ANNALES

DE

L'INSTITUT FOURIER

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Tome 56, nº 7 (2006), p. 2093-2113.

<http://aif.cedram.org/item?id=AIF_2006__56_7_2093_0>

Ann. Inst. Fourier, Grenoble
56, 7 (2006) 2093-2113

CONTINUED FRACTIONS AND TRANSCENDENTAL NUMBERS

by Boris ADAMCZEWSKI, Yann BUGEAUD & Les DAVISON

Abstract. — The main purpose of this work is to present new families of transcendental continued fractions with bounded partial quotients. Our results are derived thanks to combinatorial transcendence criteria recently obtained by the first two authors in [3].

Résumé. — L'objet principal de ce travail est de donner de nouvelles familles de fractions continues transcendantes dont la suite des quotients partiels est bornée. Les démonstrations de nos résultats reposent sur les critères combinatoires de transcendance récemment obtenus par les deux premiers auteurs dans [3].

1. Introduction

It is widely believed that the continued fraction expansion of every irrational algebraic number α either is eventually periodic (and we know that this is the case if and only if α is a quadratic irrational), or it contains arbitrarily large partial quotients. Apparently, this question was first considered by Khintchine in [22] (see also [6, 39, 41] for surveys including a discussion on this subject). A preliminary step towards its resolution consists in providing explicit examples of transcendental continued fractions.

The first result of this type goes back to the pioneering work of Liouville [26], who constructed transcendental real numbers with a very fast growing sequence of partial quotients. Indeed, the so-called "Liouville inequality" implies the transcendence of real numbers with very large partial quotients. Replacing it by Roth's theorem yields refined results, as shown by Davenport and Roth [15]. In [4], the argument of Davenport and Roth

Keywords: Continued fractions, transcendental numbers, subspace theorem.
Math. classification: 11J81, 11J70, 68R15.

is slightly improved and Roth's theorem is replaced by a more recent result of Evertse [19] to obtain the best known result of this type. Note that the constant e, whose continued fraction expansion is given by

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots, 1, 1, 2n, 1, 1, \dots],$$

(see for instance [23]) provides an explicit example of a transcendental number with unbounded partial quotients; however, its transcendence does not follow from the criteria from [26, 15, 4].

At the opposite side, there is a quest for finding explicit examples of transcendental continued fractions with bounded partial quotients. The first examples of such continued fractions were found by Maillet [28] (see also Section 34 of Perron [30]). The proof of Maillet's results is based on a general form of the Liouville inequality which limits the approximation of algebraic numbers by quadratic irrationals. They were subsequently improved upon by A. Baker [10, 11], who used the Roth theorem for number fields obtained by LeVeque [24]. Later on, Davison [16] applied a result of W. M. Schmidt [36], saying in a much stronger form than Liouville's inequality that a real algebraic number cannot be well approximable by quadratic numbers, to show the transcendence of some specific continued fractions (see Section 4). With the same auxiliary tool, M. Queffélec [32] established the nice result that the Thue–Morse continued fraction is transcendental (see Section 5). This method has then been made more explicit, and combinatorial transcendence criteria based on Davison's approach were given in [7, 17, 13, 25].

Recently, Adamczewski and Bugeaud [3] obtained two new combinatorial transcendence criteria for continued fractions that are recalled in Section 2. The main novelty in their approach is the use of a stronger Diophantine result of W. M. Schmidt [37, 38], commonly known as the Subspace Theorem. After some work, it yields considerable improvements upon the criteria from [7, 17, 13, 25]. These allowed them to prove that the continued fraction expansion of every real algebraic number of degree at least three cannot be "too simple", in various senses. It is the purpose of the present work to give further applications of their criteria to well-known (families of) continued fractions. In addition, we provide a slight refinement of their second criterion. Note also that a significative improvement of the results of Maillet and of Baker mentioned above is obtained in [4] thanks to a similar use of the Subspace Theorem.

The present paper is organized as follows. In Section 2, we state the two main transcendence criteria of [3], namely Theorems 2.1 and 2.2. Then, Section 3 is devoted to a slight sharpening of Theorem 2.2. In the subsequent Sections, we give various examples of applications of these transcendence criteria. We first solve in Section 4 a problem originally tackled by Davison in [16] and later considered by several authors in [7, 17, 13]. The Rudin–Shapiro and the Baum–Sweet continued fractions are proved to be transcendental in Section 5. Then, Sections 6 and 7 are respectively devoted to folded continued fractions and continued fractions arising from perturbed symmetries (these last sequences were introduced in [29, 14]). In the last Section, we show through the study of another family of continued fractions that, in some cases, rather than applying roughly Theorem 2.2, it is much better to go into its proof and to evaluate continuants carefully.

2. The transcendence criteria

In this Section, we recall the transcendence criteria, namely Theorems 2.1 and 2.2, obtained by the first two authors in [3].

Before stating the new criteria, we need to introduce some notation. Let $\mathcal{A}$ be a given set, not necessarily finite. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. For any positive integer k, we write W^k for the word $W \dots W$ (k times repeated concatenation of the word W). More generally, for any positive rational number x, we denote by W^x the word $W^{[x]}W'$, where W' is the prefix of W of length $\lceil (x-[x])|W| \rceil$. Here, and in all what follows, $[y]$ and $\lceil y \rceil$ denote, respectively, the integer part and the upper integer part of the real number y. For example, if W denotes the word 232243, then $W^{3/2}$ is the word 232243232. Let $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ be a sequence of elements from $\mathcal{A}$, that we identify with the infinite word $a_1 a_2 \dots a_\ell \dots$. Let w be a rational number with $w > 1$. We say that $\mathbf{a}$ satisfies Condition $(\star)_w$ if $\mathbf{a}$ is not eventually periodic and if there exists a sequence of finite words $(V_n)_{n \geqslant 1}$ such that:

- For any $n \geqslant 1$, the word V_n^w is a prefix of the word $\mathbf{a}$;
- The sequence $(|V_n|)_{n \geqslant 1}$ is increasing.

Roughly speaking, $\mathbf{a}$ satisfies Condition $(\star)_w$ if $\mathbf{a}$ is not eventually periodic and if there exist infinitely many "non-trivial" repetitions (the size of which is measured by w) at the beginning of the infinite word $a_1 a_2 \dots a_\ell \dots$

A first transcendence criterion for "purely" stammering continued fractions is given in [3].

Theorem 2.1. — *Let* $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ *be a sequence of positive integers. Let* $(p_\ell/q_\ell)_{\ell \geqslant 1}$ *denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

If there exists a rational number $w \geqslant 2$ *such that* $\mathbf{a}$ *satisfies Condition* $(\star)_w$*, then* α *is transcendental. If there exists a rational number* $w > 1$ *such that* $\mathbf{a}$ *satisfies Condition* $(\star)_w$*, and if the sequence* $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ *is bounded (which is in particular the case when the sequence* $\mathbf{a}$ *is bounded), then* α *is transcendental.*

Unfortunately, in the statement of Theorem 2.1, the repetitions must appear at the very beginning of $\mathbf{a}$. When this is not the case, but the repetitions occur not too far from the beginning of $\mathbf{a}$, then we have another criterion.

Keep the above notation. Let w and w' be non-negative rational numbers with $w > 1$. We say that $\mathbf{a}$ satisfies Condition $(\star\star)_{w,w'}$ if $\mathbf{a}$ is not eventually periodic and if there exist two sequences of finite words $(U_n)_{n \geqslant 1}$, $(V_n)_{n \geqslant 1}$ such that:

- For any $n \geqslant 1$, the word $U_n V_n^w$ is a prefix of the word $\mathbf{a}$;
- The sequence $(|U_n|/|V_n|)_{n \geqslant 1}$ is bounded from above by w';
- The sequence $(|V_n|)_{n \geqslant 1}$ is increasing.

We are now ready to present a transcendence criterion for (general) stammering continued fractions, as stated in [3].

Theorem 2.2. — *Let* $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ *be a sequence of positive integers. Let* $(p_\ell/q_\ell)_{\ell \geqslant 1}$ *denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ *is bounded and set*

$$M = \limsup_{\ell \to +\infty} q_\ell^{1/\ell}$$

and

$$m = \liminf_{\ell \to +\infty} q_\ell^{1/\ell}.$$

Let w *and* w' *be non-negative real numbers with*

$$w > w' \left(2 \frac{\log M}{\log m} - 1 \right) + \frac{\log M}{\log m}. \tag{2.1}$$

If $\mathbf{a}$ *satisfies Condition* $(\star\star)_{w,w'}$*, then* α *is transcendental.*

It turns out that condition (2.1) *can be slightly weakened by means of a careful consideration of continuants. This is the purpose of Section 3.*

3. A slight sharpening of Theorem 2.2

In this Section, we establish the following improvement of Theorem 2.2.

THEOREM 3.1. — *Let* $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ *be a sequence of positive integers. Let* $(p_\ell/q_\ell)_{\ell \geqslant 1}$ *denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell \geqslant 1}$ *is bounded and set*

$$M = \limsup_{\ell \to +\infty} q_\ell^{1/\ell}$$

and

$$m = \liminf_{\ell \to +\infty} q_\ell^{1/\ell}.$$

Let w *and* w' *be non-negative real numbers with*

$$w > w' \left(2 \frac{\log M}{\log m} - 1 \right) + 1. \tag{3.1}$$

If $\mathbf{a}$ *satisfies Condition* $(\star\star)_{w,w'}$*, then* α *is transcendental.*

We note that the right hand side of (3.1) is always smaller than or equal to the right hand side of (2.1).

Before proceeding with the proof, we recall some useful facts on continuants. For positive integers $a_1, \ldots, a_m$, denote by $K_m(a_1, \ldots, a_m)$ the denominator of the rational number $[0; a_1, \ldots, a_m]$. It is commonly called a *continuant*.

LEMMA 3.2. — *For any positive integers* $a_1, \ldots, a_m$ *and any integer* k *with* $1 \leqslant k \leqslant m-1$*, we have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) &\leqslant K_m(a_1, \ldots, a_m) \\ &\leqslant 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m). \end{aligned}$$

Proof. — See Kapitel 1 from Perron's book [30]. □

Proof of Theorem 3.1 — We follow step by step the proof of Theorem 2 from [3], with a single modification. We keep the notation from [3]. Recall that, for any $n \geqslant 1$, we set $r_n = |U_n|$ and $s_n = |V_n|$.

Let n be a positive integer. Let $\delta > 0$ be a (small) real number. Since $w > 1$ and $r_n \leqslant w' s_n$, we get

$$\frac{2r_n + \delta s_n}{r_n + (w-1)s_n} \leqslant \frac{2w' s_n + \delta s_n}{w' s_n + (w-1)s_n} = \frac{2w' + \delta}{w' + w - 1} < \frac{\log m}{\log M},$$

by (3.1), if δ is sufficiently small. Consequently, there exist positive real numbers η and η' with $\eta < 1$ such that

$$2(1+\eta)(1+\eta')r_n + \eta(1+\eta')s_n < (1-\eta')(r_n + ws_n - s_n)\frac{\log m}{\log M}, \tag{3.2}$$

for any $n \geqslant 1$.

In the course of the proof of Theorem 4.1 from [3] we had to bound from above the quantity $q_{r_n}^{1+\eta}\, q_{r_n+s_n}^{1+\eta}\, q_{r_n+[ws_n]}^{-1}$. Our new observation is that the estimate

$$q_{r_n+s_n}\, q_{r_n+[(w-1)s_n]} \ll q_{r_n}\, q_{r_n+[ws_n]}$$

follows from Lemma 3.2 (here and below, the numerical constant implied by $\ll$ does not depend on n). Consequently, we get

$$q_{r_n}^{1+\eta}\, q_{r_n+s_n}^{1+\eta}\, q_{r_n+[ws_n]}^{-1} \ll q_{r_n}^{2+\eta}\, q_{r_n+s_n}^{\eta}\, q_{r_n+[(w-1)s_n]}^{-1}. \tag{3.3}$$

Assuming n sufficiently large, we then have

$$q_{r_n} \leqslant M^{(1+\eta')r_n}, \quad q_{r_n+s_n} \leqslant M^{(1+\eta')(r_n+s_n)},$$

and

$$q_{r_n+[(w-1)s_n]} \geqslant m^{(1-\eta')(r_n+ws_n-s_n)},$$

with η' as in (3.2). Consequently, we get

$$\begin{aligned} &q_{r_n}^{2+\eta}\, q_{r_n+s_n}^{\eta}\, q_{r_n+[(w-1)s_n]}^{-1} \\ &\qquad\leqslant M^{2(1+\eta)(1+\eta')r_n + \eta(1+\eta')s_n}\, m^{-(1-\eta')(r_n+ws_n-s_n)} \leqslant 1, \end{aligned}$$

by our choice of η and η'. It then follows from (3.2) and (3.3) that

$$q_{r_n}^{1+\eta}\, q_{r_n+s_n}^{1+\eta}\, q_{r_n+[ws_n]}^{-1} \ll 1,$$

and, with the notation from [3], we get the upper bound

$$\prod_{1 \leqslant j \leqslant 4} |L_j(\underline{z}_n)| \ll (q_{r_n}\, q_{r_n+s_n})^{-\eta}$$

for any positive integer n. We then conclude as in that paper. □

4. Davison's continued fractions

Let θ be an irrational number with $0 < \theta < 1$. Under some mild assumptions, Davison [16] established the transcendence of the real number $\alpha_\theta = [0; d_1, d_2, \ldots]$, with $d_n = 1 + ([n\theta] \bmod 2)$ for any $n \geqslant 1$. These extra assumptions were subsequently removed in [7]. Then, Davison [17] and Baxa [13] studied the more general question of the transcendence of the real number $\alpha_{k,\theta} = [0; d_1, d_2, \ldots]$, where $k \geqslant 2$ is an integer and $d_n = 1 + ([n\theta] \bmod k)$ for any $n \geqslant 1$. The two authors obtained some partial results but their methods did not allow them to cover all the cases. It turns out that Theorem 2.1 yields a complete answer to this question.

THEOREM 4.1. — *Let θ be an irrational number with $0 < \theta < 1$ and let k be an integer at least equal to 2. Let $\mathbf{d} = (d_n)_{n\geqslant 1}$ be defined by $d_n = 1+([n\theta] \bmod k)$ for any $n \geqslant 1$. Then, the number $\alpha_{k,\theta} = [0; d_1, d_2, \ldots]$ is transcendental.*

In order to prove Theorem 4.1, we need two auxiliary results (Lemmas 4.3 and 4.4 below), which will be deduced from the following proposition obtained in [17]. Throughout this Section, $\theta = [0; a_1, a_2, \ldots, a_n, \ldots]$ denotes an irrational number in $(0, 1)$ and $(p_n/q_n)_{n\geqslant 0}$ is the sequence of its convergents.

PROPOSITION 4.2. — *For any non-negative integers n and r with $1 \leqslant r \leqslant q_{n+1} - 1$, we have*

$$[(q_n + r)\theta] = p_n + [r\theta].$$

Our first auxiliary result is the following.

LEMMA 4.3. — *For any integers n, r and s with $n \geqslant 1$, $0 \leqslant s \leqslant a_{n+1}$ and $1 \leqslant r \leqslant q_n + q_{n-1} - 1$, we have*

$$[(sq_n + r)\theta] = sp_n + [r\theta].$$

Proof. — The proof goes by induction on s. If $s = 0$, the statement is a tautology. Let us assume that the result holds for a given s with $0 \leqslant s < a_{n+1}$. Then, we have $s + 1 \leqslant a_{n+1}$ and

$$(s + 1)q_n + r \leqslant a_{n+1}q_n + q_n + q_{n-1} - 1 = q_{n+1} + q_n - 1.$$

Hence, $sq_n + r \leqslant q_{n+1} - 1$ and Proposition 4.2 implies that

$$[((s + 1)q_n + r)\theta] = [(q_n + (sq_n + r))\theta] = p_n + [(sq_n + r)\theta].$$

By our inductive assumption, we thus obtain

$$[((s + 1)q_n + r)\theta] = p_n + sp_n + [r\theta] = (s + 1)p_n + [r\theta],$$

concluding the proof of the lemma. □

Lemma 4.4. — *For any integers n, ℓ and r with $n \geqslant 1$, $\ell \geqslant 0$ and $1 \leqslant r \leqslant q_{n+1} - 1$, we have*

$$[(q_{n+\ell} + q_{n+\ell-1} + \ldots + q_n + r)\theta] = p_{n+\ell} + p_{n+\ell-1} + \ldots + p_n + [r\theta].$$

Proof. — The proof goes by induction on ℓ. The case $\ell = 0$ is given by Proposition 4.2 Let us assume that the desired result holds for a given non-negative integer ℓ. Let n and r be as in the statement of the lemma. Since $q_{m+2} \geqslant q_{m+1} + q_m$ holds for any non-negative integer m, we easily get that

$$q_{n+\ell} + \ldots + q_{n+1} + q_n + r \leqslant q_{n+\ell+2} - 1.$$

Then, we infer from Proposition 4.2 that

$$\begin{aligned}[(q_{n+\ell+1} + q_{n+\ell} + \ldots + q_n + r)\theta] &= [(q_{n+\ell+1} + (q_{n+\ell} + \ldots + q_n + r))\theta] \\ &= p_{n+\ell+1} + [(q_{n+\ell} + \ldots + q_n + r))\theta].\end{aligned}$$

Using the inductive assumption, this shows that the desired result holds for $\ell + 1$ and proves the lemma. □

Proof of Theorem 4.1 — We are now ready to prove Theorem 4.1 by showing that the sequence $(d_n)_{n\geqslant 1}$ satisfies Condition $(\star)_w$ for a suitable real number $w > 1$. We first remark that the sequence $\mathbf{d}$ is not eventually periodic since θ is irrational. We have to distinguish two cases.

First, let us assume that there are infinitely many integers n such that $a_{n+1} \geqslant k$. For such an n, we infer from Lemma 4.3 with $s = k$ that

$$[(kq_n + r)\theta] \equiv [r\theta] \bmod k, \text{ for } 1 \leqslant r \leqslant q_n + q_{n-1} - 1. \tag{4.1}$$

Set $V_n = d_1 \ldots d_{kq_n}$ and $W_n = d_{kq_n+1} d_{kq_n+2} \ldots d_{(k+1)q_n+q_{n-1}-1}$ and view V_n and W_n as words on the alphabet $\{1, 2, \ldots, k\}$. It follows from (4.1) that W_n is a prefix of V_n. Furthermore, since

$$\frac{|W_n|}{|V_n|} = \frac{q_n + q_{n-1} - 1}{kq_n} \geqslant \frac{1}{k},$$

the infinite word $\mathbf{d}$ begins in $V_n^{1+1/k}$. Thus, $\mathbf{d}$ satisfies Condition $(\star)_{1+1/k}$.

Now, let us assume that there exists an integer n_0 such that $a_n < k$ for all $n \geqslant n_0$. Let $n \geqslant n_0$ be an integer. At least two among the $k+1$ integers $p_n, p_n + p_{n+1}, \ldots, p_n + p_{n+1} + \ldots + p_{n+k}$ are congruent modulo k. Consequently, there exist integers $n' \geqslant n+1$ and ℓ, with $0 \leqslant \ell \leqslant k-1$ and $n' + \ell \leqslant n + k$, such that

$$p_{n'+\ell} + p_{n'+\ell-1} + \ldots + p_{n'} \equiv 0 \bmod k.$$

Set $N = q_{n'+\ell} + q_{n'+\ell-1} + \ldots + q_{n'}$. Then, Lemma 4.4 implies that

$$[(N+r)\theta] \equiv [r\theta] \mod k, \tag{4.2}$$

for $1 \leqslant r \leqslant q_{n'+1} - 1$. Set $V_n = d_1 \ldots d_N$ and $W_n = d_{N+1} \ldots d_{N+q_{n'+1}-1}$ and view V_n and W_n as words on the alphabet $\{1, 2, \ldots, k\}$. Then, (4.2) implies that W_n is a prefix of V_n. Since $q_{n'+\ell} \leqslant k^\ell q_{n'}$ (this follows from the assumption $a_m < k$ for any $m \geqslant n_0$), we obtain

$$\frac{|W_n|}{|V_n|} = \frac{q_{n'+1} - 1}{N} \geqslant \frac{q_{n'}}{kq_{n'+\ell}} \geqslant \frac{1}{k^{\ell+1}} \geqslant \frac{1}{k^k}.$$

It thus follows that the infinite word $\mathbf{d}$ begins in V_n^{1+1/k^k}.

Consequently, the sequence $\mathbf{d}$ satisfies in both cases Condition $(\star)_{1+1/k^k}$. By Theorem 2.1, the real number $\alpha_{k,\theta}$ is transcendental, as claimed. □

5. Automatic continued fractions

Let $k \geqslant 2$ be an integer. An infinite sequence $\mathbf{a} = (a_n)_{n \geqslant 0}$ is said to be k-automatic if a_n is a finite-state function of the base-k representation of n. This means that there exists a finite automaton starting with the k-ary expansion of n as input and producing the term a_n as output. Finite automata are one of the most basic models of computation and take place at the bottom of the hierarchy of Turing machines. A nice reference on this topic is the book of Allouche and Shallit [9]. We refer the reader to it for more details about this notion.

Motivated by the Hartmanis-Stearns problem [21], the following question was addressed in [3]: do there exist algebraic numbers of degree at least three whose continued fraction expansion can be produced by a finite automaton?

The first result towards this problem is due to Queffélec [32], who proved the transcendence of the Thue–Morse continued fractions. A more general statement can be found in [33]. As it is shown in [3], the transcendence of a large class of automatic continued fractions can be derived from Theorem 2.1 and 2.2. In the present Section, we show how our transcendence criteria apply to two others emblematic automatic sequences: the Rudin–Shapiro and the Baum–Sweet sequences.

Before proving such results, we recall some classical facts about morphisms and morphic sequences. For a finite set $\mathcal{A}$, we denote by $\mathcal{A}^\star$ the free monoid generated by $\mathcal{A}$. The empty word ε is the neutral element of $\mathcal{A}^\star$. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite sets. An application from $\mathcal{A}$ to $\mathcal{B}^\star$ can be uniquely extended to a homomorphism between the free monoids $\mathcal{A}^\star$ and $\mathcal{B}^\star$. We call morphism from $\mathcal{A}$ to $\mathcal{B}$ such a homomorphism. If there is a positive integer k such that each element of $\mathcal{A}$ is mapped to a word of length k, then the morphism is called *k-uniform* or simply *uniform*.

A morphism σ from $\mathcal{A}$ into itself is said to be prolongable if there exists a letter a such that $\sigma(a) = aW$, where the word W is such that $\sigma^n(W)$ is a non-empty word for every $n \geqslant 0$. In that case, the sequence of finite words $(\sigma^n(a))_{n\geqslant 1}$ converges in $\mathcal{A}^{\mathbf{Z}_{\geqslant 0}}$ (endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$) to an infinite word $\mathbf{a}$. Such an infinite word is clearly a fixed point for the map σ.

5.1. The Rudin–Shapiro continued fractions

Let $\varepsilon = (\varepsilon_n)_{n\geqslant 0}$ be a sequence with values in the set $\{+1, -1\}$. It is not difficult to see that

$$\sup_{\theta\in[0,1]} \left| \sum_{0\leqslant n<N} \varepsilon_n e^{2i\pi n\theta} \right| \geqslant \sqrt{N},$$

for any positive integer N. In 1950, Salem asked the following question, related to some problems in harmonic analysis: do there exist a sequence $\varepsilon = (\varepsilon_n)_{n\geqslant 0}$ in $\{+1, -1\}^{\mathbf{Z}_{\geqslant 0}}$ and a positive constant c such that

$$\sup_{\theta\in[0,1]} \left| \sum_{0\leqslant n<N} \varepsilon_n e^{2i\pi n\theta} \right| \leqslant c\sqrt{N} \tag{5.1}$$

holds for any positive integer N? A positive answer to this problem was given by Shapiro [40] and Rudin [35], who provided an explicit solution which is now known as the Rudin–Shapiro sequence. This sequence is a famous example of a 2-automatic sequence and can be defined as follows: r_n is equal to $+1$ (respectively -1) if the number of occurrences of the pattern "11" in the binary representation of n is even (respectively odd). Theorem 2.1 yields the transcendence of the Rudin–Shapiro continued fractions.

THEOREM 5.1. — *Let a and b two distinct positive integers, and let $\mathbf{r} = (r_n)_{n\geqslant 0}$ be the Rudin–Shapiro sequence on the alphabet $\{a, b\}$ (that is the symbol 1 is replaced by a and the symbol -1 is replaced by b in the usual Rudin–Shapiro sequence). Then, the real number $\alpha = [0; r_0, r_1, r_2, \ldots]$ is transcendental.*

Proof. — We first infer from (5.1) that the Rudin–Shapiro sequence is not eventually periodic. We present now a useful description of this sequence. Let σ be a morphism defined from $\{1,2,3,4\}^*$ into itself by: $\sigma(1)=12$, $\sigma(2)=42$, $\sigma(3)=13$ and $\sigma(4)=43$. Let

$$\mathbf{u}=1242434213\ldots$$

be the fixed point of σ begining with 1 and let φ be the morphism defined from $\{1,2,3,4\}^\star$ to $\{a,b\}^\star$ by: $\varphi(1)=\varphi(2)=a$ and $\varphi(3)=\varphi(4)=b$. It is known (see for instance [20], Ch. 5) that

$$\mathbf{r}=\varphi(\mathbf{u}). \tag{5.2}$$

Since $\sigma^5(1)=1242434213$, $\mathbf{u}$ begins with $V^{1+1/8}$, where $V=12424342$. Then, it follows from (5.2) that for any positive integer n, the word $\mathbf{r}$ begins with $\varphi(\sigma^n(V^{1+1/8}))$. The morphism σ being a uniform morphism, we easily check that

$$\varphi(\sigma^n(V^{1+1/8}))=(\varphi(\sigma^n(V)))^{1+1/8}.$$

The sequence $\mathbf{r}$ thus satisfies the condition $(\star)_{1+1/8}$. This ends the proof thanks to Theorem 2.1. □

5.2. The Baum–Sweet continued fractions

In 1976, Baum and Sweet [12] proved that, unlike what is expected in the real case, the function field $\mathbf{F}_2((X^{-1}))$ contains a cubic element (over $\mathbf{F}_2(X)$) with bounded partial quotient in its continued fraction expansion. This element is $\sum_{n\geqslant 0} s_nX^{-n}$, where s_n is equal to 0 if the binary representation of n contains at least one string of 0's of odd length and s_n is equal to 1 otherwise. The sequence $\mathbf{s}=(s_n)_{n\geqslant 0}$ is now usually referred to as the Baum–Sweet sequence. It follows from [5, 2] that, for any integer $b\geqslant 2$, the real number $\sum_{n\geqslant 0} s_n/b^n$ is transcendental. Here, we use Theorem 3.1 to prove a similar result for the continued fraction expansion.

Theorem 5.2. — *Let a and b be distinct positive integers, and let $\mathbf{s}=(s_n)_{n\geqslant 0}$ be the Baum–Sweet sequence on the alphabet $\{a,b\}$ (that is, the symbol 0 is replaced by a and the symbol 1 is replaced by b in the usual Baum–Sweet sequence). Then, the real number $\alpha=[0;s_0,s_1,s_2,\ldots]$ is transcendental.*

Proof. — Let us first remark that the sequence $\mathbf{s}$ is not eventually periodic. Indeed, as shown in [12], the formal power series $\sum_{n\geqslant 0} s_nX^{-n}$ is a cubic element over $\mathbf{F}_2(X)$, thus, it is not a rational function.

Let us now recall a useful description of the Baum–Sweet sequence on the alphabet $\{a,b\}$. Let σ be the morphism defined from $\{1,2,3,4\}^\star$ into itself by: $\sigma(1) = 12$, $\sigma(2) = 32$, $\sigma(3) = 24$ and $\sigma(4) = 44$. Let also φ be the morphism defined from $\{1,2,3,4\}^\star$ to $\{a,b\}^\star$ by: $\varphi(1) = \varphi(2) = b$ and $\varphi(3) = \varphi(4) = a$. Let

$$\mathbf{u} = 123224323244\ldots$$

denote the fixed point of σ begining with 1. It is known (see for instance [9], Ch. 6) that

$$\mathbf{s} = \varphi(\mathbf{u}). \tag{5.3}$$

Observe that $\mathbf{u}$ begins in the word $UV^{3/2}$, where $U = 1$ and $V = 232243$. Since the morphism σ is uniform, it follows from (5.3) that, for any positive integer n, the word $\mathbf{s}$ begins with $U_n V_n^{3/2}$, where $U_n = \varphi(\sigma^n(U))$ and $V_n = \varphi(\sigma^n(V))$. In particular, we have $|U_n|/|V_n| = 1/6$ and $\mathbf{s}$ thus satisfies Condition $(\star\star)_{3/2,1/6}$. We further observe that the frequency of a in the word U_n tends to 1 as n tends to infinity. By Theorem 5 from [7], this implies that the sequence $(q_\ell)_{\ell\geqslant 1}^{1/\ell}$ converges, where q_ℓ denotes the denominator of the ℓ-th convergent of α, for any positive integer ℓ. Thus, with the notation of Theorem 3.1, we have $M = m$. Since $3/2 > 1 + 1/6$, we derive from Theorem 3.1 that α is transcendental. This finishes the proof of our theorem. □

6. Folded continued fractions

Numerous papers, including the survey [18], are devoted to paperfolding sequences. In this Section we consider folded continued fractions and we prove that they are always transcendental.

A sheet of paper can be folded in half lengthways in two ways: right half over left (the positive way) or left half over right (the negative way). After having been folded an infinite number of times, the sheet of paper can be unfolded to display an infinite sequence of creases formed by hills and valleys. For convenience, we will denote by $+1$ the hills and by -1 the valleys. The simplest choice, that is to fold always in the positive way, gives the well-known regular paperfolding sequence over the alphabet $\{+1,-1\}$. More generally, if $\mathbf{e} = (e_n)_{n\geqslant 0}$ is in $\{+1,-1\}^{\mathbf{Z}_{\geqslant 0}}$, the associated paperfolding sequence on the alphabet $\{+1,-1\}$ is obtained accordingly to the sequence $\mathbf{e}$ of folding instructions, that is, the n-th fold is positive if $e_n = +1$ and it is negative otherwise.

Among the numerous studies concerned with paperfolding sequences, much attention has been brought on the way (quite intriguing) they are related to the continued fraction expansion of some formal power series.

Indeed, it can be shown that for any sequence $\mathbf{e} = (e_n)_{n\geqslant 0} \in \{+1,-1\}^{\mathbf{Z}_{\geqslant 0}}$ of folded instructions, the continued fraction expansion of the formal power series $X\sum_{n\geqslant 0} e_n X^{-2^n}$ can be deduced from the associated paperfolding sequence $\mathbf{f}$. As a consequence, the authors of [31] precisely described the continued fraction expansion of the real number $\xi = 2\sum_{n\geqslant 0} e_n 2^{-2^n}$. In particular, they proved that such an expansion is also closely related to the sequence $\mathbf{f}$ and called it a "folded continued fraction". However, we point out that these "folded continued fractions" are not the same as those we consider in Theorem 6.1. Real numbers such as ξ were shown to be transcendental in [27] thanks to the so-called Mahler method (the observation that Ridout's Theorem [34] also implies that such numbers are transcendental was for instance done in [1]). Consequently, we get the transcendence of a family of continued fractions whose shape arises from paperfolding sequences; a fact mentioned in [31]. The following result has the same flavour though it is obtained in a totally different way. It deals with another family of continued fractions arising from paperfolding sequences.

Theorem 6.1. — *Let a and b be two positive distinct integers, $(e_n)_{n\geqslant 0} \in \{+1,-1\}^{\mathbf{Z}_{\geqslant 0}}$ be a sequence of folding instructions and let $\mathbf{f} = (f_n)_{n\geqslant 0}$ be the associated paperfolding sequence over the alphabet $\{a,b\}$ (that is, the symbol $+1$ is replaced by a and the symbol -1 is replaced by b in the usual paperfolding sequence associated with $(e_n)_{n\geqslant 0}$). Then, the number $\alpha = [0; f_0, f_1, f_2, \ldots]$ is transcendental.*

Proof. — Let $\mathbf{f} = (f_n)_{n\geqslant 0}$ be the paperfolding sequence on the alphabet $\{a,b\}$ associated with the sequence $(e_n)_{n\geqslant 0}$ in $\{+1,-1\}^{\mathbf{Z}_{\geqslant 0}}$ of folding instructions. We first notice that $\mathbf{f}$ is not eventually periodic since no paperfolding sequence is eventually periodic (see [18]).

We present now another useful description of the sequence $\mathbf{f}$. Let $\mathcal{F}_i$, $i \in \{+1,-1\}$, be the map defined from the set $\{+1,-1\}^\star$ into itself by

$$\mathcal{F}_i : \quad w \longmapsto wi - (\overline{w}),$$

where $\overline{w_1 w_2 \ldots w_m} := w_m w_{m-1} \ldots w_1$ denotes the mirror image of the word $w_1 w_2 \ldots w_m$, and $-(w_1 w_2 \ldots w_m) := w_1' w_2' \ldots w_m'$, with $w_i' = +1$ (resp. $w_i' = -1$) if $w_i = -1$ (resp. $w_i = +1$). Let φ be the morphism defined from $\{+1,-1\}^\star$ to $\{a,b\}^\star$ by $\varphi(+1) = a$ and $\varphi(-1) = b$. One can easily verify (see for instance [18]) that

$$\mathbf{f} = \lim_{n\to+\infty} \varphi(\mathcal{F}_{e_0}\mathcal{F}_{e_1}\ldots\mathcal{F}_{e_n}(\varepsilon)), \tag{6.1}$$

where ε denotes the empty word.

Let $n \geqslant 2$ be an integer and set $V_n = \mathcal{F}_{e_2} \dots \mathcal{F}_{e_n}(\varepsilon)$. By (6.1), the finite word

$$F_n = \varphi(\mathcal{F}_{e_0}\mathcal{F}_{e_1} \dots \mathcal{F}_{e_n}(\varepsilon)) = \varphi(\mathcal{F}_{e_0}\mathcal{F}_{e_1}(V_n))$$

is a prefix of $\mathbf{f}$. Moreover, we have

$$\mathcal{F}_{e_0}\mathcal{F}_{e_1}(V_n) = \mathcal{F}_{e_0}(V_n e_1 - \overline{V_n}) = (V_n e_1 - \overline{V_n})e_0 - (\overline{V_n e_1 - \overline{V_n}}).$$

This gives

$$\mathcal{F}_{e_0}\mathcal{F}_{e_1}(V_n) = (V_n e_1 - \overline{V_n})e_0 V_n - (\overline{V_n e_1}).$$

In particular, the word

$$\varphi(V_n e_1(-\overline{V_n})e_0 V_n)$$

is a prefix of $\mathbf{f}$. Moreover, we have

$$\varphi(V_n e_1(-\overline{V_n})e_0 V_n) = (\varphi(V_n e_1(-\overline{V_n})e_0))^{w_n},$$

where $w_n = 1 + |\varphi(V_n)|/(2|\varphi(V_n)| + 2)$. Thus, $\mathbf{f}$ begins with the word $(\varphi(V_n e_1(-\overline{V_n})e_0))^{5/4}$. Since $|\varphi(V_n)|$ tends to infinity with n, this proves that $\mathbf{f}$ satisfies Condition $(\star)_{5/4}$. Consequently, Theorem 2.1 implies that the real number $\alpha = [0; f_0, f_1, f_2, \dots]$ is transcendental. This ends the proof. □

7. Generalized perturbed symmetry systems

A perturbed symmetry is a map defined from $\mathcal{A}^\star$ into itself by $S_X(W) = WX\overline{W}$, where $\mathcal{A}$ is a finite set, W and X are finite words on $\mathcal{A}$, and $\overline{W}$ is defined as in the previous Section. Since $S_X(W)$ begins in W, we can iterate the map S_X to obtain the infinite sequence

$$S_X^\infty(W) = WX\overline{W}XW\overline{XW}\dots$$

Such sequences were introduced in [29, 14], where it is proved that they are periodic if and only if the word X is a palindrome, that is, $X = \overline{X}$.

More recently, Allouche and Shallit [8] generalized this notion as follows. For convenience, for any finite word W, we set $W^E := W$ and $W^R := \overline{W}$. Let k be a positive integer. Let $X_1, X_2, \dots, X_k$ be finite words of the same length s, and let $e_1, e_2, \dots, e_k$ be in $\{E, R\}$. Then, the associated generalized perturbed symmetry is defined by

$$S(W) = W \prod_{i=1}^{k} X_i W^{e_i}.$$

Again, since $S(W)$ begins in W, we get an infinite word $S^\infty(W)$ by iterating S. In [8], the authors proved that $S^\infty(W)$ is $(k+1)$-automatic and gave a necessary and sufficient condition for this sequence to be eventually periodic.

We introduce now a generalization of this process. First, let us assume in the previous definition that the X_i are finite and possibly empty words, without any condition on their length. We can similarly define a map S that we again call a generalized perturbed symmetry. If all the X_i have the same length, as previously, we may then call this map a uniform generalized perturbed symmetry. We point out that the definition we consider here is more general than the one given in [8]. A generalized perturbed symmetry system is defined as a 4-tuple $(\mathcal{A}, W, \mathcal{S}, (S_n)_{n\geqslant 0})$, where $\mathcal{A}$ is a finite set, $W \in \mathcal{A}^\star$, $W \neq \varepsilon$, $\mathcal{S}$ is a finite set of generalized perturbed symmetries and $(S_n)_{n\geqslant 0} \in \mathcal{S}^{\mathbf{Z}_{\geqslant 0}}$. Then, it is easily verified that the sequence of finite words

$$S_n S_{n-1} \dots S_0(W)$$

converges to an infinite sequence, which we call the sequence produced by the generalized perturbed symmetry system $(\mathcal{A}, W, \mathcal{S}, (S_n)_{n\geqslant 0})$. When $\mathcal{A}$ is a subset of $\mathbf{Z}_{\geqslant 1}$, we obtain an infinite sequence of positive integers that we can view as the continued fraction expansion of a real number.

Theorem 7.1. — *Generalized perturbed symmetry systems generate either quadratic or transcendental continued fractions.*

Proof. — For any integer $n \geqslant 0$, we consider a generalized perturbed symmetry S_n associated with the parameters

$$X_{1,n}, X_{2,n}, \dots, X_{k_n,n}$$

and

$$e_{1,n}, e_{2,n}, \dots, e_{k_n,n} \text{ in } \{E, R\}.$$

Let $\mathbf{u}$ be the sequence generated by the generalized perturbed symmetry system $(\mathcal{A}, W, \mathcal{S}, (S_n)_{n\geqslant 0})$. For every positive integer n we define the finite word

$$W_n = S_{n-1} \dots S_0(W).$$

Then, W_n is a prefix of $\mathbf{u}$ and the sequence $(|W_n|)_{n\geqslant 1}$ tends to infinity. To prove Theorem 7.1, we will show that there exists a rational w greater than 1, such that $\mathbf{u}$ satisfies Condition $(\star)_w$. In order to do this, we have to distinguish two different cases.

First, assume that for infinitely many integers n, we have $e_{1,n} = E$. In this case, $S_n(W_n)$ begins in $W_n X_{1,n} W_n^{e_{1,n}} = W_n X_{1,n} W_n$, thus the sequence $\mathbf{u}$ begins in $W_n X_{1,n} W_n$. Since $|W_n|$ tends to infinity and $|X_{1,n}|$ lies in a finite set, we have that $\mathbf{u}$ begins in $(W_n X_{1,n})^{3/2}$ as soon as n is large enough. Thus, $\mathbf{u}$ satisfies Condition $(\star)_{3/2}$.

Now, assume that there exists an integer n_0 such that for all $n \geqslant n_0$, we have $e_{1,n} = R$. Let n be an integer greater than n_0. Then, $S_n(W_n)$ begins in $W_n X_{1,n} W_n^R$, where

$$W_n = S_{n-1}(W_{n-1}) = W_{n-1} \prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}.$$

This implies that $S_n(W_n)$ begins with

$$\begin{aligned} &\Big(W_{n-1} \prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big) X_{1,n} \Big(W_{n-1} \prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big)^R \\ &= \Big(W_{n-1} \prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big) X_{1,n} \Big(\prod_{i=2}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big)^R \\ &\qquad\qquad \Big(W_{n-1} X_{1,n-1} W_{n-1}^{e_{1,n-1}}\Big)^R. \end{aligned}$$

Since $n > n_0$, we have $e_{1,n-1} = R$. Consequently, $S_n(W_n)$ begins in

$$\begin{aligned} &W_{n-1}\Big(\prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big) X_{1,n} \Big(\prod_{i=1}^{k_{n-1}} X_{i,n-1} W_{n-1}^{e_{i,n-1}}\Big)^R \\ &\qquad\qquad W_{n-1} =: W_{n-1} Y_{n-1} W_{n-1}. \end{aligned}$$

Since both the k_j and the $X_{i,j}$ lie in a finite set, and since $|W_n|$ tends to infinity with n, we get that $S_n(W_n)$, and thus $\mathbf{u}$, begins in $(W_{n-1}Y_{n-1})^{1+\frac{1}{(3k)}}$ for n large enough, where we have set $k = \max\{k_n : n \geqslant 1\}$. This implies that $\mathbf{u}$ satifies Condition $(\star)_{1+1/(3k)}$, and we conclude the proof by applying Theorem 2.1. □

8. Inside the proof of Theorem 3.1: an example of the use of continuants

The purpose of this Section is to point out that, in some cases, rather than applying roughly Theorem 3.1, it is much better to go into its proof and to evaluate continuants carefully. In order to illustrate this idea, we introduce a new family of continued fractions.

Throughout this Section, we use the following notation. Let $k \geqslant 3$ be an odd integer. Let $b_1, \ldots, b_k$ be positive integers with $b_1 < \ldots < b_k$. We consider words defined on the alphabet $\Sigma = \{b_1, \ldots, b_k\}$. The character b_j denotes either the letter b_j, or the integer b_j, according to the context. For any finite word W on Σ and for $j = 1, \ldots, k$, denote by $|W|_{b_j}$ the number of occurrences of the letter b_j in W. Set

$$\Sigma^+_{\text{equal}} = \{W \in \Sigma^+ : |W|_{b_j} = |W|_{b_1} \quad \text{for } 2 \leqslant j \leqslant k\}.$$

Let $\lambda > 1$ be real and, for any $n \geqslant 1$, let W_n be in Σ^+_{equal} with $|W_{n+1}| > \lambda|W_n|$. Consider the infinite word $\mathbf{a} = (a_\ell)_{\ell \geqslant 1}$ defined by

$$\mathbf{a} = W_1 W_2^2 W_3^2 \ldots W_n^2 \ldots$$

Observe that $\mathbf{a}$ is either eventually periodic, or it satisfies Condition $(\star)_{w,w'}$ with $w = 2$ and $w' = 2/(\lambda - 1)$.

If $\mathbf{a}$ is not eventually periodic, then Theorem 3.1 implies the transcendence of the real number $\alpha = [0; a_1, a_2, \ldots]$, provided λ is sufficiently large in terms of b_k. It turns out that the method of proof of Theorem 3.1 is flexible enough to yield in some cases much better results: we can get a condition on λ that does not depend on the values of the b_j's.

THEOREM 8.1. — *Keep the preceding notation. Assume furthermore that $|W_n|$ is odd for all sufficiently large n and that $\lambda > 3.26$. Then, the real number*

$$\alpha = [0; a_1, a_2, \ldots]$$

is either quadratic, or transcendental.

In order to establish Theorem 8.1, we need three lemmas. We keep the notation from Section 3 of [7]. In particular, for an integer matrix A, we denote by $\rho(A)$ its spectral radius and by $||A||$ its L^2-norm. Recall that $\rho(A) = ||A||$ when A is symmetrical. Our first auxiliary result is extracted from [7]. In all what follows, we set $\gamma = 0.885$.

LEMMA 8.2. — *If $A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$ where a and b are distinct positive integers, then we have*

$$\rho(AB) > \big(\rho(A)\rho(B)\big)^\gamma.$$

For $j = 1, \ldots, k$, set $B_j = \begin{pmatrix} b_j & 1 \\ 1 & 0 \end{pmatrix}$. Set also

$$X = \frac{1}{k} \sum_{j=1}^{k} \log \rho(B_j).$$

For any finite word V on Σ, denote by $K(V)$ the corresponding continuant.

LEMMA 8.3. — *If V is in Σ^+_{equal}, then we have*

$$\frac{1}{|V|}\log K(V) \leqslant X.$$

Proof. — Let $V = d_1 d_2 \dots d_m$ be a finite word defined over the alphabet Σ^+_{equal}. Set $p_{m-1}/q_{m-1} = [0; d_1, \dots, d_{m-1}]$ and $p_m/q_m = [0; d_1, \dots, d_m]$. Then, we have $K(V) = q_m$ and

$$K(V) \leqslant \left\| \begin{pmatrix} q_m & q_{m-1} \\ p_m & p_{m-1} \end{pmatrix} \right\|.$$

Setting $h_j = |V|_{b_j}$ for $j = 1, \dots, k$, it follows from the theory of continued fractions that

$$K(V) \leqslant ||B_1||^{h_1} \dots ||B_k||^{h_k}.$$

Since the B_j's are symmetrical and $h_1 = \dots = h_k = h$, we have

$$K(V)^{1/h} \leqslant \rho(B_1) \dots \rho(B_k).$$

Hence, the proof. □

Our last auxiliary result is the following.

LEMMA 8.4. — *If V is in Σ^+_{equal} with $|V|$ odd, then we have*

$$\frac{1}{|V|}\log K(V) > \gamma X - \frac{\log 4}{|V|}.$$

Proof. — We use repeatedly a particular case of Theorem 3.4 from [17]. It asserts that if W is the product of an odd number m of matrices $B_1, \dots, B_k$, each of which occurring exactly ℓ times, then we have

$$\operatorname{tr}(W) \geqslant \rho(B_1 B_k)^\ell \operatorname{tr}(W'),$$

where W' is the product arising from W by replacing the matrices B_1 and B_k by the identity matrix. As usual, $\operatorname{tr}(M)$ denotes the trace of the matrix M.

With the notation of the proof of Lemma 8.3, we then get that

$$\begin{aligned} K(V) &\geqslant \frac{1}{2}\operatorname{tr}\begin{pmatrix} q_m & q_{m-1} \\ p_m & p_{m-1} \end{pmatrix} \\ &\geqslant \frac{1}{2}\rho(B_1 B_k)^h \dots \rho(B_{(k-1)/2} B_{(k+3)/2})^h \operatorname{tr}(B^h_{(k+1)/2}) \\ &\geqslant \frac{1}{4}\rho(B_1 B_k)^h \dots \rho(B_{(k-1)/2} B_{(k+3)/2})^h \rho(B^h_{(k+1)/2}) \\ &> \frac{1}{4}\big(\rho(B_1) \dots \rho(B_k)\big)^{\gamma h}, \end{aligned}$$

by Lemma 8.2. The lemma follows. □

We have now all the tools needed to establish Theorem 8.1.

Proof of Theorem 8.1 — For any $n \geqslant 2$, set

$$U_n = W_1 W_2^2 \ldots W_{n-1}^2 \quad \text{and} \quad V_n = W_n.$$

Clearly, $\mathbf{a}$ begins in $U_n V_n^2$. Denote by $K(U_n)$ and by $K(V_n)$ the continuants associated to the words U_n and V_n, respectively. In view of Lemma 3.2 and of (3.3), the theorem is proved as soon as we establish that there exists a positive real number ε such that

$$\log K(V_n) > (1+\varepsilon) \log K(U_n), \tag{8.1}$$

for any sufficiently large integer n.

To prove (8.1), we first infer from Lemmas 3.2 and 8.3 that

$$\log K(U_n) < \log K(V_1) + 2 \sum_{j=2}^{n-1} \log K(V_j) + 2n < 2X \sum_{j=1}^{n-1} |V_j| + 2n.$$

Consequently, we get

$$\frac{1}{|V_n|} \log K(U_n) < \frac{2X}{\lambda - 1} + \frac{2n}{|V_n|}. \tag{8.2}$$

On the other hand, Lemma 8.4 gives us that

$$\frac{1}{|V_n|} \log K(V_n) > \gamma X - \frac{\log 4}{|V_n|}. \tag{8.3}$$

We then infer from (8.2) and (8.3) that (8.1) is satisfied for some positive ε as soon as we have $\gamma > 2/(\lambda - 1)$, that is, $\lambda > 1 + 2\gamma^{-1} = 3.25\ldots$ This completes the proof of the theorem. □

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