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Marcia EDSON & Luca Q. ZAMBONI

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ON THE NUMBER OF PARTITIONS OF AN INTEGER IN THE m-BONACCI BASE

by Marcia EDSON & Luca Q. ZAMBONI

Abstract. — For each $m \geqslant 2$, we consider the m-bonacci numbers defined by $F_k = 2^k$ for $0 \leqslant k \leqslant m-1$ and $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ for $k \geqslant m$. When $m = 2$, these are the usual Fibonacci numbers. Every positive integer n may be expressed as a sum of distinct m-bonacci numbers in one or more different ways. Let $R_m(n)$ be the number of partitions of n as a sum of distinct m-bonacci numbers. Using a theorem of Fine and Wilf, we obtain a formula for $R_m(n)$ involving sums of binomial coefficients modulo 2. In addition we show that this formula may be used to determine the number of partitions of n in more general numeration systems including generalized Ostrowski number systems in connection with Episturmian words.

Résumé. — Pour $m \geqslant 2$, on définit les nombres de m-bonacci $F_k = 2^k$ pour $0 \leqslant k \leqslant m-1$ et $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ pour $k \geqslant m$. Dans le cas $m = 2$, on retrouve les nombres de Fibonacci. Chaque entier positif n s'écrit comme une somme distincte de nombres de m-bonacci d'une ou plusieurs façons. Soit $R_m(n)$ le nombre de partitions de n en base m-bonacci. En utilisant un théorème de Fine et Wilf on déduit une formule pour $R_m(n)$ comme somme de coefficients binomiaux modulo 2. De plus, nous montrons que cette formule peut-être utilisée pour déterminer le nombre de partitions de n dans des systèmes généraux de numération incluant les systèmes de nombres d'Ostrowski généralisés associés aux suites episturmiennes.

1. Introduction and Preliminaries

For each $m \geqslant 2$, we define the m-bonacci numbers by $F_k = 2^k$ for $0 \leqslant k \leqslant m-1$ and $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ for $k \geqslant m$. When $m = 2$, these are the usual Fibonacci numbers. We denote by $\{0,1\}^*$ the set of all words $w = w_1 w_2 \cdots w_k$ with $w_i \in \{0,1\}$. Each positive integer n may be expressed as a sum of distinct m-bonacci in one or more different ways.

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That is we can write $n = \sum_{i=1}^{k} w_i F_{k-i}$ where $w_i \in \{0,1\}$ and $w_1 = 1$. We call the associated $\{0,1\}$-word $w_1 w_2 \cdots w_k$ a *representation* of n. One way of obtaining such a representation is by applying the "greedy algorithm". This gives rise to a representation of n of the form $w = w_1 w_2 \cdots w_k$ with the property that w does not contain m consecutive 1's. Such a representation of n is necessarily unique and is called the *m-Zeckendorff representation* of n, denoted $Z_m(n)$ [13]. For example, taking $m = 2$ and applying the greedy algorithm to $n = 50$ we obtain $50 = 34 + 13 + 3 = F_7 + F_5 + F_2$ which gives rise to the representation $Z_2(50) = 10100100$. A $\{0,1\}$-word w beginning in 1 and having no occurrences of 1^m will be called a *m-Zeckendorff word.*

Other representations arise from the fact that an occurrence of 10^m in a given representation of n may be replaced by 01^m to obtain another representation of n, and conversely. Thus a number n has a unique representation in the m-bonacci base if and only if $Z_m(n)$ does not contain any occurrences of 0^m. For example, again taking $m = 2$ and $n = 50$ we obtain the following 6 representations (arranged in decreasing lexicographic order):

$$\begin{array}{c} 10100100 \\ 10100011 \\ 10011100 \\ 10011011 \\ 1111100 \\ 1111011 \end{array}$$

We are interested in the sequence $R_m(n)$ which counts the number of distinct partitions of n in the m-bonacci base. More precisely, given $n \in \mathbb{Z}^{>0}$ we set

$$\Omega_m(n) = \{w = w_1 w_2 \cdots w_k \in \{0,1\}^* \,|w_1 = 1 \text{ and } n = \sum_{i=1}^{k} w_i F_{k-i}\}$$

and put $R_m(n) = \#\Omega_m(n)$. For $w \in \Omega_m(n)$ we will sometimes write $R_m(w)$ for $R_m(n)$. Also we let $R_m^{\leqslant}(w)$ denote the number of representations of n which are less or equal to w in the lexicographic order. As $Z_m(n)$ is the largest representation of n with respect to the lexicographic order, it follows that $R_m(n) = R^{\leqslant}(Z_m(n))$.

In a 1968 paper L. Carlitz [3] studied the multiplicities of representations of n as sums of distinct Fibonacci numbers; he obtained recurrence relations for $R_2(n)$ and explicit formulae for $R_2(n)$ in the case $Z_2(n)$ contains $1, 2$ or 3 Fibonacci numbers. He states in the paper however that a general formula for the number of partitions of n in the Fibonacci base appears to be very complicated. In [1] J. Berstel derives a formula for $R_2(n)$ as a product of 2×2 matrices (see Proposition 4.1 in [1]). Recently, P. Kocábová, Z. Masácová, and E. Pelantová [10] extended Berstel's result to $R_m(n)$ for all $m \geqslant 2$ again as a product of 2×2 matrices.

In this paper we give a formula for $R_m(n)$ involving sums of binomial coefficients modulo 2. Our proof makes use of the well known Fine and Wilf Theorem [4]. In order to state our main result, we first consider a special factorization of $Z_m(n)$: Either $Z_m(n)$ contains no occurrences of 0^m (in which case $R_m(n) = 1$), or $Z_m(n)$ can be factored uniquely in the form

$$Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW$$

where

- $V_1, V_2, \ldots, V_N$ and W do not contain any occurrences of 0^m.
- 0^{m-1} is not a suffix of $V_1, V_2, \ldots, V_N$.
- Each U_i is of the form

$$U_i = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m$$

with $x_i \in \{0, 1\}$.

We shall refer to this factorization as the *principal factorization* of $Z_m(n)$ and call the U_i *indecomposable factors*. We observe that in the special case of $m = 2$, the factors V_i are empty. Each indecomposable factor U_i may be coded by a positive integer r_i whose base 2 expansion is $1x_kx_{k-1} \cdots x_0$, in other words $r_i = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$.
Given a positive integer r whose base 2 expansion is $1x_kx_{k-1} \cdots x_0$, we set

$$[r] = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m.$$

We now state our main result:

THEOREM 1.1. — *Let $m \geqslant 2$. Given a positive integer n, let $Z_m(n) = V_1U_1V_2U_2 \cdots V_NU_NW$ be the principal factorization of the m-Zeckendorff representation of n as defined above. Then the number of distinct partitions of n as sums of distinct m-bonacci numbers is given by*

$$R_m(n) = \prod_{i=1}^{N} \sum_{j=0}^{r_i} \binom{2r_i - j}{j} \pmod 2$$

where $[r_i] = U_i$ for each $1 \leqslant i \leqslant N$.

2. Proof of Theorem 1.1

Let $Z_m(n) = V_1U_1V_2U_2\cdots V_NU_NW$ be the principal factorization of $Z_m(n)$ described above. Then the number of partitions of n is simply the product of the number of partitions of each indecomposable factor:

$$R_m(n) = \prod_{i=1}^{N} R_m(U_i). \tag{2.1}$$

In fact, any representation of n as a sum of distinct m-bonacci numbers may be factored in the form

$$V_1U_1'V_2U_2'\cdots V_NU_N'W$$

where for each $1 \leqslant i \leqslant N$, U_i' is an equivalent representation of U_i. To see this we first observe that since the V_i and W contain no 0^m, we have $R_m(V_i) = R_m(W) = 1$. So the only way that V_i or W could change in an alternate representation of n would be as a result of a neighboring indecomposable factor. If V_i contains an occurrence of 1, then since V_i does not end in 0^{m-1} the last occurrence of 1 in V_i can never be followed by 0^m. In other words the last 1 in V_i can never move into the U_i that follows. If V_i contains no occurrences of 1, then $V_i = 0^r$ with $r < m-1$. Since the indecomposable factor U_{i-1} preceding V_i ends in Km many consecutive 0's (for some $K \geqslant 1$), any equivalent representation of U_{i-1} either ends in 0^m or in 1^m, and since V_i does not begin in 0^m, any representation of U_{i-1} terminating in 1^m will never be followed by 0^m. In other words, no 1 in U_{i-1} can ever move into V_i or in the following U_i. A similar argument applies to the indecomposable factor U_N preceding W.

Thus in view of (2.1) above, in order to prove Theorem 1.1, it remains to show that for each positive integer $r = 1\cdot 2^{k+1} + x_k \cdot 2^k + \cdots x_1 \cdot 2 + x_0$, we have

$$R_m([r]) = \sum_{j=0}^{r} \binom{2r-j}{j} \pmod 2. \tag{2.2}$$

For each positive integer n there is a natural decomposition of the set $\Omega_m(n)$ of all partitions of n in the m-bonacci base: Let F be the largest m-bonacci number less or equal to n. We denote by $\Omega_m^+(n)$ the set of all partitions of n involving F and $\Omega_m^-(n)$ the set of all partitions of n not involving F, and set $R_m^+(n) = \#\,\Omega_m^+(n)$ and $R_m^-(n) = \#\,\Omega_m^-(n)$. Clearly

$$R_m(n) = R_m^+(n) + R_m^-(n).$$

We will make use of the following recursive relations:

LEMMA 2.1. — *Let* $U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m$ *with* $x_i \in \{0,1\}$. *Then*

$$R_m^+(10^{m-1}10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m) = R_m(U) = R_m^+(U) + R_m^-(U)$$
$$R_m^-(10^{m-1}10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m) = R_m^-(U)$$
$$R_m^+(10^{m-1}00^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m) = R_m^+(U)$$
$$R_m^-(10^{m-1}00^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m) = R_m(U) = R_m^+(U) + R_m^-(U)$$

Proof. — It is easy to see that $w \in \Omega_m^+(10^{m-1}U)$ if and only if w is of the form $w = 10^{m-1}w'$ for some $w' \in \Omega_m(U)$. Whence $R_m^+(10^{m-1}U) = R_m(U)$. Similarly, $w \in \Omega_m^-(10^{m-1}U)$ if and only if w is of the form $w = 01^mw'$ for some $w' \in \Omega_m^-(U)$. Whence $R_m^-(10^{m-1}U) = R_m^-(U)$. A similar argument applies to the remaining two identities. □

Fix a positive integer $r = 1\cdot 2^{k+1} + x_k\cdot 2^k + \cdots x_1\cdot 2 + x_0$. The above lemma can be used to compute $R_m([r])$ as follows: We construct a tower of $k+2$ levels $L_0, L_1, \cdots, L_{k+1}$, where each level L_i consists of an ordered pair (a,b) of positive integers. We start with level 0 by setting $L_0 = (1,1)$. Then L_{i+1} is obtained from L_i according to the value of x_i. If $L_i = (a,b)$, then $L_{i+1} = (a, a+b)$ if $x_i = 0$, and $L_{i+1} = (a+b, b)$ if $x_i = 1$. It follows from the above Lemma that $L_{k+1} = (R_m^+([r]), R_m^-([r]))$. Hence $R_m([r])$ is the sum of the entries of level L_{k+1}.

By the well known Fine and Wilf Theorem [4], given a pair of relatively prime numbers (p,q), there exists a $\{0,1\}$-word w of length $p+q-2$ (unique up to isomorphism) having periods p and q, and if p and q are both greater than 1, then this word contains both 0's and 1'; in other words $1 = \gcd(p,q)$ is not a period. We call such a word a *Fine and Wilf word* relative to (p,q). Moreover it can be shown (see [12] for example) that if both p and q are greater than 1, then the suffixes of w of lengths p and q begin in different symbols. We denote by $FW(p,q)$ the unique Fine and Wilf word relative to (p,q) with the property that its suffix of length p begins in 0 and its suffix of length q begins in 1.

We now apply this to the ordered pair $(p,q) = (R_m^+([r]), R_m^-([r]))$. It is well known that $FW(R_m^+([r]), R_m^-([r]))01$ is given explicitly by the following composition of morphisms:

$$FW(R_m^+([r]), R_m^-([r]))01 = \tau_{x_0}\circ\tau_{x_1}\circ\cdots\circ\tau_{x_k}(01)$$

where

$$\begin{array}{ll} \tau_0(0) = 0 & \tau_0(1) = 01 \\ \tau_1(0) = 10 & \tau_1(1) = 1 \end{array}$$

(see for instance [5, 12]).

Let

$$\alpha(r) = |FW(R_m^+([r]), R_m^-([r]))01|_1$$

and

$$\beta(r) = |FW(R_m^+([r]), R_m^-([r]))01|_0$$

in other words, $\alpha(r)$ is the number of occurrences of 1 in

$$FW(R_m^+([r]), R_m^-([r]))01$$

and $\beta(r)$ the number of 0's in

$$FW(R_m^+([r]), R_m^-([r]))01.$$

In summary:

$$\begin{aligned}
R_m([r]) &= R_m^+([r]) + R_m^-([r]) \\
&= R_m^+([r]) + R_m^-([r]) - 2 + 2 \\
&= |FW(R_m^+([r]), R_m^-([r]))| + 2 \\
&= |FW(R_m^+([r]), R_m^-([r]))01| \\
&= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)| \\
&= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1 + |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_0 \\
&= \alpha(r) + \beta(r) \\
&= |\tau_1 \circ \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1 \\
&= \alpha(2r+1).
\end{aligned}$$

The key step in the proof of Theorem 1.1 is to replace above the sum of the periods $R_m^+([r]) + R_m^-([r])$ of the Fine and Wilf word $FW(R_m^+([r]), R_m^-([r]))$ by the sum of the number of occurrences of 0's and 1's in $FW(R_m^+([r]), R_m^-([r]))01$. The following basic identities are readily verified:

- $\alpha(1) = \beta(1) = 1$.
- $\alpha(2r) = \alpha(r)$.
- $\beta(2r) = \alpha(r) + \beta(r)$.
- $\alpha(2r+1) = \alpha(r) + \beta(r)$.
- $\beta(2r+1) = \beta(r)$.
- $\beta(r) = \alpha(r+1)$.

Summarizing we have

Proposition 2.2. — *Let $U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_0 0^m$ with $x_i \in \{0,1\}$. Let r be the number whose base 2 expansion is given by $1x_kx_{k-1}\cdots x_0$. Then $R_m(U) = \alpha(2r+1)$ where the sequence $\alpha(r)$ is defined recursively by:*

- $\alpha(1) = 1$
- $\alpha(2r) = \alpha(r)$
- $\alpha(2r+1) = \alpha(r) + \alpha(r+1)$.

We now consider a new function $\psi(r)$ defined by $\psi(1) = 1$, and for $r \geqslant 1$

$$\psi(r+1) = \sum_{j=0}^{2j \leqslant r} \binom{r-j}{j} \pmod 2.$$

We will show that $\psi(r)$ and $\alpha(r)$ satisfy the same recursive relations, namely: $\psi(2r) = \psi(r)$ and $\psi(2r+1) = \psi(r) + \psi(r+1)$. Thus $\alpha(r) = \psi(r)$ for each r thereby establishing formula (2.2).

We shall make use of the following lemma:

Lemma 2.3. — $\binom{n}{k} \pmod 2 = \binom{2n+1}{2k} \pmod 2 + \binom{2n}{2k+1} \pmod 2$.

Proof. — This follows immediately from the so-called Lucas' identities: $\binom{2n}{2k+1} = 0 \pmod 2$ for $0 \leqslant k \leqslant n-1$, and $\binom{n}{k} = \binom{2n+1}{2k} \pmod 2$ for $0 \leqslant k \leqslant n$. □

Proposition 2.4. — *For $r \geqslant 0$ we have $\psi(2r+2) = \psi(r+1)$ and for $r \geqslant 1$ we have $\psi(2r+1) = \psi(r) + \psi(r+1)$.*

Proof. — By Lemma 2.3 we have

$$\begin{aligned}
\psi(r+1) &= \sum_{j=0}^{2j \leqslant r} \binom{r-j}{j} \pmod 2 \\
&= \sum_{j=0}^{2j \leqslant r} \left(\binom{2r-2j+1}{2j} \pmod 2 + \binom{2r-2j}{2j+1} \pmod 2 \right) \\
&= \sum_{i=0}^{r} \binom{2r+1-i}{i} \pmod 2 \\
&= \psi(2r+2).
\end{aligned}$$

As for the second recursive relation we have

$$\begin{aligned}
\psi(2r+1) &= \sum_{j=0}^{r} \binom{2r-j}{j} \pmod 2 \\
&= \sum_{i=0}^{2i \leqslant r} \binom{2r-2i}{2i} \pmod 2 + \sum_{i=0}^{2i \leqslant r-1} \binom{2r-2i-1}{2i+1} \pmod 2
\end{aligned}$$

But

$$\begin{aligned}\binom{2r-2i}{2i} \pmod 2 &= \frac{(2r-2i)!}{(2i)!(2r-4i)!} \pmod 2\\ &= \frac{(2r-2i+1)!}{(2i)!(2r-4i+1)!} \pmod 2\\ &= \binom{2r-2i+1}{2i} \pmod 2\\ &= \binom{r-i}{i} \pmod 2 \quad \text{by Lemma 2.3.}\end{aligned}$$

Hence

$$\sum_{i=0}^{2i\leqslant r} \binom{2r-2i}{2i} \pmod 2 = \sum_{i=0}^{2i\leqslant r} \binom{r-i}{i} \pmod 2 = \psi(r+1).$$

Similarly

$$\begin{aligned}\binom{2r-2i-1}{2i+1} \pmod 2 &= \frac{(2r-2i-1)!}{(2i+1)!(2r-4i-2)!} \pmod 2\\ &= \frac{(2r-2i-1)!}{(2i)!(2r-4i-1)!} \pmod 2\\ &= \binom{2r-2i-1}{2i} \pmod 2\\ &= \binom{r-1-i}{i} \pmod 2 \quad \text{by Lemma 2.3.}\end{aligned}$$

Hence

$$\sum_{i=0}^{2i\leqslant r-1} \binom{2r-2i-1}{2i+1} \pmod 2 = \sum_{i=0}^{2i\leqslant r-1} \binom{r-1-i}{i} \pmod 2 = \psi(r).$$

It follows that $\psi(2r+1) = \psi(r) + \psi(r+1)$. □

Having established that $\alpha(r) = \psi(r)$ for each $r \geqslant 1$, we deduce that:

Corollary 2.5. — *Let $U = 10^{m-1}x_k0^{m-1}x_{k-1}\cdots 0^{m-1}x_00^m$ with $x_i \in \{0,1\}$. Let r be the number whose base 2 expansion is given by $1x_kx_{k-1}\cdots x_0$. Then $R_m(U) = \sum_{j=0}^{r}\binom{2r-j}{j} \pmod 2$.*

This concludes our proof of Theorem 1.1.

3. Concluding Remarks

3.1. A formula for $R_m^{\leqslant}(w)$

Our proof applies more generally to give a formula for $R_m^{\leqslant}(w)$ for each representation w of n. In other words, given $w \in \Omega_m(n)$, then either w does not contain any occurrences of 0^m (in which case $R_m^{\leqslant}(w) = 1$) or w may be factored in the form

$$w = V_1 U_1 V_2 U_2 \cdots V_N U_N W$$

where the V_i and W do not contain any occurrences of 0^m and the V_i do not end in 0^{m-1}, and where the U_i are of the form

$$U_i = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m$$

with $x_i \in \{0,1\}$. Each factor U_i is coded by a positive integer r_i whose base 2 expansion is $1x_k x_{k-1} \cdots x_0$. It is easy to see that any representation of n less or equal to w may be factored in the form

$$V_1 U_1' V_2 U_2' \cdots V_N U_N' W$$

where for each $1 \leqslant i \leqslant N$, U_i' is an equivalent representation of U_i. Hence $R_m^{\leqslant}(w) = \prod_{i=1}^{N} R_m(U_i)$ from which it follows that

$$R_m^{\leqslant}(w) = \prod_{i=1}^{N} \sum_{j=0}^{r_i} \binom{2r_i - j}{j} \pmod{2}.$$

3.2. Episturmian numeration systems

Let A be a finite non-empty set. Associated to an infinite word $\omega = \omega_1\omega_2\omega_3 \ldots \in A^{\mathbb{N}}$ is a non-decreasing sequence of positive integers $\mathcal{E}(\omega) = E_1, E_2, E_3, \ldots$ defined recursively as follows: $E_1 = 1$, and for $k \geqslant 1$, the quantity E_{k+1} is defined by the following rule: If $\omega_{k+1} \neq \omega_j$ for each $1 \leqslant j \leqslant k$, then set

$$E_{k+1} = 1 + \sum_{j=1}^{k} E_j.$$

Otherwise let $\ell \leqslant k$ be the largest integer such that $\omega_{k+1} = \omega_\ell$, and set

$$E_{k+1} = \sum_{j=\ell}^{k} E_j.$$

In particular we note that $E_{k+1} = E_k$ if and only if $\omega_{k+1} = \omega_k$.

Set $\mathcal{N}(\omega) = \{E_k | k \geqslant 1\}$. For $E \in \mathcal{N}(\omega)$ let $k \geqslant 1$ be such that $E = E_k$. We define $\sigma(E) = \omega_k$ and say that E is *based* at $\omega_k \in A$. We also define the quantity $\rho(E)$, which we call the *multiplicity* of E, by

$$\rho(E) = \#\{i \geqslant 1 | E = E_i\}.$$

We can write $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots\}$ where for each $i \geqslant 1$ we have $x_i < x_{i+1}$. Thus we have that $\omega = \sigma(x_1)^{\rho(x_1)} \sigma(x_2)^{\rho(x_2)} \ldots$.

It can be verified that the set $\mathcal{N}(\omega)$ defines a numeration system (see [8]). More precisely, each positive integer n may be written as a sum of the form

$$n = m_k x_k + m_{k-1} x_{k-1} + \cdots + m_1 x_1 \tag{3.1}$$

where for each $1 \leqslant i \leqslant k$ we have $0 \leqslant m_i \leqslant \rho(x_i)$ and $m_k \geqslant 1$. While such a representation of n is not necessarily unique, one way of obtaining such a representation is to use the "greedy algorithm". In this case we call the resulting representation the Zeckendorff representation of n and denote it $Z_\omega(n)$. We call the above numeration system a *generalized Ostrowski system* or an *Episturmian numeration system.* In fact, the quantities E_i are closely linked to the lengths of the palindromic prefixes of the characteristic Episturmian word associated to the directive sequence ω (see [6, 7, 8, 9]). In case $\#A = 2$, this is known as the Ostrowski numeration system (see [1, 2, 11]). In case $A = \{1, 2, \ldots, m\}$ and ω is the periodic sequence $\omega = (1, 2, 3, \ldots, m,)^\infty$, then the resulting numeration system is the m-bonacci system defined earlier.

Given an infinite word $\omega = \omega_1 \omega_2 \omega_3 \ldots \in A^{\mathbb{N}}$, we are interested in the number of distinct ways of writing each positive integer n as a sum of the form (3.1). More precisely, denoting by $\hat{A}$ the set $\{\hat{a} | a \in A\}$, we set $R_\omega(n) = \#\Omega_\omega(n)$ where $\Omega_\omega(n)$ is the set of all expressions of the form

$$\widehat{\sigma(x_k)}^{m_k} \sigma(x_k)^{\rho(x_k)-m_k} \widehat{\sigma(x_{k-1})}^{m_{k-1}} \sigma(x_{k-1})^{\rho(x_{k-1})-m_{k-1}} \cdots \widehat{\sigma(x_1)}^{m_1} \sigma(x_1)^{\rho(x_1)-m_1} \tag{3.2}$$

in $(A \cup \hat{A})^*$, such that $n = m_k x_k + m_{k-1} x_{k-1} + \cdots + m_1 x_1$ where $\mathcal{N}(\omega) = \{x_1, x_2, x_3, \ldots | 1 = x_1 < x_2 < x_3 \ldots\}$ and where $0 \leqslant m_i \leqslant \rho(x_i)$ and $m_k \geqslant 1$.(1) For $w \in \Omega_\omega(n)$ we sometimes write $R_\omega(w)$ for $R_\omega(n)$.

Just as in the previous section, we begin with a unique special factorization of the Zeckendorff representation of n. In this case, this factorization

(1) Our notation here differs somewhat from that of Justin and Pirillo in [8]. For instance, in [8] the authors use the notation $\bar{a}$ for in lieu of our $\hat{a}$. Also instead of the expression (3.2), they consider the reverse of this word.

was originally defined by Justin and Pirillo (see Theorem 2.7 in [8]):

$$Z_\omega(n) = V_1U_1V_2U_2\cdots V_NU_NW$$

where for each $1 \leqslant i \leqslant N$ we have that U_i is a *a_i-based maximal semigood multiblock* for some $a_i \in A$. Moreover any other representation of n may be factored in the form

$$Z_\omega(n) = V_1U_1'V_2U_2'\cdots V_NU_N'W$$

where U_i' is an equivalent representation of U_i (see Theorem 2.7 in [8]). Thus as before (see (2.1)) we have

$$R_\omega(n) = \prod_{i=1}^{N} R_\omega(U_i).$$

For each $1 \leqslant i \leqslant N$ the factor U_i corresponds to a sum of the form

$$m_Kx_K + m_{K-1}x_{K-1} + \cdots + m_kx_k$$

for some $K > k$ with $m_K \neq 0$, and for each $K \geqslant j \geqslant k$ we have that if $m_j \neq 0$, then $\sigma(x_j) = a_i$ [8]. In other words the only "accented" symbol occurring in U_i is a_i, i.e., $U_i \in (A \cup \{\hat{a}_i\})^*$.

Associated to U_i is a $\{0,1\}$-word $\nu(U_i) = \nu_K\nu_{K-1}\ldots\nu_k$ where $\nu_K = 10$, $\nu_j = \varepsilon$ (the empty word) if $\sigma(x_j) \neq a_i$, $\nu_j = 10$ if $\sigma(x_j) = a_i$ and $m_j = \rho(x_j)$, $\nu_j = 010$ if $\sigma(x_j) = a_i$ and $0 < m_j < \rho(x_j)$ and $\nu_j = 00$ if $\sigma(x_j) = a_i$ and $m_j = 0$.

By comparing the matrix formulation given in Corollary 2.11 in [8] used to compute $R_\omega(U_i)$ with the matrix formulation given in Proposition 4.1 in [1], we leave it to the reader to verify the following:

Proposition 3.1. — $R_\omega(U_i) = R_2(\nu(U_i))$.

In other words computing the multiplicities of representations in a generalized Ostrowski numeration system may be reduced to a computation of the multiplicities of representations in the Fibonacci base.

Example 3.2. — We consider the example originally started in Berstel's paper [1] and later revisited by Justin and Pirillo as Example 2.3 in [8] of the Ostrowski numeration system associated to the infinite word $\omega = a, a, b, b, a, a, a, b, b, a, a, b, b, a, a, a, b, \ldots$. It is readily verified that

$$\mathcal{N}(\omega) = \{1, 3, 7, 24, 55, 134, 323, \ldots\},$$

$\sigma(1) = \sigma(7) = \sigma(55) = \sigma(323) = a$, $\sigma(3) = \sigma(24) = \sigma(134) = b$, and $\rho(1) = 2$, $\rho(3) = 2$, $\rho(7) = 3$, $\rho(24) = 2$, $\rho(55) = 2$, $\rho(134) = 2$, $\rho(323) = 3$.

Applying the greedy algorithm we obtain the following representation of the number 660

$$660 = 2(323) + 0(134) + 0(55) + 0(24) + 2(7) + 0(3) + 0(1).$$

So $Z_\omega(660) = \hat{a}\hat{a}abbaabb\hat{a}\hat{a}abbaa$. which is a semigood multiblock based at a. We deduce that

$$\nu(Z_\omega(660)) = 10 \cdot \varepsilon \cdot 00 \cdot \varepsilon \cdot 010 \cdot \varepsilon \cdot 00$$

or simply $\nu(Z_\omega(660)) = 100001000$.

Following the algorithm of Corollary 2.11 of [8] due to Justin and Pirillo, we obtain $q_1 = 2$, $q_2 = 4$, $p_1 = 2$, $p_2 = 2$, $c_1 = c_2 = 1$, so that

$$R_\omega(660) = (1,0)\begin{pmatrix}0 & 2\\ 0 & 3\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = 6$$

In contrast, applying the algorithm in Proposition 4.1 of [1] due to Berstel to the Zeckendorff word $\nu(Z_\omega(660)) = 100001000$, we obtain $d_1 = 4$, $d_2 = 3$ so that

$$R_2(\nu(Z_\omega(660))) = (1,1)\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = 6$$

as required(2)

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Marcia EDSON & Luca Q. ZAMBONI
University of North Texas
Department of Mathematics
PO Box 311430
Denton, TX 76203-1430 (USA)
mre0006@unt.edu
luca@unt.edu