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# STALKING STAGGERINGLY LARGE NUMBERS

**by Michel MENDÈS FRANCE**

This special issue represents the proceedings of a very successful meeting organised by Jean-Louis Verger-Gaugry in 2005 which took place in Grenoble. The unifying theme was Numeration Systems and this in turn touches upon a huge variety of topics such as for example Dynamical Systems, Automata Theory, Fractals and Tilings, Number Theory, Language Theory and even Physics !

When prefacing a book, I understand it is traditional to survey the different articles it contains. I shall depart from the usual custom for at least two reasons. First of all, the reader can easily glance through the table of contents together with all the abstracts both in French and English. Secondly, I would very much like to take the opportunity of saying a few words about Jacques Harthong (1948 – 2005) who died recently at the early age of 57. Very discrete and modest, he never really took pains to advertise his own work though apart from many papers on mathematics, physics and even philosophy, he wrote a book on Quantum Mechanics [4] and a book on Probability Theory [6]. Those who knew his work admired and praised him greatly for his originality and the depth of his insight. He very definitely was an exceptional mathematician.

If I mention him here in this Preface, it is because some 20 years ago he wrote a beautiful paper “Le Continu et l'Ordinateur” [5] which concerns Numeration Systems and more specifically huge numbers and their representation, a topic which more than 2000 years ago Archimedes got involved in [2].

We all naively think that given any large integer $x$ and its decimal representation, we can easily write down those integers less than $x$. Jacques

Harthong remarks that this is not so. Following D. Knuth, define recursively

$$\begin{aligned} a \uparrow b &= a^b \\ a \uparrow\uparrow b &= a \uparrow (a \uparrow b) \\ a \uparrow^n b &= a \uparrow^{n-1} (a \uparrow^{n-1} b). \end{aligned}$$

For example $2 \uparrow^3 2 = 2^{2^{2^{2^2}}}$ (4 exponents). With this notation there is no problem in writing down staggeringly large integers: $x = 10 \uparrow^{1000} 10$. Now consider the interval $[x, 2x]$ for example. Both extremities are simply expressed by a string of digits with a huge tail of 0's. Similarly, there is no problem in representing $x+1, x+2, \ldots, \frac{3}{2}x, \ldots, 2x-2, 2x-1$. But we will never be able to fill in the dots convincingly.

The problem is to represent a "generic" number $y$ between $x$ and $2x$ say. Its digits form a random sequence à la Kolmogorov. The shortest algorithm which generates the sequence has precisely as many steps $l(y)$ as the length $L(y)$ of the sequence. Harthong defines the entropy of an integer $z$ to be the ratio

$$l(z)/L(z) \leqslant 1.$$

He argues that for most integers in $[x, 2x]$ the entropy is maximal whereas it is practically 0 for expressible integers. Since $l(z+1) \approx l(z)$, neighbors of high entropy numbers have high entropy and low entropy numbers are surrounded by low entropy numbers so that their set looks like a Cantor set. Huge expressible integers are therefore extremely sparse. This statement holds whatever the choice of the numeration system. Needless to say, Harthong's analysis is much more precise and pertinent.

As sparse as they may be, big numbers occur here and there in the scientific literature. Here is a short pot-pourri. We already mentioned Archimedes. He estimated the number of grains of sand in the Universe to be something like $10^{60}$, ([2], p. 373). The number of different configurations for a perfect gas at normal pressure and temperature in an ordinary sized bottle is of order $10^{10^{23}}$, ([3], p. 8). The largest known prime number $2^{25964951} - 1$ was discovered by Martin Novack in 2005. The Japanese seem to have a word to express $10^{88}$, namely "*muryontaisuu*" which, I am told by Tadashi Tokieda, means a large number of nothing, ([1], p. 16). I am then reminded of some medieval philosopher or scientist who claimed that geometrical points are so incredibly tiny that if you were to bring all the points of the Universe together, they would easily hold in a thimble. Mathematics meets surrealism…

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