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**Beyond the classical Weyl and Colin de Verdière's formulas for Schrödinger operators with polynomial magnetic and electric fields**

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**BEYOND THE CLASSICAL WEYL  
AND COLIN DE VERDIÈRE'S FORMULAS  
FOR SCHRÖDINGER OPERATORS  
WITH POLYNOMIAL MAGNETIC  
AND ELECTRIC FIELDS**

by Mitya BOYARCHENKO & Sergei LEVENDORSKI

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ABSTRACT. — We present a pair of conjectural formulas that compute the leading term of the spectral asymptotics of a Schrödinger operator on  $L^2(\mathbb{R}^n)$  with quasi-homogeneous polynomial magnetic and electric fields. The construction is based on the orbit method due to Kirillov. It makes sense for any nilpotent Lie algebra and is related to the geometry of coadjoint orbits, as well as to the growth properties of certain “algebraic integrals,” studied by Nilsson. By using the direct variational method, we prove that the formulas give the correct answer not only in the “regular” cases where the classical formulas of Weyl or Colin de Verdière are applicable but in many “irregular” cases, with different types of degeneration of potentials.

RÉSUMÉ. — Nous donnons deux formules conjecturelles pour calculer le terme dominant du comportement asymptotique du spectre d'un opérateur de Schrödinger agissant dans  $L^2(\mathbb{R}^n)$  avec des polynômes quasi-homogènes comme champs électriques et magnétiques. La construction se base sur la méthode des orbites de Kirillov, et s'applique donc à n'importe quelle algèbre de Lie nilpotente. Elle est liée à la géométrie des orbites coadjointes et à certaines “intégrales algébriques” étudiées par Nilsson. En utilisant la méthode de variation directe, nous démontrons que nos formules sont correctes non seulement dans le cas régulier où s'appliquent les formules de Weyl ou Colin de Verdière, mais aussi dans certains cas “irréguliers” avec différents types de dégénérescence des potentiels.

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## 1. Introduction

### 1.1.

Let

$$(1.1) \quad H = H(a) + V = - \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} + \sqrt{-1} \cdot a_j(x) \right)^2 + V(x)$$

be a Schrödinger operator in  $\mathbb{R}^n$  with a real semi-bounded electric potential  $V$  and magnetic potential  $a(x) = (a_1(x), \dots, a_n(x)) \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ . If the electric potential  $V$  regularly grows at infinity, it is well-known that  $H(0) + V$  is a self-adjoint operator with discrete spectrum, and the counting function of the spectrum obeys the classical Weyl formula

$$(1.2) \quad N(\lambda, H(0) + V) \sim (2\pi)^{-n} \int_{\mathfrak{a}(x, \xi) < \lambda} dx d\xi.$$

Here  $\mathfrak{a}(x, \xi) = \|\xi\|^2 + V(x)$  is the symbol of  $H(0) + V$ , and  $f(\lambda) \sim g(\lambda)$  means that  $f(\lambda)/g(\lambda) \rightarrow 1$  as  $\lambda \rightarrow +\infty$ . One easily rewrites (1.2) in the form

$$(1.3) \quad N(\lambda, H(0) + V) \sim (2\pi)^{-n} |v_n| \int_{\mathbb{R}^n} (\lambda - V(x))_+^{n/2} dx,$$

where  $|v_n|$  is the volume of the unit ball of  $\mathbb{R}^n$  and  $a_+ = \max\{0, a\}$  (see e.g., [33, 34]). The classical Weyl formula is applicable to many classes of operators, and in its initial form, it was related to the (Dirichlet or Neumann) Laplacian on a bounded domain  $\Omega$ , with the symbol  $\mathfrak{a}(x, \xi)$ . For the Laplacian, the equation (1.3) is valid with  $V(x) = 0$  and the integration over  $\Omega$ ; hence, (1.3) allows one “to hear the area of the drum”. If more information about the spectrum is available, then one can “hear” much more about the geometry of a “drum”: see [14, 8].

In [35, 32, 36, 17, 9], it is shown that the spectrum of  $H(0) + V$  can be discrete even if  $V$  does not grow in some directions, and for wide classes of degenerate potentials, the leading term of the asymptotics of  $N(\lambda, H(a) + V)$  is computed. The results of these papers agree with the general “uncertainty principle” stated in [7]; it seems that this principle provides upper and lower bounds, but it is difficult to use it to study spectral asymptotics. Note that in many cases, asymptotic formulas are non-classical in the sense that they do not agree with the “classical” formula (1.2). Three cases are possible: the classical Weyl formula holds (the so-called *weak degeneration* case); an analog of the classical Weyl formula with the operator-valued symbol parameterized by points of a set with a measure inherited from  $T^*\mathbb{R}^n$

is valid (*strong degeneration case*); the classical Weyl formula fails but the leading term of the asymptotics is expressed in terms of an auxiliary scalar function, and no operator-valued symbol is involved (*intermediate degeneration case*). In simple strong degeneration cases, an operator-valued symbol is parameterized by the cotangent bundle over a manifold of degeneration of  $V$ , call it  $M$ , and the operator-valued analog of (1.2) is of the form

$$(1.4) \quad N(\lambda, H(0) + V) \sim (2\pi)^{-n+r} \int_{T^*M} N(\lambda, \mathbf{a}(x', \xi')) dx' d\xi',$$

where  $r = \text{codim}M$ , and for each  $(x', \xi') \in T^*M$ ,  $\mathbf{a}(x', \xi')$  is an operator in  $L^2(\mathbb{R}^r)$ . Similar types of asymptotic formulas hold for many other classes of differential operators, pseudodifferential operators and boundary value problems (see [17, 18, 19, 20] and the bibliography therein).

**1.2.**

If  $V = 0$  and the magnetic tensor  $B = [b_{jk}]$ ,  $b_{jk}(x) = \partial_k a_j(x) - \partial_j a_k(x)$ , grows regularly at infinity, the leading term of the asymptotics was obtained in [5] (see also [37, 13]):

$$(1.5) \quad N(\lambda, H(a)) \sim \int_{\mathbb{R}^n} v_B(x)(\lambda) dx,$$

where  $v_B(\lambda)$  is defined as follows. Let  $\text{rank } B = 2r$ , and let  $b_1 \geq b_2 \geq \dots \geq b_r > 0$  be the positive eigenvalues of  $iB$ . Then

$$v_B(\lambda) = (2\pi)^{-n+r} |v_{n-2r}| b_1 \cdots b_r \sum_{n_1, \dots, n_r \geq 0} \left( \lambda - \sum_{j=1}^r (2n_j + 1) \cdot b_j \right)_+^{n/2-r}.$$

Note that  $B$ ,  $r$  and the  $b_j$ 's depend on  $x$ . However, in the case of a Schrödinger operator with polynomial potentials, there is a dense open subset of  $\mathbb{R}^n$  of full measure on which  $B(x)$  has maximal rank, so one can replace the integral in (1.5) by the integral over this subset. Then  $r$  will remain constant throughout the integration.

**1.3.**

In the general case, only upper and lower bounds for  $N(\lambda, H(a) + V)$  are known [26]. They are given in terms of a function  $\Psi^* = \Psi_{a,V}^*$  constructed in [10]; for polynomial  $V(\geq 0)$  and  $b_{jk}$ ,

$$(1.6) \quad \Psi^*(x) = \sum_{\alpha} |\partial^\alpha V(x)|^{1/(|\alpha|+2)} + \sum_{\alpha, j, k} |\partial^\alpha b_{jk}(x)|^{1/(|\alpha|+2)}.$$

It was shown that under certain conditions - in particular, for polynomial  $V$  and  $b_{jk}$  - the following statements hold: ([10]) *the spectrum of  $H(a)+V$  is discrete if and only if*

$$(1.7) \quad \Psi^*(x) \rightarrow +\infty \quad \text{as} \quad |x| \rightarrow +\infty,$$

and ([26]) *if (1.7) holds, then there exists  $C > 0$  such that*

$$(1.8) \quad C^{-1}G(C^{-1}\lambda, \Psi^*, \mathbb{R}^n) \leq N(\lambda, H(a) + V) \leq CG(C\lambda, \Psi^*, \mathbb{R}^n),$$

where

$$(1.9) \quad G(\lambda, \Psi^*, U) = \int_U (\lambda - \Psi^*(x)^2)_+^{n/2} dx.$$

Thus in the case  $B \neq 0$  not growing in some directions, the leading term of the asymptotics is unknown apart from a special case of a Schrödinger operator (and Dirac operator) in 2D with homogeneous potentials [20].

### 1.4.

Notice the difference between the formulas (1.2), (1.4) and (1.5): the first two are written in an invariant form, whereas the last one is similar to (1.3), which is a realization of the invariant formula (1.2). This observation suggests that there should be an invariant formula of which (1.5) is a realization. Moreover, one should expect that there is a general formula, with (1.2), (1.4) and (1.5) as special cases. The following observations indicate the direction where one should look for such a formula.

Let  $H$  be a Schrödinger operator (1.1) with polynomial potentials. We consider the real Lie algebra  $\mathfrak{g}$  of polynomial differential operators on  $\mathbb{R}^n$  generated by the operators  $L_j = \partial_j + \sqrt{-1} \cdot a_j(x)$  and the operator  $L_0$  of multiplication by  $\sqrt{-1} \cdot V(x)$ . It is easy to see that  $\mathfrak{g}$  is a finite dimensional nilpotent Lie algebra. We prove in Theorem 2.1 that, after possibly replacing  $H$  with a gauge equivalent operator (a process which changes neither the spectrum of  $H$  nor the Lie algebra  $\mathfrak{g}$ , up to isomorphism), the “tautological” representation of  $\mathfrak{g}$  on  $L^2(\mathbb{R}^n)$  lifts to a unitary representation  $\rho$  of the corresponding connected and simply connected nilpotent Lie group  $G = \exp \mathfrak{g}$ . Moreover,  $H$  has discrete spectrum if and only if  $\rho$  is irreducible. Note that  $H$  can be naturally viewed as the image of the element  $H^\circ := -\sum_{j=1}^n L_j^2 - \sqrt{-1} \cdot L_0$  of  $\mathcal{U}(\mathfrak{g})_{\mathbb{C}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$ , the complexification of the universal enveloping algebra of  $\mathfrak{g}$ , under the representation of  $\mathfrak{g}$  induced by  $\rho$ .

Assume that  $\sigma(H) = \sigma_d(H)$ , so  $\rho$  is irreducible. The orbit method, due to Kirillov [15], provides a natural one-to-one correspondence between (unitary equivalence classes of) unitary irreducible representations of  $G$  and orbits of the coadjoint action of  $G$  on  $\mathfrak{g}^*$ . In particular, we let  $\Omega_\rho \subset \mathfrak{g}^*$  denote the coadjoint orbit corresponding to  $\rho$ . Suppose now that the magnetic potential  $a = 0$ , and that  $V(x)$  grows regularly at infinity. The values of the symbol  $\mathfrak{a}(x, \xi)$  appearing in the classical Weyl formula (1.2) can be interpreted as the images of  $H^\circ$  in a family of representations of  $G$  on the one-dimensional space  $L^2(\mathbb{R}^0)$ . The family is parameterized by points of the orbit  $\Omega_\rho$ , and the measure  $(2\pi)^{-n} dx d\xi$  coincides with the canonical (Kostant) measure on  $\Omega_\rho$ .

On the other hand, assume that  $V = 0$  and the magnetic tensor  $B(x)$  grows regularly at infinity. It can (and will) be shown that the formula of Colin de Verdière (1.5) can be written in the form

$$(1.10) \quad N(\lambda, H) \sim \int_Q N(\lambda, H_\Theta) d\nu(\Theta) \quad \text{as } \lambda \rightarrow +\infty,$$

where  $H_\Theta$  is the image of  $H^\circ$  in a certain unitary irreducible representation of  $G$  on  $L^2(\mathbb{R}^r)$ ,  $Q$  is a manifold parametrizing a family of such representations, and the measure  $d\nu(\Theta)$  can be obtained in the following way. Let  $\tilde{Q} \subset \mathfrak{g}^*$  be the union of the orbits corresponding to the representations parameterized by the points of  $Q$ . There is a natural “projection map”  $p : \Omega_\rho \rightarrow \tilde{Q}$ , such that the pushforward  $\tilde{\nu}$  of the canonical measure on  $\Omega_\rho$  is a  $G$ -invariant measure on  $\tilde{Q}$ . One can decompose  $\tilde{\nu}$  as an integral of the canonical measures on the orbits contained in  $\tilde{Q}$ , with respect to a certain “quotient” measure on  $Q = \tilde{Q}/G$ . Then we take  $\nu$  to be this quotient measure.

Let us explain the case  $n = 2$  in detail. The magnetic tensor must be of the form

$$B(x) = \begin{pmatrix} 0 & b(x) \\ -b(x) & 0 \end{pmatrix},$$

where  $b(x)$  is a polynomial, and since  $B(x)$  grows regularly at infinity, we may assume without loss of generality that  $b(x) > 0$  for  $\|x\| \gg 0$ . Note that the eigenvalues of  $\sqrt{-1} \cdot B(x)$  are  $\pm b(x)$ .

The Lie algebra  $\mathfrak{g}$  is generated by the operators  $L_1 = \partial_1 + \sqrt{-1} \cdot a_1(x)$  and  $L_2 = \partial_2 + \sqrt{-1} \cdot a_2(x)$ , which satisfy  $[L_1, L_2] = \sqrt{-1} \cdot b(x)$ . Let us write  $P_0 = \sqrt{-1} \cdot b(x)$ , and let  $P_1, \dots, P_N$  be an arbitrary basis of the vector space spanned by all mixed partial derivatives of  $P_0$  of all positive orders (i.e., not including  $P_0$  itself). Thus  $\{L_1, L_2, P_0, P_1, \dots, P_N\}$  is a basis of  $\mathfrak{g}$ .

We can now define a map  $p : \Omega_\rho \rightarrow \mathfrak{g}^*$  by the formulas

$$(1.11) \quad p(f)(L_1) = f(L_1), \quad p(f)(L_2) = f(L_2),$$

$$(1.12) \quad p(f)(P_0) = f(P_0), \quad p(f)(P_j) = 0 \text{ for } 1 \leq j \leq N.$$

We will now show that if  $\tilde{Q}$  is taken to be the image of this map, then  $\tilde{Q}$  is  $G$ -stable, and the pushforward measure  $\tilde{\nu} = p_*(\mu_{\Omega_\rho})$  is  $G$ -invariant (where  $\mu_{\Omega_\rho}$  is the Kostant measure on the orbit  $\Omega_\rho$ ). Moreover, if  $Q = \tilde{Q}/G$  and  $\nu$  is the measure on  $Q$  induced by  $\tilde{\nu}$ , then the right hand side of (1.10) coincides with the right hand side of Colin de Verdière’s formula.

It follows from Propositions 2.10 and 2.11 that the orbit  $\Omega_\rho$  admits a parameterization  $\phi : \mathbb{R}^4 \rightarrow \mathfrak{g}^*$  given by

$$\phi(\xi_1, \xi_2, y_1, y_2)(L_t) = \xi_t \text{ for } t = 1, 2,$$

and

$$\phi(\xi_1, \xi_2, y_1, y_2)(P_j) = -\sqrt{-1} \cdot P_j(y_1, y_2) \text{ for } 0 \leq j \leq N;$$

moreover, we have  $\mu_{\Omega_\rho} = (2\pi)^{-2} \cdot \phi_*(d\xi dy)$ , where  $d\xi dy$  denotes the Lebesgue measure on  $\mathbb{R}^4$ . Let  $\mathfrak{a}$  denote the subspace of  $\mathfrak{g}$  spanned by  $P_1, \dots, P_N$ ; it is clearly an ideal of  $\mathfrak{g}$ . By definition, the image of the map  $p$  is contained in the annihilator of this ideal in  $\mathfrak{g}^*$ , which we can identify with  $(\mathfrak{g}/\mathfrak{a})^*$ . Now  $\mathfrak{g}/\mathfrak{a}$  has basis  $\{X, Y, Z\}$ , where  $X, Y$  and  $Z$  are the images of  $L_1, L_2$  and  $P_0$  under the quotient map  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ . They satisfy the relations  $[X, Y] = Z, [X, Z] = [Y, Z] = 0$ , so we see that  $\mathfrak{g}/\mathfrak{a}$  is the 3-dimensional Heisenberg algebra. Let us use the basis  $\{X, Y, Z\}$  to identify  $(\mathfrak{g}/\mathfrak{a})^*$  with  $\mathbb{R}^3$  in the obvious way. Then the composition  $\psi = p \circ \phi : \mathbb{R}^4 \rightarrow (\mathfrak{g}/\mathfrak{a})^*$  is given by  $\psi(\xi, y) = (\xi_1, \xi_2, b(y))$ , and we are interested in the measure  $\tilde{\nu} = (2\pi)^{-2} \psi_*(d\xi dy)$ . It is well known that there are two types of coadjoint orbits in  $(\mathfrak{g}/\mathfrak{a})^*$ : the two-dimensional orbits given by  $f(Z) = c$ , where  $c$  is a nonzero constant, and the zero-dimensional orbits, namely, points of the plane defined by  $f(Z) = 0$ . In particular, we see that  $\psi(\mathbb{R}^4)$  is a union of coadjoint orbits, so the set  $p(\Omega_\rho)$  is  $G$ -stable. Moreover, if  $c \neq 0$  is fixed, then the functions  $u : f \mapsto f(X)$  and  $v : f \mapsto f(Y)$  are coordinates on the coadjoint orbit defined by  $f(Z) = c$ , and the Kostant measure on this orbit is given by  $\mu_c = (2\pi)^{-1} c^{-1} \cdot dudv$ . Consequently, the pushforward measure  $\tilde{\nu}$  can be decomposed as an integral of the Kostant measures  $\mu_c$  in the following way:

$$\tilde{\nu} = \int_{\mathbb{R}} \mu_c d\nu(c),$$

where  $\nu$  is the measure on  $\mathbb{R}$  obtained as the pushforward of the measure  $(2\pi)^{-1} b(y) dy$  via the map  $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ . (A fortiori, this implies that  $\tilde{\nu}$  is  $G$ -invariant. Note also that we have ignored the plane  $f(Z) = 0$  in

the computation above, which can be done because it has measure zero with respect to  $\tilde{\nu}$ .) Finally, the representation of  $\mathfrak{g}/\mathfrak{a}$  corresponding to the orbit  $f(Z) = c$  can be realized in the space  $L^2(\mathbb{R})$  in such a way that  $X \mapsto d/dx$  and  $Y \mapsto \sqrt{-1} \cdot cx$ . Under this representation, the sublaplacian  $-(X^2 + Y^2)$  maps to the operator  $-\Delta_x + c^2x^2$ , whose spectrum can be computed explicitly: it consists of eigenvalues of the form  $(2m + 1)c$ , each having multiplicity 1, where  $m$  runs through all nonnegative integers. We now have all the ingredients needed to make sense out of the right hand side of (1.10), and we see that it becomes

$$(2\pi)^{-1} \cdot \int_{\mathbb{R}^2} N(\lambda, -\Delta_x + y^2x^2) \cdot b(y)dy,$$

which coincides with the right hand side of Colin de Verdière's formula.

The classical Weyl formula also can be written in the form (1.10), with  $Q$  parameterizing a family of one-dimensional representations (in this case,  $Q = \tilde{Q}$ , so one does not need to decompose the pushforward measure). It is tempting to conjecture that for any magnetic Schrödinger operator with discrete spectrum one can find a family of irreducible representations of  $G$  and the pushforward measure  $d\nu(\Theta)$  on  $Q$  such that (1.10) holds. As it turns out, this construction can be realized in many cases albeit not all, and the first goal of the paper is to suggest a general way of constructing the family  $Q$  and the pushforward measure  $d\nu(\Theta)$ . Naturally enough (*cf.* generalizations of the classical Weyl formula in [17, 18, 19]), we have two similar (but a bit different) algorithms: one for the strong degeneration case, and one for the weak and intermediate degeneration case; and in the intermediate degeneration case, one has to introduce additional logarithmic factors into (1.10). To verify our conjecture for several classes of magnetic Schrödinger operators, we use a modification of the variational technique from [17, 18, 19, 20].

## 1.5.

Let us keep the same notation as above, and write  $\mu_{\Omega_\rho}$  for the canonical (Kostant) measure on the orbit  $\Omega_\rho$ . In trying to turn the vague ideas of the previous paragraph into a precise formula that applies to Schrödinger operators with degenerate potentials, one meets two considerable difficulties. The first one lies in the fact that there seems to be no natural general way of defining a projection map  $p : \Omega_\rho \rightarrow \tilde{Q} \subset \mathfrak{g}^*$  such that the pushforward  $p_*(\mu_{\Omega_\rho})$  will always be a  $G$ -invariant measure. The second difficulty,



which is more serious, is that in certain cases (such as the intermediate degeneration example studied in §5.2) there exists an asymptotic formula of the form (1.10) (with additional logarithmic factors), but the measure  $\nu$  cannot be obtained from a *pushforward* measure arising from a process described above.

Thus, one has to look for a different construction of the subset  $\tilde{Q} \subset \mathfrak{g}^*$  and the  $G$ -invariant measure  $\tilde{\nu}$  on  $\tilde{Q}$ . In our paper we suggest a construction which has the advantage of being canonical (*i.e.*, independent of any choices). Moreover, the measure  $\tilde{\nu}$  it provides is automatically  $G$ -invariant. Thus, both problems mentioned in the last paragraph are solved at once. To the best of our knowledge, no similar construction has been used in this or any related context before.

Let us give a brief description of our idea. For each  $\lambda > 0$ , we let  $\mu_\lambda = \mu_{\lambda, \Omega_\rho}$  denote the positive Borel measure on  $\mathfrak{g}^*$  defined by  $\mu_\lambda(A) = \mu_{\Omega_\rho}(\Omega_\rho \cap \lambda \cdot A)$  for every Borel subset  $A \subset \mathfrak{g}^*$ . Note that  $\mu_\lambda$  is supported on  $\lambda^{-1} \cdot \Omega_\rho$ , which is another coadjoint orbit in  $\mathfrak{g}^*$ . Now  $\Omega_\rho$  is closed in  $\mathfrak{g}^*$ , and there is a coordinate system on  $\Omega_\rho$  which identifies  $\Omega_\rho$  with  $\mathbb{R}^{2n}$ , such that  $\mu_{\Omega_\rho}$  corresponds to the usual Lebesgue measure under this identification (both of these statements hold for arbitrary nilpotent Lie algebras). In particular, we see that each  $\mu_\lambda$  can be identified with a positive linear functional on the space  $C_c(\mathfrak{g}^*)$  of compactly supported continuous functions on  $\mathfrak{g}^*$ . Note also that, if  $A$  is a neighborhood of 0 in  $\mathfrak{g}^*$ , then, as  $\lambda \rightarrow +\infty$ , the sets  $\Omega_\rho \cap \lambda \cdot A$  exhaust all of  $\Omega_\rho$ ; thus,  $\mu_\lambda(A) \rightarrow +\infty$ . Let us now suppose that there exists a function  $f(\lambda)$  such that the functionals  $f(\lambda) \cdot \mu_\lambda \in C_c(\mathfrak{g}^*)^*$  have a nonzero weak- $*$  limit  $f_\infty \in C_c(\mathfrak{g}^*)^*$ . By the Riesz representation theorem,  $f_\infty$  corresponds to a positive Borel measure  $\mu_\infty$  on  $\mathfrak{g}^*$ . We define  $\tilde{Q} = \text{supp}(\mu_\infty)$ , and  $\tilde{\nu} = \mu_\infty|_{\tilde{Q}}$ . Then  $\tilde{Q}$  is a conical  $G$ -invariant subset of  $\mathfrak{g}^*$ , and the  $G$ -invariance of  $\tilde{\nu}$  is automatic, since each of the measures  $\mu_\lambda$  is  $G$ -invariant.

## 1.6.

For simplicity, we will refer to the construction described in the previous paragraph as the *scaling construction*. Due to its “homogeneous” nature, it is not surprising that in applying the construction to the computation of spectral asymptotics of Schrödinger operators, one has to require a certain homogeneity condition on the potentials. We will say, somewhat imprecisely, that (1.1) is a *Schrödinger operator with quasi-homogeneous potentials* if  $V(x)$  and  $B(x)$  are *quasi-homogeneous* polynomials of the same

weight; this means that there exists an  $n$ -tuple of positive rational numbers  $\gamma = (\gamma_1, \dots, \gamma_n)$  such that for all  $t \in \mathbb{R}$ ,  $t > 0$ , and all  $x \in \mathbb{R}^n$ , we have

$$V(t^{\gamma_1} x_1, \dots, t^{\gamma_n} x_n) = t \cdot V(x) \quad \text{and} \quad B(t^{\gamma_1} x_1, \dots, t^{\gamma_n} x_n) = t \cdot B(x).$$

Similarly, if  $V(x)$  and all the components  $b_{jk}(x)$  of the magnetic tensor are homogeneous polynomials of the same degree, we will refer to  $H = H(a) + V$  as a *Schrödinger operator with homogeneous potentials*.

We will prove that in the quasi-homogeneous situation where the classical formulas of Weyl and Colin de Verdière are applicable, our construction gives the same result as the “pushforward” construction described in §1.4. On the other hand, in the intermediate degeneration examples that we have studied, it also produces the “correct” measure space  $(Q, \nu)$ , even though the pushforward construction no longer applies.

### 1.7.

We remark that our scaling construction makes sense for any nilpotent Lie algebra. Indeed, let  $\mathfrak{g}$  be a finite dimensional nilpotent Lie algebra over  $\mathbb{R}$  and  $\Omega \subset \mathfrak{g}^*$  a coadjoint orbit. It is known (cf. e.g., [2], Ch. I) that  $\Omega$  is a closed (in fact, Zariski closed) submanifold of  $\mathfrak{g}^*$ . Moreover, it follows from the explicit parameterization obtained in [3] (see also [1, 29, 39]) that there exists a polynomial map  $\varphi : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$  which is a diffeomorphism onto  $\Omega$ , and such that under this diffeomorphism  $\mu_\Omega$  corresponds to the standard Lebesgue measure on  $\mathbb{R}^{2n}$ .

As before, for every  $\lambda > 0$ , we define a positive Borel measure  $\mu_\lambda$  on  $\mathfrak{g}^*$  by

$$\mu_\lambda(A) = \mu_\Omega(\Omega \cap \lambda \cdot A) = \text{meas}(\varphi^{-1}(\lambda \cdot A)),$$

where  $\text{meas}$  is the Lebesgue measure. Since  $\varphi$  is proper, we see that  $C_c(\mathfrak{g}^*) \subset L^1(d\mu_\lambda)$  for each  $\lambda > 0$ . In particular, once again, we can identify  $\mu_\lambda$  with a positive linear functional on  $C_c(\mathfrak{g}^*)$ , and the rest of our construction goes through without any changes. It will be apparent from the computations of explicit examples in Sections 3 and 4 below that the scaling construction is closely related to the geometry of the embedding  $\Omega \hookrightarrow \mathfrak{g}^*$ .

Furthermore, let us choose an arbitrary Euclidean structure on  $\mathfrak{g}^*$  and a corresponding orthonormal basis, and let  $(P_1(x), \dots, P_t(x))$  be the coordinates of  $\varphi(x)$  with respect to this basis. Fix  $A \subset \mathfrak{g}^*$  to be the unit ball around the origin. We have seen in §1.5 that the growth of  $\mu_\lambda$  as  $\lambda \rightarrow +\infty$

is related to the behavior of the function

$$G(\lambda) = \mu_\lambda(A) = \text{meas} \left\{ x \in \mathbb{R}^{2n} \mid \sum_{j=1}^t P_j(x)^2 \leq \lambda^2 \right\}.$$

Now  $P(x) := \sum P_j(x)^2$  is a polynomial function on  $\mathbb{R}^{2n}$  with  $P(x) \rightarrow +\infty$  as  $\|x\| \rightarrow \infty$ , and it follows from the results of Nilsson on the growth of “algebraic integrals” [27, 28] that there exist positive reals  $c, C, \alpha$  and a nonnegative integer  $\beta$  such that

$$(1.13) \quad C^{-1} \cdot \lambda^\alpha \cdot (\log \lambda)^\beta \leq G(\lambda) \leq C \cdot \lambda^\alpha \cdot (\log \lambda)^\beta \quad \text{for all } \lambda > c.$$

This result is important for the formulation of our conjectures. (The work of Nilsson was used, in a similar situation, by Manchon [22].)

### 1.8.

The idea of applying representation-theoretic methods to the study of partial differential operators is not new (see, for example, [11] and the references therein). Several authors have studied extensions of the known results about Schrödinger operators to the differential operators arising from unitary representations of general nilpotent Lie groups. In [21], an analogue of (1.8) for the image under an irreducible representation of the “sublaplacian” on a stratified nilpotent Lie algebra was obtained. Manchon in [22] has generalized the approximate spectral projection method of Tulovskii and Shubin [38] to prove a Weyl-type asymptotic formula for elliptic operators associated to representations of arbitrary nilpotent Lie groups. In [23, 24, 25], this result was generalized to arbitrary Lie groups, more precisely, to the representations corresponding to closed tempered coadjoint orbits for which Kirillov’s character formula is valid. Notice, however, that [22, 23, 24, 25] use the initial form of the approximate spectral projection method, which requires high regularity of the symbol. In particular, if a degeneration of any kind is present, this form of the approximate spectral projection method does not work at all. For a general version of the approximate spectral projection method, and applications to various classes of degenerate and hypoelliptic operators, see [17, 18, 19].

### 1.9.

The plan of the paper is as follows. In Section 2, we recall the main necessary definitions, and formulate our conjectures. We also state several

useful theorems and propositions which are needed to explain our construction and apply it to computing the leading term of the asymptotics of Schrödinger operators (cf. Sections 3 and 4). The proofs of these results are given in Appendix A. We do not claim any originality here: some of these theorems and propositions can be deduced from either the general results on nilpotent Lie groups [3, 2] or the previous work on applications of representation theory to differential operators [11, 21], while others are straightforward extensions of known facts. However, there are several reasons for including a complete account of these results.

First, we explicitly isolate the class of Lie algebras that are “responsible” for Schrödinger operators with polynomial magnetic and electric fields: these are precisely the nilpotent Lie algebras  $\mathfrak{g}$  such that the commutator  $[\mathfrak{g}, \mathfrak{g}]$  is abelian. Indeed, if  $H$  is a Schrödinger operator (1.1) with polynomial potentials  $a, V$ , and  $\mathfrak{g}$  is the associated Lie algebra as defined in §1.4, then it is clear that  $\mathfrak{g}$  enjoys this property. A “converse” to this statement is given in §2.1. It seems that this observation has not explicitly appeared in the literature before.

Moreover, most of the works relating differential operators to representation theory of nilpotent Lie groups deal only with *stratified* Lie algebras (cf. [11, 21]), i.e., Lie algebras  $\mathfrak{g}$  admitting a decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$  as a direct sum of vector subspaces, such that  $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$  ( $\mathfrak{g}_j = (0)$  for  $j > s$ ), and  $\mathfrak{g}$  is generated by  $\mathfrak{g}_1$  as a Lie algebra. However, there are situations where the Lie algebra arising from a Schrödinger operator with polynomial potentials admits no natural grading, as the simple example studied in Section 3 already shows. On the other hand, the theory we develop in Section 2 makes no use of a grading on  $\mathfrak{g}$ .

Lastly, we state the results of Section 2 in exactly the form one needs to be able to use our conjectural formula for practical computations. The fact that we are only dealing with a special class of nilpotent Lie algebras allows us to give rather simple proofs and make our text essentially self-contained. In some sense, this is easier than trying to deduce all the results in the form we need from the works of other authors.

In Section 3, we use an example of the Schrödinger operator in 2D with zero electric potential and magnetic tensor  $b(x) = x_1^2 - x_2$  (this is an example of strong degeneration), both to illustrate in detail the use of our conjectural formula, and to explain the direct variational method of the calculation of the asymptotics of the spectrum, which can be applied to many other classes of Schrödinger operators with degenerate potentials. In Section 4, we study the weak degeneration case for operators without

either magnetic or electrical potential, and deduce from our conjecture the classical Weyl formula and Colin de Verdière's formula, respectively. At the end of the section, using the direct variational method, we prove a general theorem for Schrödinger operators with polynomial electric and magnetic fields, which gives the leading term of the asymptotics under fairly weak conditions on the function  $\Psi^*$ . These conditions are satisfied in many cases of weak and intermediate degeneration but not in the strong degeneration case. In particular, we prove that in the case of a quasi-homogeneous electric potential, the classical Weyl formula holds if and only if the integral in this formula converges; and our general conjectural formula also gives the classical Weyl formula if and only if this condition is satisfied. In Section 5, we consider the Schrödinger operator in 2D with magnetic tensor  $b(x) = x_1^k x_2^l$  and zero electric potential. In the case  $k \neq l$  we have the strong degeneration, and in the case  $k = l$  - the intermediate one. Finally, we briefly mention the example of the Schrödinger operator in 3D,  $H = -\Delta + x_1^{2k} x_2^{2l} x_3^{2p}$ , with  $p < k \leq l$ . The most technical parts of the proofs are delegated to Appendix B.

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We would like to thank the referee of this paper for suggesting several improvements, correcting misprints, and providing us with some references missing from the first version of the paper. We are especially thankful to the referee for suggesting an alternate construction of the limiting measure  $\mu_\infty$  presented in §2.2.

### 1.11. Notation and terminology

Our interest lies mainly in the degenerate cases where the classical formulas of Weyl or Colin de Verdière are not applicable. However, as we show in §4.1 and §4.2, our formula works just as well in the regular situations. In terms of the distinction between “strong degeneration” and “weak/intermediate degeneration”, the cases with no degeneration at all belong to the latter type.

We use the following notation:  $\mathbb{N} = \{1, 2, 3, \dots\}$  (the set of natural numbers),  $\mathbb{Z}_+ = \{n \in \mathbb{Z} | n \geq 0\}$ ,  $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ ,  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ . If  $S$  is any finite set, we write  $\#S$  for the number of elements of  $S$ . If  $d \in \mathbb{Z}_+$ , we denote by  $|v_d|$  the volume of the unit ball in  $\mathbb{R}^d$  (our convention is that  $|v_0| = 1$ ). For  $a \in \mathbb{R}$ , we write  $a_+ = \max\{a, 0\}$ .

## 2. Main results and conjectures

### 2.1. Schrödinger operators and unitary representations

As explained in the introduction, our goal is to write down a conjectural formula for the leading term of the spectral asymptotics of a Schrödinger operator with quasi-homogeneous polynomial potentials. However, in this subsection the quasi-homogeneity condition plays no role. Thus, we fix a Schrödinger operator  $H = H(a) + V$  with polynomial potentials  $a, V$ . We define  $\mathfrak{g}_H$ , as in the introduction, to be the real Lie algebra generated by the polynomial differential operators  $L_j = \partial_j + \sqrt{-1} \cdot a_j(x)$  ( $1 \leq j \leq n$ ) and  $L_0 = \sqrt{-1} \cdot V(x)$ . It is clear that  $\mathfrak{g}_H$  is a finite dimensional nilpotent Lie algebra. Moreover, the commutator  $[\mathfrak{g}_H, \mathfrak{g}_H]$  consists only of multiplication operators, and thus  $[\mathfrak{g}_H, \mathfrak{g}_H]$  is an abelian ideal of  $\mathfrak{g}_H$ .

Recall that two Schrödinger operators,  $H = H(a) + V$  and  $H' = H(a') + V'$ , are said to be *gauge equivalent* if  $V = V'$  and the corresponding magnetic tensors are the same:  $B = B'$ . By Poincaré's lemma, the last condition is equivalent to the existence of a differentiable function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $a'_j(x) = a_j(x) + \partial_j \phi(x)$  for all  $1 \leq j \leq n$ . If such a  $\phi$  exists, it is easy to check that the unitary operator  $\exp(i\phi)$  conjugates  $H$  into  $H'$ ; in particular,  $H$  and  $H'$  have the same spectrum. On the other hand, if  $H, H'$  are Schrödinger operators with polynomial potentials that are gauge equivalent, then the corresponding Lie algebras  $\mathfrak{g}_H, \mathfrak{g}_{H'}$  are isomorphic, because the commutation relations in  $\mathfrak{g}_H$  depend only on  $V(x), b_{jk}(x)$  and their derivatives.

By the “tautological representation” of  $\mathfrak{g}_H$  we will mean the representation of  $\mathfrak{g}_H$  on  $L^2(\mathbb{R}^n)$  by (unbounded) skew-adjoint operators that takes every element of  $\mathfrak{g}_H$  to the polynomial differential operator it represents. We note that, unlike the case of finite dimensional representations, the problem of lifting the tautological representation to a unitary representation of the connected and simply connected nilpotent Lie group  $G = \exp \mathfrak{g}_H$  is not trivial. We will address this issue in the theorem below.

From a more abstract point of view, let  $\mathfrak{g}$  be an arbitrary finite dimensional nilpotent Lie algebra over  $\mathbb{R}$  such that  $[\mathfrak{g}, \mathfrak{g}]$  is abelian. A *sublaplacian* for  $\mathfrak{g}$  is an element  $S \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  which has the form  $S = -(L_1^2 + \cdots + L_N^2) - \sqrt{-1} \cdot L_0$ , where  $L_0, L_1, \dots, L_N \in \mathfrak{g}$  are linearly independent elements that generate  $\mathfrak{g}$  as a Lie algebra, and  $L_0$  commutes with  $[\mathfrak{g}, \mathfrak{g}]$ . Note that we have extended the standard definition of a sublaplacian (which does not contain the  $L_0$  term) to include the case of a Schrödinger operator with nonzero electric potential. Then we have the following result.

**THEOREM 2.1.** — a) *Every unitary irreducible representation of  $G = \exp \mathfrak{g}$  has a natural realization in a space  $L^2(\mathbb{R}^n)$ ,  $n \leq N$ , such that each element of  $\mathfrak{g}$  maps to a polynomial differential operator of order  $\leq 1$ ;  $L_0$  and all elements of  $[\mathfrak{g}, \mathfrak{g}]$  map to multiplication operators; and  $S$  maps to a Schrödinger operator with polynomial potentials which has discrete spectrum if the image of  $-\sqrt{-1} \cdot L_0$  is a polynomial that is bounded below.*

b) *Conversely, if  $H$  is a Schrödinger operator (1.1) with polynomial potentials, there exists a Schrödinger operator  $H_0$  with polynomial potentials which is gauge equivalent to  $H$ , such that if  $\mathfrak{g} = \mathfrak{g}_{H_0}$  and  $S \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  is the element corresponding to  $H_0$ , then the tautological representation of  $\mathfrak{g}$  on  $L^2(\mathbb{R}^n)$  can be lifted to a unitary irreducible representation of  $G = \exp \mathfrak{g}$ , which is irreducible if and only if  $H$  has discrete spectrum.*

It is important to have a concrete realization of each of the representations of  $\mathfrak{g}$  that arises from a unitary irreducible representation of  $G$ . These will be discussed in detail in §2.4.

*Remark 2.2.* — As pointed out by the referee, it follows from part (b) of this theorem that the tautological representation of the Lie algebra  $\mathfrak{g}_H$  corresponding to *any* Schrödinger operator (1.1) with polynomial potentials already lifts to a unitary representation of  $G = \exp \mathfrak{g}_H$ . Namely, if  $H_0$  is as in the theorem and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a differentiable function such that  $\exp(i\phi)$  conjugates  $H$  into  $H_0$ , then  $\exp(i\phi)$  also conjugates the tautological representation of  $\mathfrak{g}_H$  into the tautological representation of  $\mathfrak{g}_{H_0}$ . However, it is clear that  $\mathfrak{g}_H$  and  $\mathfrak{g}_{H_0}$  are isomorphic as abstract Lie algebras, and if

$\rho$  is the irreducible unitary representation of  $G$  that lifts the tautological representation of  $\mathfrak{g}_{H_0}$ , then conjugating  $\rho$  by  $\exp(i\phi)$  produces an equivalent representation of  $G$  which lifts the tautological representation of  $\mathfrak{g}_H$ .

*Remark 2.3.* — We would like to explain the reason for our change of notation: we use  $S$  to denote a sublaplacian in  $\mathfrak{g}$ , whereas we use  $H^\circ$  everywhere else to denote the sublaplacian that naturally corresponds to a Schrödinger operator  $H$ . The reason is that in our proof of Theorem 2.1, as well as some other proofs presented in Appendix A, we often use the letter  $H$  to denote a closed subgroup of the group  $G = \exp \mathfrak{g}$ . This notation is used throughout the literature on representation theory, so it does not make sense to change it. On the other hand, we do not wish to confuse a subgroup  $H$  with a Schrödinger operator  $H$ .

## 2.2. Preliminary version of the conjecture

Let us now formulate a preliminary version of our conjecture. Let  $H$  be a Schrödinger operator (1.1) with discrete spectrum and quasi-homogeneous polynomial potentials, and let  $\mathfrak{g} = \mathfrak{g}_H$  be the associated Lie algebra. Since we are interested in  $\sigma(H)$ , we may assume, by Theorem 2.1, that the tautological representation of  $\mathfrak{g}$  lifts to a unitary representation of  $G$  on  $L^2(\mathbb{R}^n)$ ; moreover, this representation is then irreducible, whence corresponds to a coadjoint orbit  $\Omega \subset \mathfrak{g}^*$ . Let  $\mu_\Omega$  be the Kostant measure on  $\Omega$ ; for the precise normalization, see Definition 2.8. Then we have the “dilates”  $\mu_\lambda$  of the measure  $\mu_\Omega$ , as defined in the introduction:  $\mu_\lambda(A) = \mu_\Omega(\Omega \cap \lambda \cdot A)$ , for every Borel subset  $A \subseteq \mathfrak{g}^*$ . Furthermore,  $H$  naturally defines an element  $H^\circ \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$ , and the definition of  $\mathfrak{g}$  implies that  $H^\circ$  is a sublaplacian for  $\mathfrak{g}$ . For any coadjoint orbit  $\Theta \subset \mathfrak{g}^*$ , we denote by  $H_\Theta$  the image of  $H^\circ$  in the unitary irreducible representation of  $G$  that corresponds to  $\Theta$  via Kirillov’s theory. By Theorem 2.1, each  $H_\Theta$  can be naturally realized as a Schrödinger operator with polynomial potentials.

**CONJECTURE 1.** There exist a positive real number  $\alpha$  and a nonnegative integer  $\beta$  such that the weak limit  $\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \cdot (\log \lambda)^{-\beta} \cdot \mu_\lambda$  exists and is nonzero. Then  $\mu_\infty$  is automatically  $G$ -invariant; let  $Q = (\text{supp } \mu_\infty)/G$ , and let  $\rho : \text{supp } \mu_\infty \rightarrow Q$  be the natural projection. Let  $\nu$  be the measure on  $Q$  such that for every nonnegative Borel-measurable function  $F$  on  $\mathfrak{g}^*$ , we have

$$\int_{\mathfrak{g}^*} F d\mu_\infty = \int_Q d\nu(q) \cdot \int_{\rho^{-1}(q)} F(x) d\mu_q(x),$$



where  $d\mu_q$  denotes the Kostant measure corresponding to the orbit  $\rho^{-1}(q)$  (the existence of  $\nu$  is proved in Proposition 2.15). Then there exists a constant  $\kappa \geq 1$  such that

$$(2.1) \quad N(\lambda, H) \sim \kappa \cdot (\log \lambda)^\beta \cdot \int_Q N(\lambda, H_\Theta) d\nu(\Theta) \quad \text{as } \lambda \rightarrow +\infty.$$

We are grateful to the referee for suggesting the following two remarks, which improve the exposition.

*Remark 2.4.* — It is possible to construct canonically a  $G$ -invariant “limiting measure”  $\mu_\infty$  which coincides with the weak limit used in Conjecture 1 whenever the latter limit exists. This allows one to reformulate the conjecture in such a way that only the existence of a constant  $\kappa$  such that (2.1) holds remains conjectural. One can then reformulate Conjecture 2 below in a similar way. However, in all the examples considered in this paper we will verify Conjecture 1 in the form stated above. The only reason for doing so is that we believe that it is technically slightly more convenient to use the definition of  $\mu_\infty$  as a weak limit.

To explain this canonical construction of  $\mu_\infty$ , which is defined without any additional assumptions, we use the notation of §1.7. Thanks to (1.13), for every bounded Borel subset  $A$  of  $\mathfrak{g}^*$ , there exist positive reals  $c, C, \alpha$  and a nonnegative integer  $\beta$  such that  $\mu_\lambda(A) \leq C\lambda^\alpha(\log \lambda)^\beta$  for all  $\lambda > c$  (the constants  $c$  and  $C$  depend on  $A$ , but  $\alpha$  and  $\beta$  do not). Define a measure  $\tilde{\mu}_\lambda = \lambda^{-\alpha}(\log \lambda)^{-\beta}\mu_\lambda$ ; then we see that the function  $f_A$  defined by

$$f_A(\lambda) = \begin{cases} \tilde{\mu}_{|\lambda|}(A) & \text{if } |\lambda| > c, \\ 0 & \text{if } |\lambda| \leq c \end{cases}$$

is a bounded function  $\mathbb{R} \rightarrow \mathbb{R}$ . It is known that the action of the affine group of transformations  $x \mapsto ax + b$  on the real line is amenable, *i.e.*, there exists a mean  $m$  on the space of bounded functions  $\mathbb{R} \rightarrow \mathbb{R}$  which is invariant under this action and satisfies  $m(1) = 1$ . We then define  $\mu_\infty(A) = m(f_A)$ , and extend  $\mu_\infty$  (uniquely) by sigma-additivity to all Borel subsets  $A \subseteq \mathfrak{g}^*$ .

*Remark 2.5.* — The construction of the previous remark is, in fact, valid for any coadjoint orbit  $\Omega \subset \mathfrak{g}^*$  for any nilpotent Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . Moreover, by construction, the support of  $\mu_\infty$  is contained in the *asymptotic cone* to the orbit  $\Omega$ , *i.e.*, the boundary of the union of the dilates  $\lambda^{-1} \cdot \Omega$  for all  $\lambda \geq 1$ .

### 2.3. Precise version of the conjecture

We now formulate a more precise form of our conjecture—one that essentially provides a formula for the constant  $\kappa$  that appears in (2.1). To that end, we introduce the function

$$(2.2) \quad \Phi^*(x) = \sum_{\alpha} |\partial^{\alpha} V(x)|^{1/2} + \sum_{\alpha, j, k} |\partial^{\alpha} b_{jk}(x)|^{1/2};$$

this is to be compared with the function  $\Psi^*$  defined by (1.6). If, for example,  $V \equiv 0$  and  $B(x)$  grows regularly at infinity, then the terms corresponding to  $\alpha = 0$  dominate both  $\Psi^*$  and  $\Phi^*$ , so we see that these two functions have the same asymptotic behavior as  $\|x\| \rightarrow \infty$ . However, in general, it may happen that the function  $\Psi^*(x)$  grows slower than the function  $\Phi^*(x)$ .

We keep the same notation and assumptions as in Conjecture 1. In particular, since  $H$  has discrete spectrum, both  $\Phi^*$  and  $\Psi^*$  tend to  $+\infty$  as  $\|x\| \rightarrow \infty$ , so the following functions are well defined:

$$G_1(\lambda) = \text{meas} \{x \in \mathbb{R}^n \mid \Phi^*(x) \leq \lambda\}$$

and

$$G_2(\lambda) = \text{meas} \{x \in \mathbb{R}^n \mid \Psi^*(x) \leq \lambda\},$$

where  $\text{meas}$  stands for the usual Lebesgue measure.

**CONJECTURE 2.** Assume that  $H$  is a Schrödinger operator on  $L^2(\mathbb{R}^n)$  with discrete spectrum and quasi-homogeneous potentials. Let  $(Q, \nu)$  be defined as in Conjecture 1. Then one of the following situations occurs.

a) We have  $G_2(\lambda)/G_1(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow +\infty$ . This is the *strong degeneration case*. Then Conjecture 1 is valid with the normalization constant  $\kappa = 1$ .

b) We have  $G_2(\lambda) = O(G_1(\lambda))$  as  $\lambda \rightarrow +\infty$ . This is the *weak/intermediate degeneration case*. Then there exists a limit  $\lim_{\lambda \rightarrow +\infty} G_2(\lambda)/G_1(\lambda)$ , and Conjecture 1 is valid with  $\kappa$  equal to the value of this limit.

### 2.4. Concrete realization of representations

Until the end of the section, the quasi-homogeneity assumption will play no role. Let  $\mathfrak{g}$  be a real finite dimensional nilpotent Lie algebra such that  $[\mathfrak{g}, \mathfrak{g}]$  is abelian, and let  $S = -(L_1^2 + \cdots + L_N^2) - \sqrt{-1} \cdot L_0 \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  be a sublaplacian. We wish to obtain concrete realizations of the representations of  $\mathfrak{g}$  induced by unitary irreducible representations of  $G = \exp \mathfrak{g}$ . Let  $\mathfrak{h} \subseteq \mathfrak{g}$  be a Lie subalgebra, and  $H = \exp \mathfrak{h}$  the corresponding connected and

simply connected subgroup of  $G$ . (Schrödinger operators do not appear until the end of the section, so the notation should not cause any confusion.) Fix  $f \in \mathfrak{g}^*$ . We say that  $\mathfrak{h}$  is *subordinate to  $f$*  if  $f|_{[\mathfrak{h}, \mathfrak{h}]} \equiv 0$ . Under this condition,  $f$  defines a unitary character  $\chi_f$  of  $H$  via  $\chi_f(\exp h) = \exp(i \cdot f(h))$ . Thus we may form the induced representation  $\rho_{f, \mathfrak{h}} = \text{Ind}_H^G(\chi_f)$ . The construction of induced representations is reviewed in §A.1.

Kirillov's classification [15] of unitary irreducible representations of  $G$  can be summarized as follows. Let us say that  $\mathfrak{h}$  is a *polarization* of  $\mathfrak{g}$  at  $f$  if  $\mathfrak{h}$  is of maximal dimension among the subalgebras of  $\mathfrak{g}$  that are subordinate to  $f$ . Then  $\rho_{f, \mathfrak{h}}$  is irreducible if and only if  $\mathfrak{h}$  is a polarization at  $f$ . Moreover, in this case,  $\rho_{f, \mathfrak{h}}$  does not depend on the choice of  $\mathfrak{h}$ , up to unitary equivalence. Also, at every  $f \in \mathfrak{g}^*$  there exists at least one polarization. Thus, we write  $\rho_f = \rho_{f, \mathfrak{h}}$  for any choice of a polarization  $\mathfrak{h}$  at  $f$ . Finally, every unitary irreducible representation of  $G$  is unitarily equivalent to  $\rho_f$  for some  $f \in \mathfrak{g}^*$ , and  $\rho_{f_1}, \rho_{f_2}$  are unitarily equivalent if and only if  $f_1, f_2$  lie in the same coadjoint orbit of  $G$ .

Let us define the alternating bilinear form

$$(2.3) \quad B_f : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}, \quad B_f(X, Y) = f([X, Y]).$$

Thus, a subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  is subordinate to  $f$  if and only if  $\mathfrak{h}$  is isotropic with respect to  $B_f$ . One can prove that  $\mathfrak{h}$  is a polarization at  $f$  if and only if  $\mathfrak{h}$  is isotropic with respect to  $B_f$  and is maximal among all linear subspaces of  $\mathfrak{g}$  with this property. In particular, all polarizations at  $f$  have the same dimension,

$$\dim \mathfrak{h} = \frac{1}{2} \cdot (\dim \mathfrak{g} + \dim \mathfrak{g}(f)),$$

where

$$\mathfrak{g}(f) = \text{Ker } B_f = \{X \in \mathfrak{g} \mid f([X, Y]) = 0 \forall Y \in \mathfrak{g}\}.$$

In our situation, we can give an elementary proof of the existence of polarizations of a special form:

LEMMA 2.6. — *Let  $\mathfrak{g}, S$  be as above, and  $f \in \mathfrak{g}^*$ . Then there exists a polarization  $\mathfrak{h}$  of  $\mathfrak{g}$  at  $f$  such that  $\mathfrak{h} \supseteq \mathfrak{g}(f) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}]$ , and hence  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ . Moreover,  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{g}(f)$ , so  $\mathfrak{g}(f)$  is an ideal of  $\mathfrak{h}$ .*

Let us now fix a subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  subordinate to  $f$ , but not necessarily a polarization at  $f$ , which satisfies the requirement of the lemma:  $\mathfrak{g}(f) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$ . Since  $L_0, L_1, \dots, L_N$  generate  $\mathfrak{g}$  as a Lie algebra, we have  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \text{span}_{\mathbb{R}}\{L_0, L_1, \dots, L_N\}$ , and hence, a fortiori,  $\mathfrak{g} = \mathfrak{h} + \text{span}_{\mathbb{R}}\{L_1, \dots, L_N\}$ . After reindexing, we may assume that for some  $0 \leq n \leq N$ , the elements  $L_1, \dots, L_n$  form a complementary basis to  $\mathfrak{h}$  in  $\mathfrak{g}$ . (We

allow  $n = 0$ , which means that  $\mathfrak{h} = \mathfrak{g}$ .) For every element  $h \in \mathfrak{h}$ , let us define a real polynomial  $p_h(x)$  in  $n$  variables  $x = (x_1, \dots, x_n)$  by

$$(2.4) \quad p_h(x) = \sum_{\alpha_1, \dots, \alpha_n \geq 0} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot f((\text{ad } L_1)^{\alpha_1} \cdots (\text{ad } L_n)^{\alpha_n}(h)) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

PROPOSITION 2.7. — *There exists a realization of the representation  $\rho_{f, \mathfrak{h}} = \text{Ind}_H^G(\chi_f)$  of the Lie group  $G$  in the space  $L^2(\mathbb{R}^n, dm)$  (where  $dm$  is the Lebesgue measure) such that the induced representation of  $\mathfrak{g}$  takes every  $h \in \mathfrak{h}$  to the operator of multiplication by  $\sqrt{-1} \cdot p_h(x)$ , and takes  $L_j$ , for  $1 \leq j \leq n$ , to the operator  $\partial_j + \sqrt{-1} \cdot a_j(x)$ , where  $a_j(x) \in \mathbb{R}[x_1, \dots, x_n]$  is a certain polynomial.*

The practical applications of this proposition are based on the obvious analogy between (2.4) and the usual Taylor's formula.

## 2.5. Coadjoint orbits and Kostant measures

Let  $G$  be any connected Lie group, and  $\mathfrak{g}$  its Lie algebra. If  $f \in \mathfrak{g}^*$ , we denote by  $G(f)$  the stabilizer of  $f$  in  $G$  (with respect to the coadjoint action), and by  $\mathfrak{g}(f)$  the Lie algebra of  $G(f)$ . If  $\Omega \subset \mathfrak{g}^*$  is a coadjoint orbit, then for any point  $f \in \Omega$ , the orbit map  $G \rightarrow \Omega$ ,  $g \mapsto (\text{Ad}^* g)(f)$ , identifies  $\Omega$  with the homogeneous space  $G/G(f)$ , and hence identifies the tangent space  $T_f\Omega$  with the quotient  $\mathfrak{g}/\mathfrak{g}(f)$ . The notation is consistent with the one used in §2.4: if  $B_f$  is the alternating bilinear form on  $\mathfrak{g}$  given by  $B_f(X, Y) = f([X, Y])$ , then it is easy to see that  $\mathfrak{g}(f)$  is precisely the kernel of  $B_f$ . Moreover,  $B_f$  induces an alternating bilinear *nondegenerate* form  $\omega_f$  on  $\mathfrak{g}/\mathfrak{g}(f) \cong T_f\Omega$ . One then proves the following facts (see, e.g., [2], Ch. II):

- 1) the forms  $\omega_f$  vary smoothly with  $f$ , thus defining a nondegenerate differential 2-form  $\omega_\Omega$  on  $\Omega$ ;
- 2) the form  $\omega_\Omega$  is closed, and thus a symplectic form on  $\Omega$ ;
- 3) the form  $\omega_\Omega$  is  $G$ -invariant.

DEFINITION 2.8. — *The form  $\omega_\Omega$  is called the canonical symplectic form on the orbit  $\Omega$ . The Kostant measure (or the canonical measure) on the orbit  $\Omega$  is the positive Borel measure  $\mu_\Omega$  associated with the volume form*

$$(2\pi)^{-n} \cdot \frac{1}{n!} \cdot \omega_\Omega^n \quad \left(n = \frac{1}{2} \dim \Omega\right).$$

(Note that  $\dim \Omega$  is even because  $\Omega$  admits a symplectic form.)

It is clear that the Kostant measure is  $G$ -invariant. In the remainder of this subsection we obtain an explicit parameterization of the coadjoint orbits for the Lie algebras of the type considered in §2.1, and we derive formulas for the corresponding canonical symplectic forms and Kostant measures. We note that explicit parameterizations of the dual space of a (not necessarily nilpotent) Lie algebra have been studied by various authors: see, e.g., [31]. More recently, a very fine stratification of  $\mathfrak{g}^*$  for nilpotent  $\mathfrak{g}$  has been obtained in [1, 3, 29]. We will use a result from [3] in the next subsection.

In our subsequent computations (especially the ones that appear in the concrete examples of Sections 3, 4 and 5) we will implicitly use the following result. The proof is completely straightforward and is therefore omitted. Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{a} \subset \mathfrak{g}$  an ideal. Write  $\mathfrak{a}^\perp$  for the annihilator of  $\mathfrak{a}$  in  $\mathfrak{g}^*$ . The quotient map  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$  induces an isomorphism of vector spaces  $(\mathfrak{g}/\mathfrak{a})^* \xrightarrow{\cong} \mathfrak{a}^\perp \hookrightarrow \mathfrak{g}^*$ . Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$ , and  $A \subset G$  the closed connected normal subgroup corresponding to  $\mathfrak{a}$ . The adjoint action of  $G$  on  $\mathfrak{g}$  leaves  $\mathfrak{a}$  stable, whence  $G$  also acts on  $\mathfrak{g}/\mathfrak{a}$  and on  $(\mathfrak{g}/\mathfrak{a})^*$ . Then we have the following:

PROPOSITION 2.9. — a) *The isomorphism  $(\mathfrak{g}/\mathfrak{a})^* \rightarrow \mathfrak{a}^\perp$  above is  $G$ -equivariant, and the action of  $G$  on  $(\mathfrak{g}/\mathfrak{a})^*$  factors through the quotient group  $G/A$ ; thus the  $G$ -orbits in  $(\mathfrak{g}/\mathfrak{a})^*$  are the same as the coadjoint orbits of  $G/A$  in  $(\mathfrak{g}/\mathfrak{a})^*$ .*

b) *If  $\Omega \subset \mathfrak{g}^*$  is any coadjoint orbit, then either  $\Omega \cap \mathfrak{a}^\perp = \emptyset$ , or  $\Omega \subset \mathfrak{a}^\perp$ . In the latter case,  $\Omega$  is the image of a coadjoint orbit in  $(\mathfrak{g}/\mathfrak{a})^*$ . Moreover, the canonical symplectic form and the Kostant measure on  $\Omega$  are the same whether we regard  $\Omega$  as a coadjoint orbit for  $G$  or as a coadjoint orbit for  $G/A$ .*

c) *If  $G$  is simply connected and nilpotent, then the bijection between the coadjoint orbits in  $\mathfrak{g}^*$  that meet  $\mathfrak{a}^\perp$  and the coadjoint orbits in  $(\mathfrak{g}/\mathfrak{a})^*$ , defined above, corresponds, via Kirillov's theory, to the natural bijection between the unitary irreducible representations of  $G$  that are trivial on  $A$ , and all unitary irreducible representations of  $G/A$ .*

We return to the situation considered in §2.4. Thus  $G$  is a connected and simply connected nilpotent Lie group with Lie algebra  $\mathfrak{g}$  such that  $[\mathfrak{g}, \mathfrak{g}]$  is abelian. Fix a point  $f_0 \in \mathfrak{g}^*$ . We wish to parameterize the  $G$ -orbit  $G \cdot f_0 \subset \mathfrak{g}^*$ . As before, we assume we are given a sublaplacian  $S = -(L_1^2 + \cdots + L_N^2) - \sqrt{-1} \cdot L_0$  for  $\mathfrak{g}$ , and we let  $\mathfrak{h}$  be a real polarization of  $\mathfrak{g}$  at  $f_0$  provided by Lemma 2.6:  $\mathfrak{g}(f_0) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$ . Furthermore, we

suppose that for some  $1 \leq n \leq N$ ,  $L_1, \dots, L_n$  is a complementary basis for  $\mathfrak{h}$  in  $\mathfrak{g}$ .

From now on, we also assume that  $\mathfrak{h}$  is an *abelian* ideal of  $\mathfrak{g}$ . To justify this assumption, we note that since  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ , so is  $\mathfrak{a} := [\mathfrak{h}, \mathfrak{h}]$ ; on the other hand, by the definition of a polarization,  $f_0$  annihilates  $\mathfrak{a}$ . Thus,  $f_0$  induces a linear functional  $\bar{f}_0$  on  $\mathfrak{g}/\mathfrak{a}$ . By Proposition 2.9, the canonical inclusion  $(\mathfrak{g}/\mathfrak{a})^* \hookrightarrow \mathfrak{g}^*$  gives an isomorphism of the coadjoint orbit of  $\bar{f}_0$  in  $(\mathfrak{g}/\mathfrak{a})^*$  onto the coadjoint orbit of  $f_0$  in  $\mathfrak{g}^*$ ; moreover, this isomorphism preserves the canonical symplectic form and the Kostant measure. Lastly, note that since  $\mathfrak{a} \subset \mathfrak{g}(f)$  by Lemma 2.6, it is clear that  $\mathfrak{h}/\mathfrak{a}$  is a maximal isotropic subspace of  $\mathfrak{g}/\mathfrak{a}$  with respect to the form  $B_{\bar{f}_0}$ . Thus, from the point of view of either the coadjoint orbit of  $f_0$ , or of the corresponding unitary irreducible representation, nothing is lost by passing from  $\mathfrak{g}$  to  $\mathfrak{g}/\mathfrak{a}$ .

PROPOSITION 2.10. — *With the notation above, assume that  $\mathfrak{h}$  is abelian. The map*

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathfrak{g}^*$$

defined by

$$\langle \varphi(\xi, x), L_j \rangle = \xi_j \quad \text{for } 1 \leq j \leq n$$

and

$$\begin{aligned} & \langle \varphi(\xi, x), Y \rangle \\ &= \sum_{\alpha_1, \dots, \alpha_n \geq 0} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot f_0((\text{ad } L_1)^{\alpha_1} \cdots (\text{ad } L_n)^{\alpha_n}(Y)) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \end{aligned}$$

for all  $Y \in \mathfrak{h}$  is a diffeomorphism of  $\mathbb{R}^{2n}$  onto the coadjoint orbit of  $f_0$  in  $\mathfrak{g}^*$ .

By a slight abuse of notation, we identify  $\Omega$  with  $\mathbb{R}^{2n}$  using the diffeomorphism  $\varphi$ , and in particular, we view  $(\xi, x)$  as coordinates on the orbit  $\Omega$ . Let us define polynomials  $b_{jk}(x)$  by

$$b_{jk}(x) = \langle \varphi(0, x), [L_j, L_k] \rangle;$$

note that if  $\mathfrak{g}$  arises from a Schrödinger operator with polynomial potentials, and if  $f_0$  restricts to the linear functional  $\sqrt{-1} \cdot P(x) \mapsto P(0)$  on the subspace of  $\mathfrak{g}$  consisting of multiplication operators, then the  $b_{jk}(x)$  are precisely the components of the magnetic tensor of the operator. We will prove the following:

PROPOSITION 2.11. — *The canonical symplectic form and the Kostant measure on the orbit  $\Omega$  are given by*

$$(2.5) \quad \omega_\Omega = \sum_{j=1}^n d\xi_j \wedge dx_j + \sum_{1 \leq j < k \leq n} b_{kj}(x) dx_j \wedge dx_k$$

and

$$\mu_\Omega = (2\pi)^{-n} \cdot d\xi_1 \cdots d\xi_n dx_1 \cdots dx_n.$$

In other words, if we identify  $\mu_\Omega$  with its extension by zero to  $\mathfrak{g}^*$ , then we can write

$$(2.6) \quad \mu_\Omega = (2\pi)^{-n} \cdot \varphi_*(d\xi_1 \cdots d\xi_n dx_1 \cdots dx_n),$$

where  $\varphi_*$  denotes the pushforward by the map  $\varphi : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$ .

### 2.6. Polynomial measures

In this subsection we collect a few results that will be used in the computations of Sections 3, 4 and 5, and a few others that help motivate our conjectures in §2.2 and §2.3.

Let  $\mathbb{R}^N$  and  $\mathbb{R}^n$  be two Euclidean spaces. A *polynomial measure* on  $\mathbb{R}^N$  is a Borel measure of the form  $\mu = c_0 \cdot \phi_*(dm)$ , where  $c_0$  is a positive constant,  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^N$  is a polynomial map, and  $dm$  is the Lebesgue measure on  $\mathbb{R}^n$ . Recall that the *pushforward measure*  $\phi_*(dm)$  is defined by the formula

$$(\phi_*(dm))(A) := m(\phi^{-1}(A)),$$

for every Borel subset  $A \subseteq \mathbb{R}^N$ . We say that  $\mu$  is a *regular polynomial measure* if it is finite on compact sets, or, equivalently, if  $\phi^{-1}(K)$  has finite Lebesgue measure for every compact subset  $K \subset \mathbb{R}^N$  (a sufficient condition for this is that  $\phi$  is a proper map, but this condition is not necessary). In practice, we will often think of regular polynomial measures as positive linear functionals on the space  $C_c(\mathbb{R}^N)$  of all compactly supported continuous functions on  $\mathbb{R}^N$ . Explicitly,  $\mu$  corresponds to the linear functional  $\Lambda$  defined by

$$\Lambda(F) = \int_{\mathbb{R}^N} F d\mu = c_0 \int_{\mathbb{R}^n} F(\phi(x)) dx,$$

where the second integral is the usual Lebesgue integral. Positivity here refers to the statement that if  $F \in C_c(\mathbb{R}^N)$  is nonnegative, then  $\Lambda(F) \geq 0$ . It follows from Propositions 2.10 and 2.11 that if  $\mathfrak{g}$  is a nilpotent Lie algebra such that  $[\mathfrak{g}, \mathfrak{g}]$  is abelian, then the Kostant measure  $\mu_\Omega$  corresponding to each coadjoint orbit  $\Omega \subset \mathfrak{g}^*$  is a regular polynomial measure on  $\mathfrak{g}^*$ . (Notice,

however, that this statement holds for an arbitrary nilpotent Lie algebra  $\mathfrak{g}$ , without the assumption that  $[\mathfrak{g}, \mathfrak{g}]$  is abelian.)

Suppose that  $\{\mu_j\}_{j=1}^\infty$  is a sequence of regular polynomial measures on  $\mathbb{R}^N$ . We will say that this sequence has a *weak limit* if the limit

$$\Lambda(F) := \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} F d\mu_j$$

exists for every  $F \in C_c(\mathbb{R}^N)$ . Note that, if this is the case, then the limit automatically defines a positive linear functional on the space  $C_c(\mathbb{R}^N)$ . It then follows from the Riesz Representation Theorem that  $\Lambda$  itself corresponds to a regular Borel measure  $\mu$  on  $\mathbb{R}^N$ . We then say that the sequence  $\{\mu_j\}$  *converges weakly* to  $\mu$ . The following two results on weak limits will be used in Sections 3, 4 and 5.

**PROPOSITION 2.12.** — *Suppose that  $N = N_1 + N_2$ , that we are given a polynomial map  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^{N_1}$ , and a sequence of polynomial maps  $\psi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{N_2}$ . Assume moreover that  $\phi^{-1}(K)$  has finite Lebesgue measure for every compact subset  $K \subset \mathbb{R}^{N_1}$ , and that the sequence  $\psi_j$  converges pointwise to a polynomial map  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{N_2}$ . Form the polynomial maps  $\tilde{\phi}_j : \mathbb{R}^n \rightarrow \mathbb{R}^N$  and  $\tilde{\phi} : \mathbb{R}^n \rightarrow \mathbb{R}^N$  by defining*

$$\tilde{\phi}_j(x) = (\phi(x), \psi_j(x)) \quad \text{and} \quad \tilde{\phi}(x) = (\phi(x), \psi(x)) \quad \forall x \in \mathbb{R}^n.$$

*Further, set  $\mu_j = (\tilde{\phi}_j)_*(dm)$  and  $\mu = \tilde{\phi}_*(dm)$ , where  $dm$  is the Lebesgue measure on  $\mathbb{R}^n$ . Then  $\mu_j, \mu$  are regular polynomial measures on  $\mathbb{R}^N$ , and  $\mu_j \rightarrow \mu$  weakly.*

As we will see, Proposition 2.12 is useful for carrying out the computation of the limiting measure  $\mu_\infty$  that appears in our conjectured formula in the cases of strong or weak degeneration. In the intermediate degeneration case, the computation turns out to be slightly more complicated, and the following result (Proposition 2.13) is needed. We keep the same notation as in §2.6. We say that a positive Borel measure on  $\mathbb{R}^N$  is *c-finite* if it is finite on compact sets; such a measure is then automatically regular. We define weak limits for the c-finite measures in the same way as for polynomial measures. Further, we introduce the following notation. Given  $a, b \in \mathbb{R}^N$ , we write

$$[a, b) = \{x \in \mathbb{R}^N \mid a_j \leq x_j < b_j \text{ for } 1 \leq j \leq n\}.$$

Also, for  $x \in \mathbb{R}^N$ , we set  $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$ . Finally, put  $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}^N$ .



PROPOSITION 2.13. — Consider a family of  $c$ -finite positive Borel measures  $\{\mu_\lambda\}_{\lambda>0}$  on  $\mathbb{R}^N$ . Assume that there exists a countable set  $E \subset \mathbb{R}$  such that for each  $a, b \in \mathbb{R}^N$  with  $a_j \notin E, b_j \notin E$  for all  $1 \leq j \leq N$ , there exists a limit

$$\nu([a, b]) = \lim_{\lambda \rightarrow +\infty} \mu_\lambda([a, b]).$$

Then there exists a unique  $c$ -finite positive Borel measure  $\mu_\infty$  on  $\mathbb{R}^N$  such that

$$(2.7) \quad \int_{\mathbb{R}^N} F(x) d\mu_\infty(x) = \lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^N} F(x) d\mu_\lambda(x) \quad \forall F \in C_c(\mathbb{R}^N).$$

Moreover, if  $\nu_\infty$  is any  $c$ -finite positive Borel measure on  $\mathbb{R}^N$  such that  $\nu_\infty([a, b]) = \nu([a, b])$  for all  $[a, b]$  as above, then (2.7) is satisfied with  $\nu_\infty$  in place of  $\mu_\infty$ ; in particular,  $\nu_\infty = \mu_\infty$ .

Let us now recall a result of Nilsson [27] and explain its relation to our conjectural formula. With the same notation as above, let  $\mu = \phi_*(dm)$  be a regular polynomial measure on  $\mathbb{R}^N$ , where  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^N$  is a polynomial map given by  $\phi(x) = (p_1(x), \dots, p_N(x))$ , and  $dm$  is the Lebesgue measure on  $\mathbb{R}^n$ . Given  $\lambda \in \mathbb{R}, \lambda > 0$ , we define a polynomial measure  $\mu_\lambda$  on  $\mathbb{R}^N$  by  $\mu_\lambda(A) = \mu(\lambda^{-1} \cdot A)$ , for each Borel subset  $A \subseteq \mathbb{R}^N$ . In particular, let  $P(x) = \sum_{j=1}^N p_j(x)^2$ . If  $A = B_1(0)$  is the unit ball around the origin in  $\mathbb{R}^N$ , then  $\mu_\lambda(B_1(0)) = \text{meas}\{x \in \mathbb{R}^n \mid P(x) \leq \lambda^2\}$ . We now state a result which is a special case of Theorem 1 in [27]; the latter, in turn, is based on the results of [28].

THEOREM 2.14 (Nilsson). — Let  $P(x)$  be a real polynomial on  $\mathbb{R}^n$  such that  $P(x) \rightarrow +\infty$  as  $\|x\| \rightarrow \infty$ , and set

$$G(\lambda) = \text{meas}\{x \in \mathbb{R}^n \mid P(x) \leq \lambda\}.$$

Then there exist positive reals  $c, C, \alpha$  and a nonnegative integer  $\beta$  such that

$$C^{-1} \cdot \lambda^\alpha \cdot (\log \lambda)^\beta \leq G(\lambda) \leq C \cdot \lambda^\alpha \cdot (\log \lambda)^\beta \quad \text{for all } \lambda > c.$$

It is clear that  $\alpha$  and  $\beta$  are uniquely determined by the polynomial  $P(x)$ . If  $P = \sum_j p_j^2$  is as above, and  $\alpha, \beta$  are as in the theorem, then we see that  $\mu_\lambda(B_1(0)) = O(\lambda^{2\alpha} \cdot (\log \lambda)^\beta)$  as  $\lambda \rightarrow +\infty$ . But then, since the Lebesgue measure is a doubling measure, we obtain that for every fixed  $R > 0$ , we have  $\mu_\lambda(B_R(0)) = O(\lambda^{2\alpha} \cdot (\log \lambda)^\beta)$  as  $\lambda \rightarrow +\infty$ . So, the existence of a nonzero weak limit of  $\lambda^{-2\alpha} \cdot (\log \lambda)^{-\beta} \cdot \mu_\lambda$  as  $\lambda \rightarrow +\infty$  is plausible, which explains part of the statements of our Conjectures 1 and 2.

The final topic of the section is the decomposition of the invariant measure on the dual space of a Lie algebra into an integral of the Kostant measures. Let  $\mathfrak{g}$  be an arbitrary finite dimensional nilpotent Lie algebra over  $\mathbb{R}$ , and  $\mathfrak{g}^*$  its dual space, with the coadjoint action of the Lie group  $G = \exp \mathfrak{g}$ . Let  $\mathfrak{g}^*/G$  denote the space of coadjoint orbits, and  $\rho : \mathfrak{g}^* \rightarrow \mathfrak{g}^*/G$  the quotient map. Write  $\mathcal{M}$  for the  $\sigma$ -algebra on  $\mathfrak{g}^*/G$  defined by  $A \in \mathcal{M} \iff \rho^{-1}(A) \subseteq \mathfrak{g}^*$  is a Borel subset. The following result is interesting in its own right.

**PROPOSITION 2.15.** — *Let  $\mu$  be a positive  $c$ -finite Borel measure on  $\mathfrak{g}^*$  which is invariant under the coadjoint action of  $G$ . Then there exists a unique positive measure  $\nu$  on  $\mathcal{M}$  such that the equation*

$$\int_{\mathfrak{g}^*} F(x) d\mu(x) = \int_{\mathfrak{g}^*/G} d\nu(y) \cdot \int_{\rho^{-1}(y)} F(x) d\mu_y(x),$$

holds for every nonnegative Borel-measurable function  $F$  on  $\mathfrak{g}^*$ , where  $\mu_y$  is the Kostant measure corresponding to the coadjoint orbit  $\rho^{-1}(y) \subset \mathfrak{g}^*$ .

### 3. An inhomogeneous example

#### 3.1. Application of the conjectural formula

In this section we consider in detail the two-dimensional Schrödinger operator

$$H = -\frac{\partial^2}{\partial x_1^2} - \left( \frac{\partial}{\partial x_2} + \sqrt{-1} \cdot (x_1^3/3 - x_1 x_2) \right)^2$$

with zero electric potential and magnetic tensor  $b(x) = x_1^2 - x_2$ . Note that  $b(x)$  is quasi-homogeneous, but not homogeneous. Furthermore,  $b(x)$  vanishes along the parabola  $x_2 = x_1^2$  and, in particular, does not grow at infinity in some directions (so the formula of Colin de Verdière is not applicable). However, the condition of the criterion of [10] is satisfied, so  $H$  has discrete spectrum. In this subsection we will apply our conjecture to compute the leading term of the asymptotics of  $N(\lambda, H)$  as  $\lambda \rightarrow +\infty$ . The formula we obtain will then be verified rigorously using analytical methods.

First we must decide if we are in the strong degeneration case or the intermediate degeneration case. To that end, we consider the functions  $\Phi^*$ ,  $\Psi^*$  given by (2.2), (1.6). In our situation they are

$$\Phi^*(x) = |x_1^2 - x_2|^{1/2} + |2x_1|^{1/2} + c_1$$

and

$$\Psi^*(x) = |x_1^2 - x_2|^{1/2} + |2x_1|^{1/3} + c_2,$$

where  $c_1, c_2$  are constants. We then have

$$\begin{aligned} G_1(\lambda) &= \text{meas}\{x \in \mathbb{R}^2 \mid \Phi^*(x) \leq \lambda\} \\ &= \text{meas}\{x \in \mathbb{R}^2 \mid |x_1^2 - x_2|^{1/2} + |2x_1|^{1/2} \leq \lambda - c_1\} \\ &= \frac{1}{3} \cdot (\lambda - c_1)^4 \sim \lambda^4/3 \quad \text{as } \lambda \rightarrow +\infty \end{aligned}$$

(the straightforward computation using Fubini’s theorem is omitted). Similarly,

$$\begin{aligned} G_2(\lambda) &= \text{meas}\{x \in \mathbb{R}^2 \mid \Psi^*(x) \leq \lambda\} \\ &= \text{meas}\{x \in \mathbb{R}^2 \mid |x_1^2 - x_2|^{1/2} + |2x_1|^{1/3} \leq \lambda - c_2\} \\ &= \frac{1}{5} \cdot (\lambda - c_2)^5 \sim \lambda^5/5 \quad \text{as } \lambda \rightarrow +\infty. \end{aligned}$$

Thus, we are in the strong degeneration case.

*Remark 3.1.* — In general, it will be difficult or impossible to obtain precise formulas for the functions  $G_1(\lambda)$  and  $G_2(\lambda)$ , such as above. However, it is usually not very hard to either prove an estimate showing that  $G_2(\lambda)/G_1(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow +\infty$ , or find  $\lim_{\lambda \rightarrow +\infty} G_2(\lambda)/G_1(\lambda)$  in case the limit exists and is finite.

Next we let  $\mathfrak{g} = \mathfrak{g}_H$  be the Lie algebra associated to  $H$ , as in Section 2. Let us fix, once and for all, the following basis of  $\mathfrak{g}$ :

$$\begin{aligned} L_1 &= \partial_1, \quad L_2 = \partial_2 + \sqrt{-1} \cdot (x_1^3/3 - x_1x_2), \\ X &= \sqrt{-1} \cdot (x_1^2 - x_2), \quad Y = \sqrt{-1} \cdot x_1, \quad Z = \sqrt{-1}. \end{aligned}$$

We use this basis to identify  $\mathfrak{g}^*$  with  $\mathbb{R}^5$ ; explicitly, the identification is given by

$$f \mapsto (f(L_1), f(L_2), f(X), f(Y), f(Z)).$$

Write  $\mathfrak{h} = \text{span}_{\mathbb{R}}\{X, Y, Z\}$ , and define a linear functional  $f_0 : \mathfrak{g} \rightarrow \mathbb{R}$  by  $f_0(L_j) = 0$  for  $j = 1, 2$ , and  $f_0(\sqrt{-1} \cdot P(x)) = P(0)$  for all  $\sqrt{-1} \cdot P(x) \in \mathfrak{h}$ . It follows from Proposition 2.7 that the induced representation  $\rho_{f_0, \mathfrak{h}}$  coincides with the tautological representation of  $\mathfrak{g}$  (at least up to changing the action of  $L_1$  and  $L_2$ , so that  $H$  is replaced by a gauge-equivalent operator). In particular,  $\rho_{f_0, \mathfrak{h}}$  is irreducible (by Theorem 2.1), so  $\mathfrak{h}$  is a polarization of  $\mathfrak{g}$  at  $f_0$ . Let  $\Omega = G \cdot f_0$  be the coadjoint orbit of  $f_0$  in  $\mathfrak{g}^*$ , and let  $\mu_\Omega$  be the corresponding Kostant measure. From now on, to simplify notation, we will implicitly identify  $\mu_\Omega$  with its extension by zero to all of  $\mathfrak{g}^*$ .

By Proposition 2.11, the orbit  $\Omega$  is parameterized by the map  $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^5 \cong \mathfrak{g}^*$  given by  $\varphi(\xi, x) = (\xi_1, \xi_2, x_1^2 - x_2, x_1, 1)$ , and, moreover, we have

$$\mu_\Omega = (2\pi)^{-2} \cdot \varphi_*(d\xi dx).$$

To apply our scaling construction, we fix  $F \in C_c(\mathfrak{g}^*)$  and study the integrals

$$\int_{\mathbb{R}^4} F(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2, \lambda^{-1}(x_1^2 - x_2), \lambda^{-1}x_1, \lambda^{-1}) d\xi dx$$

for  $\lambda \gg 0$ . Making the change of variables  $\xi_j = \lambda\xi'_j$ ,  $\lambda^{-1}(x_1^2 - x_2) = x'_1$ ,  $\lambda^{-1}x_1 = x'_2$ , the integral above becomes

$$\lambda^4 \cdot \int_{\mathbb{R}^4} F(\xi'_1, \xi'_2, x'_1, x'_2, \lambda^{-1}) d\xi' dx'.$$

Now  $\lambda^{-1} \rightarrow 0$  as  $\lambda \rightarrow +\infty$ , and, using Proposition 2.12, we find that there exists a weak limit

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-4} \cdot \mu_\lambda = (2\pi)^{-2} \cdot \psi_*(d\eta dy),$$

where  $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^5 \cong \mathfrak{g}^*$  is given by  $(\eta, y) \mapsto (\eta_1, \eta_2, y_1, y_2, 0)$ .

In particular,  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp \subset \mathfrak{g}^*$ , where  $\mathfrak{a} = \mathbb{R} \cdot Z$  is an ideal of  $\mathfrak{g}$ . Using Proposition 2.9, we pass from  $\mathfrak{g}$  to the quotient algebra  $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ . Naturally, we use the following basis of  $\bar{\mathfrak{g}}$ :  $\bar{L}_1, \bar{L}_2, \bar{X}, \bar{Y}$ , where the bar denote the image of an element of  $\mathfrak{g}$  under the quotient map. The commutation relations in  $\bar{\mathfrak{g}}$  are determined by  $[\bar{L}_1, \bar{L}_2] = \bar{X}$ ,  $[\bar{L}_1, \bar{X}] = \bar{Y}$ , and all the other brackets are zero. Furthermore, we see that  $\mu_\infty$  is, up to the multiple  $(2\pi)^{-2}$ , the Lebesgue measure on  $\bar{\mathfrak{g}}^*$ . In particular, the set of elements  $f \in \bar{\mathfrak{g}}^*$  that vanish on  $\bar{Y}$  has measure zero with respect to  $\mu_\infty$ , and is also invariant under the coadjoint action because  $\bar{Y}$  is central in  $\bar{\mathfrak{g}}$ . Therefore this set can be ignored in the subsequent computations.

Fix an element  $f \in \bar{\mathfrak{g}}^*$  such that  $f(\bar{Y}) \neq 0$ . It is clear that  $\text{span}_{\mathbb{R}}\{\bar{L}_2, \bar{X}, \bar{Y}\}$  is a polarization of  $\bar{\mathfrak{g}}$  at  $f$ . Using Proposition 2.10, we see that the coadjoint orbit of  $f$  in  $\bar{\mathfrak{g}}^*$  is parameterized by the map

$$\varphi_f : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \cong \bar{\mathfrak{g}}^*,$$

$$(\zeta, z) \mapsto (\zeta, f(\bar{L}_2) + z f(\bar{X}) + z^2 f(\bar{Y}), f(\bar{X}) + 2z f(\bar{Y}), f(\bar{Y})).$$

Moreover, by Proposition 2.11, the corresponding Kostant measure  $\mu_f = \mu_{\Omega_f}$  is given by  $\mu_f = (2\pi)^{-1} \cdot (\varphi_f)_*(d\zeta dz)$ . Now we must find a convenient parameterization of the space  $Q$  that appears in our conjectures, which, in this case, is  $\bar{\mathfrak{g}}^*/\bar{G}$ . Given a “generic” coadjoint orbit  $\Theta \subset \bar{\mathfrak{g}}^*$  (i.e., one that does not meet  $\bar{Y}^\perp$ ), it follows from the previous paragraph that there is a

unique  $f \in \Theta$  with  $f(\overline{X}) = 0$ . Moreover,  $\Theta$  is then determined uniquely by the value  $f(\overline{L}_2)$ . In other words, if we define a map

$$\overline{\mathfrak{g}}^* \setminus \overline{Y}^\perp \rightarrow \mathbb{R}^\times \times \mathbb{R}$$

by

$$f \mapsto \left( f(\overline{Y}), f(\overline{L}_2) - \frac{3}{4} \cdot \frac{f(\overline{X})^2}{f(\overline{Y})} \right),$$

then the fibers of this map are precisely the generic coadjoint orbits in  $\overline{\mathfrak{g}}^*$ . In coordinate form, the map is given by  $(z_1, z_2, z_3, z_4) \mapsto (z_4, z_2 - (3z_3^2)/(4z_4))$ . This map clearly admits a smooth section given by  $(a, b) \mapsto (0, b, 0, a)$ . Thus, we have identified  $Q$  with  $\mathbb{R}^\times \times \mathbb{R}$  (up to a set of measure zero, which we have ignored in the beginning).

Now the Kostant measure for the coadjoint orbit corresponding to the point  $(a, b) \in \mathbb{R}^\times \times \mathbb{R}$  is given by

$$\mu_{(a,b)} = (2\pi)^{-1} \cdot (\varphi_{(a,b)})_* (d\zeta dz),$$

where  $\varphi_{(a,b)} : (\zeta, z) \mapsto (\zeta, az^2 + b, 2az, a)$ . To complete our construction, we must decompose the measure  $\mu_\infty$  as an integral of the measures  $\mu_{(a,b)}$  with respect to a certain measure  $\nu$  on  $\mathbb{R}^\times \times \mathbb{R}$ . More explicitly, this means that we must find a measure  $\nu$  such that, for every  $F \in C_c(\mathbb{R}^3 \times \mathbb{R}^\times)$ , we have

$$\begin{aligned} (2\pi)^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^\times} F(\zeta, az^2 + b, 2az, a) d\zeta dz d\nu(a, b) \\ = (2\pi)^{-2} \int_{\mathbb{R}^3 \times \mathbb{R}^\times} F(\eta_1, \eta_2, y_1, y_2) d\eta dy. \end{aligned}$$

Using the change of variables  $\xi_1 = \zeta, \xi_2 = az^2 + b, x_1 = 2az, x_2 = a$ , we see that the equation above is satisfied if we put  $d\nu(a, b) = \pi^{-1} \cdot |a| da db$ .

Finally, we substitute the result into (2.1). Using Proposition 2.7 to find the image of  $-\overline{L}_1^2 - \overline{L}_2^2$  in the unitary irreducible representations of  $\overline{G} = \exp \overline{\mathfrak{g}}$ , we obtain the following result:

$$(3.1) \quad N(\lambda, H) \sim \frac{1}{\pi} \int_{\mathbb{R}^\times \times \mathbb{R}} |a| da db \cdot N\left(\lambda, -\frac{\partial^2}{\partial z^2} + (az^2 + b)^2\right),$$

or equivalently, in the form which will appear naturally in the result of the direct variational argument at the end of the section,

$$(3.2) \quad N(\lambda, H) \sim \frac{1}{\pi} \int_0^\infty da \int_{-\infty}^{+\infty} db \cdot N\left(\lambda, -\frac{d^2}{dz^2} + (\sqrt{a}z^2 + b)^2\right).$$

Making the change of variables  $z \mapsto \lambda^{-1/2}z$ ,  $\xi_2 \mapsto \lambda^{1/2}\xi_2$ ,  $a \mapsto \lambda^{3/2}a$ , we find that

$$(3.3) \quad N(\lambda, H) \sim \kappa(H)\lambda^{7/2},$$

where

$$(3.4) \quad \kappa(H) = \frac{1}{\pi} \int_0^\infty da \int_{-\infty}^{+\infty} db \cdot N \left( 1, -\frac{d^2}{dz^2} + (\sqrt{a}z^2 + b)^2 \right).$$

The convergence of the integral (3.4) will be proved in Appendix B.

### 3.2. Verification of (3.3): the general outline

In the rest of the section we will give a proof of (3.3) which is independent of our main conjectures.

We modify the variational method of computing the leading term of the spectral asymptotics applied to Schrödinger operators without magnetic fields in [17, 19], and Schrödinger operators in 2D with homogeneous magnetic potentials, in [20]. The proofs are based on several standard variational lemmas; see, e.g., the Appendix in [18] for the list. We will construct open sets  $U_j \subset U_{1j}$  in  $\mathbb{R}^n$  and non-negative functions  $\psi_j \in C^1(\mathbb{R}^n)$ ,  $j = 0, \pm 1, \pm 2, \dots$ , such that

$$(3.5) \quad \cup_j \overline{U_j} = \mathbb{R}^n;$$

$$(3.6) \quad U_j \cap U_k = \emptyset;$$

$$(3.7) \quad \forall x \in \mathbb{R}^n, \quad \text{card}\{j \mid x \in U_{1j}\} \leq C;$$

$$(3.8) \quad \text{supp } \psi_j \subset U_{1j};$$

$$(3.9) \quad |\nabla \psi_j(x)|^2 \leq \epsilon \Psi^*(x)^2,$$

where  $C, \epsilon$  are independent of  $j$ , and  $\epsilon$  is small, and

$$(3.10) \quad \sum_j \psi_j^2(x) = 1, \quad x \in \mathbb{R}^n.$$

Let  $N_D(\lambda, A, U)$  stand for the counting function of the Dirichlet problem for an operator  $A$  on an open set  $U$ . Since the open sets  $U_j$  do not intersect (pairwise), the standard variational argument gives an estimate from below

$$(3.11) \quad N(\lambda; H(a) + V) \geq \sum_j N_D(\lambda, H(a) + V, U_j).$$

To derive a similar upper bound, we use the IMS-localization formula (see e.g., [6])

$$(3.12) \quad H(a) + V = \sum_j \psi_j(H(a) + V)\psi_j - \sum_{j \geq 0} |\nabla \psi_j|^2.$$

Applying (3.9) and (3.7) to last term on the RHS, we obtain

$$(3.13) \quad H(a) + V \geq \sum_j \psi_j(H(a) + V - C\epsilon(\Psi^*)^2)\psi_j.$$

Employing the standard variational considerations, we derive from (3.13) and (3.8) an estimate from above

$$(3.14) \quad N(\lambda; H(a) + V) \leq \sum_j N_D(\lambda, H(a) + V - C\epsilon(\Psi^*)^2, U_{1j}).$$

We will construct the partition of unity, depending on  $\lambda$ , so that  $\epsilon = \epsilon(\lambda) \rightarrow +0$  as  $\lambda \rightarrow +\infty$ . Then on the strength of the lower bound obtained in [10], [26]:

$$(3.15) \quad H(a) + V \geq c(\Psi^*)^2,$$

one should expect that the leading term of the asymptotics of the RHS in (3.14) will not change if the error term  $C\epsilon(\Psi^*)^2$  is omitted. Hence, the lower bound (3.11) and upper bound (3.14) may have the same leading term of the asymptotics (which is necessary if we want to derive the leading term of the asymptotics of  $N(\lambda, H(a) + V)$  from (3.11) and (3.14)) provided the difference  $U_{1j} \setminus U_j$  is relatively small w.r.t.  $U_j$ . However, (3.9) implies that this difference cannot be arbitrarily small. Moreover, (3.9) prevents the sets  $U_j$  themselves from being too small, which is desirable in the next step: the “freezing” of the electric and magnetic tensor at a point  $x^j \in U_j$ . Indeed, the larger the size of  $U_{1j}$ , the larger is the error due to the freezing.

These general arguments explain that one should construct the partition of unity carefully, by taking into account the competing requirements described above. The next general observation explains the difference between the non-degenerate cases (as in [5]), and the degenerate ones, from the point of view of this scheme. If  $U_j$  is close to the degeneration set, where  $\Psi_0 := V^{1/2} + \sum_{jk} |b_{jk}|^{1/2}$  is zero (call this set  $\Sigma$ ), then one cannot freeze  $V$  and  $B$  completely, and still have a small error; but one can freeze  $V$  and  $B$  in the directions tangent to  $\Sigma$ . When convenient, one may freeze  $V$  and  $B$  in directions close to tangent ones. In the special case which we consider in this section,  $\Sigma = \{(x_1, x_2) \mid x_1^2 = x_2\}$ , and in a vicinity of  $x^j \in \Sigma$ , it is convenient to freeze the dependence on  $x_2$ . Notice also that in order to justify the freezing procedure, one has to work with the magnetic potential

$a$ , and not with the magnetic tensor only. This requires a careful choice of a gauge transformation.

The final general remark is: if it is known (or anticipated) that one or the other term in (3.11) and/or (3.14) does not contribute to the leading term of the asymptotics, then one may omit the corresponding term in (3.11), and obtain a crude upper bound for the term in (3.14). In particular, from the classical results for the Laplacian in a bounded domain, it follows that a ball  $U_0 = B(0, M_0)$  does not contribute to the leading term of the asymptotics, if  $M_0 = M_0(\lambda)$  grows slower than any power of  $\lambda$ . (Here,  $B(y, r)$  denotes an open ball of radius  $r$  centered at  $y$ .) We will use this observation when we construct the partition of unity. Further, in all examples, which we considered, in the weak and intermediate degeneration cases, only the  $U_j$  which are separated from  $\Sigma$  contribute to the leading term of the asymptotics; on the contrary, in the strong degeneration case, only the  $U_j$  adjacent to  $\Sigma$  do. The same holds for many other classes of degenerate operators (see [17]-[20]). Hence, in the former case, it suffices to obtain a crude estimate from above for terms with  $U_j$  adjacent to  $\Sigma$ , and in the latter, for terms with  $U_j$  separated from  $\Sigma$ . In the special case which we consider in this section, the degeneration is strong.

### 3.3. Construction of a partition of unity

We adopt the standard policy on constants: unless otherwise stated,  $C, c, C_1, c_1$ , etc., denote positive constants (different at different stages of the proof), which can be chosen the same for all  $\lambda$  and  $j$ , and the other parameters they can depend on. The dependence on  $n$  is possible but  $n$  is fixed. (In fact, in our special case,  $n = 2$ , but we will use  $n$  since the scheme below works in the general case as well.)

We will construct open sets  $U_j \subset U_{1j}$  and functions  $\chi_j$ , which enjoy the properties (3.5)-(3.9). In addition,

$$(3.16) \quad 0 \leq \chi_j(x) \leq 1,$$

and

$$(3.17) \quad \chi_j(x) = 1, \quad x \in U_j.$$

Then the functions

$$(3.18) \quad \psi_j = \frac{\chi_j}{(\sum_k \chi_k^2)^{1/2}}$$

satisfy (3.8)-(3.10).



For large  $M$  and  $M_0$ , denote by  $\Sigma_M$  the intersection of the union of balls  $B(y, M\Psi^*(y)^{-1})$ , over all  $y \in \Sigma$ , with the set  $\{x \mid \|x\| > M_0\}$ , and set  $\Sigma_{-M} = \mathbb{R}^n \setminus \overline{\Sigma_M \cup U_0}$ . For the general scheme to work, it is convenient that  $M$  tends to infinity with  $\lambda$  but slower than any power of  $\lambda$  or  $|x|$  provided the latter is not in  $U_0$ ; and  $M_0$  should grow slower than any power of  $\lambda$ . For instance,  $M_0 = M_0(\lambda) = \log \lambda$  and  $M = M(\lambda) = \log \log \lambda$  will do. Notice that we will use the notation  $\Sigma_{\pm M}$  not only with the subscript  $\pm M$ , but with  $\pm cM$  or  $\pm cM$  as well. We will also need a function which grows slower than any power of  $M$ ; so we introduce  $M_1 = \log M$ .

The function  $\chi_0$  is a “smoothed” indicator function of  $U_0$ : we take  $\mu \in C^\infty(\mathbb{R})$  such that  $\mu(x) = 1, x \leq 1, \mu(x) = 0, x \geq 2$ , and set  $\chi_0(x) = \mu(M_1^{-1/2}(|x| - M_0))$ ,  $U_{1,0} = B(0, M_0 + 2M_1^{1/2})$ . Next, we construct the sets and functions such that  $U_j \subset \Sigma_{-M}$  (we use negative subscripts for these functions), and finally, the sets and functions such that  $U_j \subset \Sigma_M$  (we use positive subscripts for these functions).

Set  $\Psi = \Psi^* - \Psi_0$ . The first useful observation is

LEMMA 3.2. — *For any  $c > 0$ , there exist  $C, p > 0$  such that for all  $x \in \Sigma_{-cM}$ ,*

$$(3.19) \quad \Psi(x) \leq CM^{-p}\Psi_0(x).$$

*Proof.* — In the case  $V(x) = 0, B(x) = x_1^2 - x_2$ , we have

$$\begin{aligned} \Psi^*(x) &= |x_1^2 - x_2|^{1/2} + |2x_1|^{1/3} + 1, \\ \Psi_0(x) &= |x_1^2 - x_2|^{1/2}, \\ \Psi(x) &= |2x_1|^{1/3} + 1, \end{aligned}$$

and for  $x \in \Sigma$ ,

$$(3.20) \quad \Psi^*(x) = |2x_1|^{1/3} + 1 = |2^2x_2|^{1/6} + 1.$$

Let  $x \in \Sigma_{-M}$ . If  $|x_1| \leq 2x_2^{1/2}$ , then

$$\begin{aligned} \sqrt{|x_1^2 - x_2|} &\geq \sqrt{|x_1| - x_2^{1/2}} \sqrt{|x_1| + x_2^{1/2}} \geq M^{1/2}x_2^{-1/4+1/12} \\ &= M^{1/2}x^{1/2} \geq M^{1/2}(|x_1|^{1/3} + 1)/2, \end{aligned}$$

and if  $|x_1| \geq 2x_2^{1/2}$ , then  $|x_1^2 - x_2|^{1/2} \geq |x_1| \geq M_0^{2/3}(|x_1|^{1/3} + 1)/2$ . □

Notice that if the degeneration is complicated, the verification of (3.19) may be more involved.

The estimate (3.19) being established, the proof of the next crucial lemma is the same in any dimension, and for any (polynomial)  $V$  and  $B$ .

LEMMA 3.3. — For any  $c, C > 0$ , there exist  $C_1, c_1 > 0$  such that if  $x \in \Sigma_{-cM}$ , and  $|x - y| \leq CM_1\Psi_0(x)^{-1}$ , then  $y \in \Sigma_{-c_1M}$ ,

$$(3.21) \quad \Psi_0(x)(1 - C_1M^{-p}) \leq \Psi_0(y) \leq \Psi_0(x)(1 + C_1M^{-p}),$$

and each of the functions  $b_{jk}, V$  satisfies the estimate

$$(3.22) \quad |W(x) - W(y)| \leq C_1M^{-p}\Psi_0(x)^2.$$

*Proof.* — The Taylor formula gives

$$|W(x) - W(y)| \leq C_3 \sum_{|\alpha|>0} |W^{(\alpha)}(x)|(M_1\Psi_0(x)^{-1})^{|\alpha|},$$

and by definition of  $\Psi$ , for  $|\alpha| > 0$ ,

$$|W^{(\alpha)}(x)| \leq \Psi(x)^{|\alpha|+2}.$$

Applying (3.19), and taking into account that  $M_1 = \log M$ , we obtain (3.22). The estimate (3.21) and inclusion  $y \in \Sigma_{-c_1M}$  follow from (3.22).  $\square$

The estimate (3.21) implies that  $g_x(z) = M_1^{-2}\Psi_0(x)^2|z|^2$  is a slowly varying metric on  $\Sigma_{-cM}$  in the sense of Section 18.4 of [12], uniformly in  $\lambda \geq \lambda_0$ : if  $|x - y| \leq M_1\Psi_0(x)^{-1}$ , then  $g_x(\cdot)/C_1 \leq g_y(\cdot) \leq C_1g_x(\cdot)$ . The proof of Lemma 18.4.4 in [12] provides constants  $0 < c_1 < c_2 \leq 1$ , and for each  $\lambda$ , points  $x^j \in \Sigma_{-M}$ ,  $j = -1, -2, \dots$ , such that the balls  $B_{j,c_2} := B(x^j, c_2M_1\Psi_0(x^j)^{-1})$  cover  $\Sigma_{-M}$ , and the balls  $B_{j,c_1}$  do not intersect (pairwise). Set  $U_{-1} = B_{-1,c_2}$ , and for  $j \leq -2$ , set  $U_j = B_{j,c_2} \setminus \cup_{j < k < 0} \bar{B}_{k,c_2}$ . Then intersect  $U_j$  with  $\{x \mid |x| > M_0\}$  (which is the interior of the complement to  $U_0$ ), and keep the same notation,  $U_j$ , for the intersection. By construction, for non-positive  $j \neq k, j, k \leq 0$ , (3.6) holds.

For  $j \leq -1$ , and a subset  $V$  of  $U_j$ , denote by  $V_c$  the 1-neighborhood of  $V$  in the metric  $c^2\Psi_0(x^j)^{-2}|\cdot|^2$ , and introduce  $U_{1j} = (U_j)_{M_1^{1/2}}$ . From the properties of  $B_{j,c_k}, k = 1, 2$ , and (3.21), we derive that there exist  $C, c > 0$  such that for all  $\lambda \geq \lambda_0$  and  $j \leq -1$ ,

$$(3.23) \quad \Sigma_{-M} \subset \cup_{j \leq -1} \bar{U}_j \subset \cup_{j \leq -1} \bar{U}_{1j} \subset \Sigma_{-cM};$$

$$(3.24) \quad \text{meas}(U_{1j} \setminus U_j)_{M_1^{1/2}} \leq CM_1^{n-1/2}\Psi_0(x^j)^{-n};$$

$$(3.25) \quad \forall x \in \mathbb{R}^n, \quad \text{card}\{j \mid x \in U_{1j}\} \leq C;$$

$$(3.26) \quad B(x^j, cM_1\Psi_0(x^j)^{-1}) \subset U_j \subset U_{1j} \subset B(x^j, CM_1\Psi_0(x^j)^{-1}).$$

Fix  $\chi \in C_0^\infty(B(0, 2))$  such that  $0 \leq \chi \leq 1$ ,  $\chi(x) = 1$  for  $|x| \leq 1$ , and for  $\alpha \in \mathbb{Z}^n$  and  $j \leq -1$ , define

$$\begin{aligned} \chi_{0j,\alpha}(x) &= \chi(2M_1^{-1/2}\Psi_0(x^j)x + \alpha), \\ \chi_{j,\alpha} &= \frac{\chi_{0j,\alpha}}{\sum_{\beta \in \mathbb{Z}^n} \chi_{0j,\beta}}, \end{aligned}$$

and

$$(3.27) \quad \chi_j = \sum_{\alpha: \text{supp } \chi_{j,\alpha} \cap U_j \neq \emptyset} \chi_{j,\alpha}.$$

It is immediate from (3.23)–(3.26) that  $U_j, U_{1j}$  and  $\chi_j$  satisfy (3.6)–(3.8), (3.16), and (3.17). In addition,

$$(3.28) \quad |\partial^\alpha \chi_j| \leq C_\alpha (M_1^{1/2} \Psi_0(x^j))^{-|\alpha|}, \quad \forall \alpha \in \mathbb{Z}_+^n,$$

where the constants  $C, C_\alpha$  are independent of  $\lambda$  and  $j$ .

Now we construct  $\chi_j, j \geq 1$ . In the case under consideration,  $\Sigma_M$  is a disjoint union of two sets; we denote the one contained in the half-plane  $x_1 > 0$  by  $\Sigma_M^+$ , and the other one by  $\Sigma_M^-$ . The functions related to the former will have odd subscripts, and the ones related to the latter—even subscripts. Clearly, the construction is the same for both parts; we consider  $\Sigma_M^+$ . Change the variable  $z = x_1 - \sqrt{x_2}$ ; then  $\Sigma^+$  becomes a coordinate half-axis. Construct points  $x^j \in \Sigma^+ \setminus U_0$  and open intervals  $U_{0j} \subset \Sigma^+ \setminus U_0, j = 1, 3, \dots$ , such that

$$(3.29) \quad U_{0j} \cap U_{0k} = \emptyset, \quad \forall j \neq k;$$

$$(3.30) \quad \cup_j \overline{U_{0j}} \supset \Sigma^+ \setminus U_0;$$

$$(3.31) \quad B(x^j, cM_1\Psi^*(x^j)^{-1}) \subset U_{0j} \subset B(x^j, CM_1\Psi^*(x^j)^{-1}).$$

Here  $B(a, r)$  denotes a ball in the manifold of degeneration  $\Sigma^+ \subset \{z = 0, x_2 > 0\}$ . For  $j \geq 1$ , and a subset  $V$  of  $\Sigma^+ \setminus U_0$ , denote by  $V_c$  the 1-neighborhood of  $V$  in the metric  $c^2\Psi^*(x^j)^{-2}|\cdot|^2$ . Evidently,

$$(3.32) \quad \forall x_2 \in \Sigma^+ \setminus U_0, \quad \text{card}\{j \mid x_2 \in (U_{0j})_{M_1^{1/2}}\} \leq C.$$

Further, construct functions  $\chi_{0j}, j = 1, 3, \dots$ , which satisfy (3.16) and the following conditions

$$(3.33) \quad \chi_{0j}(x_2) = 1, \quad x_2 \in U_{0j};$$

$$(3.34) \quad \text{supp } \chi_{0j} \subset (U_{0j})_{M_1^{1/2}};$$

$$(3.35) \quad \forall x_2, \quad \text{card } \{j \mid x_2 \in (U_{0j})_{M_1^{1/2}}\} \leq C;$$

$$(3.36) \quad |\chi_{0j}^{(s)}| \leq C_\alpha (M_1^{1/2} \Psi^*(x^j))^{-s}, \quad \forall s \in \mathbb{Z}_+,$$

where the constants  $C, C_\alpha$  are independent of  $\lambda$  and  $j$ . In our 2D-example,  $\dim \Sigma = 1$ , and the construction of open sets  $U_{0j}$  and functions  $\chi_{0j}$  is straightforward; if  $\dim \Sigma > 1$ , one checks that the metric  $M_1^{-2} \Psi^*(x)^2 |\cdot|^2$  is slowly varying on  $\Sigma \setminus U_0$ , uniformly in  $\lambda \geq \lambda_0$ , and argues as in the construction of  $x^j, U_j, j \leq -1$ , above.

Set  $U_j = \{(z, x_2) \mid x_2 \in U_{0j}, |z| \leq M/\Psi^*(x^j)\}$ ,  $U_{1j} = \{(z, x_2) \mid x_2 \in (U_{0j})_{M_1^{1/2}}, |z| \leq 2M/\Psi^*(x^j)\}$ . Further, take  $\mu \in C^\infty(\mathbb{R})$ ,  $\mu(z) = 1, z \leq 1$ ,  $\mu(z) = 0, z \geq 1$ , and for  $j = 1, 3, \dots$ , set  $\chi_j(z, x_2) = \chi_{0j}(x_2) \mu(M_1^{-1/2} \Psi^*(x^j) (|z| - M))$ . Finally, repeat these constructions with  $\Sigma^-$  and  $j = 2, 4, \dots$ . Clearly, for  $j \geq 1$ , (3.6)–(3.8), (3.16) and (3.17) hold, and an analog of (3.28) is

$$(3.37) \quad |\partial^\alpha \chi_j| \leq C_\alpha (M_1^{-1/2} \Psi^*(x^j))^{|\alpha|}, \quad \forall \alpha \in \mathbb{Z}_+^n.$$

Now it is clear that properties (3.5)–(3.8), (3.16) and (3.17) hold for all  $j \in \mathbb{Z}$ , and that uniformly in  $j \in \mathbb{Z}$  and  $\lambda$ , (3.9) holds with  $\epsilon = C_1 M_1^{-1/2}$ . Applying (3.18), we finish the construction of open sets  $U_j \subset U_{1j}$  and functions  $\psi_j$  with properties (3.5)–(3.10); hence, we have bounds (3.11) and (3.14). It is easy to show that there exist  $c, C_1 > 0$  such that

$$(3.38) \quad \langle (H(a) + V - C\epsilon(\Psi^*))u, u \rangle \geq c \langle (-\Delta - C_1 M_0^p)u, u \rangle, \quad u \in C_0^\infty(U_0),$$

therefore from the classical estimate for the counting function of the Dirichlet Laplacian, we obtain a bound

$$(3.39) \quad N_D(\lambda; H(a) + V - C\epsilon(\Psi^*)^2; U_{1j}) \leq C_2 M^q \lambda^2,$$

where  $C_2, q$  are independent of  $\lambda$ . Since  $M = \log \lambda$ , (3.39) and (3.3) imply that the  $j = 0$  terms in (3.11) and (3.14) do not contribute to the leading term of the asymptotics.

### 3.4. Estimates for individual terms in (3.11) and (3.14), for $j \leq -1$ : the case of general $V$ and $B$

The proof is valid for the general case provided Lemma 3.2 and Lemma 3.3 have been proved already. First, we make a gauge transformation used in [10, 26]. Set

$$(3.40) \quad a_{x^j}(x) = \sum_\alpha \frac{(x - x^j)^\alpha}{\alpha!(|\alpha| + 2)} (\partial^\alpha B)(x^j) \cdot (x - x^j),$$

and write  $V$  in the form

$$(3.41) \quad V(x) = \sum_{\alpha} \frac{(x - x^j)^{\alpha}}{\alpha!} (\partial^{\alpha} V)(x^j).$$

One easily checks that  $da_{x^j} = B$ , therefore there exists a gauge transformation  $u \mapsto \exp(-i\phi_j)u$  such that

$$(3.42) \quad \langle H(a)u, u \rangle = \langle H(a_{x^j}) \exp(-i\phi_j)u, \exp(-i\phi_j)u \rangle,$$

and so we may study the  $j \leq -1$ -terms in (3.11) and (3.14) with  $a_{x^j}$  instead of  $a$ . On  $U_{1j} (\supset U_j)$ , the absolute value of each term in (3.40) with  $|\alpha| \geq 1$  admits an upper bound via

$$C(M_1^{1/2} \Psi_0(x^j)^{-1})^{|\alpha|+1} \Psi(x^j)^{|\alpha|+2};$$

using (3.19), we find an upper bound via  $M_1^{-1} \Psi^*(x^j)$ . Similarly, on the same set, for each of the terms in (3.41) with  $|\alpha| \geq 1$ , we obtain an upper bound via  $M_1^{-2} \Psi^*(x^j)^2$ .

Introduce  $a_{x^j}^0(x) = \frac{1}{2} B(x^j) \cdot (x - x^j)$ ,  $V_{x^j}^0(x) = V(x^j)$ ,  $H_{x^j} = H(a_{x^j}^0) + V_{x^j}^0$ , and label by  $\epsilon, \epsilon_1, \dots$ , any function of  $\lambda$ , which tends to 0 as  $\lambda \rightarrow +\infty$ . Using the estimates for  $|\alpha| \geq 1$  terms, which we just obtained, and (3.15) and (3.21), we conclude that on  $C_0^{\infty}(U_{1j})$ ,

$$(1 - \epsilon_1)H_{x^j} \leq H(a) + V - C\epsilon \Psi^*(x)^2 \leq H(a) + V \leq H_{x^j}(1 + \epsilon_1).$$

It follows that

$$(3.43) \quad N_D(\lambda(1 - \epsilon_2); H_{x^j}, U_j) \leq N_D(\lambda; H(a) + V, U_j),$$

and

$$(3.44) \quad N_D(\lambda; H(a) + V - C\epsilon \Psi^*(x)^2, U_{1j}) \leq N_D(\lambda(1 + \epsilon_2); H_{x^j}, U_{1j}).$$

For  $j \leq -1$ ,  $H_{x^j}$  is a Schrödinger operator with uniform magnetic potential and constant electric potential. By using coverings of  $U_j$  and  $U_{1j}$  by cubes of size  $M^{1/2} \Psi_0(x^j)^{-1}$ , repeating the proof of Theorem 3.1 in [5], and using (3.24) and (3.26), we obtain the estimates

$$(3.45) \quad (1 - \epsilon_3) \text{meas } U_j \cdot \nu_{B(x^j)}(\lambda(1 - \epsilon_3) - V(x^j)) \leq N_D(\lambda(1 - \epsilon_2), H_{x^j}, U_j),$$

and

$$(3.46) \quad N_D(\lambda(1 + \epsilon_2); H_{x^j}, U_{1j}) \leq (1 + \epsilon_3) \text{meas } U_j \nu_{B(x^j)}(\lambda(1 + \epsilon_3) - V(x^j)).$$

It follows from (3.15) and (3.21) that there exists  $\epsilon_4$  such that for all  $x \in U_{1j}$ ,

$$\nu_{B(x)}(\lambda(1 - \epsilon_4) - V(x)) \leq \nu_{B(x^j)}(\lambda(1 \pm \epsilon_2) - V(x^j)) \leq \nu_{B(x)}(\lambda(1 + \epsilon_4) - V(x)),$$

therefore by gathering (3.43)-(3.46), we obtain the estimates

$$(3.47) \quad (1-\epsilon_3) \int_{\Sigma_{-M}} \nu_{B(x)}(\lambda(1-\epsilon_4)-V(x))dx \leq \sum_{j \leq -1} N_D(\lambda, H(a)+V, U_j),$$

and

$$(3.48) \quad \sum_{j \leq -1} N_D(\lambda, H(a) + V, U_{1j}) \leq (1 + \epsilon_3) \int_{\Sigma_{-M}} \nu_{B(x)}(\lambda(1 + \epsilon_4) - V(x))dx.$$

**3.5. Estimate for the contribution of  $j \leq -1$ -terms: the case**

$$V = 0, B(x) = x_1^2 - x_2$$

Since the degeneration is strong (this can be guessed or deduced from the general conjectural formula as in §3.1), the  $j \leq -1$  terms should not contribute to the leading term of the asymptotics, and therefore it suffices to obtain a crude upper bound. In view of (3.3), it suffices to show that the RHS in (3.48) is  $o(\lambda^{7/2})$ . We have

$$\nu_{B(x)}(\lambda) = (2\pi)^{-1} |x_1^2 - x_2| \cdot \text{card}\{k \mid (2k + 1)|x_1^2 - x_2| < \lambda\},$$

therefore

$$(3.49) \quad \sum_{j \leq -1} N_D(\lambda, H(a), U_{1j}) \leq C\lambda \text{meas } U(\lambda),$$

where  $U(\lambda) = \{x \in \Sigma_{-M} \mid |x_1^2 - x_2| \leq \lambda\}$ . Thus, we need to show that  $\text{meas } U(\lambda) = o(\lambda^{5/2})$ . Clearly, the measure of the part of  $U(\lambda)$  below the line  $x_2 = 1$  admits a bound via  $\text{meas}\{x \mid x_1^2 + |x_2| < \lambda\} \leq C\lambda^{3/2}$ , therefore it suffices to obtain an upper bound for the measure of the set  $V(\lambda)$  defined by the following inequalities:  $x_1 > 0, x_2 > 1, |x_1 - \sqrt{x_2}| \geq Mx_2^{-1/6}, |x_2 - x_1^2| \leq \lambda$ . The last two inequalities taken together imply that on  $V(\lambda)$ ,  $x_1 + \sqrt{x_2} \leq \lambda M^{-1}x_2^{1/6}$ , and therefore,  $x_2 \leq (\lambda/M)^3$ .

The set  $V(\lambda)$  is the union of  $V_1(\lambda)$  defined by  $x_2 > 1, 0 < x_1 < \sqrt{x_2} - Mx_2^{-1/6}, x_2 - x_1^2 \leq \lambda$ , and  $V_2(\lambda)$  defined by  $x_2 > 1, x_1 > \sqrt{x_2} + Mx_2^{-1/6}, x_1^2 - x_2 \leq \lambda$ . On  $V_2(\lambda)$ ,  $\sqrt{x_2} + Mx_2^{-1/6} \leq x_1 \leq \sqrt{\lambda + x_2}$ , therefore

$$\begin{aligned} \text{meas } V_2(\lambda) &\leq \int_1^{(\lambda/M)^3} dx_2 \cdot (\sqrt{\lambda + x_2} - \sqrt{x_2}) \\ &\leq \lambda^{3/2} + \lambda \int_\lambda^{(\lambda/M)^3} dx_2 \cdot x_2^{-1/2} \leq CM^{-3/2}\lambda^{5/2}, \end{aligned}$$

which is  $o(\lambda^{5/2})$ . On  $V_1(\lambda)$ ,  $\sqrt{x_2 - \lambda} < x_1 < \sqrt{x_2} - Mx_2^{-1/6}$ , and essentially the same calculations give  $\text{meas } V_1(\lambda) = o(\lambda^{5/2})$  as well.

**3.6. Estimates for individual terms in (3.11) and (3.14), for  $j \geq 1$**

Now Lemma 3.20 is no longer available, and we cannot freeze  $V$  and  $B$  at  $x = x^j$ . However, we can freeze the coordinates along  $\Sigma$ . In the case under consideration,  $V = 0$ , and we may assume that  $a_1(x) = 0, a_2(x) = x_1^3/3 - x_1x_2$ . By applying the Taylor formula at  $x^j = (y_1, y_2) \in \Sigma_M^+$ , we obtain

$$x_1^3/3 - x_1x_2 = a_{21}(x) + a_{22}(x) + a_{x^j}(x),$$

where  $a_{21}(x) := y_1^3/3 - y_1y_2 - y_1(x_2 - y_2)$  can be gauged away,  $a_{22}(x) := -(x_1 - y_1)(x_2 - y_2)/2$  is small on  $U_{1j}$ :

$$|a_{22}(x)| \leq CM(1 + y_2^{1/6})^{-1}M_1(1 + y_2^{1/6})^{-1} = O(M_1^{-\infty}),$$

and  $a_{x^j}(x) := y_1(x_1 - y_1)^2 = y_2^{1/2}(x_1 - y_2^{1/2})^2$ . Denote  $D_j = -i\partial_j$ , and introduce

$$H_{x^j} = H(a_{x^j}) = D_1^2 + (D_2 + y_2^{1/2}(x_1 - y_2^{1/2})^2)^2 = D_z^2 + (D_{x_2} + y_2^{1/2}z^2)^2.$$

We have (3.43) and (3.44) with  $V = 0$ . In  $(z, x_2)$  coordinates,

$$U_j = (-M\Psi^*(x^j)^{-1}, M\Psi^*(x^j)^{-1}) \times U_{0j},$$

and

$$U_{1j} = (-2M\Psi^*(x^j)^{-1}, 2M\Psi^*(x^j)^{-1}) \times (U_{0j})_{M_1^{1/2}},$$

therefore,

$$(3.50) \quad N_D(\lambda(1 + \epsilon_2); H_{x^j}, U_{1j}) \leq N_D(\lambda(1 + \epsilon_2); H_{x^j}, \mathbb{R} \times (U_{0j})_{M_1^{1/2}}).$$

The operator  $H_{x^j}$  on the RHS can be realized as the operator on  $(U_{0j})_{M_1^{1/2}}$ , with the constant operator-valued symbol, and the same arguments as in the case of Schrödinger operators without magnetic potential used in [17, 19], and for 2D-Schrödinger operators with homogeneous potentials in [20], show that the RHS admits an upper bound

$$(3.51) \quad \begin{aligned} & N_D(\lambda(1 + \epsilon_2); H_{x^j}, \mathbb{R} \times (U_{0j})_{M_1^{1/2}}) \\ & \leq (2\pi)^{-1}(1 + \epsilon) \int_{(U_{0j})_{M_1^{1/2}}} dx_2 \int_{\mathbb{R}} d\xi_2 N(\lambda(1 + \epsilon), D_z^2 + (\eta_2 + y_2^{1/2}z^2)^2). \end{aligned}$$

Since  $\text{meas}((U_{0j})_{M_1^{1/2}} \setminus U_{0j}) = o(\text{meas } U_{0j})$ , we may replace the integration over  $(U_{0j})_{M_1^{1/2}}$  with the one over  $U_{0j}$  (and change the  $\epsilon$  in front of the integral). Further, on  $U_{0j}$ ,  $|x_2 - y_2| \leq M\Psi^*(x^j)^{-1}$  is small, therefore we may replace  $y_2$  with  $x_2$  (and change the  $\epsilon$  in the integrand). After that we

sum up all the terms  $j \geq 1$ , and taking into account the symmetry of  $B(x)$  and  $\Sigma$ , derive from (3.44), (3.50), and (3.51)

$$(3.52) \quad \sum_{j \geq 1} N_D(\lambda; H(a) + V - C\epsilon\Psi^*(x)^2, U_{1j}) \leq \frac{1 + \epsilon}{\pi} \int_{M_0}^\infty dx_2 \int_{\mathbb{R}} d\xi_2 N(\lambda(1 + \epsilon), D_z^2 + (\eta_2 + x_2^{1/2}z^2)^2).$$

We replace  $M_0$  with 0 (the inequality remains valid, of course), and change the variables  $z \mapsto (\lambda(1 + \epsilon))^{-1/2}z$ ,  $\xi_2 \mapsto (\lambda(1 + \epsilon))^{1/2}\xi_2$ ,  $x_2 \mapsto (\lambda(1 + \epsilon))^3x_2$ ; the RHS becomes  $\kappa(H)\lambda^{7/2}(1 + \epsilon)^{9/2}$ , where  $\kappa(H)$  is given by (3.4). Since the sum of the terms with the labels  $j \leq 0$  grows slower than  $\lambda^{7/2}$ , we conclude that

$$(3.53) \quad N(\lambda, H) \leq (1 + \epsilon)\kappa(H)\lambda^{7/2}$$

(with a new  $\epsilon$ ). To finish the proof of (3.3), we need to obtain a lower bound of the form

$$(3.54) \quad N(\lambda, H) \geq (1 - \epsilon)\kappa(H)\lambda^{7/2}.$$

Construct a partition of unity  $\phi_1^2 + \phi_2^2 = 1$  on  $\mathbb{R}$  with properties  $0 \leq \phi_j \leq 1$ ,  $\phi_1(z) = 1$  for  $|z| \leq M/(2\Psi^*(x^j))$ ,  $\phi_2(z) = 1$  for  $|z| \geq M/(\Psi^*(x^j))$ , and  $|\phi'_j(z)| \leq CM^{-1}/\Psi^*(x^j)$ . If  $\epsilon_2$  decays slower than  $\epsilon_1$  and any negative power of  $M_1$ , then we can use this partition of unity to obtain an upper bound

$$(3.55) \quad N_D(\lambda(1 - \epsilon_3); H_{x^j}, \mathbb{R} \times U_{0j}) \leq N_D(\lambda(1 - \epsilon_2); H_{x^j}, U_j) + N_D(\lambda(1 - \epsilon_2); H_{x^j}, \{|z| \geq M/(2\Psi^*(x^j))\} \times U_{0j}).$$

Similarly to (3.52), we derive

$$(3.56) \quad \sum_{j \geq 1} N_D(\lambda(1 - \epsilon_3); H_{x^j}, \mathbb{R} \times U_{0j}) \geq \frac{1 - \epsilon_4}{\pi} \int_{M_0}^\infty dx_2 \int_{\mathbb{R}} d\xi_2 \cdot N(\lambda(1 - \epsilon_4), D_z^2 + (\xi_2^2 + x_2^{1/2}z^2)^2).$$

At the end of the section, we show that

$$(3.57) \quad \sum_{j \geq 1} N_D(\lambda(1 - \epsilon_3); H_{x^j}, \{|z| \geq M/(2\Psi^*(x^j))\} \times U_{0j}) = o(\lambda^{7/2}),$$

and the argument used to derive (3.3) from (3.2) and prove the convergence of the integral (3.4) gives

$$(3.58) \quad \int_0^{M_0} dx_2 \int_{\mathbb{R}} d\xi_2 \cdot N(\lambda; D_z^2 + (\xi_2 + x_2^{1/2}z^2)^2) = o(\lambda^{7/2}).$$



Estimates (3.55)–(3.58) taken together give an estimate from below

(3.59)

$$\begin{aligned} & \sum_{j \geq 1} N_D(\lambda(1 - \epsilon_2); H_{x^j}, U_j) \\ & \geq \frac{1 - \epsilon_4}{\pi} \int_0^\infty dx_2 \int_{\mathbb{R}} d\xi_2 \cdot N(\lambda(1 - \epsilon_4), D_z^2 + (\xi^2 + x_2^{1/2} z^2)^2) + o(\lambda^{7/2}). \end{aligned}$$

Changing the variables  $z \mapsto (\lambda(1 - \epsilon_4))^{-1/2} z$ ,  $\xi_2 \mapsto (\lambda(1 - \epsilon_4))^{1/2} \xi_2$ ,  $x_2 \mapsto (\lambda(1 - \epsilon_4))^3 x_2$ , we obtain (3.54).

The proof of (3.3) is complete.

### 3.7. Proof of (3.57)

$H_{x^j}$  is a Schrödinger operator with magnetic tensor  $2y_2^{1/2} z$  (and  $V = 0$ ), where  $y_2$  is a constant (the second coordinate of the fixed point  $x^j$ ). For this operator,  $\Psi_0(z, x_2) = |2y_2^{1/2} z|^{1/2}$ , and  $\Psi(z, x_2) = |2y_2^{1/2} z|^{1/3} + 1$ . On the set  $\{|z| \geq M/(2\Psi^*(x^j))\} \times U_{0j}$ , we have  $|z| \geq cMy_2^{-1/6}$ ; therefore

$$\Psi_0(z, x_2)/\Psi(z, x_2) \geq c_2 y_2^{1/12} |z|^{1/2} \geq c_3 M^{1/2},$$

whence we can repeat the argument used in §3.3 and §3.4 for the localization of  $H(a) + V$  on  $\Sigma_{-M}$  and derive an upper bound of the form

$$\begin{aligned} & N_D(\lambda(1 - \epsilon_3); H_{x^j}, \{|z| \geq M/(2\Psi^*(x^j))\} \times U_{0j}) \\ & \leq C \text{meas}(U_{1j}) \cdot \int \int_{|z| \geq cMy_2^{-1/6}} \nu_{2y_2^{1/2} z}(\lambda) dz dy_2. \end{aligned}$$

Clearly,  $\nu_{2y_2^{1/2} z}(\lambda) = 2y_2^{1/2} |z| \cdot \#\{k \mid (2k + 1)2y_2^{1/2} |z| < \lambda\}$  is bounded, and vanishes unless  $c_2 My_2^{-1/6} / 2 \leq |z| \leq \lambda y_2^{-1/2}$ . Summing w.r.t.  $j \geq 1$ , we obtain that the LHS in (3.57) admits an upper bound via the measure of the set  $\{(z, y_2) \mid y_2 > 1, c_2 My_2^{-1/6} \leq |z| \leq \lambda y_2^{-1/2}\}$ . It is easy to see that this measure is  $O((\lambda/M)^3) = o(\lambda^{7/2})$ .

## 4. Classical formulass

### 4.1. The classical Weyl formula

In this subsection we consider a Schrödinger operator  $H = H(0) + V$  with zero magnetic potential and quasi-homogeneous weakly degenerate electric potential  $V(x)$ . We aim at the following result.

PROPOSITION 4.1. — Assume that  $V(x) \geq 0$  is a quasi-homogeneous polynomial, and that the integral on the RHS of the classical Weyl formula (1.2) converges. Then the formula for the leading term of the asymptotics of  $N(\lambda, H)$  provided by our conjecture coincides with the classical Weyl formula.

We begin the proof of Proposition 4.1 with some general estimates that will also be useful for us later. Observe that Fubini's theorem easily implies that

$$\text{meas}\{(\xi, x) \in \mathbb{R}^{2n} \mid \|\xi\|^2 + V(x) \leq \lambda\} < \infty \quad \text{for all } \lambda > 0$$

if and only if

$$\text{meas}\{x \in \mathbb{R}^n \mid V(x) \leq \lambda\} < \infty \quad \text{for all } \lambda > 0.$$

Now suppose  $V(x)$  is quasi-homogeneous of weight  $\gamma = (\gamma_1, \dots, \gamma_n)$ ,  $\gamma_j \in \mathbb{Q}$ ,  $\gamma_j > 0$ . Introduce the “quasi-dilation”  $\delta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,

$$\delta_\lambda : x = (x_1, \dots, x_n) \mapsto (\lambda^{\gamma_1} x_1, \dots, \lambda^{\gamma_n} x_n).$$

Thus  $V(\delta_\lambda x) = \lambda \cdot V(x)$  for all  $\lambda > 0$ ,  $x \in \mathbb{R}^n$ .

LEMMA 4.2. — If  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_j \in \mathbb{Z}$ ,  $\alpha_j \geq 0$ , then

$$(\partial^\alpha V)(\delta_\lambda x) = \begin{cases} \lambda^{1-\langle \alpha, \gamma \rangle} \cdot \partial^\alpha V(x) & \text{if } \langle \alpha, \gamma \rangle \leq 1, \\ 0 & \text{if } \langle \alpha, \gamma \rangle > 1, \end{cases}$$

where  $\langle \alpha, \gamma \rangle = \alpha_1 \gamma_1 + \dots + \alpha_n \gamma_n$ .

*Proof.* — This follows by induction on  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , observing that if  $P(x)$  is a polynomial such that  $P(\delta_\lambda x) = \lambda^\beta \cdot P(x)$  for some  $\beta > 0$ , then

$$\frac{\partial P}{\partial x_j}(\delta_\lambda x) = \lambda^{-\gamma_j} \cdot \frac{\partial}{\partial x_j} (P(\delta_\lambda x)) = \lambda^{\beta-\gamma_j} \cdot \frac{\partial P}{\partial x_j}(x).$$

□

LEMMA 4.3. — Let  $P(x), Q(x) \geq 0$  be measurable functions on  $\mathbb{R}^n$ , finite almost everywhere, such that  $\text{meas}\{P(x) \leq \lambda\}$  and  $\text{meas}\{Q(x) \leq \lambda\}$  are finite for each  $\lambda > 0$ . Fix  $\gamma = (\gamma_1, \dots, \gamma_n)$ ,  $\gamma_j \in \mathbb{R}$ ,  $\gamma_j > 0$ , and define the quasi-dilation  $\delta_\lambda$  as before. Assume that there exist constants  $0 < \epsilon < 1$  and  $C > 0$  such that  $P(\delta_\lambda x) = \lambda \cdot P(x)$  and  $Q(\delta_\lambda x) \leq C\lambda^\epsilon \cdot Q(x)$  for all  $x \in \mathbb{R}^n$ . Then

$$\text{meas}\{P(x) + Q(x) \leq \lambda\} \sim \text{meas}\{P(x) \leq \lambda\} = \lambda^{|\gamma|} \cdot \text{meas}\{P(x) \leq 1\}$$

as  $\lambda \rightarrow +\infty$ , where  $|\gamma| = \gamma_1 + \dots + \gamma_n$ .

The proof is given in Appendix B.

COROLLARY 4.4. — Let  $V \geq 0$  satisfy the assumptions of Proposition 4.1. Introduce the functions  $\Phi^*(x), \Psi^*(x)$  as in Section 2:

$$\Phi^*(x) = \sum_{\alpha \geq 0} |\partial^\alpha V(x)|^{1/2}, \quad \Psi^*(x) = \sum_{\alpha \geq 0} |\partial^\alpha V(x)|^{1/(2+|\alpha|)}.$$

Then

$$\lim_{\lambda \rightarrow +\infty} \frac{\text{meas}\{\Phi^*(x) \leq \lambda\}}{\text{meas}\{V(x)^{1/2} \leq \lambda\}} = 1 = \lim_{\lambda \rightarrow +\infty} \frac{\text{meas}\{\Psi^*(x) \leq \lambda\}}{\text{meas}\{V(x)^{1/2} \leq \lambda\}}.$$

Proof. — Define

$$\Phi(x) = \sum_{\alpha > 0} |\partial^\alpha V(x)|^{1/2}, \quad \Psi(x) = \sum_{\alpha > 0} |\partial^\alpha V(x)|^{1/(2+|\alpha|)}.$$

If  $\epsilon = \max_{1 \leq j \leq n} \{1 - \gamma_j\} < 1$ , then by Lemma 4.2, we have

$$\Phi(\delta_{\lambda^2} x) \leq \lambda^\epsilon \cdot \Phi(x) \quad \text{and} \quad \Psi(\delta_{\lambda^2} x) \leq \lambda^\epsilon \cdot \Psi(x)$$

for all  $\lambda > 1$ , and since  $V(\delta_{\lambda^2} x)^{1/2} = \lambda \cdot V(x)^{1/2}$ , Lemma 4.3 applies.  $\square$

We see that we are in case (b) of Conjecture 2. Moreover, the normalizing factor  $G_2(\lambda)/G_1(\lambda)$  is asymptotically equal to 1. It remains to compute the quotient measure  $\mu_\infty$ . The first step here is very similar to the first step in the computation presented in §3.1.

Let  $\mathfrak{g}$  be the Lie algebra associated to the operator  $H = -\Delta + V$ . Choose a basis  $P_1(x), \dots, P_K(x)$  of the vector space spanned by all mixed partial derivatives (of all orders) of  $V(x)$ , *not including*  $V(x)$  itself. Then we obtain a basis  $\{L_1, \dots, L_n, v, p_1, \dots, p_K\}$  of  $\mathfrak{g}$  such that the tautological representation of  $\mathfrak{g}$  maps  $L_j \mapsto \partial/\partial x_j, v \mapsto \sqrt{-1} \cdot V(x), p_k \mapsto \sqrt{-1} \cdot P_k(x)$ . As in §3.1, we use this basis to identify  $\mathfrak{g}^*$  with  $\mathbb{R}^{n+K+1}$ . Write  $\mathfrak{h} = \text{span}_{\mathbb{R}}\{v, p_1, \dots, p_K\} \subset \mathfrak{g}$ , and define a linear functional  $f_0 : \mathfrak{g} \rightarrow \mathbb{R}$  by  $f_0(L_j) = 0$  ( $1 \leq j \leq n$ ),  $f_0(v) = V(0) = 0, f_0(p_k) = P_k(0)$  ( $1 \leq k \leq K$ ). It follows from Proposition 2.7 that the induced representation  $\rho_{f_0, \mathfrak{h}}$  coincides with the tautological representation of  $\mathfrak{g}$  (at least up to changing the action of  $L_1$  and  $L_2$ , so that  $H$  is replaced by a gauge-equivalent operator). In particular,  $\rho_{f_0, \mathfrak{h}}$  is irreducible (by Theorem 2.1), so  $\mathfrak{h}$  is a polarization of  $\mathfrak{g}$  at  $f_0$ . Let  $\Omega = G \cdot f_0$  be the coadjoint orbit of  $f_0$  in  $\mathfrak{g}^*$ , and let  $\mu_\Omega$  be the corresponding Kostant measure. Again, to simplify notation, we will implicitly identify  $\mu_\Omega$  with its extension by zero to all of  $\mathfrak{g}^*$ .

By Proposition 2.11, the orbit  $\Omega$  is parameterized by the map  $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n+K+1} \cong \mathfrak{g}^*, (\xi, x) \mapsto (\xi_1, \dots, \xi_n, V(x), P_1(x), \dots, P_K(x))$ , and, moreover, we have  $\mu_\Omega = (2\pi)^{-n} \cdot \varphi_*(d\xi dx)$ . To find  $\mu_\infty$ , we fix  $F \in C_c(\mathfrak{g}^*)$  and

consider the integral

$$\int_{\mathbb{R}^{2n}} F(\lambda^{-1}\xi_1, \dots, \lambda^{-1}\xi_n, \lambda^{-1}V(x), \lambda^{-1}P_1(x), \dots, \lambda^{-1}P_K(x)) d\xi dx.$$

The change of variables  $\xi = \lambda\xi', x = \delta_\lambda x'$  transforms the integral into

$$\lambda^{n+|\gamma|} \cdot \int_{\mathbb{R}^{2n}} F(\xi'_1, \dots, \xi'_n, V(x'), \lambda^{-1}P_1(\delta_\lambda x'), \dots, \lambda^{-1}P_K(\delta_\lambda x')) d\xi' dx'.$$

By assumption,  $\text{meas}\{x' \in \mathbb{R}^n \mid V(x') \leq \lambda\}$  is finite for every  $\lambda > 0$ . Now Fubini's theorem implies that for every compact subset  $\mathcal{K} \subset \mathbb{R}^{n+1}$ , we have

$$\text{meas}\{(\xi', x') \in \mathbb{R}^{2n} \mid (\xi', V(x')) \in \mathcal{K}\} < \infty.$$

On the other hand, for a fixed  $x' \in \mathbb{R}^n$  and for every  $k$ , we have  $\lambda^{-1}P_k(\delta_\lambda x') \rightarrow 0$  as  $\lambda \rightarrow +\infty$  by Lemma 4.2. Thus Proposition 2.12 applies, and we see that there exists a weak limit

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-(n+|\gamma|)} \cdot \mu_\lambda,$$

given by

$$(4.1) \quad \int_{\mathfrak{g}^*} F d\mu_\infty = (2\pi)^{-n} \cdot \int_{\mathbb{R}^{2n}} F(\xi'_1, \dots, \xi'_n, V(x'), 0, \dots, 0) d\xi dx$$

for every  $F \in C_c(\mathfrak{g}^*)$ . In particular,  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp \subset \mathfrak{g}^*$  of the ideal

$$\mathfrak{a} = \text{span}_{\mathbb{R}}\{p_1, \dots, p_K\} \subset \mathfrak{g}.$$

But  $\mathfrak{g}/\mathfrak{a}$  is clearly abelian, so all unitary irreducible representations of the corresponding Lie group  $\exp(\mathfrak{g}/\mathfrak{a})$  are 1-dimensional and are obtained by exponentiating linear functionals on  $\mathfrak{g}/\mathfrak{a}$ . Given  $f \in \mathfrak{a}^\perp \cong (\mathfrak{g}/\mathfrak{a})^*$ , the image of  $H^\circ = -(L_1^2 + \dots + L_n^2) - \sqrt{-1} \cdot v \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  under the representation corresponding to  $f$  is the scalar  $f(L_1)^2 + \dots + f(L_n)^2 + f(v)$ . Thus we see that our conjecture produces the following formula:

$$N(\lambda, H) \sim (2\pi)^{-n} \cdot \text{meas}\{(\xi', x') \in \mathbb{R}^{2n} \mid \|\xi'\|^2 + V(x') \leq \lambda\} \quad \text{as } \lambda \rightarrow +\infty,$$

which coincides with the classical Weyl formula (1.2).

*Remark 4.5.* — We can now justify the remark of §1.4 of the Introduction. Indeed, note that the measure  $\mu_\infty$  given by (4.1) can also be obtained simply by taking the pushforward of the canonical measure  $\mu_\Omega$  with respect to the projection  $\mathfrak{g}^* \rightarrow \mathfrak{g}^*$  induced by the map  $\mathfrak{g} \rightarrow \mathfrak{g}$  taking  $L_j \mapsto L_j, v \mapsto v$  and  $p_k \mapsto 0$  for  $1 \leq k \leq K$ . Furthermore, if we simply apply Conjecture 1 (i.e., formula (2.1)) with the normalization factor  $\kappa$  equal to 1, then we immediately recover Weyl's formula. Also, this “derivation” of Weyl's formula does not rely on the assumption that  $V$  is quasi-homogeneous. On the other

hand, the computation presented above (via the scaling construction) uses the quasi-homogeneity assumption in an essential way.

### 4.2. Colin de Verdière’s formula

In this subsection we consider a Schrödinger operator  $H = H(a)$  with zero electric potential and polynomial magnetic tensor  $B(x) = [b_{jk}(x)]_{j,k=1}^n$  such that  $\|B(x)\| \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ . We suppose that  $B$  is quasi-homogeneous of weight  $\gamma = (\gamma_1, \dots, \gamma_n)$  and use the notation  $\delta_\lambda$  in the same way as was done in §4.1. Our goal is to show that the formula provided by our conjecture coincides with the formula (1.5) of Colin de Verdière (see also [5], Théorème 4.1).

We begin by observing that since  $\|B(x)\| \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ , we have, a fortiori,  $\text{meas}\{\sum_{j,k} |b_{jk}(x)|^{1/2} \leq \lambda\} < \infty$  for all  $\lambda > 0$ , and therefore the same type of argument as in §4.1 above applies. Thus we are in case (b) of Conjecture 2 (in fact, there is no degeneration at all), and, moreover, the normalizing factor  $G_2(\lambda)/G_1(\lambda)$  is asymptotically equal to 1.

Next, let  $\mathfrak{g}$  be the Lie algebra corresponding to  $H$ . We have the elements  $L_j = \partial/\partial x_j + \sqrt{-1} \cdot a_j(x) \in \mathfrak{g}$ . We will denote by  $B_{jk}$  the elements of  $\mathfrak{g}$  that map to  $\sqrt{-1} \cdot b_{jk}(x)$  under the tautological representation of  $\mathfrak{g}$ . We also choose a basis  $P_1(x), \dots, P_K(x)$  of the space of polynomials spanned by all mixed partial derivatives of all *nonzero* orders of all the  $b_{jk}(x)$ , and we let  $p_k \in \mathfrak{g}$  denote the element mapping to  $\sqrt{-1} \cdot P_k(x)$  under the tautological representation of  $\mathfrak{g}$ . As before, up to replacing  $H$  with a gauge equivalent Schrödinger operator, the tautological representation of  $\mathfrak{g}$  lifts to a unitary irreducible representation of  $G = \exp \mathfrak{g}$ , which corresponds to the coadjoint orbit  $\Omega \subset \mathfrak{g}^*$  parameterized by the map  $\varphi : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$  defined by

$$\varphi(\xi, x)(L_j) = \xi_j, \quad \varphi(\xi, x)(B_{jk}) = b_{jk}(x), \quad \varphi(\xi, x)(p_k) = P_k(x).$$

Note that the polynomials  $b_{jk}(x)$  are not linearly independent, so that  $\{L_1, \dots, L_n, B_{jk}, p_k\}$  spans  $\mathfrak{g}$  but is not a basis of  $\mathfrak{g}$ . Still, the formula above is meaningful and correct. Moreover, the canonical measure  $\mu_\Omega$  is given by

$$\mu_\Omega = (2\pi)^{-n} \cdot \varphi_*(d\xi dx).$$

Now, using quasi-homogeneity as in §4.1, we see that there exists a weak limit

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-n-|\gamma|} \cdot \mu_\lambda = (2\pi)^{-n} \cdot \psi_*(d\xi dx),$$

where  $\psi : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$  is defined by

$$\psi(\xi, x)(L_j) = \xi_j, \quad \psi(\xi, x)(B_{jk}) = b_{jk}(x), \quad \psi(\xi, x)(p_k) = 0.$$

In particular,  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp$  of the ideal  $\mathfrak{a} \subset \mathfrak{g}$  spanned by  $p_1, \dots, p_K$ . Let  $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ , and let  $\bar{L}_j, \bar{B}_{jk}$  denote the images of the elements  $L_j, B_{jk}$  in  $\bar{\mathfrak{g}}$ . Thus  $[\bar{L}_j, \bar{L}_k] = \bar{B}_{jk}$  in  $\bar{\mathfrak{g}}$ , and the elements  $\bar{B}_{jk}$  are central in  $\bar{\mathfrak{g}}$ .

*Remark 4.6.* — As in Remark 4.5, we note that the measure  $\mu_\infty$  could have been obtained from the canonical measure  $\mu_\Omega$  as the pushforward with respect to a suitable projection map  $\mathfrak{g}^* \rightarrow \mathfrak{a}^\perp$ . The computation of the “quotient measure”  $\nu$  presented below does not make use of the quasi-homogeneity assumption. In fact, if we take  $Q = (\text{supp } \mu_\infty)/G$ , then (1.10) becomes precisely the formula of Colin de Verdière (1.5).

The rest of the computation is based on the following two lemmas. The proof of Lemma 4.7 is a simple exercise in linear algebra, and the proof of Lemma 4.8 is a straightforward application of Fubini’s theorem. Therefore both proofs are omitted.

LEMMA 4.7. — *Let  $O(n, \mathbb{R})$  denote the Lie group of orthogonal  $n \times n$  matrices over  $\mathbb{R}$ , and let  $\mathfrak{o}(n, \mathbb{R})$  be its Lie algebra, i.e., the space of all skew-symmetric  $n \times n$  matrices over  $\mathbb{R}$ . Then there exists a measurable map*

$$\mathfrak{C} : \mathfrak{o}(n, \mathbb{R}) \longrightarrow O(n, \mathbb{R})$$

such that for every  $B \in \mathfrak{o}(n, \mathbb{R})$ , the matrix  $\mathfrak{C}(B) \cdot B \cdot \mathfrak{C}(B)^T$  is block-diagonal, with  $r$  blocks along the diagonal of the form  $\begin{pmatrix} 0 & b_j \\ -b_j & 0 \end{pmatrix}$ , where  $b_1 \geq b_2 \geq \dots \geq b_r > 0$ , and which has all the other elements equal to 0.

LEMMA 4.8. — *Let  $r, d \geq 1$  be integers, and let  $\lambda, b_1, \dots, b_r > 0$  be fixed real numbers. Then we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \# \left\{ m_1, \dots, m_r \in \mathbb{Z}_+ \mid \sum_{j=1}^r (2m_j + 1)b_j \leq \lambda - \|\xi\|^2 \right\} d\xi \\ = |v_d| \cdot \sum_{m_j \geq 0} \left( \lambda - \sum_{j=1}^r (2m_j + 1)b_j \right)_+^{d/2}. \end{aligned}$$

Let  $2r$  be the maximal possible rank of  $B(x)$ , as  $x$  runs through all of  $\mathbb{R}^n$ . The set of points  $x \in \mathbb{R}^n$  where  $B(x)$  has rank less than  $2r$  is defined by a collection of polynomial equations, and hence has Lebesgue measure 0. Thus it can be ignored both in our conjectural asymptotic formula, and in

the formula of Colin de Verdière. Put  $U = \{x \in \mathbb{R}^n \mid \text{rank } B(x) = 2r\}$ . For each  $x \in U$ , we let  $b_1(x) \geq \dots \geq b_r(x) > 0$  denote the positive eigenvalues of  $\sqrt{-1} \cdot B(x)$ . Also, let  $\mathfrak{C}$  be the measurable map provided by Lemma 4.7, and set  $\mathcal{L} = \text{span}_{\mathbb{R}}\{L_1, \dots, L_n\} \subset \mathfrak{g}$ . For every point  $x \in U$ , we have a basis  $\{L'_k(x)\}_{k=1}^n$  of  $\mathcal{L}$  defined by  $L'_k(x) = \sum_{j=1}^n p_{kj}(x)L_j$ , where  $p_{kj}(x)$  are the entries of the matrix  $\mathfrak{C}(B(x))$ . Thus we can define a new map  $\psi' : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$  by

$$\psi'(\xi, x)(L'_j(x)) = \xi_j, \quad \psi'(\xi, x)(B_{jk}) = b_{jk}(x), \quad \psi'(\xi, x)(p_k) = 0,$$

where  $p_k \in \mathfrak{g}$  is the element that maps to  $\sqrt{-1} \cdot P_k(x)$  under the tautological representation of  $\mathfrak{g}$ . Note that  $\psi'$  is no longer a polynomial map, but it is still measurable. Since the matrix  $\mathfrak{C}(B(x))$  is orthogonal, it is easy to check that  $\mu_{\infty} = (2\pi)^{-n} \cdot \psi'_*(d\xi dx)$ , and also  $L'_1(x)^2 + \dots + L'_n(x)^2 = L_1^2 + \dots + L_n^2$  in  $\mathcal{U}(\mathfrak{g})$  for all  $x \in U$ .

By abuse of notation, we view  $\psi'$  as a map  $\mathbb{R}^{2n} \rightarrow \bar{\mathfrak{g}}^*$ , where  $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$  is defined as above. Given  $\xi \in \mathbb{R}^n$ , let us write  $\xi' = (\xi_1, \dots, \xi_{2r})$  and  $\xi'' = (\xi_{2r+1}, \dots, \xi_n)$ . It is easy to check, using the definition of  $\psi'$  and Propositions 2.10 and 2.11 that, for fixed  $\xi'' \in \mathbb{R}^{n-2r}$  and  $x \in U$ , the map  $\psi_{\xi'', x} : \mathbb{R}^{2r} \rightarrow \bar{\mathfrak{g}}^*$  defined by  $\psi_{\xi'', x}(\xi') = \psi'(\xi', \xi'', x)$  parameterizes a single coadjoint orbit  $\Theta_{\xi'', x} \subset \bar{\mathfrak{g}}^*$  whose associated canonical measure  $\mu_{\xi'', x}$  is given by

$$\mu_{\xi'', x} = (2\pi)^{-r} \cdot b_1(x)^{-1} \dots b_r(x)^{-1} \cdot (\psi_{\xi'', x})_*(d\xi_1 \dots d\xi_{2r}).$$

Now it is clear that as the quotient measure space  $(Q, \nu)$  we can take  $Q = \mathbb{R}^{n-2r} \times U$  and

$$\nu(\xi'', x) = b_1(x) \dots b_r(x) \cdot d\xi_{2r+1} \dots d\xi_n dx_1 \dots dx_n.$$

Moreover, the representation of  $\bar{\mathfrak{g}}$  corresponding to the orbit  $\Theta_{\xi'', x}$  can be realized in  $L^2(\mathbb{R}^r)$  in such a way that  $L_j \mapsto \partial/\partial y_j$  for  $1 \leq j \leq r$ ,  $L_j \mapsto \sqrt{-1} \cdot b_{j-r}(x) \cdot y_{j-r}$  for  $r + 1 \leq j \leq 2r$ , and  $L_j \mapsto \sqrt{-1} \cdot \xi_j$  for  $2r + 1 \leq j \leq n$ . The rest follows immediately from Lemma 4.8 and the formula

$$\sigma(-\Delta_y + \sum_{j=1}^r b_j^2 y_j^2 + \|\xi''\|^2) = \left\{ \sum_{j=1}^r (2m_j + 1) b_j + \|\xi''\|^2 \mid m_1, \dots, m_r \in \mathbb{Z}_+ \right\}$$

which is straightforward from the well-known formula for the spectrum of the quantum harmonic oscillator in 1D.

### 4.3. Weak and Intermediate Degeneration Cases: A General Theorem

The results of this subsection hold unconditionally (*i.e.*, they are proved independently of our main conjectures). For  $U \subset \mathbb{R}^n$ , define  $G(\lambda, \Psi^*, U)$  by (1.9), and set

$$(4.2) \quad F(\lambda, a, V, U) = \int_U \nu_{B(x)}(\lambda - V(x)) dx.$$

For  $d \gg 1$ , define  $W_{-d} = \{x \mid \Psi(x) \leq d^{-1} \Psi^*(x)\}$ ,  $W_d = \mathbb{R}^n \setminus W_{-d}$ . The next theorem states that if for large  $d$ , the sets  $W_d$  are relatively small in a certain sense, then the classical Weyl formula (when there is no magnetic field) or Colin de Verdière's formula hold but with the integration over  $W_{-d}$ , where  $d$  is a fixed positive number.

**THEOREM 4.9.** — *Let (1.7) hold, and let there exist a function  $M \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ , such that for any  $C > 0$ ,*

$$(4.3) \quad G(C\lambda, \Psi^*, W_M) = o(G(\lambda, \Psi^*, \mathbb{R}^n)) \quad \text{as } \lambda \rightarrow +\infty.$$

Then

- a) *there exists a function  $\epsilon = \epsilon(\lambda) \rightarrow 0$  as  $\lambda \rightarrow +\infty$  such that for any  $d > 0$ ,*

$$(4.4) \quad (1-\epsilon)F(\lambda(1-\epsilon), a, V, W_{-d}) \leq N(\lambda, H(a)+V) \leq (1+\epsilon)F(\lambda(1+\epsilon), a, V, W_{-d});$$

- b) *if, in addition, for any  $\epsilon = \epsilon(\lambda) \rightarrow 0$  as  $\lambda \rightarrow +\infty$ ,*

$$(4.5) \quad F(\lambda(1+\epsilon), a, V, W_{-d}) \sim F(\lambda, a, V, W_{-d}), \quad \text{as } \lambda \rightarrow +\infty,$$

then

$$(4.6) \quad N(\lambda, H(a) + V) \sim F(\lambda, a, V, W_{-d}) \quad \text{as } \lambda \rightarrow +\infty.$$

*Proof.* — As  $M \rightarrow +\infty$ , the set  $W_{-M}$  satisfies the crucial condition (3.19) for  $\Sigma_{-M}$ , which was needed to construct sets  $U_j, U_{1j}$  and functions  $\chi_j, j \leq -1$  (see (3.27)) satisfying properties (3.6)-(3.9), and (3.16)-(3.17). Set  $\chi_0 = 1 - \sum_{j \leq -1} \chi_j$ , and define  $\psi_j = \chi_j / \sum_{j \leq 0} \chi_j^2$ ,  $U_0 = \{x \mid \chi_0(x) > 0\}$ ,  $U_{1,0} = W_{CM}$ , where  $C > 0$  sufficiently large so that  $\text{supp } \chi_0 \subset W_{CM}$  (if  $W_M = \emptyset$  for large  $M$ , this step is not needed, and only  $j \leq -1$  are involved). Then all the conditions (3.5)-(3.10) are satisfied, and we have the estimates (3.11) and (3.14) for  $N(\lambda; H(a) + V)$ . The  $j \leq -1$  terms are treated exactly as §3.4, and we derive estimates (3.47) and (3.48). In view



of (4.3), to finish the proof, it suffices to show that there exist constants  $C, C_1, C_2$  such that

$$(4.7) \quad N_D(\lambda; H(a) + V, W_M) \leq C \cdot G(C_1\lambda, \Psi^*, W_{C_2M}).$$

With  $\mathbb{R}^n$  in place of  $W_M$  and  $W_{C_2M}$ , this is the estimate obtained in [26]. Since the proof in [26] is obtained by using an appropriate partition of unity, it can be used to derive (4.7), and we are done.  $\square$

If the potentials are quasi-homogeneous, and  $\Psi_0$  has no zero outside the origin, then for large  $d$ ,  $U_d = \emptyset$  (hence, (4.3) holds). Due to the quasi-homogeneity,  $F(\lambda, a, V, \mathbb{R}^n)$  is of the form  $const \cdot \lambda^p$ , hence (4.5) is satisfied, and (4.6) gives

$$(4.8) \quad N(\lambda, H(a) + V) \sim F(\lambda, a, V, \mathbb{R}^n) \quad \text{as } \lambda \rightarrow +\infty.$$

In particular, we recover the classical Weyl formula or Colin de Verdière’s formula (for the no degeneration case). If  $\Psi_0$  has zeroes outside the origin, we apply the following result, whose proof is given in Appendix B.

LEMMA 4.10. — *Let  $V(x)$  and  $B(x)$  be quasi-homogeneous of the same weight  $\gamma = (\gamma_1, \dots, \gamma_n)$ . Assume also that*

$$(4.9) \quad \text{meas}\{(\xi, x) \in \mathbb{R}^{2n} \mid \Psi_0(x)^2 + \|\xi\|^2 \leq \lambda\} < \infty \quad \forall \lambda > 0.$$

Then

$$(4.10) \quad \text{meas}\{\Psi_0(x)^2 + \|\xi\|^2 \leq \lambda\} \sim \text{meas}\{\Psi^*(x)^2 + \|\xi\|^2 \leq \lambda\} \quad \text{as } \lambda \rightarrow +\infty,$$

and (4.3) holds.

THEOREM 4.11. — *Let  $V$  be quasi-homogeneous,  $a = 0$ , and (4.9) hold. Then the classical Weyl formula holds.*

*Proof.* — Under assumptions of the theorem, the conclusion of Lemma 4.10 holds, and from Theorem 4.9, we know that the estimate (4.4) holds as well. If  $V$  is non-degenerate, then  $W_{-d} = \mathbb{R}^n$ , and since there exist  $\kappa(V)$  and  $p > 0$  such that  $F(\lambda; 0, V, \mathbb{R}^n) \sim \kappa(V)\lambda^p$ , the estimate (4.4) gives the classical Weyl formula. If  $V$  is degenerate, we notice that if  $\delta \rightarrow 0$  sufficiently slowly, and  $C$  is sufficiently large, then outside a ball  $B(0, C/\epsilon)$ , the  $\delta_\lambda$ -invariant  $\epsilon$ -neighborhood of the degeneration set  $\Sigma$ , call it  $CW_\epsilon$ , will contain  $W_d$ . Hence,  $F(\lambda; 0, V, W_d) \leq F(\lambda; 0, V, CW_\epsilon) + O(\epsilon^{-1}\lambda^n)$  as  $\epsilon \rightarrow 0$ , uniformly in  $\lambda > 1$ . But  $F(\lambda; 0, V, CW_\epsilon) = o(\lambda^p)$ . Hence, on the LHS and RHS of (4.4), we may replace  $W_{-d}$  with  $\mathbb{R}^n$ , and add  $o(\lambda^p)$ . After that, the classical Weyl formula is immediate.  $\square$

## 5. Examples of Schrödinger operators with degenerate homogeneous potentials

### 5.1. An example with strong degeneration

We consider a 2-dimensional Schrödinger operator with zero electric potential and magnetic tensor  $b(x) = b_{12}(x) = x_1^k x_2^l$ , where  $k > l \geq 1$  (the case  $k = l$  is considered in §5.2, and leads to substantially different computations). The corresponding Lie algebra  $\mathfrak{g}$  has a natural basis of the form

$$\{L_1, L_2\} \cup \{L^{(ij)}\}_{0 \leq i \leq k, 0 \leq j \leq l},$$

where, in the tautological representation of  $\mathfrak{g}$ , we have:

$$\begin{aligned} L_1 &\mapsto \frac{\partial}{\partial x_1} + \sqrt{-1} \cdot a_1(x), & L_2 &\mapsto \frac{\partial}{\partial x_2} + \sqrt{-1} \cdot a_2(x), \\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} &= x_1^k x_2^l, & L^{(ij)} &\mapsto \sqrt{-1} \cdot x_1^i x_2^j. \end{aligned}$$

(Since this is a faithful representation, these formulas determine the commutation relations between the basis elements  $L_1, L_2, L^{(ij)}$  of  $\mathfrak{g}$ .) Our Schrödinger operator is the image of the element  $H^\circ = -L_1^2 - L_2^2 \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  under this representation. As usual, we assume that the tautological representation of  $\mathfrak{g}$  lifts to a unitary irreducible representation of  $G = \exp \mathfrak{g}$ , and we write  $\Omega \subset \mathfrak{g}^*$  for the corresponding coadjoint orbit, and  $\mu_\Omega$  for its canonical measure, extended by zero to a measure on  $\mathfrak{g}^*$ . Our goal is the following

PROPOSITION 5.1. — *We have an asymptotic formula*

$$(5.1) \quad \begin{aligned} N(\lambda, H) &\sim \frac{1}{\pi} \cdot \lambda^{(l+k+2)/2l} \\ &\times \int_0^{+\infty} dx_2 \int_{-\infty}^{+\infty} d\xi_2 N \left( 1, -\frac{d^2}{dy^2} + \left( x_2^l \cdot \frac{y^{k+1}}{k+1} + \xi_2 \right)^2 \right) \end{aligned}$$

as  $\lambda \rightarrow +\infty$ .

The direct variational proof of (5.1) is an evident modification of the proof in Section 3. The goal of this subsection is to derive Proposition 5.1 from Conjecture 2. We leave it to the reader to check that  $H$  exhibits strong degeneration, so that we are in case (a) of Conjecture 2. The computations are rather straightforward, and are easier, for example, than the ones involved in Lemma 5.3 below whose detailed proof is given in Appendix B.

In what follows, the notation will become quite messy. To simplify it a little bit, we choose an ordering of our basis of  $\mathfrak{g}$  as follows:

$$L_1, L_2, L^{(k,l)}, L^{(k-1,l)}, \dots, L^{(1,l)}, L^{(0,l)},$$

$$L^{(k,l-1)}, \dots, L^{(0,l-1)}, \dots, L^{(k,0)}, \dots, L^{(0,0)}.$$

Using this ordering, we identify  $\mathfrak{g}^*$  with  $\mathbb{R}^{2+(k+1)(l+1)}$ . This particular identification will be used throughout our computation.

It follows from Propositions 2.10 and 2.11 that the orbit  $\Omega$  is parameterized by the map  $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^{2+(k+1)(l+1)} \cong \mathfrak{g}^*$  defined by

$$\begin{aligned} \varphi(\xi_1, \xi_2, x_1, x_2) = & (\xi_1, \xi_2, x_1^k x_2^l, x_1^{k-1} x_2^l, \dots, x_1 x_2^l, x_2^l, \\ & x_1^k x_2^{l-1}, \dots, x_2^{l-1}, \dots, x_1^k, \dots, x_1, 1), \end{aligned}$$

and we have

$$\mu_\Omega = (2\pi)^{-2} \cdot \varphi_*(d\xi_1 d\xi_2 dx_1 dx_2).$$

Now we fix a continuous function  $F$  with compact support on  $\mathfrak{g}^* \cong \mathbb{R}^{2+(k+1)(l+1)}$ , and for  $\lambda > 0$ , we consider the integral

(5.2)

$$\begin{aligned} \lambda^{-2-1/l} \cdot \int_{\mathfrak{g}^*} F d\mu_\lambda = & (2\pi)^{-2} \cdot \lambda^{-2-1/l} \\ & \times \int_{\mathbb{R}^4} F(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2, \lambda^{-1}x_1^k x_2^l, \dots, \lambda^{-1}) d\xi_1 d\xi_2 dx_1 dx_2. \end{aligned}$$

We make the following change of variables:  $\xi'_1 = \lambda\xi_1$ ,  $\xi'_2 = \lambda^{-1}\xi_2$ ,  $x'_1 = x_1$ ,  $x'_2 = \lambda^{-1/l} \cdot x_2$ . This kills the factor  $\lambda^{-2-1/l}$ , and the integral becomes

(5.3)

$$\begin{aligned} (2\pi)^{-2} \int_{\mathbb{R}^4} F(\xi'_1, \xi'_2, (x'_1)^k (x'_2)^l, \dots, (x'_2)^l, \lambda^{-1/l} \cdot (x'_1)^k (x'_2)^{l-1}, \dots, \lambda^{-1}) \\ \times d\xi'_1 d\xi'_2 dx'_1 dx'_2. \end{aligned}$$

It is straightforward to prove that, for a fixed  $R > 0$ , the set of points  $(\xi'_1, \xi'_2, x'_1, x'_2) \in \mathbb{R}^4$  such that  $\|\xi'\| \leq R$  and  $|x'_1|^k |x'_2|^l \leq R, \dots, |x'_2|^l \leq R$  has finite Lebesgue measure (this is where the assumption that  $k > l$  is used in a crucial way). Now Proposition 2.12 implies that the limit as  $\lambda \rightarrow +\infty$  of the above integral is equal to

$$(2\pi)^{-2} \int_{\mathbb{R}^4} F(\xi'_1, \xi'_2, (x'_1)^k (x'_2)^l, \dots, (x'_2)^l, 0, 0, \dots, 0) d\xi'_1 d\xi'_2 dx'_1 dx'_2,$$

which implies that there exists a weak limit

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-2-1/l} \cdot \mu_\lambda = (2\pi)^{-2} \cdot \psi_*(d\xi_1 d\xi_2 dx_1 dx_2),$$

where  $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^{2+(k+1)(l+1)} \cong \mathfrak{g}^*$  is given by

$$\psi(\xi_1, \xi_2, x_1, x_2) = (\xi_1, \xi_2, x_1^k x_2^l, x_1^{k-1} x_2^l, \dots, x_1 x_2^l, x_2^l, 0, 0, \dots, 0).$$

Next we must compute the “relevant” coadjoint orbits, i.e., those that contribute to the RHS of our conjectural formula. We begin by observing

that  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp \subset \mathfrak{g}^*$ , where  $\mathfrak{a} \subset \mathfrak{g}$  is the ideal spanned by all the elements  $L^{(ij)}$  with  $0 \leq i \leq k$ ,  $0 \leq j \leq l - 1$ . Let us identify the quotient algebra  $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$  explicitly. It has a basis which we will denote (abusing notation) by  $(\frac{\partial}{\partial w}, w^{k+1}, w^k, \dots, w, 1)$ . The commutation relations are obvious from the notation, and the quotient map  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$  is determined by  $L_1 \mapsto \frac{\partial}{\partial w}$ ,  $L_2 \mapsto \frac{w^{k+1}}{k+1}$ , and  $L^{(jl)} \mapsto w^j$  for  $0 \leq j \leq k$ . Once again, we will use this chosen basis to identify  $\bar{\mathfrak{g}}^*$  with  $\mathbb{R}^{k+3}$ . By abuse of notation, we will identify  $\mu_\infty$  with its restriction to  $\bar{\mathfrak{g}}^* \cong \mathfrak{a}^\perp \subset \mathfrak{g}^*$ . Explicitly, we have

$$\mu_\infty = (2\pi)^{-2} \cdot \psi_*(d\xi_1 d\xi_2 dx_1 dx_2),$$

where  $\psi : \mathbb{R}^4 \rightarrow \mathbb{R}^{k+3}$  is given by

$$\psi(\xi_1, \xi_2, x_1, x_2) = (\xi_1, (k + 1)\xi_2, x_1^k x_2^l, \dots, x_1 x_2^l, x_2^l).$$

Note that the set

$$\psi^{-1}(\{f \in \bar{\mathfrak{g}}^* \mid f(1) = 0\}) = \{(\xi_1, \xi_2, x_1, x_2) \in \mathbb{R}^4 \mid x_2 = 0\}$$

has zero Lebesgue measure. Hence, for the rest of our computation, we can throw away the “non-generic” elements  $f \in \bar{\mathfrak{g}}^*$  (i.e., such that  $f(1) = 0$ ), and concentrate our attention on the map (which we still denote by  $\psi$ )

$$\psi : \mathbb{R}^3 \times \mathbb{R}^\times \longrightarrow \bar{\mathfrak{g}}_{gen}^*,$$

$$(\xi_1, \xi_2, x_1, x_2) \longmapsto (\xi_1, (k + 1)\xi_2, x_1^k x_2^l, \dots, x_1 x_2^l, x_2^l),$$

where  $\bar{\mathfrak{g}}_{gen}^* = \{f \in \bar{\mathfrak{g}}^* \mid f(1) \neq 0\}$ , and  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ . Thus, the “relevant” orbits for us are those which are contained in  $\psi(\mathbb{R}^3 \times \mathbb{R}^\times)$  (we will see below that this set is actually stable under the adjoint action on  $\bar{\mathfrak{g}}^*$ ).

Given any generic  $f \in \bar{\mathfrak{g}}_{gen}^*$ , it is immediate that  $\text{span}_{\mathbb{R}}\{w^{k+1}, w^k, \dots, 1\}$  is a polarization of  $\bar{\mathfrak{g}}$  at  $f$ . Hence it follows from Proposition 2.10 that for any fixed point  $(\xi_2, x_2) \in \mathbb{R} \times \mathbb{R}^\times$ , the coadjoint orbit in  $\bar{\mathfrak{g}}^*$  through  $\psi(0, \xi_2, 0, x_2)$  is parameterized by the map

$$(\eta, y) \mapsto \psi'(\eta, \xi_2, y, x_2) := (\eta, (k + 1)\xi_2 + x_2^l y^{k+1}, x_2^l y^k, \dots, x_2^l y, x_2^l).$$

Moreover, by Proposition 2.11, the canonical measure on this orbit in terms of this parameterization is given by  $(2\pi)^{-1} d\eta dy$ . Now consider the map  $\theta : \mathbb{R}^3 \times \mathbb{R}^\times \rightarrow \mathbb{R}^3 \times \mathbb{R}^\times$  given by

$$\theta(\xi_1, \xi_2, x_1, x_2) = \left( \xi_1, \xi_2 - \frac{x_1^{k+1} x_2^l}{k + 1}, x_1, x_2 \right).$$

It is clear that  $\theta$  is a diffeomorphism preserving the Lebesgue measure; moreover,  $\psi = \psi' \circ \theta$ . Therefore  $\mu_\infty = (2\pi)^{-2} \cdot \psi'_*(d\eta d\xi_2 dy dx_2)$ . Now we see that, tautologically, the image of  $\psi'$  (which is the same as the image of

$\psi$ ) is invariant under the coadjoint action on  $\bar{\mathfrak{g}}^*$ , and the space of coadjoint orbits in the image of  $\psi'$  is naturally parameterized by the points  $(\xi_2, x_2) \in \mathbb{R} \times \mathbb{R}^\times$ , with the quotient measure  $\nu$  given by  $\nu = (2\pi)^{-1} d\xi_2 dx_2$ . By Proposition 2.7, the representation of  $\bar{\mathfrak{g}}$  corresponding to the coadjoint orbit parameterized by  $(\xi_2, x_2)$  can be realized in the space  $L^2(\mathbb{R}^1)$  as follows:

$$\begin{aligned} \partial/\partial w &\mapsto d/dy, & w^{k+1} &\mapsto \sqrt{-1} \cdot ((k+1)\xi_2 + x_2^l y^{k+1}), \\ w^j &\mapsto \sqrt{-1} \cdot x_2^l y^j && \text{for } 0 \leq j \leq k. \end{aligned}$$

Finally, we see that our conjectural formula (2.1) gives the following answer:

$$N(\lambda, H) \sim (2\pi)^{-1} \int_{\mathbb{R} \times \mathbb{R}^\times} N\left(\lambda, -\frac{d^2}{dy^2} + \left(x_2^l \cdot \frac{y^{k+1}}{k+1} + \xi_2\right)^2\right) d\xi_2 dx_2$$

as  $\lambda \rightarrow +\infty$ . To reduce this equation to the form (5.1), we make the change of variables  $y \mapsto \lambda^{-1/2}y$ ,  $x_2 \mapsto \lambda^{(k+2)/2l}x_2$ ,  $\xi_2 \mapsto \lambda^{1/2}\xi_2$ , and use symmetry to replace  $\int_{\mathbb{R} \times \mathbb{R}^\times} dx_2 d\xi_2$  with  $2 \cdot \int_0^\infty dx_2 \int_{-\infty}^\infty d\xi_2$ .

### 5.2. An example with intermediate degeneration

In this subsection we consider a Schrödinger operator  $H$  on  $L^2(\mathbb{R}^2)$  with zero electric potential and magnetic tensor  $b(x) = b_{12}(x) = x_1^k x_2^k$  (since we are working in 2D, the magnetic tensor has only one relevant component). We assume that  $k \geq 1$ . We will use Conjecture 2 to derive

PROPOSITION 5.2. — *We have the following asymptotic formula, as  $\lambda \rightarrow +\infty$ ,*

$$(5.4) \quad N(\lambda, H) \sim \frac{2}{\pi k} \cdot \lambda^{1+1/k} \log \lambda \cdot \sum_{j=0}^\infty (2j+1)^{-1-1/k}.$$

(Note that the equation (5.4) can be derived from Theorem 4.9.)

The functions (2.2), (1.6) associated to the operator  $H$  are given by

$$\Phi^*(x) = \sum_{j,l=0}^k \left| \frac{k!}{(k-j)!} \cdot \frac{k!}{(k-j)!} \cdot x_1^{k-j} x_2^{k-l} \right|^{1/2}$$

and

$$\Psi^*(x) = \sum_{j,l=0}^k \left| \frac{k!}{(k-j)!} \cdot \frac{k!}{(k-j)!} \cdot x_1^{k-j} x_2^{k-l} \right|^{1/(2+j+l)}.$$

Define  $G_1(\lambda)$ ,  $G_2(\lambda)$  as in §2.3. We begin with the following

LEMMA 5.3. — *We have*

$$G_1(\lambda) \sim \frac{8}{k} \cdot \lambda^{2/k} \log \lambda \quad \text{and} \quad G_2(\lambda) \sim \frac{8(k+1)}{k} \cdot \lambda^{2/k} \log \lambda,$$

and hence  $G_2(\lambda)/G_1(\lambda) \rightarrow k+1$  as  $\lambda \rightarrow +\infty$ .

The proof is given in Appendix B.

The lemma implies that we are in the intermediate degeneration case, and the constant  $\kappa$  in our formula (2.1) can be replaced by  $k+1$ . Thus we have to compute  $\mu_\infty$  and the quotient measure  $\nu$ . We will use the same notation as in §5.1 (indeed, the only difference in the setup is that  $k=l$ , which, however, does not affect the representation-theoretic part of our discussion). In particular, we have the basis  $\{L_1, L_2, L^{(k,k)}, \dots, L^{(0,0)}\}$  of  $\mathfrak{g}$  which yields an identification of  $\mathfrak{g}^*$  with  $\mathbb{R}^{2+(k+1)^2}$ . Moreover, the orbit  $\Omega \subset \mathfrak{g}^*$  corresponding to the tautological representation of  $\mathfrak{g}$  is parameterized by the map

$$\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^{2+(k+1)^2} \cong \mathfrak{g}^*, \quad (\xi_1, \xi_2, x_1, x_2) \mapsto (\xi_1, \xi_2, x_1^k x_2^k, \dots, 1),$$

and the canonical measure on  $\Omega$  is given by

$$\mu_\Omega = (2\pi)^{-2} \cdot \varphi_*(d\xi_1 d\xi_2 dx_1 dx_2).$$

In Appendix B, we use Proposition 2.13 to prove

LEMMA 5.4. — *There exists a weak limit*

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-2-1/k} (\log \lambda)^{-1} \cdot \mu_\lambda.$$

Moreover,  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp \subset \mathfrak{g}^*$  of the ideal

$$\mathfrak{a} = \text{span}_{\mathbb{R}} \{L^{(i,j)} \mid i < k \text{ or } j < k\}.$$

In particular, we can view  $\mu_\infty$  as a measure on  $\bar{\mathfrak{g}}^*$ , where  $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$  is a 3-dimensional Heisenberg algebra. If  $(\eta_1, \eta_2, y)$  are the coordinates on  $\bar{\mathfrak{g}}^*$  defined by the images of the elements  $L_1, L_2, L^{(k,k)} \in \mathfrak{g}$  in  $\bar{\mathfrak{g}}$ , then

$$\mu_\infty = \frac{1}{2\pi^2 k^2} \cdot |y|^{\frac{1-k}{k}} d\eta_1 d\eta_2 dy.$$

Now the image of  $L^{(k,k)}$  in  $\bar{\mathfrak{g}}$  is central, whence the set  $\{y=0\} \subset \bar{\mathfrak{g}}^*$  is invariant under the coadjoint action and has  $\mu_\infty$ -measure zero. Thus it can be ignored in our computation. The complement of this set is a union of 2-dimensional coadjoint orbits parameterized by points  $c \in \mathbb{R}^\times$ . The orbit corresponding to such a  $c$  is defined by  $y=c$ , and the corresponding canonical measure is  $\mu_c = (2\pi)^{-1} \cdot |c|^{-1} d\eta_1 d\eta_2$  (we may view  $(\eta_1, \eta_2)$  as coordinates on the orbit). We see immediately that the measure space  $(Q, \nu)$  which appears in our conjectural formula is given by  $Q = \mathbb{R}^\times$ ,  $\nu = (\pi k^2)^{-1}$ .

$|y|^{1/k} dy$ . Combining this result with Lemmas 5.3 and 5.4, we see that (2.1) becomes

$$(5.5) \quad N(\lambda, H) \sim \frac{k+1}{\pi k^2} \cdot \log \lambda \cdot \int_{-\infty}^{\infty} |y|^{1/k} N\left(\lambda, -\frac{\partial^2}{\partial x^2} + y^2 x^2\right) dy.$$

To reduce this to the formula of Proposition 5.2, we recall that the eigenvalues of the operator  $-\partial/\partial x^2 + y^2 x^2$  (for  $y \neq 0$ ) are given by  $\{(2j+1)y \mid j \in \mathbb{Z}_+\}$ , and Fubini's theorem implies that for any  $\lambda > 0$ ,

$$\begin{aligned} \int_0^{\infty} y^{1/k} \cdot N(\lambda, -\partial/\partial x^2 + y^2 x^2) dy &= \int_0^{\infty} y^{1/k} \cdot \#\{j \in \mathbb{Z}_+ \mid (2j+1)y \leq \lambda\} dy \\ &= \sum_{j=0}^{\infty} \int_{\{y \geq 0 \mid (2j+1)y \leq \lambda\}} y^{1/k} dy \\ &= \frac{k}{k+1} \lambda^{1+1/k} \sum_{j=0}^{\infty} (2j+1)^{-1-1/k}. \end{aligned}$$

Substituting this into (5.5), we obtain (5.4).

### 5.3. A three-dimensional example

As our last example we briefly discuss the Schrödinger operator  $H = -\Delta + x_1^{2k} x_2^{2l} x_3^{2p}$  on  $L^2(\mathbb{R}^3)$ , where  $1 \leq p < k \leq l$ . Then the set of degeneration,  $\Sigma$ , is the union of coordinate planes  $\Sigma^{jk}$ ,  $j \neq k \in \{1, 2, 3\}$ . The scheme of Section 3 should be modified as follows.

After  $U_0, U_{1,0}$  and  $\chi_0$  are constructed, the sets  $U_j, U_{1j}$  and functions  $\chi_j$  should be constructed separately for  $\Sigma_{-M}$ , then for  $M$ -neighborhoods of coordinate planes (in the metric  $\Psi^*(x)^2|\cdot|^2$ ) but outside  $M^{1/2}$ -neighborhoods (in the same metric) of coordinate axis, and finally, for  $M^{1/2}$ -neighborhoods of coordinate axis. Under the condition  $p < k \leq l$ , the leading contribution comes from the sets adjacent to the axis  $\{l = 0, k = 0\}$ , and we derive

$$(5.6) \quad N(\lambda, H(0) + V) \sim (2\pi)^{-1} \int_{-\infty}^{+\infty} dy_3 \int_{-\infty}^{+\infty} d\eta_3 \cdot N(\lambda, \eta_3^2 - \Delta_{x_1, x_2} + y_3^{2p} x_1^{2k} x_2^{2l}).$$

Making substitutions  $x_1 \mapsto |y_3|^{-\alpha} x_1$ ,  $x_2 \mapsto |y_3|^{-\alpha} x_2$ ,  $\eta_3 \mapsto |y_3|^\alpha \eta_3$ , where  $\alpha = p/(k+l+1)$ , we obtain

$$N(\lambda, H(0) + V) \sim \kappa(V) \lambda^{(\alpha+1)/2\alpha},$$

where

$$(5.7) \quad \kappa(V) = \frac{1}{\pi} \int_0^{+\infty} dy_3 \cdot y_3^\alpha \int_{-\infty}^{+\infty} d\eta_3 \cdot N(y_3^{-2\alpha}, \eta_3^2 - \Delta_{x_1, x_2} + x_1^{2k} x_2^{2l}).$$

Let  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  be the eigenvalues of the operator  $H_1 = -\Delta + x_1^{2k} x_2^{2l}$  in  $L^2(\mathbb{R}^2)$ . Then

$$\begin{aligned} \kappa(V) &= \frac{1}{\pi} \int_0^{+\infty} dy_3 \cdot y_3^\alpha \int_{-\infty}^{+\infty} d\eta_3 \cdot \#\{j \mid \eta_3^2 < y_3^{-2\alpha} - \lambda_j\} \\ &= \sum_j \frac{2}{\pi} \int_0^{+\infty} dy_3 \cdot y_3^\alpha ((y_3^{-2\alpha} - \lambda_j)_+)^{1/2}. \end{aligned}$$

Under the summation sign, change the variable  $y_3 \mapsto \lambda_j^{-1/2\alpha} y_3$ :

$$(5.8) \quad \kappa(V) = \sum_{j \geq 1} \lambda_j^{-1/2\alpha} \kappa_1(V),$$

where

$$\begin{aligned} \kappa_1(V) &= \frac{2}{\pi} \int_0^1 dy_3 \cdot y_3^\alpha (y_3^{-2\alpha} - 1)^{1/2} = \frac{2}{\pi} \int_0^1 dy_3 \cdot (1 - y_3^{2\alpha})^{1/2} \\ &= \frac{1}{\pi\alpha} B(1/2\alpha, 3/2). \end{aligned}$$

We need to prove the convergence of the series in (5.8). If  $k < l$ , then similarly to (5.1), we have

$$N(\lambda; H_1) \sim \kappa(k, l) \lambda^{(l+k+1)/2l},$$

where  $\kappa(k, l) > 0$  is independent of  $\lambda$ , and hence,  $\lambda_j \sim \text{const} \cdot j^{2l/(l+k+1)}$ , as  $j \rightarrow +\infty$ . Since  $-(2l/(l+k+1))/(2\alpha) = -l/p < -1$ , the series converges. If  $k = l$ , then similarly to (5.4),

$$N(\lambda; H_1) \sim \kappa(l) \cdot \log \lambda \cdot \lambda^{1+1/2l},$$

where  $\kappa(l) > 0$  is independent of  $\lambda$ , and hence,  $\lambda_j \sim \text{const} \cdot j^{2l/(2l+1)} / \log j$ , as  $j \rightarrow +\infty$ . Since  $-(2l/(2l+1))/(2\alpha) = -l/p < -1$ , the series converges.

We now wish to derive (5.6) from our conjectural formula. The computations are completely analogous to, and, in a way, simpler (due to the absence of the magnetic potential) than those in §5.1; therefore we restrict ourselves to outlining the main steps. First one verifies without difficulty that we are in the strong degeneration case of Conjecture 2. The Lie algebra  $\mathfrak{g}$  associated to  $H$  has a basis

$$\{L_1, L_2, L_3\} \cup \{L^{(jst)} \mid 0 \leq j \leq 2k, 0 \leq s \leq 2l, 0 \leq t \leq 2p\},$$

where, in the tautological representation of  $\mathfrak{g}$ , we have  $L_j \mapsto \partial/\partial x_j$  ( $j = 1, 2, 3$ ) and  $L^{(jst)} \mapsto \sqrt{-1} \cdot x_1^j x_2^s x_3^t$ . The canonical measure on the orbit corresponding to the tautological representation of  $\mathfrak{g}$  is given by  $\mu_\Omega = (2\pi)^{-3} \cdot \varphi_*(d\xi dx)$ , where  $\varphi : \mathbb{R}^6 \rightarrow \mathfrak{g}^*$  is defined by

$$\langle \varphi(\xi, x), L_j \rangle = \xi_j \quad (j = 1, 2, 3), \quad \langle \varphi(\xi, x), L^{(jst)} \rangle = x_1^j x_2^s x_3^t.$$



As in §5.1, one finds that there exists a weak limit

$$\mu_\infty = \lim_{\lambda \rightarrow +\infty} \lambda^{-3-1/(2p)} \mu_\lambda = (2\pi)^{-3} \cdot \psi_*(d\xi dx),$$

where  $\psi : \mathbb{R}^6 \rightarrow \mathfrak{g}^*$  is defined by

$$\begin{aligned} \langle \psi(\xi, x), L_j \rangle &= \xi_j \quad (j = 1, 2, 3), \\ \langle \psi(\xi, x), L^{(j,s,2p)} \rangle &= x_1^j x_2^s x_3^{2p}, \quad \langle \psi(\xi, x), L^{(jst)} \rangle = 0 \text{ if } t < 2p. \end{aligned}$$

In particular,  $\mu_\infty$  is supported on the annihilator  $\mathfrak{a}^\perp \subset \mathfrak{g}^*$  of the ideal

$$\mathfrak{a} = \text{span}_{\mathbb{R}} \{ L^{(jst)} \mid t < 2p \} \subset \mathfrak{g}.$$

If we denote by “bar” the quotient map  $\mathfrak{g} \rightarrow \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ , then we have a direct sum decomposition of Lie algebras  $\bar{\mathfrak{g}} = \mathbb{R} \cdot \bar{L}_3 \oplus \mathfrak{g}'$ , where  $\mathfrak{g}' = \text{span}_{\mathbb{R}} \{ \bar{L}_1, \bar{L}_2, \bar{L}^{(j,s,2p)} \}$  is isomorphic to the Lie algebra associated to the 2D Schrödinger operator  $-\Delta_{x_1, x_2} + x_1^{2k} x_2^{2l}$ . Using this and the explicit formula for  $\mu_\infty$ , it is easy to see that Conjecture 2(a) predicts (5.6).

## Appendix A. Representation theory

### A.1. Review of induced representations for nilpotent Lie groups

The orbit method invented by Kirillov is based on the notion of an induced representation. Let  $G$  be a Lie group and  $H \subset G$  a closed subgroup. To every unitary representation  $U$  of  $H$  one can associate a unitary representation  $T = \text{Ind}_H^G(U)$  of  $G$ . In general, the construction of  $T$  is somewhat complicated, due to the fact that the quotient space  $G/H$  might not possess a  $G$ -invariant measure, and the projection map  $G \rightarrow G/H$  might have no continuous sections. However, if  $G$  is nilpotent, the situation becomes much simpler. Let us assume that  $G$  is a connected and simply connected Lie group, and  $H \subset G$  a closed connected subgroup ( $H$  is then also simply connected). Let  $\mathfrak{h} \subset \mathfrak{g}$  be the corresponding Lie algebras. A *coexponential basis* for  $\mathfrak{h}$  in  $\mathfrak{g}$  is a set of elements  $X_1, \dots, X_n \in \mathfrak{g}$  such that the map

$$\mathfrak{h} \times \mathbb{R}^n \rightarrow G, \quad (\xi, t_1, \dots, t_n) \mapsto \exp(t_1 X_1 + \dots + t_n X_n) \cdot \exp(\xi)$$

is a diffeomorphism. Such a basis always exists. It can be constructed as follows. Considering the adjoint action of  $\mathfrak{h}$  on  $\mathfrak{g}/\mathfrak{h}$  and using the fact that  $\mathfrak{h}$  is a nilpotent Lie algebra, we see that there exists a subspace  $\mathfrak{a} \supset \mathfrak{h}$  of  $\mathfrak{g}$  such that  $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{h}$ , and  $\mathfrak{h}$  has codimension 1 in  $\mathfrak{a}$ . It is then immediate that  $\mathfrak{a}$  is a subalgebra of  $\mathfrak{g}$ , and  $\mathfrak{h}$  is an ideal of  $\mathfrak{a}$ . Next we apply this construction to  $\mathfrak{a}$  in place of  $\mathfrak{h}$ , and continue inductively. We find that there is a chain

$\mathfrak{h} = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{g}$  of subalgebras of  $\mathfrak{g}$  such that each  $\mathfrak{a}_j$  is an ideal of codimension 1 in  $\mathfrak{a}_{j+1}$ . Now if we choose arbitrary elements  $X_j \in \mathfrak{a}_j$  such that  $X_j \notin \mathfrak{a}_{j-1}$  for all  $1 \leq j \leq n$ , then it is not hard to show, by induction on  $n$ , that  $\{X_j\}$  is a coexponential basis for  $\mathfrak{h}$  in  $\mathfrak{g}$ .

We fix one coexponential basis  $\{X_j\}$  for  $\mathfrak{h}$  in  $\mathfrak{g}$ . Let  $X = G/H$  denote the quotient space, let  $\pi : G \rightarrow X$  be the natural projection, and let  $e : \mathbb{R}^n \rightarrow G$  be defined by  $e(t) = \exp(t_1 X_1 + \cdots + t_n X_n)$ . The composition  $\mathbb{R}^n \xrightarrow{e} G \xrightarrow{\pi} X$  is a diffeomorphism. Composing the inverse of this diffeomorphism with the map  $e$  defines a smooth section  $s : X \rightarrow G$  of the projection  $\pi$ . We also fix a  $G$ -invariant positive Borel measure  $\nu$  on  $X$ ; one can prove that since  $G$  is nilpotent, such a measure always exists, and is unique up to a positive constant multiple. If  $U$  is a unitary representation of  $H$  in a (complex) Hilbert space  $\mathcal{H}$ , then the induced representation  $T = \text{Ind}_H^G(U)$  of  $G$  is realized in the space  $L^2(X, \mathcal{H}; \nu)$  of square-integrable functions  $F : X \rightarrow \mathcal{H}$  with respect to  $\nu$ , according to the following explicit formula:

$$(g \cdot F)(x) = U(s(x)^{-1} \cdot g \cdot s(g^{-1}x))F(g^{-1}x) \quad (g \in G, x \in X).$$

Here,  $g^{-1}x$  denotes the action of the element  $g^{-1} \in G$  on the point  $x \in X$ . A more detailed discussion of induced representations can be found, e.g., in [15], §1 or [2], Chapter V. Note that Kirillov in [15] uses the right homogeneous  $G$ -space  $H \backslash G$  for the construction of the induced representation, whereas we have used  $G/H$ , as in [2]; of course, the two approaches are equivalent.

## A.2. Description of unitary representations

In this subsection we prove the results contained in §2.1 and §2.4. We begin with the

*Proof of Lemma 2.6.* — By definition,  $\mathfrak{g}(f)$  is the kernel of the form  $B_f$ , and  $\mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}]$  is isotropic with respect to  $B_f$  because it is abelian. Hence  $\mathfrak{g}(f) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}]$  is an isotropic subspace of  $\mathfrak{g}$  with respect to  $B_f$ . Let  $\mathfrak{h}$  be any maximal isotropic subspace of  $\mathfrak{g}$  containing  $\mathfrak{g}(f) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}]$ . Then  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$  (and a fortiori a subalgebra) because  $\mathfrak{h}$  contains  $[\mathfrak{g}, \mathfrak{g}]$ . Moreover, we claim that  $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{g}(f)$ . Let  $x, y \in \mathfrak{h}$  and  $z \in \mathfrak{g}$ . Then Jacobi's identity implies that

$$[z, [x, y]] = [[z, x], y] + [x, [z, y]].$$

But  $[z, x] \in \mathfrak{h}$  and  $[z, y] \in \mathfrak{h}$  because  $\mathfrak{h}$  is an ideal, whence  $f$  annihilates both terms on the RHS of the last equation, as  $\mathfrak{h}$  is isotropic with respect to  $B_f$ . Since  $z \in \mathfrak{g}$  is arbitrary, we find that  $[x, y] \in \mathfrak{g}(f)$ , as desired.  $\square$

Our strategy in what follows will be to give a proof of Theorem 2.1 and obtain the explicit formula of Proposition 2.7 as a by-product of our discussion. We use the notation introduced in §2.4 (cf. especially the paragraph preceding the statement of Proposition 2.7).

Define  $\mathfrak{a}_k = \mathfrak{h} \oplus \mathbb{R}L_1 \oplus \dots \oplus \mathbb{R}L_k$ ; since  $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}]$ , we see that  $\mathfrak{h} = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots \subset \mathfrak{a}_n = \mathfrak{g}$  is a chain of ideals of  $\mathfrak{g}$ , each of codimension 1 in the next. It follows easily that  $L_1, \dots, L_n$  is a *coexponential basis* for  $\mathfrak{h}$  in  $\mathfrak{g}$ , i.e., the map

$$\varphi : H \times \mathbb{R}^n \rightarrow G, \quad (h, x) \mapsto \exp(-(x_1L_1 + \dots + x_nL_n)) \cdot h,$$

is a diffeomorphism. To simplify our formulas, we introduce the following notation: if  $x \in \mathbb{R}^n$ , then  $x.L = x_1L_1 + \dots + x_nL_n \in \mathcal{L}_0$ . Since  $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}]$ , we see that  $G/H$  is an abelian Lie group. Hence the composition

$$\pi := \text{proj}_2 \circ \phi^{-1} : G \xrightarrow{\phi^{-1}} H \times \mathbb{R}^n \xrightarrow{\text{proj}_2} \mathbb{R}^n$$

is a Lie group homomorphism with kernel  $H$ , so we obtain an identification  $G/H \cong \mathbb{R}^n$  (as Lie groups). Moreover, if  $s : \mathbb{R}^n \rightarrow G$  is the map  $s(x) = \exp(-x.L)$ , then the composition  $\mathbb{R}^n \xrightarrow{s} G \xrightarrow{\pi} \mathbb{R}^n$  is the identity on  $\mathbb{R}^n$ . Also, the  $G$ -action on  $G/H \cong \mathbb{R}^n$  factors through the action of  $\mathbb{R}^n$  on itself by translations, so the Lebesgue measure on  $\mathbb{R}^n$  is  $G$ -invariant. It follows from §A.1 above that  $\text{Ind}_H^G(\chi_f)$  can be realized as a unitary representation of  $G$  on  $L^2(\mathbb{R}^n)$  defined by the following explicit formula:

$$(g \cdot F)(x) = \chi_f(s(x)^{-1} \cdot g \cdot s(g^{-1}x)) \cdot F(g^{-1}x),$$

where  $g \in G$ ,  $F \in L^2(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n$ , and  $g^{-1}x$  denotes the action of  $g^{-1}$  on  $x$ . From the previous remarks, we see that  $g^{-1}x = -\pi(g) + x$ , so we can rewrite our formula as follows:

$$(A.1) \quad (g \cdot F)(x) = \chi_f(s(x)^{-1} \cdot g \cdot s(-\pi(g) + x)) \cdot F(-\pi(g) + x).$$

The proof of Theorem 2.1(a) is now a straightforward computation. Since  $\mathfrak{g} = \mathfrak{h} \oplus \text{span}\{L_1, \dots, L_n\}$ , we analyze separately the action of  $\mathfrak{h}$  and of the elements  $L_j$  on  $L^2(\mathbb{R}^n)$ . First, if  $h \in H = \exp \mathfrak{h}$ , then  $\pi(h) = 0$ , so we see from (A.1) that  $h$  maps to the operator of multiplication by the function  $x \mapsto \chi_f(s(x)^{-1}hs(x))$ . Differentiating with respect to  $h$ , we find that an element  $\xi \in \mathfrak{h}$  maps to the operator of multiplication by the function  $i \cdot f((\text{Ad } s(x)^{-1})\xi)$ . By definition,  $s(x)^{-1} = s(-x) = \exp(x.L)$ . Thus, to find the image of  $\xi$  explicitly, we must compute  $\exp(x_1 \cdot \text{ad } L_1 + \dots + x_n \cdot \text{ad } L_n)(\xi)$ . Now, even though the operators  $\text{ad } L_j$  do not commute in general, each of their iterated commutators will have the form  $\text{ad } Z$  for

some  $Z \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$ . Hence, we will have  $f([Z, \xi]) = 0$ . So we find that

$$\begin{aligned}
 & \text{(A.2)} \\
 & f((\text{Ad } s(x)^{-1})\xi) \\
 &= \sum_{\alpha_1, \dots, \alpha_n \geq 0} \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot f((\text{ad } L_1)^{\alpha_1} \cdots (\text{ad } L_n)^{\alpha_n}(\xi)) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
 \end{aligned}$$

Next we compute the action of  $L_j$  on  $L^2(\mathbb{R}^n)$ , for  $1 \leq j \leq n$ . If  $g = s(-y) \in G$ , then (A.1) becomes

$$(s(-y) \cdot F)(x) = \chi_f(s(x)^{-1} \cdot s(-y) \cdot s(y+x)) \cdot F(y+x).$$

Differentiating with respect to  $y_j$  at  $y = 0$ , we obtain, using the product rule,

$$(L_j \cdot F)(x) = \frac{\partial F}{\partial x_j}(x) + F(x) \cdot \frac{\partial}{\partial y_j} \Big|_{y=0} \left( \chi_f(s(x)^{-1} \cdot s(-y) \cdot s(y+x)) \right).$$

The second term on the RHS has the form  $\sqrt{-1} \cdot a_j(x) \cdot F(x)$ , where  $a_j(x)$  is a certain real polynomial which can easily be found by using the known formula for the differential of the exponential map  $\exp : \mathfrak{g} \rightarrow G$  at an arbitrary point of  $\mathfrak{g}$ . However, we are not interested in the exact formula for  $a_j(x)$ . We have shown that the elements  $\xi \in \mathfrak{h}$  (and in particular  $L_0, L_{n+1}, \dots, L_N$ ) map to multiplication operators of the form  $\sqrt{-1} \cdot p_\xi(x)$  on  $L^2(\mathbb{R}^n)$ , where  $p_\xi(x)$  is a real polynomial, and that  $L_j$  maps to  $\partial_j + \sqrt{-1} \cdot a_j(x)$  for  $j = 1, \dots, n$ . This proves all the assertions of Theorem 2.1(a) except for the discreteness of spectrum. To complete the proof, we apply the following

**PROPOSITION A.1.** — *A Schrödinger operator (1.1) with polynomial potentials and with  $V \geq 0$  has discrete spectrum if and only if there is no rotation of the coordinate axes in  $\mathbb{R}^n$  making  $V(x)$  and all components of the magnetic tensor  $b_{jk}(x)$  independent of one of the coordinates.*

*Proof.* — The “only if” direction is trivial. For the “if” direction we apply the criterion of [10] stated in §1.3. Thus assume that  $H$  does not have discrete spectrum. It follows that there exists a sequence of points  $x(l) \in \mathbb{R}^n$  such that  $\|x(l)\| \rightarrow \infty$  as  $l \rightarrow \infty$  and  $\Psi^*(x(l)) \leq C'$  for a constant  $C' > 0$ . The sequence  $x(l)/\|x(l)\|$  has an accumulation point  $y$  on the unit sphere in  $\mathbb{R}^n$ ; choose any rotation of the coordinate axes in  $\mathbb{R}^n$  such that the  $x_1$ -axis maps to the ray through the point  $y$ . We will prove that this rotation makes  $V$  and all  $b_{jk}$  independent of the coordinate  $x_1$ .

To simplify the notation, we may assume from the start that  $y = e_1 = (1, 0, \dots, 0)$ . We have  $|\partial^\alpha V(x(l))| \leq C$  and  $|\partial^\alpha b_{jk}(x(l))| \leq C$  for all  $l, j, k$  and all multi-indices  $\alpha$ , for a suitable constant  $C > C'$ . Since we would

like to prove that  $V$  and  $b_{jk}$  are independent of the coordinate  $x_1$ , we may concentrate our attention on  $V$ . We use induction on the total degree  $\deg V$  of  $V(x)$ , the case  $\deg V \leq 1$  being trivial. For any  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , write  $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ . Note that each of the partial derivatives  $\partial V / \partial x_j$  satisfies the same condition as  $V$ ; thus, by induction, they are all independent of the coordinate  $x_1$ . It follows easily that either  $V(x)$  is independent of  $x_1$ , which is what we want to prove; or, up to a nonzero scalar,  $V(x) = x_1 + W(x')$  for a polynomial  $W(x') \in \mathbb{R}[x_2, \dots, x_n]$ . Observe that the sequence  $x(l)_1$  is unbounded, because  $x(l) / \|x(l)\|$  converges to  $e_1$ . Now the sequence  $x(l)'$  cannot be bounded, for otherwise the sequence  $W(x(l)')$  is also bounded and we get a contradiction. However,

$$\frac{\partial V}{\partial x_j}(x) = \frac{\partial W}{\partial x_j}(x')$$

for all  $2 \leq j \leq n$ . Thus, again by induction, there exists a rotation of the coordinates  $x'$  making all partial derivatives of  $W$  independent of one of the coordinates; without loss of generality, let  $x_2$  be the coordinate. Write  $x'' = (x_3, \dots, x_n)$ ; then we have  $W(x') = cx_2 + W_1(x'')$  for a constant  $c \in \mathbb{R}$  and a polynomial  $W_1 \in \mathbb{R}[x_3, \dots, x_n]$ . Since  $x(l) / \|x(l)\|$  converges to  $e_1$ , it is clear that the sequence  $x(l)_1 + cx(l)_2$  is unbounded, whether  $c = 0$  or  $c \neq 0$ . Thus the sequence  $x(l)''$  must be unbounded as well (by the same argument as above), and we may continue in this fashion. Clearly, after  $n$  such steps, we conclude that  $\deg V \leq 1$ , in which case the problem is trivial. □

To apply the criterion of the proposition, note that, by definition,  $b_{jk}(x)$  is the image of  $-\sqrt{-1} \cdot [L_j, L_k]$ , and  $V(x)$  is the image of  $-L_{n+1}^2 - \dots - L_N^2 - \sqrt{-1} \cdot L_0$  under the representation constructed above. We assume that  $-\sqrt{-1} \cdot L_0$  maps to a nonnegative polynomial; then  $V(x)$  is the sum of the image of  $-\sqrt{-1} \cdot L_0$  and the images of  $(\sqrt{-1} \cdot L_s)^2$  ( $n + 1 \leq s \leq N$ ). Each  $\sqrt{-1} \cdot L_s$  maps to a real polynomial. We see that if there exists a rotation of the coordinate axes with the property defined in the lemma, then it makes the images of  $[L_j, L_k]$  and  $L_0, L_s$  ( $n + 1 \leq s \leq N$ ) separately independent of one of the coordinates. But the operator  $L_j$  maps to  $\partial / \partial x_j + \sqrt{-1} \cdot a_j(x)$  for all  $1 \leq j \leq n$ , so we see that there exists a rotation of the coordinate axes as above if and only if there exists an element  $D = y_1 L_1 + \dots + y_n L_n \in \mathcal{L}_0$  of unit length such that the image of  $D$  commutes with the images of  $[L_j, L_k]$ ,  $L_0, L_s$  and all their derivatives. Since the operators  $L_0, L_1, \dots, L_N$  generate  $\mathfrak{g}$ , we see that the image of  $[D, \xi]$  in our representation is zero for each  $\xi \in \mathfrak{h}$ , and a fortiori,  $f([D, \xi]) = 0$  for all  $\xi \in \mathfrak{h}$ . But this contradicts the

assumption that  $\mathfrak{h}$  is a maximal totally isotropic subspace with respect to the form  $B_f$ .

Comparing (A.1) with (2.4), we see that we have obtained a proof of Proposition 2.7.

We now prove Theorem 2.1(b). Suppose we are given a Schrödinger operator (1.1), let  $\mathfrak{g}$  be the corresponding Lie algebra and  $S \in \mathcal{U}(\mathfrak{g})_{\mathbb{C}}$  the corresponding sublaplacian. Let  $\mathfrak{h} \subset \mathfrak{g}$  be the subspace spanned by  $\sqrt{-1} \cdot b_{jk}(x)$ ,  $\sqrt{-1} \cdot V(x)$  and all their derivatives. Thus  $\mathfrak{h}$  is an abelian ideal of  $\mathfrak{g}$  that contains  $[\mathfrak{g}, \mathfrak{g}]$ ; it can also be described as the part of  $\mathfrak{g}$  consisting of operators of order 0. Let  $f : \mathfrak{h} \rightarrow \mathbb{R}$  be the linear functional given by  $f(\sqrt{-1} \cdot P(x)) = P(0)$ , and extend  $f$  in an arbitrary way to all of  $\mathfrak{g}$ . The construction of induced representation given above works for the character  $\chi_f$  of  $H = \exp \mathfrak{h}$  defined by  $\chi_f(\exp \xi) = \sqrt{-1} \cdot f(\xi)$  without the assumption that  $\mathfrak{h}$  is a real polarization at  $f$ . Thus, we obtain a unitary representation of  $G = \exp \mathfrak{g}$  on the space  $L^2(\mathbb{R}^n)$ . Note that if we set  $L_j = \partial/\partial x_j + \sqrt{-1} \cdot a_j(x)$ , then  $\text{ad } L_j$  acts on  $\mathfrak{h}$  as  $\partial/\partial x_j$ , so comparing (A.2) to the standard Taylor's formula, we find that the representation of  $\mathfrak{g}$  we have just defined coincides with the "tautological representation" at least on the subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . In particular, since the components of the magnetic tensor lie in  $\mathfrak{h}$ , we see that the image of the sublaplacian  $S$  in this representation is at least gauge equivalent to the original Schrödinger operator (1.1). Finally, Kirillov's theory [15] implies that the representation of  $G$  we have constructed is irreducible if and only if  $\mathfrak{h}$  is a maximal totally isotropic subspace with respect to  $B_f$ . The fact that this condition is equivalent to the discreteness of spectrum of  $H$  follows from Proposition A.1 using the same argument as in the previous paragraph.

### A.3. Parameterization of coadjoint orbits and Kostant measures

Here we prove the results stated in §2.5. We keep the notation used in §2.4, §2.5 and §A.2.

*Proof of Proposition 2.10.* — Let  $H = \exp \mathfrak{h}$ ; this is an abelian normal subgroup of  $G$ . The adjoint action of  $G$  on  $\mathfrak{g}$  leaves  $\mathfrak{h}$  stable, whence  $G$  acts on  $\mathfrak{h}^*$ . Restriction of linear functionals defines a surjective  $G$ -equivariant linear map  $\pi : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ . Since  $H$  is abelian, it acts trivially on  $\mathfrak{h}^*$ , so the action of  $G$  on  $\mathfrak{h}^*$  factors through its abelian quotient  $G/H$ . In particular, if  $\Omega = G \cdot f_0 \subset \mathfrak{g}^*$  is the coadjoint orbit of  $f_0$ , then  $\pi(\Omega) \subset \mathfrak{h}^*$  is the  $G/H$ -orbit of  $f_0|_{\mathfrak{h}}$ .

On the other hand, since  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ , we see that for any  $g \in G$ ,  $\mathfrak{h}$  is a real polarization of  $\mathfrak{g}$  at  $(\text{Ad}^* g)(f_0)$ . Now Pukanszky's criterion (see [30] or [2], Chapter VI) implies that if  $f \in \mathfrak{g}^*$  is any point such that  $\mathfrak{h}$  is a real polarization at  $f$ , then the  $G$ -orbit of  $f$  contains  $f + \mathfrak{h}^\perp = \{f' \in \mathfrak{g}^* \mid f'|_{\mathfrak{h}} = f|_{\mathfrak{h}}\}$ . With the notation of the previous paragraph, we see that  $\Omega = \pi^{-1}(\pi(\Omega))$ . On the other hand, by construction,  $L_1, \dots, L_n$  is a complementary basis to  $\mathfrak{h}$  in  $\mathfrak{g}$ . So we see that it suffices to prove that the map  $\phi : \mathbb{R}^n \rightarrow \mathfrak{h}^*$  given by

$$(A.3) \quad \phi(x)(Y) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot f_0((\text{ad } L_1)^{\alpha_1} \cdots (\text{ad } L_n)^{\alpha_n}) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ for all } Y \in \mathfrak{h}$$

is a diffeomorphism of  $\mathbb{R}^n$  onto  $\pi(\Omega)$ .

Now  $\pi(\Omega)$  is the  $G/H$ -orbit of  $f_0|_{\mathfrak{h}}$  in  $\mathfrak{h}^*$ . Recall that in §A.2 we have obtained an identification of  $G/H$  with the abelian Lie group  $\mathbb{R}^n$ . More precisely, if  $s : \mathbb{R}^n \rightarrow G$  is the map defined by  $s(x) = \exp(-x.L)$ , then the composition  $\mathbb{R}^n \xrightarrow{s} G \rightarrow G/H$  is a Lie group isomorphism. Hence  $\pi(\Omega)$  can be parameterized by

$$(A.4) \quad \mathbb{R}^n \ni x \longmapsto (\text{Ad}^* \exp(-x_1 L_1 - \cdots - x_n L_n))(f_0|_{\mathfrak{h}}).$$

But  $\text{Ad}^* \exp(-x.L) = \exp(x_1 \cdot (\text{ad } L_1) + \cdots + x_n \cdot (\text{ad } L_n))$ . Since  $\mathfrak{h}$  is an abelian ideal of  $\mathfrak{g}$ , the operators  $\text{ad } L_1, \dots, \text{ad } L_n$  commute on  $\mathfrak{h}$ . Thus we immediately see that the parameterization (A.4) of  $\pi(\Omega)$  agrees with the one given by (A.3). Finally, it remains to check that  $\phi$  is a diffeomorphism, i.e., that the stabilizer of  $f_0|_{\mathfrak{h}}$  in  $G/H$  is trivial. Let  $H'$  be the stabilizer of  $f_0|_{\mathfrak{h}}$  in  $G$ . Since  $\mathfrak{h}$  is a maximally isotropic subspace with respect to the form  $B_{f_0}$ , it is easy to see that the Lie algebra of  $H'$  coincides with  $\mathfrak{h}$ . Lastly,  $H'$  is connected; this follows from the general theory of unipotent representations of nilpotent Lie groups ([2], Ch. I). □

*Proof of Proposition 2.11.* — Fix a point  $f \in \Omega$ . Since  $\mathfrak{h}$  is a maximal isotropic subspace of  $\mathfrak{g}$  with respect to  $B_f$  and  $L_1, \dots, L_n$  is a complementary basis to  $\mathfrak{h}$  in  $\mathfrak{g}$ , there exist elements  $Y_1, \dots, Y_n \in \mathfrak{h}$  such that  $B_f(L_j, Y_k) = \delta_{jk}$  (the Kronecker delta). Define

$$Z_j = -L_j + \sum_{k \neq j} f([L_k, L_j]) \cdot Y_k.$$

For each  $Y \in \mathfrak{g}$ , let us write  $\nu_Y$  for the tangent vector to  $\Omega$  at  $f$  generated by  $Y$  (via the  $G$ -action on  $\Omega$ ), and let us write  $\epsilon_Y$  for the function  $\Omega \rightarrow \mathbb{R}$ ,  $f' \mapsto f'(Y)$ . We have the following

LEMMA A.2. — *If  $Y, Z \in \mathfrak{g}$ , then  $\nu_Y(\epsilon_Z) = f([Z, Y]) = B_f(Z, Y)$ .*

The proof is a completely straightforward computation (see, e.g., [2], Chapter III). The lemma implies that  $\nu_{Y_j}(\epsilon_{L_k}) = B_f(L_k, Y_j) = \delta_{jk}$ , and also

$$\nu_{Z_j}(\epsilon_{L_k}) = B_f(L_j - \sum_{s \neq j} f([L_s, L_j]) \cdot Y_s, L_k).$$

If  $j = k$ , the last expression is trivially zero. If  $j \neq k$ , the expression equals

$$B_f(L_j, L_k) - f([L_k, L_j]) \cdot B_f(Y_k, L_k) = 0.$$

Now we wish to compute the vector field  $\partial/\partial\xi_j, \partial/\partial x_k$  on  $\Omega$  corresponding to the coordinates  $(\xi, x)$ . Note that, with the notation above, we have  $\xi_j = \epsilon_{L_j}$ . Thus, we see that

$$\nu_{Y_j}(\xi_k) = \delta_{jk} \quad \text{and} \quad \nu_{Z_j}(\xi_k) = 0 \quad \text{for all } j, k.$$

On the other hand, if  $\xi \in \mathbb{R}^n$  is fixed, then by construction, the map

$$\mathbb{R}^n \rightarrow \pi(\Omega) \subset \mathfrak{h}^*, \quad x \mapsto \pi(\varphi(\xi, x)),$$

coincides with the map

$$x \mapsto (\text{Ad}^* \exp(-x.L))(\pi(\varphi(\xi, 0)))$$

(cf. the proof of Proposition 2.10). Since  $Y_k \in \mathfrak{h}$  for all  $k$  and  $H$  acts trivially on  $\mathfrak{h}^*$ , we find that  $\exp(t \cdot Z_j)$  acts on  $\pi(\Omega) \cong \mathbb{R}^n$  as translation by  $t$  in the  $x_j$ -direction. It is then immediate that

$$\nu_{Z_j}(x_k) = \delta_{jk} \quad \text{and} \quad \nu_{Y_j}(x_k) = 0 \quad \text{for all } j, k.$$

We conclude that the values of the vector fields  $\partial/\partial\xi_j$  and  $\partial/\partial x_k$  at the point  $f \in \Omega$  are given by  $\nu_{Y_j}$  and  $\nu_{Z_k}$ , respectively. Therefore

$$\omega_\Omega(\partial/\partial\xi_j, \partial/\partial\xi_k) = B_f(Y_j, Y_k) = 0;$$

$$\omega_\Omega(\partial/\partial\xi_j, \partial/\partial x_k) = B_f(Y_j, Z_k) = B_f(L_k, Y_j) = \delta_{jk};$$

$$\omega_\Omega(\partial/\partial x_j, \partial/\partial x_k) = B_f(Z_j, Z_k) = f([L_k, L_j]).$$

This proves (2.5), and the formula for the Kostant measure  $\mu_\Omega$  follows trivially from this. □

#### A.4. Limits of polynomial measures and quotient measures

In this subsection we prove the propositions stated in §2.6.



*Proof of Proposition 2.12.* — For every compact set  $K \subset \mathbb{R}^N$ , we have  $\tilde{\phi}_j^{-1}(K), \tilde{\phi}^{-1}(K) \subseteq \phi^{-1}(K')$ , where  $K'$  is the projection of  $K$  onto the first  $N_1$  coordinates, so  $\mu_j$  and  $\mu$  are regular. Now let  $F$  be a continuous function on  $\mathbb{R}^N$  that is supported on  $K$ ; it follows that

$$\int_{\mathbb{R}^N} F d\mu_j = \int_{\phi^{-1}(K')} F(\tilde{\phi}_j(x)) dm(x)$$

and

$$\int_{\mathbb{R}^N} F d\mu = \int_{\phi^{-1}(K')} F(\tilde{\phi}(x)) dm(x),$$

where  $K'$  is defined as before. Since  $\phi^{-1}(K')$  has finite measure, and the integrands above are bounded by a constant independent of  $j$  (namely, the sup norm of  $F$ ), the Dominated Convergence Theorem applies. Thus, we may pass to the pointwise limit as  $j \rightarrow \infty$  inside the first integral, which proves the proposition.  $\square$

*Proof of Proposition 2.13.* — For the first statement, it suffices to show that the limit on the RHS of (2.7) exists and is finite for all  $F \in C_c(\mathbb{R}^N)$ . Indeed, it is then obvious that this limit defines a positive linear functional on  $C_c(\mathbb{R}^N)$ , so the Riesz representation theorem yields the existence and uniqueness of  $\mu_\infty$ .

Fix  $f \in C_c(\mathbb{R}^N)$ ,  $\epsilon > 0$ , and  $R > 0$  such that

$$\text{supp}(f) \subseteq \{x \in \mathbb{R}^N \mid \|x\|_\infty \leq R - 1\}.$$

Now  $f$  is uniformly continuous, so there exists  $0 < \delta < 1$  such that if  $x, y \in \mathbb{R}^N$  and  $\|x - y\|_\infty \leq \delta$ , then  $|f(x) - f(y)| \leq \epsilon$ . Write  $B_{\infty,R} = \{x \in \mathbb{R}^N \mid \|x\|_\infty \leq R\}$ . Then there exist finitely many points  $m_1, \dots, m_K \in \mathbb{Z}^N$  such that the sets  $[\delta \cdot m_k, \delta \cdot (m_k + \mathbf{1}))$  cover  $B_{\infty,R}$ . Clearly, for each  $y_0 \in B_{\infty,1}$ , the sets  $[\delta \cdot m_k + y_0, \delta \cdot (m_k + \mathbf{1}) + y_0)$  will cover  $B_{\infty,R-1}$ . Since  $B_{\infty,1}$  is uncountable, we can find  $y_0 \in B_{\infty,1}$  such that each of the coordinates of the finitely many points  $\{\delta \cdot m_k + y_0\}_{k=1}^K \cup \{\delta \cdot (m_k + \mathbf{1}) + y_0\}_{k=1}^K$  is not contained in the countable set  $E$ . On the other hand, we still have  $[\delta \cdot m_k + y_0, \delta \cdot (m_k + \mathbf{1}) + y_0) \subseteq B_{\infty,R+1}$ . By assumption,

$$M := \sup_{\lambda > 0} \mu_\lambda(B_{\infty,R+1}) \leq \sup_{\lambda > 0} \mu_\lambda([- (R + 1) \cdot \mathbf{1}, (R + 2) \cdot \mathbf{1}]) < +\infty.$$

For each  $1 \leq k \leq K$ , put  $P_k = [\delta \cdot m_k + y_0, \delta \cdot (m_k + \mathbf{1}) + y_0)$ , and

$$u_k = \max_{x \in \bar{P}_k} F(x), \quad v_k = \min_{x \in \bar{P}_k} F(x).$$

By construction,  $v_k \leq u_k \leq v_k + \epsilon$ , and

$$(A.5) \quad \sum_{k=1}^K v_k \cdot \mu_\lambda(P_k) \leq \int_{\mathbb{R}^N} F d\mu_\lambda \leq \sum_{k=1}^K u_k \cdot \mu_\lambda(P_k).$$

This implies that

$$\left( \limsup_{\lambda \rightarrow +\infty} \int F d\mu_\lambda \right) - \left( \liminf_{\lambda \rightarrow +\infty} \int F d\mu_\lambda \right) \leq \epsilon \cdot \sum_k \left( \lim_{\lambda \rightarrow +\infty} \mu_\lambda(P_k) \right) \leq \epsilon \cdot M.$$

Since  $M$  does not depend on  $\epsilon$ , this proves the first part of the proposition.

For the second part, we proceed as above, and combine (A.5) with the inequalities

$$\sum_k v_k \cdot \nu_\infty(P_k) \leq \int F d\nu_\infty \leq \sum_k u_k \cdot \nu_\infty(P_k),$$

which show that both  $\limsup_{\lambda \rightarrow +\infty} \int F d\mu_\lambda$  and  $\liminf_{\lambda \rightarrow +\infty} \int F d\mu_\lambda$  are within  $\epsilon \cdot M$  of  $\int F d\nu_\infty$ , for any  $\epsilon > 0$ .  $\square$

It remains to prove Proposition 2.15. We begin with a very useful standard result from measure theory. Let  $Z$  be any set. A collection  $\mathcal{A}$  of subsets of  $Z$  is called

- a  $\pi$ -system if it is closed under finite intersections;
- a  $\lambda$ -system if  $Z \in \mathcal{A}$ ; whenever  $A_1, A_2 \in \mathcal{A}$  and  $A_1 \subseteq A_2$ , we have  $A_2 \setminus A_1 \in \mathcal{A}$ ; and  $\mathcal{A}$  is closed under countable disjoint unions.

The following result is proved, e.g., in [16].

LEMMA A.3 (Lemma on  $\pi$ -systems and  $\lambda$ -systems). — *Let  $\Pi$  and  $\Lambda$  be a  $\pi$ -system and a  $\lambda$ -system on a set  $Z$ , and assume that  $\Pi \subseteq \Lambda$ . Then  $\Lambda$  contains the  $\sigma$ -algebra generated by  $\Pi$ .*

In what follows, to simplify the terminology, let us say that a *good* measure on a topological space is a nontrivial positive Borel measure which is finite on compact sets.

PROPOSITION A.4. — *Let  $G$  be a nilpotent Lie group,  $Y = G/H$  a homogeneous space for  $G$  (in particular, a manifold), and  $X$  an arbitrary (finite dimensional) manifold. Let  $G$  act on  $X \times Y$  via its action on the second factor. Then any  $G$ -invariant good measure  $\mu$  on  $X \times Y$  has the form  $\mu = \mu_X \times \mu_Y$ , where  $\mu_X$  is a Borel measure on  $X$  and  $\mu_Y$  is a good  $G$ -invariant measure on  $Y$  (unique up to a positive scalar multiple). Moreover, if  $\mu_Y$  is fixed, then  $\mu_X$  is uniquely determined and is automatically good.*

*Proof.* — It is known that since  $G$  is nilpotent, there is a good  $G$ -invariant measure  $\mu_Y$  on  $Y$ , unique up to a positive scalar multiple. Let us choose such a  $\mu_Y$  once and for all, and fix a compact set  $B \subset Y$  with  $\mu_Y(B) > 0$ . Now if  $\mu_X$  exists, it is uniquely determined, since for every Borel subset  $A \subseteq X$  we must have  $\mu_X(A) = \mu(A \times B)/\mu_Y(B)$ . This also shows that  $\mu_X$  must be good. Conversely, if we define  $\mu_X$  by this formula, it is immediate that  $\mu_X$  is a good measure on  $X$ , and we need to verify that  $\mu = \mu_X \times \mu_Y$ .

Fix a compact subset  $K \subset X$ . We then obtain a measure  $\mu^K$  on  $Y$  via  $\mu^K(C) = \mu(K \times C)$  for every Borel subset  $C \subseteq Y$ . Since  $\mu$  is good and  $G$ -invariant, so is  $\mu^K$ , and it follows that  $\mu^K = c_K \cdot \mu_Y$  for a constant  $c_K > 0$ . Taking  $C = B$  we find that  $c_K = \mu_X(K)$ , whence  $\mu(K \times C) = \mu_X(K) \cdot \mu_Y(C)$  for all compact  $K \subset X$  and all Borel  $C \subseteq Y$ . Since  $X$  and  $Y$  are manifolds, the “rectangles” of the form  $K \times C$  as above form a  $\pi$ -system which generates the Borel  $\sigma$ -algebra on  $X \times Y$ . On the other hand, the collection of Borel subsets of  $X \times Y$  on which the two measures  $\mu$  and  $\mu_X \times \mu_Y$  agree is manifestly a  $\lambda$ -system. Now Lemma A.3 completes the proof. □

It is easy to deduce Proposition 2.15 from this result. Observe first that both the existence and the uniqueness of  $\nu$  are “local” questions in the following sense. If we write  $\mathfrak{g}^*$  as a disjoint union of finitely many  $G$ -invariant Borel sets  $Z_1, Z_2, \dots, Z_N$  and prove the result with  $\mathfrak{g}^*$  and  $\mathfrak{g}^*/G$  replaced by  $Z_j$  and  $Z_j/G$ , respectively, then the general case follows. The idea is to use this remark to reduce the problem to the situation of Proposition A.4

The main result of [3] provides a stratification of  $\mathfrak{g}^*$  by locally closed (even in the Zariski topology) subsets, such that each stratum is equipped with a triple  $(r, q, p)$  of vector-valued functions having the following properties:

- every stratum is a union of coadjoint orbits, which are given by the level sets  $r = \text{const}$ ;
- for a fixed  $r = r_0$ , the pair  $(q, p)$  is a global chart for the corresponding orbit  $\Omega_{r_0}$ , such that the canonical symplectic form on the orbit is given by  $\sum_{j=1}^{\ell} dq_j \wedge dp_j$ .

It is understood that the number  $\ell$  depends on the stratum, and the functions  $r, q, p$  take values in  $\mathbb{R}^{n-2\ell}, \mathbb{R}^{\ell}, \mathbb{R}^{\ell}$ , respectively. It follows that the Kostant measure on  $\Omega_{r_0}$  also has the simple form

$$(A.6) \quad (2\pi)^{-\ell} \cdot dq_1 \cdots dq_{\ell} dp_1 \cdots dp_{\ell}$$

in the coordinates  $(q, p)$ . Now the restriction of  $\mu$  to any locally closed subset of  $\mathfrak{g}^*$  clearly remains good, and the proof is complete.

### Appendix B. Auxiliary computations

#### B.1. Proof of convergence of the integral (3.4)

Change the variables  $z \mapsto a^{-1/6}z$ ,  $b \mapsto a^{1/6}b$ , and then  $a^{7/6} = u$ , to obtain

$$\kappa(H) = \frac{6}{7\pi} \int_0^\infty du \int_{-\infty}^{+\infty} db \cdot N \left( u^{-2/7}, -\frac{d^2}{dz^2} + (z^2 + b)^2 \right).$$

Introduce  $A_0 = -d^2/dz^2 + (z^2 + b)^2$  and  $A = -d^2/dz^2 + z^4$ , as self-adjoint operators in  $L^2(\mathbb{R})$ . Since  $A_0$  is positive definite, there exist  $c, c_1 > 0$  such that for all  $b \geq -c_1$ ,  $A_0 \geq 2^{-2/7}(A + c + b^2)$ , hence

$$\begin{aligned} \int_0^\infty du \int_{-c_1}^{+\infty} db \cdot N \left( u^{-2/7}, -\frac{d^2}{dz^2} + (z^2 + b)^2 \right) \\ \leq \int_0^\infty du \int_{-c_1}^{+\infty} db \cdot N \left( (u/2)^{-2/7}, A + b^2 + c \right). \end{aligned}$$

The classical Weyl formula gives  $N(\lambda, A) \sim \text{const} \cdot \lambda^{1/2+1/4}$ , as  $\lambda \rightarrow +\infty$ , therefore we have an upper bound via

$$C_1 \int_{-c_1}^{+\infty} db \int_0^{2(c+b^2)^{-7/2}} du \left( (u/2)^{-2/7} - (c + b^2) \right)^{3/4}.$$

Change the variable  $u \mapsto 2(c+b^2)^{-7/2}u$  to obtain the product of a constant and two integrals:

$$2C_1 \int_{-c_1}^{+\infty} db (c + b^2)^{-7/2+3/4} \int_0^1 du (u^{-2/7} - 1)^{3/4},$$

which converge.

It remains to prove the convergence of the integral  $\int_{-\infty}^{-c_1} db \int_0^\infty du$ . For each  $b < 0$ , the potential  $(b + z^2)^2$  has two wells, at  $z = \pm\sqrt{-b}$ , and grows as  $|z|^4$  at infinity. The leading term of the Taylor expansion at  $\pm\sqrt{-b}$  is  $8(-b)(z \mp \sqrt{-b})^2$ , therefore  $A_0 \geq c_2(-b)^{1/2}$ , and

$$N(\lambda, A_0) \leq CN(\lambda, -d^2/dz^2 + (-b)z^2) \leq C_1\lambda(-b)^{-1/2}.$$

We conclude that the integrand below vanishes for  $b < -C_3u^{-4/7}$ , and therefore

$$\begin{aligned} \int_{-\infty}^{-c_1} db \int_0^\infty du N \left( u^{-2/7}, -\frac{d^2}{dz^2} + (z^2 + b)^2 \right) \\ \leq C_2 \int_0^{C_3} du \int_{-C_3u^{-4/7}}^{-c_1} db u^{-2/7} (-b)^{-1/2}. \end{aligned}$$

The convergence of the integral on the RHS is straightforward.

**B.2. Proof of Lemma 4.3**

Fix  $\epsilon < \epsilon' < 1$ . We have

$$\begin{aligned} \text{meas}\{P(x) + Q(x) \leq \lambda\} &\leq \text{meas}\{P(x) \leq \lambda\} = \text{meas}(\delta_\lambda(\{P(x) \leq 1\})) \\ &= \lambda^{|\gamma|} \cdot \text{meas}\{P(x) \leq 1\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{meas}\{P(x) + Q(x) \leq \lambda\} &\geq \text{meas}\{P(x) \leq \lambda - \lambda^{\epsilon'}, Q(x) \leq \lambda^{\epsilon'}\} \\ &= \lambda^{|\gamma|} \cdot \text{meas}(\delta_\lambda^{-1}(\{P(x) \leq \lambda - \lambda^{\epsilon'}, Q(x) \leq \lambda^{\epsilon'}\})) \\ &\geq \lambda^{|\gamma|} \cdot \text{meas}\{P(x) \leq 1 - \lambda^{\epsilon'-1}, Q(x) \leq C^{-1}\lambda^{\epsilon'-\epsilon}\}. \end{aligned}$$

Since  $Q$  is finite a.e., we have

$$\{P(x) \leq 1\} = \bigcup_{\lambda \rightarrow +\infty} \{P(x) \leq 1 - \lambda^{\epsilon'-1}, Q(x) \leq C^{-1}\lambda^{\epsilon'-\epsilon}\}$$

(increasing union), whence

$$\lim_{\lambda \rightarrow +\infty} \text{meas}\{P(x) \leq 1 - \lambda^{\epsilon'-1}, Q(x) \leq C^{-1}\lambda^{\epsilon'-\epsilon}\} = \text{meas}\{P(x) \leq 1\},$$

completing the proof.

**B.3. Proof of Lemma 4.10**

Introduce the dilation  $\tilde{\delta}_\lambda$  in the space  $\mathbb{R}^{2n}$  as follows:

$$\tilde{\delta}_\lambda(\xi, x) = (\lambda^{1/2}\xi, \lambda^{\gamma_1}x_1, \dots, \lambda^{\gamma_n}x_n),$$

and let  $\delta_\lambda$  on  $\mathbb{R}^n$  be defined as in §4.1. Set

$$\tilde{\Psi}^*(\xi, x) = \Psi^*(x)^2 + \|\xi\|^2, \quad \tilde{\Psi}_0(\xi, x) = \Psi_0(x)^2 + \|\xi\|^2$$

$$\text{and } \tilde{\Psi}(\xi, x) = \tilde{\Psi}^*(\xi, x) - \tilde{\Psi}_0(\xi, x).$$

Then it is easy to see that  $\tilde{\Psi}^*(\tilde{\delta}_\lambda(\xi, x)) = \lambda \cdot \tilde{\Psi}^*(\xi, x)$ , and also, using Lemma 4.2, we find that there exists  $0 < q < 1$  with  $\tilde{\Psi}(\tilde{\delta}_\lambda(\xi, x)) \leq \lambda^q \tilde{\Psi}(\xi, x)$  for all  $\lambda > 1$ . Now the proof of (4.10) is the same as the proof of Corollary 4.4.

To prove that (4.3) holds, we have to show that

$$\begin{aligned} \text{(B.1)} \quad &\text{meas}\{\Psi^*(x) < M \cdot \Psi(x), \Psi^*(x)^2 + \|\xi\|^2 \leq C\lambda\} \\ &= o(\text{meas}\{\Psi^*(x)^2 + \|\xi\|^2 \leq \lambda\}) \end{aligned}$$

as  $\lambda \rightarrow +\infty$  for every fixed  $C > 0$ , where  $M = M(\lambda)$  is a suitable function such that  $M(\lambda) \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ .

We assume that  $M$  grows slower than any power of  $\lambda$ . Certainly, if  $\Psi^*(x)^2 + \|\xi\|^2 \leq C\lambda$ , then  $\tilde{\Psi}_0(\xi, x) \leq C\lambda$ ; combining this with (4.10), we see that to prove (B.1), it suffices to check that

$$\text{meas}\{\Psi_0(x) < M \cdot \Psi(x), \tilde{\Psi}_0(\xi, x) \leq C\lambda\} = o\left(\text{meas}\{\tilde{\Psi}_0(\xi, x) \leq \lambda\}\right)$$

as  $\lambda \rightarrow +\infty$ . Now, the dilation  $\tilde{\delta}_\lambda$  scales the Lebesgue measure by a factor of  $\lambda^{|\gamma|+n/2}$ . Dividing both sides of the last equality by  $\lambda^{|\gamma|+n/2}$ , we see that we can rewrite it as

$$\begin{aligned} \text{(B.2)} \quad & \text{meas}\{\Psi_0(\delta_\lambda x) < M \cdot \Psi(\delta_\lambda x), \tilde{\Psi}_0(\xi, x) \leq C\} \\ & = o\left(\text{meas}\{\tilde{\Psi}_0(\xi, x) \leq 1\}\right) \quad \text{as } \lambda \rightarrow +\infty. \end{aligned}$$

But  $\text{meas}\{\tilde{\Psi}_0(\xi, x) \leq 1\}$  is a finite nonzero constant; on the other hand, the set on the left is contained in the set of points  $(\xi, x) \in \mathbb{R}^{2n}$  for which  $\tilde{\Psi}_0(\xi, x) \leq C$  and  $\Psi_0(x) < M\lambda^{q-1}\Psi(x)$ . Since the set where  $\tilde{\Psi}_0(\xi, x) \leq C$  has finite measure by assumption, we can take the limit as  $\lambda \rightarrow +\infty$ ; since  $M \cdot \lambda^{q-1} \rightarrow 0$  as  $\lambda \rightarrow +\infty$  by construction, we see that the limit of the LHS of (B.2) as  $\lambda \rightarrow +\infty$  is at most the measure of the set where  $\Psi_0(x) = 0$ , which is zero because the potentials are polynomial.

#### B.4. Proof of Lemma 5.3

Throughout the proof, we use the notation  $M$  for a function of  $\lambda$  that grows slower than any power of  $\log \lambda$  as  $\lambda \rightarrow +\infty$ , such as  $M = \log \log \lambda$  (the precise formula is not important). Also,  $C$  or  $C'$  appearing in an inequality will always stand for a constant which can be chosen to make the inequality work, and if the inequality in question depends on an additional variable such as  $x$ , it is to be understood that  $C$  or  $C'$  can be chosen uniformly for all  $x$ . Finally,  $h(\lambda)$  will stand for any function of  $\lambda$  such that  $h(\lambda) \rightarrow 1$  as  $\lambda \rightarrow +\infty$ . We define

$$G_0(\lambda) = \text{meas}\{x \in \mathbb{R}^2 \mid |x_1|, |x_2| \geq M, |x_1^k x_2^k| \leq \lambda^2 \cdot h(\lambda)\}.$$

One immediately computes (using Fubini's theorem) that  $G_0(\lambda) \sim (8/k) \cdot \lambda^{2/k} \log \lambda$  as  $\lambda \rightarrow +\infty$ . On the other hand, we have  $\Phi^*(x) = |x_1^k x_2^k|^{1/2} + R(x)$ , where

$$R(x) = \sum_{j>0 \text{ or } l>0} \left| \frac{k!}{(k-j)!} \cdot \frac{k!}{(k-j)!} \cdot x_1^{k-j} x_2^{k-l} \right|^{1/2}.$$

If  $|x_1| \geq M$  and  $|x_2| \geq M$ , then  $R(x) \leq (C/M) \cdot |x_1^k x_2^k|$ . Therefore, if  $|x_1| \geq M$ ,  $|x_2| \geq M$  and  $|x_1^k x_2^k| \leq \lambda^2 \cdot (1 - \frac{C}{M})$ , then  $\Phi^*(x) \leq \lambda$ . Taking  $h(\lambda) = 1 - C/M$  and applying the statement about the asymptotics of  $G_0(\lambda)$ , we see that  $G_1(\lambda) = \text{meas}\{\Phi^* \leq \lambda\}$  is bounded below by a function which has the required asymptotics as  $\lambda \rightarrow +\infty$ . On the other hand, we have

$$\begin{aligned} \{\Phi^*(x) \leq \lambda\} \subseteq \{|x_1|, |x_2| \geq M, |x_1^k x_2^k| \leq \lambda^2\} \cup \{|x_1| \leq \lambda, \Phi^*(x) \leq \lambda\} \\ \cup \{|x_2| \leq \lambda, \Phi^*(x) \leq \lambda\}. \end{aligned}$$

As above, the measure of the first set on the RHS has the required asymptotics, while the measure of the second set is bounded by

$$\text{meas}\{|x_1| \leq M, |x_2| \leq \lambda^{2/k}\} = 2M\lambda^{2/k},$$

which grows slower than  $\lambda^{2/k} \log \lambda$ . By symmetry, the same statement holds for the third set. This provides the desired upper bound and concludes the proof of the formula  $G_1(\lambda) \sim (8/k) \cdot \lambda^{2/k} \log \lambda$ .

Next we study the asymptotics of the function  $G_2(\lambda) \sim \text{meas}\{\Psi^* \leq \lambda\}$ . We will first prove that the asymptotics of  $G_2(\lambda)$  is controlled by the measure of the set where  $\Psi^*(x) \leq \lambda$  and  $\Psi^*(x)$  is dominated by the first term, in the sense that  $|x_1^k x_2^k|^{1/2} \geq M \cdot |x_1^{k-j} x_2^{k-l}|^{1/(2+j+l)}$  whenever  $j > 0$  or  $l > 0$ . To this end, we fix  $(j, l) \neq (0, 0)$  and consider the measure of the set of points  $(x_1, x_2) \in \mathbb{R}^2$  such that  $\Psi^*(x) \leq \lambda$  and

$$(B.3) \quad |x_1^k x_2^k|^{1/2} \leq M \cdot |x_1^{k-j} x_2^{k-l}|^{1/(2+j+l)}.$$

It suffices to show that this measure grows slower than  $\lambda^{2/k} \log \lambda$ , since  $G_2(\lambda) \geq G_1(\lambda)$ . By symmetry, we may assume that  $x_1, x_2 > 0$ . We may rewrite (B.3) as

$$(B.4) \quad x_1 \leq M^{\frac{2(2+j+l)}{kj+kl+2j}} \cdot x_2^{-\frac{kj+kl+2l}{kj+kl+2j}}.$$

There are two cases to consider.

1) We have  $j = l > 0$ . Set  $M' = M^{\frac{2(2+j+l)}{kj+kl+2j}}$ . Thus (B.4) becomes  $x_1 \leq M'/x_2$ . On the other hand, the condition  $\Psi^*(x) \leq \lambda$  forces, in particular,  $x_1^k \leq \lambda^{k+2}$  and  $x_2^k \leq \lambda^{k+2}$ . One easily checks that the measure of the set where these two inequalities are satisfied and where  $x_1 x_2 \leq M'$  grows as  $C \cdot M' \cdot \log \lambda \ll \lambda^{2/k} \log \lambda$ .

2) We have  $j \neq l$ . By symmetry, we may assume that  $j > l$ . So (B.4) becomes  $x_1 \leq M' \leq x_2^{-\kappa}$ , where

$$0 < \kappa = \frac{kj + kl + 2l}{kj + kl + 2j} = 1 - 2 \cdot \frac{j - l}{kj + kl + 2j} < 1.$$

One easily checks that  $\kappa \geq \frac{k}{k+2}$  (this minimum value is achieved when  $j = k, l = 0$ ). Thus  $0 < 1 - \kappa \leq \frac{2}{k+2}$ . Now  $\Psi^*(x) \leq \lambda$  implies, in particular,  $x_2 \leq \lambda^{(k+2)/k}$ . One easily bounds the measure of the set where  $0 \leq x_2 \leq 1$  and  $x_1 \leq M' \cdot x_2^{-\kappa}$  by  $CM' \ll \lambda^{2/k} \log \lambda$ , and the measure of the set where  $1 \leq x_2 \leq \lambda^{(k+2)/k}$  and  $x_1 \leq M' \cdot x_2^{-\kappa}$  by  $CM' \lambda^{2/k} \ll \lambda^{2/k} \log \lambda$ .

To complete the proof of the lemma, we have to estimate the measure of the set where  $\Psi^*(x) \leq \lambda$  and  $|x_1^k x_2^k|^{1/2} \geq M \cdot |x_1^{k-j} x_2^{k-l}|^{1/(2+j+l)}$  whenever  $j > 0$  or  $l > 0$ . Again, by symmetry, we may assume that  $0 \leq x_1 \leq x_2$ . Clearly, the set where also  $x_2 \leq 1$  does not contribute to the asymptotics. Therefore we only need to consider the set where  $x_2 \geq 1, x_2 \geq x_1 \geq M' \cdot x_2^{-\kappa}$  and  $\Psi^*(x) \leq \lambda$ . Here,  $M'$  and  $\kappa$  are defined as above, and  $(j, l)$  run over all pairs with  $0 \leq j, l \leq k$ . But since  $\kappa \geq \frac{k}{k+2}$ , all the inequalities  $x_1 \geq M' \cdot x_2^{-\kappa}$  reduce to  $x_1 \geq M' \cdot x_2^{-k/(k+2)}$  if  $x_2 \geq 1$ . Under the assumptions

$$(B.5) \quad x_2 \geq 1, \quad x_2 \geq x_1 \geq M' \cdot x_2^{-\frac{k}{k+2}},$$

it is clear that the following implications hold:

$$x_1^k x_2^k \leq (1 - C/M) \cdot \lambda^2 \implies \Psi^*(x) \leq \lambda \implies x_1^k x_2^k \leq \lambda^2.$$

Finally, one computes that for any function  $h(\lambda)$  such that  $h(\lambda) \rightarrow 1$  as  $\lambda \rightarrow +\infty$ , the measure of the set where (B.5) holds and where  $x_1^k x_2^k \leq h(\lambda) \cdot \lambda^2$  is asymptotically equal to  $\frac{k+1}{k} \cdot \lambda^{2/k} \cdot \log \lambda$ . Multiplying this by 8 (due to the fact that we only considered the set where  $x_2 \geq x_1 \geq 0$ ), we see that the proof of Lemma 5.3 is complete.

### B.5. Proof of Lemma 5.4

Note that the measure  $\mu = \mu_\Omega$  has a “trivial component”—namely,  $\mu$  is naturally a product of a measure  $\mu'$  on  $\mathbb{R}^2$  and a measure  $\mu''$  on  $\mathbb{R}^{(k+1)^2}$ , given by  $\mu' = (2\pi)^{-2} \cdot d\xi_1 d\xi_2$  and  $\mu'' = \phi_*(dx_1 dx_2)$ , where  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^{(k+1)^2}$  is defined by  $\phi(x_1, x_2) = (x_1^k x_2^k, \dots, 1)$ . It is clear that we can apply our scaling constructions to the measures  $\mu'$  and  $\mu''$  separately, obtaining measures  $\mu'_\lambda$  and  $\mu''_\lambda$  for all  $\lambda > 0$ . Trivially, there is a weak limit

$$\mu'_0 = \lim_{\lambda \rightarrow +\infty} \lambda^{-2} \mu'_\lambda = \mu' = (2\pi)^{-2} \cdot d\xi_1 d\xi_2.$$

Thus, we only need to concentrate on the measures  $\mu''_\lambda$  on  $\mathbb{R}^{(k+1)^2}$ . The proof of Lemma 5.4 will be complete if we show that there is a weak limit

$$(B.6) \quad \mu''_0 = \lim_{\lambda \rightarrow +\infty} \lambda^{-1/k} (\log \lambda)^{-1} \cdot \mu''_\lambda = \psi_* \left( \frac{2}{k^2} \cdot |y|^{\frac{1-k}{k}} dy \right),$$



where  $\psi : \mathbb{R} \rightarrow \mathbb{R}^{(k+1)^2}$  is given by  $\psi(y) = (y, 0, \dots, 0)$ . Let us define  $\mu''_0$  by the RHS of (B.6). The idea is to apply Proposition 2.13. Since the function  $|y|^{(1-k)/k}$  is integrable near 0, we see that  $\mu''_0$  is a c-finite Borel measure on  $\mathbb{R}^{(k+1)^2}$ . Let us first fix real numbers  $R_{ij} > 0$  ( $0 \leq i, j \leq k$ ) and consider for every  $\lambda > 0$  the set

$$A_\lambda = \{(x_1, x_2) \mid x_1, x_2 \geq 0, \lambda^{-1}x_1^i x_2^j \leq R_{ij} \ \forall 0 \leq i, j \leq k\}.$$

It is easy to see that when  $u > 0$  is large enough (depending only on the  $R_{ij}$ 's, but not on  $\lambda$ ), we have

$$A_\lambda \cap \{x_1 \geq u, x_2 \geq u\} = \{x_1 \geq u, x_2 \geq u, x_1^k x_2^k \leq \lambda R_{kk}\},$$

and then one computes (for fixed  $R_{ij}$  and  $u$ )

$$\text{meas}(A_\lambda) \sim \text{meas}(A_\lambda \cap \{x_1 \geq u, x_2 \geq u\}) \sim \frac{R_{kk}^{1/k}}{k} \cdot \lambda^{1/k} \log \lambda \text{ as } \lambda \rightarrow +\infty.$$

By symmetry, if

$$B_\lambda = \{(x_1, x_2) \in \mathbb{R}^2 \mid \lambda^{-1}|x_1^i x_2^j| \leq R_{ij} \ \forall 0 \leq i, j \leq k\},$$

then

$$(B.7) \quad \text{meas}(B_\lambda) \sim 4 \cdot \frac{R_{kk}^{1/k}}{k} \cdot \lambda^{1/k} \log \lambda \quad \text{as } \lambda \rightarrow +\infty.$$

In particular, the answer does not depend on  $R_{ij}$  if  $i < k$  or  $j < k$ . We now apply Proposition 2.13 with  $E = \{0\}$ , which is certainly countable. So let  $a = (a_{ij}) \in \mathbb{R}^{(k+1)^2}$  and  $b = (b_{ij}) \in \mathbb{R}^{(k+1)^2}$ , with  $a_{ij} < b_{ij}$  and  $a_{ij} \neq 0, b_{ij} \neq 0$  for all  $0 \leq i, j \leq k$ . Two situations are possible.

1) We have  $0 \notin [a_{ij}, b_{ij}]$  for some  $(i, j)$  such that either  $i < k$  or  $j < k$ . Then there exists  $\delta > 0$  such that  $[a_{ij}, b_{ij}]$  does not intersect  $[-\delta, \delta]$  and is contained in  $[-\delta^{-1}, \delta^{-1}]$ . We apply the observation of the previous paragraph with  $R_{ij} = \delta$  and  $R_{ij} = \delta^{-1}$ , and conclude that

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-1/k} \log \lambda \cdot \mu_\lambda([a, b]) = 0 = \mu''_0([a, b]).$$

2) We have  $0 \in [a_{ij}, b_{ij}]$  for all  $(i, j)$  such that either  $i < k$  or  $j < k$ . Then, as before, there exists  $\delta > 0$  such that  $[-\delta, \delta] \subset [a_{ij}, b_{ij}] \subset [-\delta^{-1}, \delta^{-1}]$  for all such  $(i, j)$ . Now if  $a_{kk} = -b_{kk}$ , then the computation (B.7) immediately implies that

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-1/k} \log \lambda \cdot \mu_\lambda([a, b]) = 4 \cdot \frac{a_{kk}}{k} = \mu''_0([a, b]).$$

The general case reduces to this one by additivity and symmetry, noting that, for example, if  $b_{kk} > a_{kk} > 0$ , then

$$[a_{kk}, b_{kk}) \cup [-b_{kk}, -a_{kk}) = [-b_{kk}, b_{kk}) \setminus [-a_{kk}, a_{kk}),$$

and if  $b_{kk} > -a_{kk} > 0$ , then

$$[a_{kk}, b_{kk}) = [a_{kk}, -a_{kk}) \cup [-a_{kk}, b_{kk}).$$

## BIBLIOGRAPHY

- [1] D. ARNAL & J.-C. CORTET, “Représentations \* des groupes exponentiels”, *J. Funct. Anal.* **92** (1990), no. 1, p. 103-135.
- [2] P. BERNAT, C. CONZE, M. DUFLO, N. LÉVY-NAHAS, M. RAIS, P. RENOARD & M. VZATIONERGNE, “Représentations des groupes de Lie résolubles”, in *Monographies*, 4, Soc. Math. de France, 1972.
- [3] P. BONNET, “Paramétrisation du dual d’une algèbre de Lie nilpotente”, *Ann. Inst. Fourier* **38** (1988), no. 3, p. 169-197.
- [4] M. BOYARCHENKO & S. LEVENDORSKII, “Generalizations of the classical Weyl and Colin de Verdière’s formulas and the orbit method”, *Proc. Natl. Acad. Sci. USA* **102** (2005), no. 16, p. 5663-5668.
- [5] Y. COLIN DE VERDIÈRE, “L’asymptotique de Weyl pour les bouteilles magnétiques”, *Comm. Math. Phys.* **105** (1986), p. 327-335.
- [6] H. L. CYCON, R. G. FROESE, W. KIRSCH & B. SIMON, *Schrödinger operators with applications to quantum mechanics and global geometry*, Springer-Verlag, Berlin, New York, Heidelberg, London, Paris, Tokyo, 1985.
- [7] C. L. FEFFERMAN, “The uncertainty principle”, *Bull. Amer. Math. Soc.* **9** (1983), p. 129-206.
- [8] C. GORDON, D. WEBB & S. WOLPERT, “One cannot hear the shape of a drum”, *Bull. Amer. Math. Soc.* **27** (1992), p. 134-138.
- [9] D. GURARIE, “Non-classical eigenvalue asymptotics for operators of Schrödinger type”, *Bull. Am. Math. Soc.* **15** (1986), no. 2, p. 233-237.
- [10] B. HELFFER & A. MOHAMED, “Caractérisation du spectre essentiel de l’opérateur de Schrödinger avec un champ magnétique”, *Ann. Inst. Fourier, Grenoble* **38** (1988), no. 2, p. 95-112.
- [11] B. HELFFER & J. NOURRIGAT, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Birkhäuser ed., Progress in Math., Boston, 1985.
- [12] L. HÖRMANDER, *The analysis of differential operators. 3*, Springer-Verlag, Berlin, New York, Heidelberg, 1985.
- [13] V. IVRII, “Estimate for the number of negative eigenvalues of the Schrödinger operator with intense field”, in *Journées Équations aux Dérivées partielles de Saint-Jean-de-Monts*, Soc. Math. France, 1987.
- [14] M. KAC, “Can one hear the shape of a drum?”, *Amer. Math. Monthly* **73** (1966), p. 1-23.
- [15] A. A. KIRILLOV, “Unitary representations of nilpotent Lie groups”, *Uspehi Mat. Nauk* **17** (1962), no. 4 (106), p. 57-110.

- [16] N. V. KRYLOV, "Introduction to the theory of diffusion processes", in *Translations of Mathematical Monographs*, 142, American Mathematical Society, Providence, RI, 1995.
- [17] S. Z. LEVENDORSKII, "Non-classical spectral asymptotics", *Russian Math. Surveys* **43** (1988), no. 1, p. 123-157.
- [18] ———, *Asymptotic distribution of eigenvalues of differential operators*, Dordrecht: Kluwer Academic Publishers, 1990.
- [19] ———, *Degenerate elliptic equations*, Dordrecht: Kluwer Academic Publishers, 1993.
- [20] ———, "Spectral properties of Schrödinger operators with irregular magnetic potentials, for a spin  $\frac{1}{2}$  particle", *J. Math. Anal. Appl.* **216** (1997), no. 1, p. 48-68.
- [21] P. LEVY-BRUHL, A. MOHAMED & J. NOURRIGAT, "Spectral theory and representations of nilpotent groups", *Bull. Amer. Math. Soc.* **26** (1992), no. 2, p. 299-303.
- [22] D. MANCHON, "Formule de Weyl pour les groupes de Lie nilpotents", *J. Reine Angew. Math.* **418** (1991), p. 77-129.
- [23] ———, "Weyl symbolic calculus on any Lie group", *Acta Appl. Math.* **30** (1993), no. 2, p. 159-186.
- [24] ———, "Opérateurs pseudodifférentiels et représentations unitaires des groupes de Lie", *Bull. Soc. Math. France* **123** (1995), no. 1, p. 117-138.
- [25] ———, "Distributions à support compact et représentations unitaires", *J. Lie Theory* **9** (1999), no. 2, p. 403-424.
- [26] A. MOHAMED & J. NOURRIGAT, "Encadrement du  $N(\lambda)$  pour des opérateurs de Schrödinger avec champ magnétique", *J. Math. Pures Appl.* **70** (1991), no. 9, p. 87-99.
- [27] N. NILSSON, "Asymptotic estimates for spectral functions connected with hypoelliptic differential operators", *Ark. Mat.* **5** (1965), p. 527-540.
- [28] ———, "Some growth and ramification properties of certain integrals on algebraic manifolds", *Ark. Mat.* **5** (1965), p. 463-476.
- [29] N. V. PEDERSEN, "On the symplectic structure of coadjoint orbits of (solvable) Lie groups and applications, part I", *Math. Ann.* **281** (1988), p. 633-669.
- [30] L. PUKANSZKY, "On the theory of exponential groups", *Trans. Amer. Math. Soc.* **126** (1967), p. 487-507.
- [31] ———, "Unitary representations of solvable Lie groups", *Ann. Sci. École Norm. Sup. (4)* (1971), no. 4, p. 457-608.
- [32] D. ROBERT, "Comportement asymptotique des valeurs propres d'opérateurs de type Schrödinger à potentiel "dégénéré"", *J. Math. Pures Appl.* **61** (1982), p. 275-300.
- [33] G. V. ROZENBLJUM, "Asymptotic behavior of the eigenvalues of the Schrödinger operator", *Mat. Sb. (N.S.)* **93** (1974), no. 135, p. 347-367, 487.
- [34] G. V. ROZENBLJUM, M. Z. SOLOMYAK & M. A. SHUBIN, "Spectral theory of differential operators", in *Contemporary problems of mathematics*, vol. 64, Itogi Nauki i Tekhniki VINITI, Moscow: VINITI, 1989.
- [35] B. SIMON, "Nonclassical eigenvalue asymptotics", *J. Funct. Anal.* **53** (1983), no. 1, p. 84-98.
- [36] M. Z. SOLOMYAK, "Asymptotics of the spectrum of the Schrödinger operator with non-regular homogeneous potential", *Math. USSR Sbornik* **55** (1986), no. 1, p. 19-37.
- [37] H. TAMURA, "Asymptotic distribution of eigenvalues for Schrödinger operators with magnetic fields", *Nagoya Math. J.* **105** (1987), p. 49-69.

- [38] V. N. TULOVSKIĬ & M. A. SHUBIN, “The asymptotic distribution of the eigenvalues of pseudodifferential operators in  $R^n$ ”, *Mat. Sb. (N.S.)* **92** (1973), no. 134, p. 571-588, 648.
- [39] M. VERGNE, “La structure de Poisson sur l’algèbre symétrique d’une algèbre de Lie nilpotente”, *Bull. SMF* **100** (1972), p. 301-335.

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