}^{\ell} T_i = \partial(Y) = W, \tag{4.1}$$

since two adjacent tiles have oppositely oriented boundaries.

Let $x$ be inside $W$. Then (4.1) implies

$$n_W(x) = \sum_{i=1}^{\ell} n_{T_i}(x) \geq 0. \tag{4.2}$$

Since $\det ([Pf(W_1), Pf(W_2)]) > 0$, Proposition 3.1 implies $n_W(x) = 1$, from which it follows that the sum in (4.2) is $\leq 1$ for all non-trace $x$. Thus any $x$ inside $W$ can be in at most one tile $t_i$, so that two tiles only can intersect along their boundaries. This means $y = t_1 + t_2 + \cdots + t_\ell$ is a positive tiling patch. \[\square\]

Conversely, if $y$ is a positive tiling patch with $\partial(y) = (x, W)$, then there exists a de Bruijn diagram $Y$ corresponding to $y$ such that $\partial(Y) = W$. This fact, which we do not use, is discussed in [5]. In general, we think of a not necessarily positive de Bruijn diagram as a generalized tiling proto-patch by positive, negative and trivial tiles. This is illustrated in Section 6.1. We denote the set of all these proto-patches by $(B^2)^*$.

### 4.3. Product diagrams and tiling substitutions

From now on we will consider only diagrams $Y$ where $\partial(Y) = [V, W]$ for $V, W \in \mathcal{F}(B)$. For the frame, we take $c$ to be the unit square. We put $V$ along the bottom, put $W$ going up the right side, put $V^{-1}$ going backwards along the top, and put $W^{-1}$ going down along the left side (see Figure 4.1 (a)).

The simplest diagram of this type is a **product diagram**, denoted $Y = V \wedge W$. To obtain it, we start with the frame described above and connect the arrows by vertical and horizontal lines (see Figure 4.1 (b)). It follows that

$$\partial(V \wedge W) = [V, W]. \tag{4.3}$$
The following “matrix notation” for a product diagram will be useful. Suppose \( V = v_1 v_1 \ldots v_\ell \) and \( W = w_1 w_2 \ldots w_n \). We write

\[
V \wedge W = \begin{bmatrix}
    v_1 \land w_n & v_2 \land w_n & \ldots & v_\ell \land w_n \\
    \vdots & \vdots & \ldots & \vdots \\
    v_1 \land w_1 & v_2 \land w_1 & \ldots & v_\ell \land w_1
\end{bmatrix}
\]

In the matrix above \( i_p \land j_q \) is the intersection of a vertical pseudo-line labeled \( i \) a horizontal pseudo-line labeled \( j \). We have \( p = 1 \) if the \( i \) arrow points up and \( p = -1 \) if the \( i \) arrow points down. Also \( q = 1 \) if the \( j \) arrow points left and \( q = -1 \) if the \( j \) arrow points right. This vertex is equivalent to \( pq(i \land j) \).

5. Cancellation

5.1. The three moves on de Bruijn diagrams

Let \( Y \) be a de Bruijn diagram. In this section we will describe three “moves” \( \mu \) that can be applied to a diagram \( Y \) to obtain a new diagram \( Y' \). Since the moves are reversible denote them \( Y \mu \leftrightarrow Y' \).

The first move \( \mu_1 \) is called the flip. In it, we slide a pseudo-line across a vertex. It is called the flip move because in a (positive) tiling it implements the “Necker cube flip” (see Figure 5.2).

The second move \( \mu_2 \) is called cancellation. It takes a pair of opposite sign adjacent vertices, \( i \land j \) and \(- (i \land j)\), and cancels them by uncrossing two loops (see Figure 5.3).

The third move \( \mu_3 \) is called trivial tile elimination. It may be used to cancel a trivial tile \( i \land i \), which occurs when two pseudo-lines with the same
Figure 5.2. The Necker cube flip, implemented by the flip move $\mu_1$.

Figure 5.3. The cancellation move $\mu_2$.

label cross (see Figure 5.4). Move $\mu_3$ is different from the other two for two reasons: (i) pseudo-lines are cut and reconnected in a different way, and (ii) it depends on the orientation.

Figure 5.4. Trivial tile elimination move $\mu_3$.

**Definition 5.1.** — Two de Bruijn diagrams are called $\mu$-equivalent if one can be obtained from the other by a finite series of moves.

**Lemma 5.2.** — If $Y$ and $Y'$ are $\mu$-equivalent de Bruijn diagrams, then $\partial(Y') = \partial(Y)$. 

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5.2. Counting tiles

For a de Bruijn diagram $Y$ based on $B = \{1, 2, \ldots, d\}$, let $M(Y)$ be the $d \times d$ matrix with $i, j$th entry $m_{i \wedge j}$ equal to the number of type $i \wedge j$ vertices that occur in $Y$. Let $e_{i \wedge j}$ be the $d \times d$ matrix with $(i \wedge j)$th entry equal to 1 and all other entries 0.

**Lemma 5.3.** — Let $Y$ and $Y'$ be de Bruijn diagrams with $\partial(Y) = \partial(Y')$.

(i) If $Y \overset{\mu_1}{\rightarrow} Y'$ (i.e., a flip) then $M(Y') = M(Y)$.

(ii) If $Y \overset{\mu_2}{\rightarrow} Y'$, where $\mu_2$ implements a cancellation of $i \wedge j$ and $j \wedge i$, then $M(Y) = M(Y') - e_{i \wedge j} - e_{j \wedge i}$.

(iii) $Y \overset{\mu_3}{\rightarrow} Y'$, where $\mu_3$ implements elimination of a trivial tile $i \wedge i$, then $M(Y') = M(Y) - e_{i \wedge i}$.

For $i \wedge j \in B^2$, $i \neq j$ let $|i \wedge j| = \text{sgn}(i \wedge j)(i \wedge j)$. That is to say, for a nontrivial tile $i \wedge j$, $|i \wedge j|$ is its positive version. Now suppose $Y$ is a de Bruijn diagram and define the vector $f^*(Y) \in \mathbb{Z}^{(2)}$ to have entries

$$(5.1) \quad f^*(Y)_{i \wedge j} = \text{sgn}(i \wedge j)(m_{|i \wedge j|} - m_{-|i \wedge j|}),$$

$i < j$ in lexicographic order.

**Corollary 5.4.** — (of Lemma 5.3) If $Y$ and $Y'$ are $\mu$-equivalent de Bruijn diagrams, then $f^*(Y) = f^*(Y')$.

**Proposition 5.5.** — Suppose $\theta$ is an endomorphism on $B = \{1, \ldots, d\}$ and let $A = L_\theta$. Then the “product tiling substitution” $\theta \wedge \theta : B^2_+ \rightarrow (B^2)^*$, defined by

$$(5.2) \quad (\theta \wedge \theta)(i \wedge j) := \theta(i) \wedge \theta(j),$$

satisfies

$$A^* = (f^*((\theta \wedge \theta)(1 \wedge 2)), f^*((\theta \wedge \theta)(1 \wedge 3)), \ldots, f^*((\theta \wedge \theta)((d - 1) \wedge d))).$$

**Remark 5.6.** — The product tiling substitution $\theta \wedge \theta$ is essentially the same as $E_1^*(\theta)$ in [4].

**Proof.** — First note that by (2.8) and (3.13) that $A^*$ has entries

$$a^*_{i \wedge j, k \wedge \ell} = \text{sgn}(i \wedge j)\text{sgn}(k \wedge \ell) \det \left( \begin{array}{cc} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell} \end{array} \right),$$

where $a_{i,j}$ are the entries of $A$. By (5.1)

$$f^*((\theta \wedge \theta)(k \wedge \ell))_{i \wedge j} = \text{sgn}(k \wedge \ell)(m_{|i \wedge j|} - m_{-|i \wedge j|}) = \text{sgn}(i \wedge j)\text{sgn}(k \wedge \ell)(m_{i \wedge j} - m_{j \wedge i}),$$

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where the $m_{i\wedge j}$ are the entries of $M((\theta \wedge \theta)(k \wedge \ell))$.

Since $L_{\theta} = A$, it follows that $m_{i\wedge j} = a_{i,k}a_{j,\ell}$ and $m_{j\wedge i} = a_{j,k}a_{i,\ell}$. Thus

$$m_{i\wedge j} - m_{j\wedge i} = \det \begin{pmatrix} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell} \end{pmatrix},$$

and $f^*((\theta \wedge \theta)(k \wedge \ell))_{i\wedge j} = a^*_{i,j,k,\ell}$. □

### 5.3. Properly ordered words

Let us write $W = i_1^{a_1}i_2^{a_2}\ldots i_\ell^{a_\ell} \in F(\mathcal{B})$, where $a_j \in \mathbb{Z}$ (i.e., $a_j = 0$ is allowed). We call $W$ efficient if whenever $i_j = i_k$, we have that $a_ja_k \geq 0$. We say $W$ is positive if $a_j \geq 0$ for all $j$, which implies $f(W) \geq 0$. Two positive words are disjoint if they have no symbols in common. A positive word has natural order if it has the form $W = 1^{a_1}2^{a_2}\ldots d^{a_d}$. In effect, we have imposed an arbitrary order on $\mathcal{B}$. In general, a word $W$ has natural order if it has the form $W = W_1^{-1}W_2$, where $W_1$ and $W_2$ are disjoint, positive natural order words.

The following construction is an elaboration of one from [13]. Let $h = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{R}^d$. Partition $\mathbb{R}^d$ into unit cubes parallel to the coordinate axes. Each such cube contains a unique vector $a + h$, where $a \in \mathbb{Z}^d$, called its label. Given a vector $b \in \mathbb{Z}^d$ we are going to define a word $W = W(b)$, called the properly ordered word for $b$.

Let $\ell$ be the line segment connecting $h$ to $b + h$. Consider the sequence of cubes that $\ell$ intersects, and let $b_0, b_1, \ldots, b_n$ be the sequence of their labels. Assume first that any pair of adjacent cubes in the sequence meets in a $d-1$ dimensional face (see Figure 5.5 (a)). Then for each $i$, $b_i - b_{i-1} = (0, 0, \ldots, 0, \pm1, 0, \ldots, 0) \in \mathbb{Z}^d$, where the $\pm1$ occurs at position $k$. We define the properly ordered word $W = W(b)$ to be the word with $i$th entry $k^{\pm1}$.

![Figure 5.5](image)

**Figure 5.5.** (a) The vector $b = (5, 2) \in \mathbb{Z}^2$ gives $W(b) = 121^321$. (b) For $b = (3, 1)$, $W' = 1*1$, with $* = 12$ proper (i.e., $* \neq 21$), and $W(b) = 1^221$. 

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Now suppose there are places in the sequence where cube $i - 1$ and cube $i$ meet along face of dimension less than $d - 1$. Construct a word $W'$ as above, but insert the symbol $*_{i}$ as a place-holder at position $i$. If the two cubes meet along a $d - m$ dimensional face, for $m > 1$, then $a_i = b_i - b_{i-1} \in \{0, 1, -1\}^d$ has $m$ nonzero entries. We write $a^+_i = a_i^+ - a_i^-$, where $a^+_i, a^-_i \geq 0$, and let $a^+_i = (a_1', a_2', \ldots, a_d')$ and $a^-_i = (a_1', a_2', \ldots, a_d')$. Define $V_+ = 1^{a_1'^2} a_2'^d \ldots d^{a_d'}$, $V_- = 1^{a_1} 2^{a_2} \ldots d^{a_d}$, and $V^*_i = (V_+)^{-1} V_+$. A crucial point is that $V^*_i$ is a naturally ordered word. The properly ordered word $W = W(b)$ is defined to be the word $W'$ with the entries $*_{i}$ replaced by the words $V^*_i$ (see Figure 5.5 (b)). As above, we can also realize $W$ as a piecewise linear curve with vertices $b_0, b_1, \ldots, b_n$ (see Figure 5.5 (b)). We call this piecewise linear curve, the broken line for $b$.

**Lemma 5.7.** — For any $b \in \mathbb{Z}^d$ the properly ordered word $W = W(b)$ satisfies $f(W) = b$. Moreover, if $0 = b_0, b_1, b_2, \ldots, b_n = b$ is the corresponding broken line, then for all $k$, $f(W_k) = b_k$, where $W_k$ denotes the $k$th prefix of $W$.

**Lemma 5.8.** — A properly ordered word $W$ is always efficient. If $W$ is properly ordered and $f(W) \in \{0, 1, -1\}^d$, then $W$ is naturally ordered.

We say an endomorphism $\theta$ is properly ordered if $\theta(i)$ is a properly ordered word for each $i \in B$. It is easy to see that for any $d \times d$ integer matrix $A$, there exists a properly ordered endomorphism $\theta$ so that $L\theta = A$.

**Definition 5.9.** — A nonsingular de Bruijn diagram $Y$ is called efficient if it has no two vertices equivalent to $i \wedge j$ and $-(i \wedge j) = j \wedge i$.

Compare this to the definition of an efficient word at the beginning of this section. We think of an efficient de Bruijn diagram as one that has been simplified. The following should be clear.

**Lemma 5.10.** — The number of vertices of type $\text{sgn}(f^*(Y)_{i \wedge j})|i \wedge j|$, $i < j$, in an efficient de Bruijn diagram $Y$ is $|f^*(Y)_{i \wedge j}|$. In particular, if $f^*(Y) \geq 0$, then $Y$ is a positive de Bruijn diagram.

### 5.4. Simplifying diagrams

**Theorem 5.11.** — Let $V, W \in \mathcal{F}(B)$ be properly ordered. Then there exists an efficient de Bruijn diagram $Y$ that is $\mu$-equivalent to $V \wedge W$.

To motivate the proof, we first explain the basic idea (including a cautionary negative example). Let $V = ij$ with $i < j$, so that $V$ is positive and
naturally ordered, and consider the diagram $V \wedge V$. Notice that $f^*(V \wedge V)$ is zero since $j \wedge i = -(i \wedge j)$. The diagram is reduced to a trivial diagram as shown in Figure 5.6. Now let $W$ be one of the other three naturally ordered two letter words. The same idea works for $V \wedge W$ where $W = j^{-1}i^{-1}$. On the other hand, for $W = j^{-1}i$, there is no need for cancellation after elimination, because both remaining vertices are type $-(i \wedge j)$. The same argument works for $W = i^{-1}j$. However, if we take the similar looking diagram $V \wedge W$, where $V = ij$ but $W = ji$, then cancellation fails (see Figure 5.7). This demonstrates the importance of natural order in the words $V_i^*$ in the definition of proper order.

The remainder of this section constitutes the proof of Theorem 5.11. The proof proceeds by induction.

Let $Y_1$ be the diagram obtained by using $\mu_3$ on $V \wedge W$ to eliminate all the trivial vertices.

Now suppose we have $Y_n$, $n \geq 1$. Take $i < j$ so that the total number of vertices of type $i \wedge j$ or $j \wedge i$ in $Y_n$ is not zero. Let $r_{i \wedge j} = |f^*(Y_n)_{i \wedge j}| \geq 0$ and $s_{i \wedge j} = \text{sgn}(f^*(Y_n))$. There are $r_{i \wedge j}$ more vertices of type $s_{i \wedge j}|i \wedge j| \ldots$
than there of type $-s_i \land j | i \land j]$. Call the latter target vertices. We would like to cancel each target vertex in $Y_n$ with one of its negatives. Let $N_n$ be the total number of target vertices in $Y_n$. If $N_n = 0$ then the diagram is efficient and the proof is done.

**Lemma 5.12.** — Let $V, W \in \mathcal{F}(B)$ be properly ordered. Suppose $Y$ is the de Bruijn diagram obtained by eliminating all the trivial vertices from $V \land W$. Then for each target vertex $v$ of type $\pm(i \land j)$ in $Y$, $i < j$, there is a unique type $\mp(i \land j)$ vertex $v'$ connected to $v$ by a pseudo-line labeled $i$ and a pseudo-line labeled $j$. Furthermore, we may assume that the $i$ line and $j$ line do not cross between $v$ and $v'$.

In other words, the two vertices are connected like the vertices in Figure 5.6 (b). We call $v_2$ the partner of $v_1$, and call $S = \{v_1, v_2\}$ a partner vertex pair.

**Proof.** — Assume without loss of generality that $i = 1$, $j = 2$. First let $V, W$ be positive, so that $V, W \in \{1, 2\}^*$. The algorithm for constructing properly ordered words commutes with the projection to the $(1, 2)$-plane. We identify this plane with $\mathbb{R}^2$, so that $f(V), f(W) \in \mathbb{Z}^2$.

In the properly ordered word algorithm, the squares surrounding $f(V)$ and $f(W)$ also contain the broken lines for $V$ and $W$ (see Figure 5.8 (a)). If $f(V)$ and $f(W)$ are not parallel, then one broken line must lie to one side of the other, although not necessarily strictly. We call this the separation property. On the other hand, if $f(V)$ and $f(W)$ are parallel, then because the symbols $*i$ were all replaced by natural order words $V_i^*$, the shorter broken line must follow the longer one for its entire length. Again, this is “separation”, although here the separation is everywhere not strict.

The separation property extends to the two piecewise linear paths defined by $VW$ and $WV$ that approximate the parallelogram spanned by $f(V)$ and $f(W)$, (see Figure 5.8 (a)). Write the words $VW = x_1x_2 \ldots x_n$ and $WV = y_1y_2 \ldots y_n$ symbolically, and denote the vertices of the corresponding broken lines by $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ respectively. For $z = (z_1, z_2) \in \mathbb{Z}^2$, define $\nu(z) = z_2 - z_1$. The separation property implies that either $\nu(x_k) \geq \nu(y_k)$ or $\nu(y_k) \geq \nu(x_k)$ for all $k = 1, 2, \ldots, n$. Lemma 5.7 implies that $\nu(f(WV_k)) \geq \nu(f(WV_k))$ or $\nu(f(WV_k)) \geq \nu(f(WV_k))$ for all $k$. For concreteness, assume that the former holds. This means that for each $k$, there are more symbols 1 in $WV_k$ than in $WV_k$ (and more symbols 2 in $WV_k$).

Let $Y$ be the diagram obtained from $V \land W$ by using $\mu_3$ to remove all the trivial tiles, and let $Y'$ be the diagram obtained from $Y$ by stretching all the pseudo-lines in $Y$ into lines (see Figure 5.8 (b)). To do this, it is
(a) Let $V = 1^221^2$ and $W = 21212$. Then $WV$ is above $VW$ since $f(W)$ is above $f(V)$. (b) The corresponding simplified diagram $Y'$. All vertices are type $1 \land 2$.

necessary to use $\mu_2$ to cancel some target vertices $v$ of type $2 \land 1$ with partners $v'$ of type $1 \land 2$. We can assign these partners in such a way that their 1- and 2-labeled pseudo-lines never cross between $v$ and $v'$.

In the boundary, the symbols in $VW$ correspond to the in-arrows encountered in going counter-clockwise along the bottom and right, and the symbols in $WV$ correspond to out-arrows, going clockwise around the left and top. By the separation property, all the 2-labeled pseudo-lines cross the 1-labeled pseudo-lines from right to left, so in $Y'$ all vertices are of type $1 \land 2$. It follows that all vertices of type $2 \land 1$ were cancelled. This shows that all the target vertices in $Y$ were assigned to partners.

So far we have assumed $V$ and $W$ are positive, which means $f(V)$ and $f(W)$ lie in the first quadrant. Essentially the same proof works whenever $f(V)$ and $f(W)$ lie in the same quadrant. If $f(V)$ and $f(W)$ lie in opposite quadrants, replace $V \land W$ with $W^{-1} \land V$, which is easily seen to be the same diagram, but $f(W^{-1}) = -f(W)$ is in the same quadrant as $f(V)$. Finally, if $f(V)$ and $f(W)$ lie in adjacent quadrants, there is no cancellation necessary.

Assume $N_n > 0$. Fix a target vertex $v_1$ in $Y_n$ and let $v_2$ be its partner, so that $S = \{v_1, v_2\}$ is a partner vertex pair. If there are no other pseudo-lines crossing the loop connecting $v_1$ and $v_2$, we can cancel using $\mu_2$ to obtain a diagram $Y_{n+1}$ with $N_{n+1} = N_n - 1$. However, there may be other pseudo-lines crossing the loop, and there may be other vertices inside the loop. Let $G$ be the vertices inside the loop, and let $I$ and $J$ be the vertices from pseudo-lines crossing the $i$ and $j$ sides of the loop. Let $H$ be the remaining vertices, outside the picture (see Figure 5.9 (a)).
Figure 5.9. The induction step: (a) The vertices in $G$ cross in an unknown way. (b) A pseudo-line $k$ in crosses $j$ twice. (c) Once there are no double-crossings, the $i$ pseudo-line moves across $G$.

Since all vertices are non-trivial, no pseudo-line labeled $i$ or $i^{-1}$ can cross the $i$ part of the loop, and likewise for $j$. However, pseudo-lines labeled $j$ or $j^{-1}$ may cross the $i$ part of the loop, and vice versa.

We may assume without loss of generality that there is no vertex $v$ of type $\pm i \land j$ in $G$. If there is, then by the previous paragraph, $v$ must be part of a partner vertex pair $S' = \{v, v'\} \subseteq G$. In this case, we consider the smaller loop between $v$ and $v'$ instead of the loop between $v_1$ and $v_2$. In a similar way, we can assume that $I$ and $J$ contain no vertices of type $\pm i \land j$.

Next, consider a pseudo-line, labeled $k$ (not $i$ or $j$), that goes through $j$ into $G$. We may assume without loss of generality that when it comes out of $G$, it goes through $i$ and not $j$. For suppose it crosses $j$ twice (see Figure 5.9 (b)). Then $J$ contains a partner vertex pair $S' = \{v, v'\}$ of types $\pm (j \land k)$. In this case, we consider the smaller loop between $v$ and $v'$ instead of the loop between $v_1$ and $v_2$. Since a similar argument holds for $i$, we may assume that all the pseudo-lines through $G$ cross both $i$ and $j$.

To complete the induction, we move the $i$ loop across $G$ using successive $\mu_1$-moves. Then we cancel the pair $S$ using $\mu_2$. The vertices in $G \cup H$ do not change. Moreover, a consequence of the previous paragraph is that the new $I$ has the same vertices as the old $I$. But since the two vertices in $S$ have been cancelled, and one was a target vertex, $N_{n+1} = N_n - 1$. The proof now follows by induction.

$\square$

6. The main result

**Theorem 6.1.** — Suppose $A$ is a $d \times d$ matrix satisfying (2.1), (2.4) and (2.9). Then there exists a properly ordered endomorphism $\theta$ with $L_{\theta} = A$, such that there is a positive tiling substitution $\Theta$ with $\partial \Theta = \theta$. 

Proof. — Let $\theta$ be a properly ordered endomorphism with $L_\theta = A$. For each $i \wedge j \in B_2^+$, Theorem 5.11 gives an efficient de Bruijn diagram $Y(i \wedge j)$ that is $\mu$-equivalent to $\theta(i) \wedge \theta(j)$.

By Corollary 5.4, $f^*(Y(i \wedge j)) = f^*(\theta(i) \wedge \theta(j))$, and by Proposition 5.5, the vector $f^*(\theta(i) \wedge \theta(j))$ is column $i \wedge j$ of $A^*$. By (2.9) (which requires (2.1) and (2.4)), $A^* \geq 0$. It follows that $f^*(Y(i \wedge j)) \geq 0$, and since $Y(i \wedge j)$ is efficient, Lemma 5.10 implies that $Y(i \wedge j)$ is positive. Define $\Theta(i \wedge j)$ to be the geometric realization of $Y(i \wedge j)$. Proposition 4.3 shows that $\Theta(i \wedge j) \in (B_2^+)^*$, i.e., $\Theta(i \wedge j)$ is a positive tiling patch. Finally we have

$$
\partial(\Theta(i \wedge j)) = \partial((\theta \wedge \theta)(i \wedge j)), \quad \text{by Lemma 5.2},
$$

$$
= \partial(\theta(i) \wedge \theta(j)), \quad \text{by (5.2)},
$$

$$
= [\theta(i), \theta(j)], \quad \text{by (4.3)},
$$

$$
= \theta([i, j]),
$$

so by (3.15) $\partial \Theta = \theta$, and $\Theta$ is a positive tiling substitution.

We define the structure matrix $L_\Theta$ of a positive tiling substitution $\Theta$ to be the $(d^2_2) \times (d^2_2)$ integer matrix whose $i \wedge j, k \wedge \ell$ entry gives the number of times $k \wedge \ell$ occurs inside $\Theta(i \wedge j)$.

Corollary 6.2. — The positive tiling substitution $\Theta$ constructed above satisfies $L_\Theta = A^*$, i.e., $(L_\theta)^* = L_\Theta$.

6.1. Example. The Ammann matrix (continued)

Let $A$ be the Ammann matrix (2.10), which by Section 2.4 satisfies (2.1), (2.4) and (2.8). Let $P$ be the projection (2.11).

To define a properly ordered endomorphism, we need for each $i$ that $\theta(i)$ is a properly ordered word with $f(\theta(i))$ equal to the $i$th column of $A$. Since the columns of $A$ belong to $\{0, 1, -1\}^4$, Lemma 5.8 implies each $\theta(i)$ should be naturally ordered:

$$
\theta(1) = 4^{-1}1^{-1}2^{-1} \quad \theta(2) = 3^{-1}2^{-1}1
$$

$$
\theta(3) = 4^{-1}3^{-1}2^{-1} \quad \theta(4) = 4^{-1}3^{-1}1^{-1}.
$$

Next we define the product substitution $(\theta \wedge \theta)(i \wedge j) = \theta(i) \wedge \theta(j)$ whose value on each of the six positive prototiles we represent as product de Bruijn diagram.
In two cases, \((\theta \land \theta)(1 \land 3)\) and \((\theta \land \theta)(4 \land 2)\), the product diagrams consist of just positive and trivial tiles. For example,

\[
(\theta \land \theta)(1 \land 3) = (4^{-1}1^{-1}2) \land (4^{-1}3^{-1}2^{-1}) = \begin{bmatrix} 4 \land 2 & 1 \land 2 & 2 \land 2 \\ 4 \land 3 & 1 \land 3 & 3 \land 2 \\ 4 \land 4 & 1 \land 4 & 4 \land 2 \end{bmatrix},
\]

in matrix form, where the entries have been simplified to positive prototiles. The resulting \(\mu\)-equivalent positive de Bruijn diagram \(Y(1 \land 3)\) is shown below (left), as well as its realization \(\Theta(1 \land 3)\) as a tiling patch (right).

Similarly, for \((\theta \land \theta)(4 \land 2)\) we obtain:

\[
(\theta \land \theta)(4 \land 2) = (4^{-1}1^{-1}2) \land (3^{-1}2^{-1}1) = \begin{bmatrix} 1 \land 4 & 1 \land 1 & -(1 \land 2) \\ 4 \land 2 & 1 \land 2 & 2 \land 2 \\ 4 \land 3 & 1 \land 3 & 3 \land 2 \end{bmatrix}.
\]

Now consider the case

\[
(\theta \land \theta)(1 \land 2) = (4^{-1}1^{-1}2) \land (3^{-1}2^{-1}1) = \begin{bmatrix} 1 \land 4 & 1 \land 1 & -(1 \land 2) \\ 4 \land 2 & 1 \land 2 & 2 \land 2 \\ 4 \land 3 & 1 \land 3 & 3 \land 2 \end{bmatrix}.
\]
Notice that the upper right $2 \times 2$ block has the form shown in Figure 5.6 (a), and can be canceled as shown in Figure 5.6 (b) and (c) to obtain the positive diagram $Y(1 \wedge 2)$ (shown left). The resulting geometric realization $\Theta(1 \wedge 2)$ is shown (right).

There are two more cases that work exactly the same way: $(\theta \wedge \theta)(3 \wedge 2)$ and $(\theta \wedge \theta)(4 \wedge 3)$. These are shown below.

The most complicated case is

$$(\theta \wedge \theta)(1 \wedge 4) = (4^{-1}1^{-1}2) \wedge (3^{-1}2^{-1}1) = \begin{bmatrix} -(1 \wedge 4) & 1 \wedge 1 & 1 \wedge 2 \\ 4 \wedge 2 & 1 \wedge 2 & 2 \wedge 2 \\ 4 \wedge 3 & 1 \wedge 3 & 3 \wedge 2 \end{bmatrix}.$$

Notice that the required cancellations are not adjacent. We use the method in the proof of Theorem 5.11 as illustrated in Figure 5.6. The steps are shown in detail below.
Finally, we will show how to iterate $\Theta$ to obtain tiling patches and tilings. We start with the tiling patch $y$, shown below, and apply $\Theta$ to obtain a sequence of patches $\Theta^N(y)$. The patch $y$ has been carefully chosen so that for each $i \wedge j \in B_4^2$, a translation of $y$ appears as a sub-patch of $\Theta^N(i \wedge j)$.
for all $N$ sufficiently large. The existence of such a patch $y$ follows from the fact that $(A^*)^N > 0$ for all $N \geq 2$. We say in such a case that $\Theta$ is a primitive tiling substitution (this means that each $k \land \ell$ appears in each $\Theta^N(i \land j)$ for all $N$ sufficiently large).

Another important property of the patch $y$ is that $y$ appears as a sub-patch of $\Theta^2(y)$, completely surrounded by other tiles. This is shown in Figure 6.1. It follows that $\Theta^{2N}(y)$ is a sub-patch of $\Theta^{2M}(y)$ for all $M > N$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.1.png}
\caption{The patches $y$ and $\Theta(y)$.}
\end{figure}
Thus the increasing limit
\[ z := \lim_{N \to \infty} \Theta^{2N}(y) \]
defines a tiling of \( \mathbb{R}^2 \). A swatch of this tiling is shown in Figure 6.4.

Figure 6.2. The patch \( \Theta^2(y) \), showing \( y \) as a sub-patch around \( 0 \).

Figure 6.3. The patch \( \Theta^3(y) \), which has the same patch \( y' \) around \( 0 \) as \( \Theta(y) \).
Figure 6.4. A swatch of the “Ammann” tiling $y$.

It is interesting to note that one obtains a different tiling $z' = \lim_{N \to \infty} \Theta^{2N+1}(y)$ by taking odd iterates. In particular, there is a different patch $y'$ around the origin. However, since $\Theta$ is primitive, both of these belong to the same “tiling space”, $X_{\Theta}$ in the notation of [16].

We call $z$ and $z'$ “Ammann tilings” although they are much less symmetric than the classical Ammann-Beenker tiling (see [17]). In fact, there are many different choices non-abelianizations $\theta$ of $A$ (i.e., not necessarily properly ordered) that result in completely different tiling substitutions $\Theta$ and different tiling spaces $X_{\Theta}$. None of them seem to be the classical Ammann-Beenker tilings.

7. The positivity of $(A^*)^N$

In addition to (2.1) and (2.4), all of the tiling results in this paper require that $A$ satisfy (2.9): $A^* \geq 0$. We begin by noting that $A$ and $A^N$, $N > 1$, have the same expanding subspace $E^u$ and the same projections $P$ to $E^u$. Thus $A^N$ satisfies (2.1) and (2.4) if and only if $A$ does. In general, we are not so concerned with the difference between $A$ and $A^N$. If $\theta$ is a non-abelianization of $A$, then $\theta^N$ is a non-abelianization of $A^N$, and we tend to think of $\theta^N$ and $\theta$ as the “same substitution”. The main result in this section is that $A^N$ may satisfy (2.9) (or rather a slightly stronger condition) even if $A$ does not.
It follows from the Binet-Cauchy Theorem (Theorem 2.3) that if $A$ satisfies (2.1) and (2.4) then
\[(A^*)^N = (A^N)^* \text{ for all } N \geq 0.\]

Instead of looking for matrices $A$ where $A^* \geq 0$, it will suffice to find $A$ so that $(A^*)^N = (A^N)^* \geq 0$. Unfortunately this is not so easy. However, it turns out to be possible to find good conditions on $A$ for a stronger conclusion: $(A^*)^N > 0$. In this case, we say $A^*$ is eventually positive.

Even in the case where $A^* \geq 0$, the eventual positivity of $A^*$ is important because by Corollary 6.2 it implies the corresponding tiling substitution $\Theta$ is primitive. In any case, if a matrix $A$ satisfying (2.1) and (2.4) is such that $(A^*)^N > 0$, then (7.1) implies that $B = A^N$ satisfies (2.1), (2.4) and (2.9). This provides lots of new examples.

**7.1. The eventual positivity of $A^*$**

A matrix $A$ is called eventually positive if $A^N > 0$ for all $N$ sufficiently large. A matrix $A$ is primitive if $A \geq 0$ and $A$ is eventually positive. Our main result in this section is the following.

**Theorem 7.1.** — Let $A$ satisfy (2.1) and (2.4). The matrix $A^*$ is eventually positive if and only if $S(A) = \pm S(A^T)$, where $S(A)$ is the matrix defined in (2.7).

Since $A$ and $A^T$ have the same eigenvalues, $A$ satisfies (2.1) if and only if $A^T$ does. The proof of Theorem 7.1 will come at the end of this section.

**Corollary 7.2.** — If $A^T = A$, then $A^*$ is eventually positive.

If $A = A^T$, then $A$ satisfies (2.4) if and only if $A^T$ does (they share the projection $P$). As we will show, however, there are also many non-symmetric examples. In the $3 \times 3$ case we get the following.

**Corollary 7.3.** — Let $B \in Sl_3(\mathbb{Z})$. Suppose $B$ is eventually positive and satisfies the following Pisot condition: $\dim(E^u(B)) = 1$. Then $A := B^{-1}$ satisfies (2.1), (2.4) and (2.9).

Combining this Theorem 6.1, we obtain a new proof of the result from [10] that under the conditions of Corollary 7.3, there exists a tiling substitution $\Theta$ with boundary $\theta$ whose structure matrix is $A$. (In [10] an additional connectivity result is also obtained.)
Proof of Corollary 7.3. — Assume without loss of generality that \( B > 0 \). The Pisot property for \( B \) implies that \( B \) has three distinct eigenvalues satisfying \( \omega_3 > 1 \) and \( |\omega_2|, |\omega_1| < 1 \). It follows that the corresponding eigenvalues of \( A \), \( \lambda_i = \omega_i^{-1} \), satisfy \( |\lambda_1|, |\lambda_2| > 1 \) and \( \lambda_3 < 1 \), so \( A \) satisfies (2.1).

Let \( v_1, v_2, v_3 \) be the corresponding eigenvectors. Since \( B > 0 \) we can assume \( v_3 = (1, a, b) \) where \( a, b > 0 \). Let \( w_1 = (-a, 1, 0) \) and \( w_2 = (-b, 0, 1) \). Since \( w_1 \cdot v_3 = w_2 \cdot v_3 = 0 \),

\[
(7.2) \quad P = (w_1, w_2)^T = \begin{pmatrix} -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}
\]

and \( C_2(P) = (b, -a, 1) \). It follows that \( S(A) = \text{diag}(1, -1, 1) \). Since \( B^T > 0 \), and since \( A^T = (B^T)^{-1} \), the same argument shows \( S(A^T) = S(A) \), and the corollary follows from Theorem 7.1. \( \square \)

7.2. Subspaces and projections

We say \( v \in \mathbb{C}^d \) is real if \( \gamma v \in \mathbb{R}^d \) for some nonzero \( \gamma \in \mathbb{C} \); otherwise we say \( v \) is complex. Let \( v_1, v_2 \in \mathbb{C}^d \) be linearly independent over \( \mathbb{C} \). If both \( v_1 \) and \( v_2 \) are real then \( \text{span}\{v_1, v_2\} \subseteq \mathbb{R}^d \) denotes span over \( \mathbb{R} \). Otherwise \( \text{span}\{v_1, v_2\} \subseteq \mathbb{C}^d \) denotes span over \( \mathbb{C} \).

In general, we let \( F = \mathbb{C} \) or \( F = \mathbb{R} \). Given a two dimensional subspace \( E = \text{span}\{v_1, v_2\} \subseteq F^d \), we call a \( 2 \times d \) matrix \( P \) satisfying (2.2) a projection to \( E \). As before, (2.3) holds, except now for \( M \in GL_2(F) \).

A subspace \( E \) is called real if it has a real basis, or equivalently, a real projection \( P \). In particular, this means there is a nonsingular \( M \) so that \( MP \) has all real entries. Defining \( R = (v_1, v_2) \), we have that (2.2) implies \( PR = I \). We put \( A_E = PAR \), which satisfies \( P|_E A = A_E P|_E \).

7.3. Jordan forms

Let \( V \in GL_d(\mathbb{C}) \) be such that \( VJ = AV \), where \( J \) is the upper triangular Jordan Canonical Form of \( A \), with \( \text{diag}(J) = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d \). The columns \( V = (v_1, \ldots, v_d) \) are a basis of ordered generalized eigenvectors for \( A \). If \( A \) is diagonalizable, these are actually eigenvectors.

**Proposition 7.4.** — Let \( A \) satisfy (2.1) and let \( w_1 \) and \( w_2 \) be the first two ordered generalized eigenvectors for \( A^T \) (i.e., corresponding to \( E^u(A^T) \)). Then \( P = (w_1, w_2)^T \) is a projection to \( E^u(A) \).
Proof. — The unique dual basis \( w'_1, \ldots, w'_d \) corresponding to \( v_1, \ldots, v_d \) is defined to satisfy \( v_i \cdot w'_j = \delta_{i,j} \). Thus \( W' := (w'_1, w'_2, \ldots, w'_d)^T \) satisfies \( W' = V^{-1} \).

We have \( A^T(W')^T = (W')^T J^T \) because \( W'A(W')^{-1} = V^{-1}AV = J \) implies \( W'A = JW' \). But \( J^T \) is the lower-triangular Jordan form for \( A \).

There exists a permutation matrix \( U \) so that \( U^TJ^T U = J \) where the first two rows (and columns) of \( U \) are either: (a) \( e_1, e_2 \), or (b) \( e_2, e_1 \), depending on whether or not \( \lambda_1 \) and \( \lambda_2 \) are in a non-trivial Jordan block (this can only happen if \( \lambda_1 = \lambda_2 \)).

Let \( W = U^T W' \), and express \( W = (w_1, \ldots, w_d)^T \). Then \( W^{-1}A^T W = (U^T W')^{-1}A^T (U^T W') = J \).

This shows \( w_1, \ldots, w_d \) are the ordered generalized eigenvectors for \( A^T \). Let \( P = (w_1, w_2)^T \). Then \( P = MP' \), where \( M = I \) in case (a), or \( M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in the case (b).

\[ \square \]

**Corollary 7.5.** — The expanding subspace \( E^u(A) \) is real.

Proof. — This is clear if \( \lambda_1 \) is real, since this is equivalent to \( \lambda_2 \) also being real.

If \( \lambda_1 \) is complex, then \( \lambda_2 = \overline{\lambda_1} \). Put \( P = (w_1, \overline{w_1})^T \). Let \( P' = MP = (\text{Re}\{w_1\}, \text{Im}\{w_1\})^T \) where \( M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \). Then \( P' \) is a real projection to \( E^u \).

\[ \square \]

### 7.4. Perron-Frobenius theory for \( A^* \)

Let \( v_1, v_2 \in \mathbb{R}^d \). Define \( v_1 \land v_2 \in \mathbb{R}^{\binom{d}{2}} \) by

\[
(7.3) \quad v_1 \land v_2 = C_2((v_1, v_2)).
\]

**Lemma 7.6.** — If \( A \) is \( d \times d \) and \( v_1, v_2 \in \mathbb{R}^d \) then

\[
C_2(A)(v_1 \land v_2) = (Av_1) \land (Av_2).
\]

Proof. —

\[
C_2(A)(v_1 \land v_2) = C_2(A)C_2((v_1, v_2)) = C_2((Av_1, Av_2)) = (Av_1) \land (Av_2).
\]

\[ \square \]
A matrix $B \in \mathbb{Z}^{p \times p}$ is called spectrally Perron (see [14]) if it has a real eigenvalue $\omega > 0$, such that for any other eigenvalue $\omega'$ of $B$, $|\omega'| < \omega$. The eigenvector $u$ corresponding to $\omega$ is called a Perron eigenvector.

**Theorem 7.7.** — (Lind and Marcus, [14]) A matrix $B$ is eventually positive if and only if it is spectrally Perron and the Perron eigenvectors $u$ and $u'$ for $B$ and $B^T$ respectively are both positive.

The “only if” direction follows from the Perron-Frobenius Theorem (see [14]). The “if” direction is Lind and Marcus [14], Exercise 11.1.9.

**Lemma 7.8.** — Let $A$ satisfy (2.1) and let $E = \text{span}\{v_1, v_2\}$ be any real or complex 2-dimensional $A$-invariant subspace. Let $P$ be a projection to $E$, and let $A_E$ be the induced matrix on $\mathbb{F}^2$. Then $u = S(A)(v_1 \wedge v_2)$ is an eigenvector for $A^*$ corresponding to the eigenvalue $\omega = \det(A_E)$. Moreover, every eigenvalue of $A^*$ is obtained this way.

**Proof.** — We have

$$RA_E = RPAR = AR.$$

Using this with (7.3) and the Binet-Cauchy Theorem (Theorem 2.3)

$$C_2(A)(v_1 \wedge v_2) = C_2(A)C_2(R) = C_2(AR) = C_2(RA_E) = \det(A_E)C_2(R) = \omega (v_1 \wedge v_2),$$

where $\omega = \det(A_E)$.

The fact that every eigenvalue of $C_2(A)$ is obtained this way follows from the Jordan canonical form for $A$.

Now put $u = S(A)(v_1 \wedge v_2)$. Then

$$A^*u = S(A)C_2(A)S(A)^2(v_1 \wedge v_2) = S(A)C_2(A)(v_1 \wedge v_2) = \omega S(A)(v_1 \wedge v_2) = \omega u.$$

□

**Proposition 7.9.** — Let $A$ be non-Pisot of order-2 and satisfy the good star property. Then $A^*$ is spectrally Perron. In particular, if $E^u = \text{span}\{v_1, v_2\}$, then $u = S(A)(v_1 \wedge v_2)$ is the Perron eigenvector and $\omega = \det(A_u)$ is the Perron eigenvalue.

**Proof.** — Since $A^* = S(A)C_2(A)S(A)$ and $A = V^{-1}JV$, the Binet-Cauchy theorem implies

$$A^* = (C_2(V)S(A))^{-1}C_2(J)(C_2(V)S(A)).$$
Now $C_2(J)$ is upper triangular so its eigenvalues are its diagonal entries. These are pairwise products of the diagonal entries of $J$, which are the eigenvalues of $A$.

Since $E^u$ is a real 2-dimensional invariant subspace there are three possibilities for $A_u$: (i) $A_u$ is diagonalizable over $\mathbb{R}$ and has two real eigenvalues $\lambda_1, \lambda_2$ with $|\lambda_1|, |\lambda_2| > 1$. If necessary, by replacing $A$ with $A^2$, we can assume $\lambda_1, \lambda_2 > 0$. Then $\omega = \lambda_1 \lambda_2 > 1$. (ii) $A_u$ is a nontrivial Jordan block with eigenvalue $\lambda$. As above, assume by replacing $A$ with $A^2$ if necessary $\lambda > 0$. Then $\omega = \lambda^2 > 1$. (iii) $A_u$ has complex eigenvalues $\lambda$ and $\bar{\lambda}$, and $\omega = \lambda \bar{\lambda} = |\lambda|^2 > 1$.

Now let $E$ be any other real or complex 2-dimensional $A$-invariant subspace, and let $\omega'$ be the corresponding eigenvalue for $A^*$. If $E \subseteq E^s$ then $\omega' = \det(A_E) = \lambda_i \lambda_j$, where $i, j > 2$, so $|\omega'| = |\lambda_i| |\lambda_j| \leq |\lambda_3|^2 < 1 < \omega$. Otherwise $A_u$ is diagonalizable and has eigenvalues $\lambda_i, \lambda_j, i \leq 2, j > 2$. Then $|\omega'| = |\lambda_i| |\lambda_j| < |\lambda_1| |\lambda_2| = \omega$. The result now follows from an application of Lemma 7.8.

**Proposition 7.10.** — Suppose $A$ satisfies (2.1). Let $v_1, v_2$ be a real basis for $E^u(A)$ and let $w_1, w_2$ be a real basis for $E^u(A^T)$. Then $A^*$ is eventually positive if and only if $S(A)(v_1 \wedge v_2)$ and $S(A^T)(w_1 \wedge w_2)$ are positive.

**Proof.** — This follows from Theorem 7.7, Proposition 7.9 and Corollary 7.4. □

**Proof of Theorem 7.1.** — Let

\[ u = S(A)(v_1 \wedge v_2) \quad \text{and} \quad u' = S(A^T)(w_1 \wedge w_2). \]

By proposition 7.10, Theorem 7.1 follows once we show

\[ u, u' > 0 \] if and only if $S(A) = \pm S(A^T).$

By Proposition 7.4, we have

\[ P = P_{E^u(A)} = (w_1, w_2)^T \quad \text{and} \quad P' = P_{E^u(A^T)} = (v_1, v_2)^T \]

where

\[ E^u(A) = \text{span}\{v_1, v_2\} \quad \text{and} \quad E^u(A^T) = \text{span}\{w_1, w_2\}. \]

By (7.3), $C_2(P) = w_1 \wedge w_2$ and $C_2(P') = v_1 \wedge v_2$. It follows from (2.7) that

\[ S(A) = \text{diag}(\text{sgn}(C_2(P))) = \text{diag}(\text{sgn}(w_1 \wedge w_2)) \]

and

\[ S(A^T) = \text{diag}(\text{sgn}(C_2(P'))) = \text{diag}(\text{sgn}(v_1 \wedge v_2)). \]

Thus $S(A) = \pm S(A^T)$ if and only if $\text{sgn}(v_1 \wedge v_2) = \text{sgn}(w_1 \wedge w_2).$
Now suppose $S(A) = S(A^T)$. Then
\[ u = S(A)(v_1 \wedge v_2) = sgn(w_1 \wedge w_2) \ast (v_1 \wedge v_2) \]
\[ = sgn(v_1 \wedge v_2) \ast (v_1 \wedge v_2) > 0, \]
and similarly $u' > 0$, where $\ast$ denotes entry-wise multiplication.

Conversely, suppose $u, u' > 0$. Then
\[ sgn(w_1 \wedge w_2) \ast (v_1 \wedge v_2) > 0, \]
which implies
\[ sgn(v_1 \wedge v_2) = \pm sgn(w_1 \wedge w_2). \]
It follows that $S(A) = S(A^T)$. \hfill \qed

8. Example: Irreducible characteristic polynomial

Consider the matrix
\[
A = \begin{pmatrix}
-1 & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 \\
-1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]
with irreducible characteristic polynomial $p(x) = x^4 + x^3 - 4x^2 + 1$. The eigenvalues are $\lambda_1 = -2.5231$, $\lambda_2 = 1.44129$, $\lambda_3 = 0.566889$, and $\lambda_4 = -0.485084$, so $A$ satisfies (2.1). Note that $A$ is symmetric, so by Corollary 7.2, $(A^*)^N \geq 0$ for some $N$ sufficiently large. We now find such an $N$.

Starting with the matrix
\[
Q = (q_1, q_2, q_3, q_4) = \begin{pmatrix}
-3.12676 & -0.252532 & -2.19713 & 0.576415 \\
-2.3925 & -0.824788 & 2.87576 & 0.188279 \\
-2.5231 & 1.44129 & 0.566889 & -0.485084 \\
1 & 1 & 1 & 1
\end{pmatrix},
\]
of eigenvectors, the projection $P$ to $E^u$ is given by
\[
(8.1) \quad P = \begin{pmatrix}
-0.141119 & -0.101064 & -0.113874 & 0.0451327 \\
-0.0660841 & -0.215836 & 0.377166 & 0.261686
\end{pmatrix},
\]
(the first two rows of $Q^{-1}$). The columns of $P$ are plotted in Figure 8.1. The matrix $A^*$ is given by
Figure 8.1. The good star of vectors for the projection $P$ in (8.1). The positive prototiles are $1 \wedge 2$, $3 \wedge 1$, $4 \wedge 1$, $3 \wedge 2$, $4 \wedge 2$, and $4 \wedge 3$.

$$A^* = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix},$$

which satisfies $A^* \leq 0$. This is sufficient for our method. By (7.1), $(A^*)^2 = (A^2)^* \geq 0$.

For the matrix $A$, we find the properly ordered morphism $\theta$ to be

$$\theta : \begin{cases} 1 & \to & 3^{-1}2^{-1}1^{-1} \\ 2 & \to & 3^{-1}1 \\ 3 & \to & 2^{-1}1^{-1}4 \\ 4 & \to & 3 \end{cases}$$

Next, we simplify all six product de Bruijn diagrams $(\theta \wedge \theta)(i \wedge j)$ to obtain the following efficient diagrams (all of which are negative):

These de Bruijn diagrams have the following realizations as tiling patches.

The gray color indicates the tiles are all negative.
\[ Y(3 \wedge 2) \quad \Theta(1 \wedge 2) \quad \Theta(3 \wedge 2) \]

\[ Y(4 \wedge 2) \quad \Theta(3 \wedge 1) \quad \Theta(4 \wedge 1) \]

\[ \Theta(3 \wedge 2) \quad \Theta(4 \wedge 2) \quad \Theta(4 \wedge 3) \]

The patch \( y \).

\[ \Theta(y) \]
Starting with the tiling patch $y$ shown below, we iterate to get patches $\Theta^N(y)$. Notice that for $N$ even, the patch is a positive tiling patch. Each of these positive patches is a sub-patch of the previous one.

Finally, we define a tiling

$$z = \lim_{N \to \infty} \Theta^{2N}(y).$$

A swatch of this tiling is shown in Figure 8.2.
Figure 8.2. The positive tiling $z = \lim_{N \to \infty} \Theta^{2N}(y)$.

BIBLIOGRAPHY


