Marcia EDSON & Luca Q. ZAMBONI

On the Number of Partitions of an Integer in the $m$-bonacci Base


<http://aif.cedram.org/item?id=AIF_2006___56_7_2271_0>
ON THE NUMBER OF PARTITIONS OF AN INTEGER
IN THE \( m \)-BONACCI BASE

by Marcia EDSON & Luca Q. ZAMBONI

Abstract. — For each \( m \geq 2 \), we consider the \( m \)-bonacci numbers defined by
\[
F_k = 2^k \quad \text{for} \quad 0 \leq k \leq m - 1 \quad \text{and} \quad F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m} \quad \text{for} \quad k \geq m.
\]
When \( m = 2 \), these are the usual Fibonacci numbers. Every positive integer \( n \) may be expressed as a sum of distinct \( m \)-bonacci numbers in one or more different ways. Let \( R_m(n) \) be the number of partitions of \( n \) as a sum of distinct \( m \)-bonacci numbers. Using a theorem of Fine and Wilf, we obtain a formula for \( R_m(n) \) involving sums of binomial coefficients modulo 2. In addition we show that this formula may be used to determine the number of partitions of \( n \) in more general numeration systems including generalized Ostrowski number systems in connection with Episturmian words.

Résumé. — Pour \( m \geq 2 \), on définit les nombres de \( m \)-bonacci \( F_k = 2^k \) pour \( 0 \leq k \leq m - 1 \) et \( F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m} \) pour \( k \geq m \). Dans le cas \( m = 2 \), on retrouve les nombres de Fibonacci. Chaque entier positif \( n \) s’écrit comme une somme distincte de nombres de \( m \)-bonacci d’une ou plusieurs façons. Soit \( R_m(n) \) le nombre de partitions de \( n \) en base \( m \)-bonacci. En utilisant un théorème de Fine et Wilf on déduit une formule pour \( R_m(n) \) comme somme de coefficients binomiaux modulo 2. De plus, nous montrons que cette formule peut-être utilisée pour déterminer le nombre de partitions de \( n \) dans des systèmes généraux de numération incluant les systèmes de nombres d’Ostrowski généralisés associés aux suites épisturmiannes.

1. Introduction and Preliminaries

For each \( m \geq 2 \), we define the \( m \)-bonacci numbers by \( F_k = 2^k \) for \( 0 \leq k \leq m - 1 \) and \( F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m} \) for \( k \geq m \). When \( m = 2 \), these are the usual Fibonacci numbers. We denote by \( \{0, 1\}^* \) the set of all words \( w = w_1 w_2 \cdots w_k \) with \( w_i \in \{0, 1\} \). Each positive integer \( n \) may be expressed as a sum of distinct \( m \)-bonacci in one or more different ways.

Keywords: Numeration systems, Fibonacci numbers, Fine and Wilf theorem, episturmian words.
That is we can write \( n = \sum_{i=1}^{k} w_i F_{k-i} \) where \( w_i \in \{0, 1\} \) and \( w_1 = 1 \). We call the associated \( \{0, 1\} \)-word \( w_1 w_2 \cdots w_k \) a representation of \( n \). One way of obtaining such a representation is by applying the “greedy algorithm”. This gives rise to a representation of \( n \) of the form \( w = w_1 w_2 \cdots w_k \) with the property that \( w \) does not contain \( m \) consecutive 1’s. Such a representation of \( n \) is necessarily unique and is called the \( m \)-Zeckendorff representation of \( n \), denoted \( Z_m(n) \) [13]. For example, taking \( m = 2 \) and applying the greedy algorithm to \( n = 50 \) we obtain \( 50 = 34 + 13 + 3 = F_7 + F_5 + F_2 \) which gives rise to the representation \( Z_2(50) = 10100100 \). A \( \{0, 1\} \)-word \( w \) beginning in 1 and having no occurrences of \( 1^m \) will be called a \( m \)-Zeckendorff word.

Other representations arise from the fact that an occurrence of \( 10^m \) in a given representation of \( n \) may be replaced by \( 01^m \) to obtain another representation of \( n \), and conversely. Thus a number \( n \) has a unique representation in the \( m \)-bonacci base if and only if \( Z_m(n) \) does not contain any occurrences of \( 0^m \). For example, again taking \( m = 2 \) and \( n = 50 \) we obtain the following 6 representations (arranged in decreasing lexicographic order):

10100100
10100011
10011100
10011011
1111100
1111011

We are interested in the sequence \( R_m(n) \) which counts the number of distinct partitions of \( n \) in the \( m \)-bonacci base. More precisely, given \( n \in \mathbb{Z}^{>0} \) we set

\[
\Omega_m(n) = \{ w = w_1 w_2 \cdots w_k \in \{0, 1\}^* | w_1 = 1 \text{ and } n = \sum_{i=1}^{k} w_i F_{k-i} \}
\]

and put \( R_m(n) = \#\Omega_m(n) \). For \( w \in \Omega_m(n) \) we will sometimes write \( R_m(w) \) for \( R_m(n) \). Also we let \( R_m^< (w) \) denote the number of representations of \( n \) which are less or equal to \( w \) in the lexicographic order. As \( Z_m(n) \) is the largest representation of \( n \) with respect to the lexicographic order, it follows that \( R_m(n) = R_m^{\leq} (Z_m(n)) \).

In a 1968 paper L. Carlitz [3] studied the multiplicities of representations of \( n \) as sums of distinct Fibonacci numbers; he obtained recurrence relations for \( R_2(n) \) and explicit formulae for \( R_2(n) \) in the case \( Z_2(n) \) contains 1, 2 or 3 Fibonacci numbers. He states in the paper however that a general formula for the number of partitions of \( n \) in the Fibonacci base appears...
to be very complicated. In [1] J. Berstel derives a formula for $R_2(n)$ as a product of $2 \times 2$ matrices (see Proposition 4.1 in [1]). Recently, P. Kocábová, Z. Masáková, and E. Pelantová [10] extended Berstel’s result to $R_m(n)$ for all $m \geq 2$ again as a product of $2 \times 2$ matrices.

In this paper we give a formula for $R_m(n)$ involving sums of binomial coefficients modulo 2. Our proof makes use of the well known Fine and Wilf Theorem [4]. In order to state our main result, we first consider a special factorization of $Z_m(n)$: Either $Z_m(n)$ contains no occurrences of $0^m$ (in which case $R_m(n) = 1$), or $Z_m(n)$ can be factored uniquely in the form

$$Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW$$

where

- $V_1, V_2, \ldots, V_N$ and $W$ do not contain any occurrences of $0^m$.
- $0^{m-1}$ is not a suffix of $V_1, V_2, \ldots, V_N$.
- Each $U_i$ is of the form

$$U_i = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m$$

with $x_i \in \{0, 1\}$.

We shall refer to this factorization as the *principal factorization* of $Z_m(n)$ and call the $U_i$ *indecomposable factors*. We observe that in the special case of $m = 2$, the factors $V_i$ are empty. Each indecomposable factor $U_i$ may be coded by a positive integer $r_i$ whose base 2 expansion is $1x_kx_{k-1} \cdots x_0$, in other words $r_i = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$.

Given a positive integer $r$ whose base 2 expansion is $1x_kx_{k-1} \cdots x_0$, we set

$$[r] = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m.$$

We now state our main result:

**Theorem 1.1.** — Let $m \geq 2$. Given a positive integer $n$, let $Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW$ be the principal factorization of the $m$-Zeckendorff representation of $n$ as defined above. Then the number of distinct partitions of $n$ as sums of distinct $m$-bonacci numbers is given by

$$R_m(n) = \prod_{i=1}^N \sum_{j=0}^{r_i} \binom{2r_i - j}{j} \pmod{2}$$

where $[r_i] = U_i$ for each $1 \leq i \leq N$. 

TOME 56 (2006), FASCICULE 7
2. Proof of Theorem 1.1

Let \( Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW \) be the principal factorization of \( Z_m(n) \) described above. Then the number of partitions of \( n \) is simply the product of the number of partitions of each indecomposable factor:

\[
R_m(n) = \prod_{i=1}^{N} R_m(U_i).
\]

In fact, any representation of \( n \) as a sum of distinct \( m \)-bonacci numbers may be factored in the form

\[
V_1U_1'V_2U_2' \cdots V_NU_N'W
\]

where for each \( 1 \leq i \leq N \), \( U_i' \) is an equivalent representation of \( U_i \). To see this we first observe that since the \( V_i \) and \( W \) contain no \( 0^m \), we have \( R_m(V_i) = R_m(W) = 1 \). So the only way that \( V_i \) or \( W \) could change in an alternate representation of \( n \) would be as a result of a neighboring indecomposable factor. If \( V_i \) contains an occurrence of \( 1 \), then since \( V_i \) does not end in \( 0^{m-1} \) the last occurrence of \( 1 \) in \( V_i \) can never be followed by \( 0^m \). In other words the last \( 1 \) in \( U_i \) can never move into the \( U_i \) that follows. If \( V_i \) contains no occurrences of \( 1 \), then \( V_i = 0^r \) with \( r < m - 1 \). Since the indecomposable factor \( U_{i-1} \) preceding \( V_i \) ends in \( Km \) many consecutive \( 0^m \)'s (for some \( K \geq 1 \)), any equivalent representation of \( U_{i-1} \) either ends in \( 0^m \) or in \( 1^m \), and since \( V_i \) does not begin in \( 0^m \), any representation of \( U_{i-1} \) terminating in \( 1^m \) will never be followed by \( 0^m \). In other words, no \( 1 \) in \( U_{i-1} \) can ever move into \( V_i \) or in the following \( U_i \). A similar argument applies to the indecomposable factor \( U_N \) preceding \( W \).

Thus in view of (2.1) above, in order to prove Theorem 1.1, it remains to show that for each positive integer \( r = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0 \), we have

\[
R_m([r]) = \sum_{j=0}^{r} \binom{2r - j}{j} \pmod{2}.
\]

For each positive integer \( n \) there is a natural decomposition of the set \( \Omega_m(n) \) of all partitions of \( n \) in the \( m \)-bonacci base: Let \( F \) be the largest \( m \)-bonacci number less or equal to \( n \). We denote by \( \Omega^+_m(n) \) the set of all partitions of \( n \) involving \( F \) and \( \Omega^-_m(n) \) the set of all partitions of \( n \) not involving \( F \), and set \( R^+_m(n) = \# \Omega^+_m(n) \) and \( R^-_m(n) = \# \Omega^-_m(n) \). Clearly

\[
R_m(n) = R^+_m(n) + R^-_m(n).
\]

We will make use of the following recursive relations:
Let $U = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m$ with $x_i \in \{0, 1\}$. Then
\[
R^m_+(10^{m-1}10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m) = R_m(U) = R^+_m(U) + R^-_m(U)
\]
\[
R^-_m(10^{m-1}10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m) = R^+_m(U)
\]
\[
R^+_m(10^{m-1}00^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m) = R^+_m(U)
\]
\[
R^-_m(10^{m-1}00^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m) = R^+_m(U) + R^-_m(U)
\]

**Proof.** It is easy to see that $w \in \Omega^+_m(10^{m-1}U)$ if and only if $w$ is of the form $w = 10^{m-1}w'$ for some $w' \in \Omega_m(U)$. Whence $R^+_m(10^{m-1}U) = R^-_m(U)$. Similarly, $w \in \Omega^-_m(10^{m-1}U)$ if and only if $w$ is of the form $w = 01^m w'$ for some $w' \in \Omega^-_m(U)$. Whence $R^-_m(10^{m-1}U) = R^-_m(U)$. A similar argument applies to the remaining two identities.

We now apply this to the ordered pair $(p, q)$, given a pair of relatively prime numbers $(p, q)$, there exists a $(0, 1)$-word $w$ of length $p+q-2$ (unique up to isomorphism) having periods $p$ and $q$, and if $p$ and $q$ are both greater than 1, then this word contains both 0’s and 1’s; in other words 1 = gcd$(p, q)$ is not a period. We call such a word a Fine and Wilf word relative to $(p, q)$. Moreover it can be shown (see [12] for example) that if both $p$ and $q$ are greater than 1, then the suffixes of $w$ of lengths $p$ and $q$ begin in different symbols. We denote by $FW(p, q)$ the unique Fine and Wilf word relative to $(p, q)$ with the property that its suffix of length $p$ begins in 0 and its suffix of length $q$ begins in 1.

We now apply this to the ordered pair $(p, q) = (R^+_m([r]), R^-_m([r]))$. It is well known that $FW(R^+_m([r]), R^-_m([r]))01$ is given explicitly by the following composition of morphisms:

\[
FW(R^+_m([r]), R^-_m([r]))01 = \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)
\]

where
\[
\begin{align*}
\tau_0(0) &= 0 & \tau_0(1) &= 1 \\
\tau_1(0) &= 10 & \tau_1(1) &= 1
\end{align*}
\]

(see for instance [5, 12]).
Let
\[ \alpha(r) = |FW(R_m^+(r)), R_m^-(r)|01, \]
and
\[ \beta(r) = |FW(R_m^+(r)), R_m^-(r)|01, \]
in other words, \( \alpha(r) \) is the number of occurrences of 1 in
\[ FW(R_m^+(r)), R_m^-(r))01 \]
and \( \beta(r) \) the number of 0’s in
\[ FW(R_m^+(r)), R_m^-(r))01. \]

In summary:

\[
R_m([r]) = R_m^+(r) + R_m^-(r) \\
= R_m^+(r) + R_m^-(r) - 2 + 2 \\
= |FW(R_m^+(r)), R_m^-(r))| + 2 \\
= |FW(R_m^+(r)), R_m^-(r))|1 \\
= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)| \\
= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|1 + |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|0 \\
= \alpha(r) + \beta(r) \\
= |\tau_1 \circ \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|1 \\
= \alpha(2r + 1).
\]

The key step in the proof of Theorem 1.1 is to replace above the sum of the periods \( R_m^+(r) + R_m^-(r) \) of the Fine and Wilf word \( FW(R_m^+(r)), R_m^-(r)) \) by the sum of the number of occurrences of 0’s and 1’s in \( FW(R_m^+(r)), R_m^-(r))01 \). The following basic identities are readily verified:

- \( \alpha(1) = \beta(1) = 1. \)
- \( \alpha(2r) = \alpha(r). \)
- \( \beta(2r) = \alpha(r) + \beta(r). \)
- \( \alpha(2r + 1) = \alpha(r) + \beta(r). \)
- \( \beta(2r + 1) = \beta(r). \)
- \( \beta(r) = \alpha(r + 1). \)

Summarizing we have

**Proposition 2.2.** — Let \( U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots0^{m-1}x_00^m \) with \( x_i \in \{0, 1\} \). Let \( r \) be the number whose base 2 expansion is given by \( 1x_kx_{k-1}\cdots x_0 \). Then \( R_m(U) = \alpha(2r+1) \) where the sequence \( \alpha(r) \) is defined recursively by:
\begin{itemize}
  \item $\alpha(1) = 1$
  \item $\alpha(2r) = \alpha(r)$
  \item $\alpha(2r + 1) = \alpha(r) + \alpha(r + 1)$.
\end{itemize}

We now consider a new function $\psi(r)$ defined by $\psi(1) = 1$, and for $r \geq 1$

\[ \psi(r + 1) = \sum_{j=0}^{2j \leq r} \binom{r - j}{j} \pmod{2}. \]

We will show that $\psi(r)$ and $\alpha(r)$ satisfy the same recursive relations, namely: $\psi(2r) = \psi(r)$ and $\psi(2r + 1) = \psi(r) + \psi(r + 1)$. Thus $\alpha(r) = \psi(r)$ for each $r$ thereby establishing formula (2.2).

We shall make use of the following lemma:

**Lemma 2.3.** \( \binom{n}{k} \pmod{2} = \left( \binom{2n+1}{2k} + \binom{2n}{2k+1} \right) \pmod{2} \).

**Proof.** This follows immediately from the so-called Lucas’ identities: \( \binom{2n}{2k+1} = 0 \pmod{2} \) for \( 0 \leq k \leq n - 1 \), and \( \binom{n}{k} = \binom{2n+1}{2k+1} \pmod{2} \) for \( 0 \leq k \leq n \). \( \square \)

**Proposition 2.4.** For $r \geq 0$ we have $\psi(2r + 2) = \psi(r + 1)$ and for $r \geq 1$ we have $\psi(2r + 1) = \psi(r) + \psi(r + 1)$.

**Proof.** By Lemma 2.3 we have

\[
\psi(r + 1) = \sum_{j=0}^{2j \leq r} \binom{r - j}{j} \pmod{2} = \sum_{j=0}^{2j \leq r} \left( \binom{2r - 2j + 1}{2j} + \binom{2r - 2j}{2j + 1} \right) \pmod{2} = \sum_{i=0}^{r} \binom{2r + 1 - i}{i} \pmod{2} = \psi(2r + 2).
\]

As for the second recursive relation we have

\[
\psi(2r + 1) = \sum_{j=0}^{2i \leq r} \binom{2r - j}{j} \pmod{2} = \sum_{i=0}^{2i \leq r} \binom{2r - 2i}{2i} \pmod{2} + \sum_{i=0}^{2i \leq r - 1} \binom{2r - 2i - 1}{2i + 1} \pmod{2}
\]
But
\[
\binom{2r - 2i}{2i} \pmod{2} = \frac{(2r - 2i)!}{(2i)!(2r - 4i)!} \pmod{2}
\]
\[
= \frac{(2r - 2i + 1)!}{(2i)!(2r - 4i + 1)!} \pmod{2}
\]
\[
= \binom{2r - 2i + 1}{2i} \pmod{2}
\]
\[
= \binom{r - i}{i} \pmod{2} \text{ by Lemma 2.3.}
\]

Hence
\[
\sum_{i=0}^{2i \leq r} \binom{2r - 2i}{2i} \pmod{2} = \sum_{i=0}^{2i \leq r} \binom{r - i}{i} \pmod{2} = \psi(r + 1).
\]

Similarly
\[
\binom{2r - 2i - 1}{2i + 1} \pmod{2} = \frac{(2r - 2i - 1)!}{(2i + 1)!(2r - 4i - 2)!} \pmod{2}
\]
\[
= \frac{(2r - 2i - 1)!}{(2i)!(2r - 4i - 1)!} \pmod{2}
\]
\[
= \binom{2r - 2i - 1}{2i} \pmod{2}
\]
\[
= \binom{r - 1 - i}{i} \pmod{2} \text{ by Lemma 2.3.}
\]

Hence
\[
\sum_{i=0}^{2i \leq r-1} \binom{2r - 2i - 1}{2i + 1} \pmod{2} = \sum_{i=0}^{2i \leq r-1} \binom{r - 1 - i}{i} \pmod{2} = \psi(r).
\]

It follows that \(\psi(2r + 1) = \psi(r) + \psi(r + 1)\).

Having established that \(\alpha(r) = \psi(r)\) for each \(r \geq 1\), we deduce that:

**Corollary 2.5.** — Let \(U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots0^{m-1}x_00^{m}\) with \(x_i \in \{0, 1\}\). Let \(r\) be the number whose base 2 expansion is given by \(1x_kx_{k-1}\cdots x_0\). Then \(R_m(U) = \sum_{j=0}^{r} \binom{2r-j}{j} \pmod{2}\).

This concludes our proof of Theorem 1.1.
3. Concluding Remarks

3.1. A formula for $R_m(w)$

Our proof applies more generally to give a formula for $R_m(w)$ for each representation $w$ of $n$. In other words, given $w \in \Omega_m(n)$, then either $w$ does not contain any occurrences of $0^m$ (in which case $R_m(w) = 1$) or $w$ may be factored in the form

$$w = V_1 U_1 V_2 U_2 \cdots V_N U_N W$$

where the $V_i$ and $W$ do not contain any occurrences of $0^m$ and the $V_i$ do not end in $0^{m-1}$, and where the $U_i$ are of the form

$$U_i = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^m x_0$$

with $x_i \in \{0, 1\}$. Each factor $U_i$ is coded by a positive integer $r_i$ whose base 2 expansion is $1 x_k x_{k-1} \cdots x_0$. It is easy to see that any representation of $n$ less or equal to $w$ may be factored in the form

$$V_1 U'_1 V_2 U'_2 \cdots V_N U'_N W$$

where for each $1 \leq i \leq N$, $U'_i$ is an equivalent representation of $U_i$. Hence $R_m(w) = \prod_{i=1}^N R_m(U_i)$ from which it follows that

$$R_m(w) = \prod_{i=1}^N \sum_{j=0}^{r_i} \left( \frac{2r_i - j}{j} \right) \pmod{2}.$$  

3.2. Episturmian numeration systems

Let $A$ be a finite non-empty set. Associated to an infinite word $\omega = \omega_1 \omega_2 \omega_3 \ldots \in A^N$ is a non-decreasing sequence of positive integers $E(\omega) = E_1, E_2, E_3, \ldots$ defined recursively as follows: $E_1 = 1$, and for $k \geq 1$, the quantity $E_{k+1}$ is defined by the following rule: If $\omega_{k+1} \neq \omega_j$ for each $1 \leq j \leq k$, then set

$$E_{k+1} = 1 + \sum_{j=1}^k E_j.$$  

Otherwise let $\ell \leq k$ be the largest integer such that $\omega_{k+1} = \omega_\ell$, and set

$$E_{k+1} = \sum_{j=\ell}^k E_j.$$  

In particular we note that $E_{k+1} = E_k$ if and only if $\omega_{k+1} = \omega_k$. 

TOME 56 (2006), FASCICULE 7
Set $\mathcal{N}(\omega) = \{E_k | k \geq 1\}$. For $E \in \mathcal{N}(\omega)$ let $k \geq 1$ be such that $E = E_k$. We define $\sigma(E) = \omega_k$ and say that $E$ is based at $\omega_k \in A$. We also define the quantity $\rho(E)$, which we call the multiplicity of $E$, by

$$\rho(E) = \#\{i \geq 1 | E = E_i\}.$$ 

We can write $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots\}$ where for each $i \geq 1$ we have $x_i < x_{i+1}$. Thus we have that $\omega = \sigma(x_1)\rho(x_1)\sigma(x_2)\rho(x_2) \ldots$.

It can be verified that the set $\mathcal{N}(\omega)$ defines a numeration system (see [8]). More precisely, each positive integer $n$ may be written as a sum of the form

$$n = m_kx_k + m_{k-1}x_{k-1} + \cdots + m_1x_1$$

where for each $1 \leq i \leq k$ we have $0 \leq m_i \leq \rho(x_i)$ and $m_k \geq 1$. While such a representation of $n$ is not necessarily unique, one way of obtaining such a representation is to use the "greedy algorithm". In this case we call the resulting representation the Zeckendorff representation of $n$ and denote it $Z_\omega(n)$. We call the above numeration system a generalized Ostrowski system or an Episturmian numeration system. In fact, the quantities $E_i$ are closely linked to the lengths of the palindromic prefixes of the characteristic Episturmian word associated to the directive sequence $\omega$ (see [6, 7, 8, 9]).

In case $\#A = 2$, this is known as the Ostrowski numeration system (see [1, 2, 11]). In case $A = \{1, 2, \ldots, m\}$ and $\omega$ is the periodic sequence $\omega = (1, 2, 3, \ldots, m,)^\infty$, then the resulting numeration system is the $m$-bonacci system defined earlier.

Given an infinite word $\omega = \omega_1\omega_2\omega_3\ldots \in A^\mathbb{N}$, we are interested in the number of distinct ways of writing each positive integer $n$ as a sum of the form (3.1). More precisely, denoting by $\hat{A}$ the set $\{\hat{a} | a \in A\}$, we set $R_\omega(n) = \#\Omega_\omega(n)$ where $\Omega_\omega(n)$ is the set of all expressions of the form

$$n = \sigma(x_k)^{m_k}\sigma(x_k)^{\rho(x_k)-m_k}\sigma(x_{k-1})^{m_{k-1}}\sigma(x_{k-1})^{\rho(x_{k-1})-m_{k-1}}\cdots \sigma(x_1)^{m_1}\sigma(x_1)^{\rho(x_1)-m_1}$$

in $(A \cup \hat{A})^*$, such that $n = m_kx_k + m_{k-1}x_{k-1} + \cdots + m_1x_1$ where $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots | 1 = x_1 < x_2 < x_3 \ldots\}$ and where $0 \leq m_i \leq \rho(x_i)$ and $m_k \geq 1$. (1) For $w \in \Omega_\omega(n)$ we sometimes write $R_\omega(w)$ for $R_\omega(n)$.

Just as in the previous section, we begin with a unique special factorization of the Zeckendorff representation of $n$. In this case, this factorization

(1) Our notation here differs somewhat from that of Justin and Pirillo in [8]. For instance, in [8] the authors use the notation $\hat{a}$ for in lieu of our $\hat{a}$. Also instead of the expression (3.2), they consider the reverse of this word.
was originally defined by Justin and Pirillo (see Theorem 2.7 in [8]):

\[ Z_\omega(n) = V_1 U_1 V_2 U_2 \cdots V_N U_N W \]

where for each \(1 \leq i \leq N\) we have that \(U_i\) is a \(a_i\)-based maximal semigood multiblock for some \(a_i \in A\). Moreover any other representation of \(n\) may be factored in the form

\[ Z_\omega(n) = V_1 U'_1 V_2 U'_2 \cdots V_N U'_N W \]

where \(U'_i\) is an equivalent representation of \(U_i\) (see Theorem 2.7 in [8]). Thus as before (see (2.1)) we have

\[ R_\omega(n) = \prod_{i=1}^{N} R_\omega(U_i). \]

For each \(1 \leq i \leq N\) the factor \(U_i\) corresponds to a sum of the form

\[ m_K x_K + m_{K-1} x_{K-1} + \cdots + m_k x_k \]

for some \(K > k\) with \(m_K \neq 0\), and for each \(K \geq j \geq k\) we have that if \(m_j \neq 0\), then \(\sigma(x_j) = a_i\) [8]. In other words the only “accented” symbol occurring in \(U_i\) is \(a_i\), i.e., \(U_i \in (A \cup \{\hat{a}_i\})^*\).

Associated to \(U_i\) is a \(\{0, 1\}\)-word \(\nu(U_i) = \nu_K \nu_{K-1} \cdots \nu_k\) where \(\nu_K = 10\), \(\nu_j = \varepsilon\) (the empty word) if \(\sigma(x_j) \neq a_i\), \(\nu_j = 10\) if \(\sigma(x_j) = a_i\) and \(m_j = \rho(x_j)\), \(\nu_j = 010\) if \(\sigma(x_j) = a_i\) and \(0 < m_j < \rho(x_j)\) and \(\nu_j = 00\) if \(\sigma(x_j) = a_i\) and \(m_j = 0\).

By comparing the matrix formulation given in Corollary 2.11 in [8] used to compute \(R_\omega(U_i)\) with the matrix formulation given in Proposition 4.1 in [1], we leave it to the reader to verify the following:

**Proposition 3.1.** — \(R_\omega(U_i) = R_2(\nu(U_i))\).

In other words computing the multiplicities of representations in a generalized Ostrowski numeration system may be reduced to a computation of the multiplicities of representations in the Fibonacci base.

**Example 3.2.** — We consider the example originally started in Berstel’s paper [1] and later revisited by Justin and Pirillo as Example 2.3 in [8] of the Ostrowski numeration system associated to the infinite word \(\omega = a, a, b, b, a, a, b, b, a, a, b, a, a, b, \ldots\). It is readily verified that

\[ \mathcal{N}(\omega) = \{1, 3, 7, 24, 55, 134, 323, \ldots\}, \]

\(\sigma(1) = \sigma(7) = \sigma(55) = \sigma(232) = a, \sigma(3) = \sigma(24) = \sigma(134) = b\), and

\(\rho(1) = 2, \rho(3) = 2, \rho(7) = 3, \rho(24) = 2, \rho(55) = 2, \rho(134) = 2, \rho(323) = 3\).
Applying the greedy algorithm we obtain the following representation of the number $660$

$$660 = 2(323) + 0(134) + 0(55) + 0(24) + 2(7) + 0(3) + 0(1).$$

So $Z_\omega(660) = \hat{a}\hat{a}\hat{a}\hat{b}a\hat{a}b\hat{a}\hat{a}b\hat{a}$. which is a semigood multiblock based at $a$. We deduce that

$$\nu(Z_\omega(660)) = 10 \cdot \varepsilon \cdot 00 \cdot \varepsilon \cdot 010 \cdot \varepsilon \cdot 00$$

or simply $\nu(Z_\omega(660)) = 100001000$

Following the algorithm of Corollary 2.11 of [8] due to Justin and Pirillo, we obtain $q_1 = 2$, $q_2 = 4$, $p_1 = 2$, $p_2 = 2$, $c_1 = c_2 = 1$, so that

$$R_\omega(660) = (1, 0) \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 6$$

In contrast, applying the algorithm in Proposition 4.1 of [1] due to Berstel to the Zeckendorff word $\nu(Z_\omega(660)) = 100001000$, we obtain $d_1 = 4$, $d_2 = 3$ so that

$$R_2(\nu(Z_\omega(660))) = (1, 1) \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6$$

as required\(^{(2)}\)

**Acknowledgements**

The second author was partially supported by a grant from the National Security Agency.

**BIBLIOGRAPHY**


\(^{(2)}\) In [1], Berstel computes $R_\omega(660)$ in a different way by using the matrix formulation of Proposition 5.1 in [1] which applies to an Ostrowski numeration system.


