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DIMENSION OF THE HARMONIC MEASURE OF NON-HOMOGENEOUS CANTOR SETS

by Athanasios BATAKIS

Abstract. — We prove that the dimension of the harmonic measure of the complementary of a translation-invariant type of Cantor sets is a continuous function of the parameters determining these sets. This results extends a previous one of the author and do not use ergotic theoretic tools, not applicables to our case.

Résumé. — Nous montrons que la dimension de la mesure harmonique du complémentaire d'ensembles de Cantor de type invariant par translation est une fonction continue des paramètres définissant ces ensembles. Ce résultat prolonge un précédent du même auteur et n'implique pas d'outils de la théorie ergotique, non-applicables dans notre configuration.

1. Introduction

The purpose of this work is to complement the study of the dimension of the harmonic measure of the complementary of (not necessarily self-similar) Cantor sets as a function of parameters assigned to these sets. In a previous work [5] we have proved that the parameters assigned to self-similar Cantor sets are continuity points for this function. A new method allows us to treat the continuity over the entire family of parameters determining these translation invariant Cantor sets. We restrain ourselves to sets in the plane for convenience, even though the proof can be applied to all "translation-invariant" Cantor sets in $\mathbb{R}^n$, $n \geqslant 2$.

Let us start by recalling the definition of the Hausdorff dimension of a measure; we will use the notation $\dim_{\mathcal{H}}$ for the Hausdorff dimension of sets.

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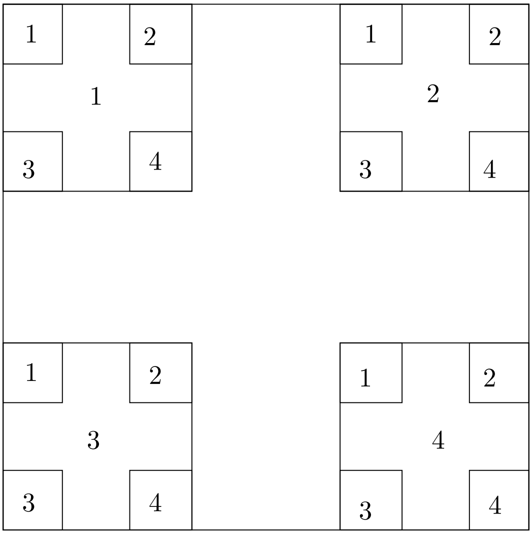

Figure 1.1. A 4-corner Cantor set and its enumeration

Definition 1.1. — *If μ is a measure on $\mathbb{K}$, we will denote by* $\dim_*(\mu)$ *the lower Hausdorff dimension of μ:*

$$\dim_* \mu = \inf\{\dim_{\mathcal{H}} E \ ; \ E \subset \mathbb{K} \textit{ and } \mu(E) > 0\}$$

and by $\dim^*(\mu)$ *the upper Hausdorff dimension of μ:*

$$\dim^* \mu = \inf\{\dim_{\mathcal{H}} E \ ; \ E \subset \mathbb{K} \textit{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

If, for a measure μ on $\mathbb{K}$, we have $\dim^*(\mu) = \dim^*(\mu)$ *then we note this common value* $\dim(\mu)$. *In the latter case the measure is called exact.*

For convenience and in order to fix ideas we consider a particular case of translation invariant Cantor sets; we study 4-corner Cantor sets constructed in the following way (see also [3]): let $\underline{A}, \overline{A}$ be two constants with $0 < \underline{A} \leqslant \overline{A} < \frac{1}{2}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $\underline{A} \leqslant a_n \leqslant \overline{A}$ for all $n \in \mathbb{N}$.

We replace the square $[0,1]^2$ by four squares of sidelength a_1 situated at the four corners of $[0,1]^2$. Each of these squares is then replaced by four squares of sidelength $a_1 a_2$ situated at its four corners. At the nth stage of the construction every square of the $(n-1)$th generation will be replaced by four squares of sidelength $a_1 ... a_n$ situated at its four corners (see figure 1.1). Let $\mathbb{K}$ be the Cantor set constructed by repeating the procedure.

Recall that the harmonic measure of a domain is supported by its boundary and can be seen as the distribution of the exit points of Brownian motion starting at some (any) point of the domain (for more details see [13], [16] and [11]). Carleson [12] has shown that for self-similar 4-corner Cantor sets (the sequence $(a_n)_{n\in\mathbb{N}}$ is constant) the dimension of the harmonic measure of their complementary is strictly smaller than 1. His proof, involving ergodic theory techniques, was improved by Makarov and Volberg [20] who showed that the dimension of the harmonic measure of any self-similar 4-corner Cantor set is strictly smaller than the dimension of the Cantor set. Volberg ([23], [24]) extended these results to a class of dynamic Cantor repellers. Other comparisons of harmonic and maximal measures for dynamical systems are proposed in [2], [18], [22]. More recently, a multifractal study of harmonic measure on simply connected domains and on Julia sets of polynomial mappings is carried out in [19], [10].

In [3] it is shown that the dimension of the harmonic measure of the complementary of 4-corner Cantor sets is strictly smaller than the Hausdorff dimension of the Cantor set, even when the sequence $(a_n)_{n\in\mathbb{N}}$ is not constant. In [4] we prove that small perturbations of the sidelength of the squares of the construction of $\mathbb{K}$ do not alterate this property. This last result can also be seen as an immediate consequence of the following theorem.

Theorem 1.2. — *Let $\mathbb{K} = \mathbb{K}_{(a_n)}$ be the 4-corners Cantor set associated to a sequence a_n and $\mathbb{K}' = \mathbb{K}_{(a'_n)}$ a second Cantor set of the same type associated to the sequence $(a'_n)_{n\in\mathbb{N}}$. Let ω and ω' be the harmonic measures of $\mathbb{R}^2 \setminus \mathbb{K}$ and $\mathbb{R}^2 \setminus \mathbb{K}'$ respectively. Then for all $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \overline{A}, \underline{A}) > 0$ such that if $|a'_n - a_n| < \delta$ for all $n \in \mathbb{N}$ then $|\dim\omega - \dim\omega'| < \epsilon$.*

When the sequence $(a_n)_{n\in\mathbb{N}}$ is constant the partial result is already established in [5] using ergodic theoretic tools, which are not applicables in the general case.

Remark 1.3. — Let $\mathcal{D} : \ell^\infty([\overline{A}, \underline{A}]) \to [0,1]$ be the function that assigns to a sequence $(a_n)_{n\in\mathbb{N}} \subset [\overline{A}, \underline{A}]$ the dimension of harmonic measure of the Cantor set associated to $(a_n)_{n\in\mathbb{N}}$. By refining the estimations in the demonstration of the theorem, we can even show that $\mathcal{D}$ is a Lipschitz continuous function. The proof of this statement is very technical but straightforward and therefore ommited.
In particular this refinement implies that if $\displaystyle\sum_{n\in\mathbb{N}} |a_n - a'_n| < \infty$ then the harmonique measures of the corresponding Cantor sets are of the same

dimension, and even equivalent when we report them onto the abstract Cantor set $\{1,2,3,4\}^{\mathbb{N}}$.

2. Notations and Preliminary results

In this section we establish some estimates on the harmonic measure of a Cantor set under perturbation, and recall some known results on the harmonic measure of Cantor-type sets. We also introduce the tools needed, such as the Hausdorff dimension and the entropy of a probability measure on a Cantor set.

Let $\mathbb{K}$ be a 4-corner Cantor set as described in the introduction. We enumerate $\mathbb{K}$ by identifying it to the abstract Cantor set $\{1,...,4\}^{\mathbb{N}}$. We denote $I_{i_1\dots i_n}$, where $i_j \in \{1,2,3,4\}$ for $1 \leqslant j \leqslant n$, the 4^n squares of the n-th generation of the construction of $\mathbb{K}$ with the enumeration shown in the figure 1.1 and the usual condition that $I_{i_1\dots i_n}$ is the "father" of the sets $I_{i_1\dots i_n i}$, $i \in \{1,2,3,4\}$. It is clear that $\overline{A} \geqslant \dfrac{\mathrm{diam} I_{i_1\dots i_n i}}{\mathrm{diam} I_{i_1\dots i_n}} = a_{n+1} \geqslant \underline{A}$, $i = 1,...,4$.

The collection of the squares of the n-th generation of the construction of $\mathbb{K}$ will be $\mathcal{F}_n = \{I_{i_1\dots i_n}\,;\, i_1,...,i_n = 1,...,4\}$, for $n \in \mathbb{N}$. For a square $I \in \mathcal{F}_n$ we note $\widehat{I}$ the "father" of I, i.e. the unique square of $\mathcal{F}_{n-1}$ containing I. If $I = I_{i_1\dots i_k} \in \mathcal{F}_k$ and $J = I_{j_1\dots j_n} \in \mathcal{F}_n$ we will note $IJ = I_{i_1\dots i_k j_1\dots j_n} \in \mathcal{F}_{n+k}$. Finally, for $x \in \mathbb{K}$ and $n \in \mathbb{N}$ let $I_n(x)$ be the unique square of $\mathcal{F}_{n-1}$ containing x.

For a domain Ω, a point $x \in \Omega$ and a Borel set $F \subset \mathbb{R}^2$ we denote by $\omega(x,F,\Omega)$ the harmonic measure of $F \cap \partial\Omega$ (for the domain Ω) assigned to the point x. Clearly, F carries no measure if it does not intersect $\partial\Omega$. If Ω is not specified it will be $\mathbb{R}^2 \setminus \mathbb{K}$ and if x is the point at infinity we will simply note $\omega(F)$. Finally, for a Borel set $E \subset \mathbb{R}^2$ we note $\dim E$ the Hausdorff dimension of the set E.

2.1. Dimension of measures

In this section we recall some known results on the dimensions of measures (see also [21], [9], [14], [25]). One can prove (see for instance [14], [17], [7]) that if μ is exact, i.e. if $\dim_* \mu = \dim^* \mu$, then

$$\dim \mu = \liminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r}\ ,\ \mu\text{-almost everywhere.} \tag{2.1}$$

If the probability measure μ is supported by a 4-corner Cantor set, the balls $B(x,r)$ can be replaced by the squares of the construction of the Cantor set (see [8], [4]):

$$\dim\mu = \liminf_{n\to\infty} \frac{\log\mu\big(I_n(x)\big)}{\log l\big(I_n(x)\big)}\ , \ \mu\text{-almost everywhere,} \tag{2.2}$$

where $l\big(I_n(x)\big)$ is the sidelength of the square $I_n(x)$ and $\underline{A}^n \leqslant l(I_n) \leqslant \overline{A}^n$.

Remark 2.1. — If μ is an arbitrary (not necessarily monodimensional) probability measure we get

$$\dim_*\mu \leqslant \liminf_{r\to 0} \frac{\log\mu\big(B(x,r)\big)}{\log r} \leqslant \dim^*\mu \quad \mu\text{-almost everywhere.} \tag{2.3}$$

Moreover $\dim^*\mu = \operatorname{supess}_\mu \liminf_{r\to 0} \frac{\log\mu\big(B(x,r)\big)}{\log r}$ and $\dim_*\mu = \operatorname{infess}_\mu \liminf_{r\to 0} \frac{\log\mu\big(B(x,r)\big)}{\log r}$.

Some results of the following section are stated without demonstration since they are already proved in [3] and in [5].

2.2. Estimating perturbations of the harmonic measure

Suppose that the 4-corner Cantor set $\mathbb{K}$ is associated to the sequence $(a_n)_{n\in\mathbb{N}}$ and let $\mathbb{K}'$ be another Cantor set associated to the sequence $(a'_n)_{n\in\mathbb{N}}$. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the collections of squares associated to $\mathbb{K}$ and $(\mathcal{F}'_n)_{n\in\mathbb{N}}$ those associated to $\mathbb{K}'$.

For $I \in \mathcal{F}_n$ and $I' \in \mathcal{F}'_m$ we will write $I \overset{\text{cod}}{\sim} I'$ if $n = m$ and if I and I' have the same encoding (with respect to the identification to the abstract Cantor set $\{1,2,3,4\}^{\mathbb{N}}$).

Finally, if $I \subset \mathbb{R}^2$ is a square of sidelength ℓ and c is a positive number we note $c\dot{I}$ the square of sidelength $c\ell$ having the same barycenter as I. If ω is the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}$ and ω' the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}'$ we have established the following theorem.

Theorem 2.2. — *(cf.* [5]*) For all $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \overline{A}, \underline{A}) > 0$ such that*

$$\sup_{n\in\mathbb{N}} |a_n - a'_n| < \delta \Rightarrow \left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I'})} - 1 \right| < \epsilon, \tag{2.4}$$

for all $I \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ with $I \overset{\text{cod}}{\sim} I'$.

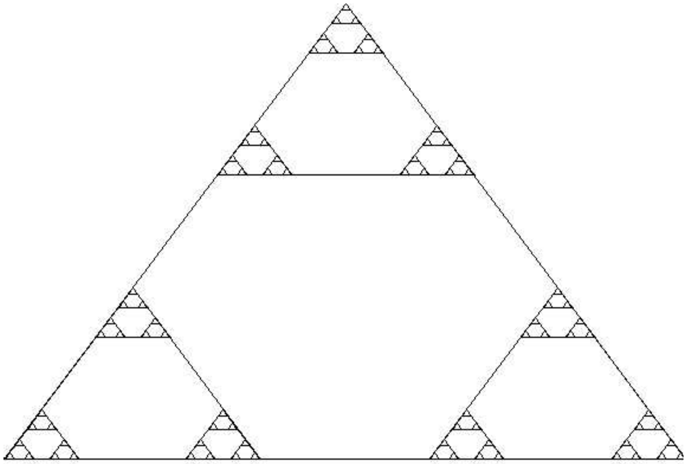

Figure 2.1. A different type of translation-invariant Cantor set

We will also need the following estimations of the harmonic measure of cylinders (see also [12] and [20] for a version adapted to self-similar sets). The proof uses the ideas already explored in [5].

LEMMA 2.3. — *For every $I, I' \in \mathcal{F}_n$, $J \in \mathcal{F}_k$ and every $L \in \mathcal{F}_m$, n, k, $m \in \mathbb{N}$*

$$\left| \frac{\omega(IJL)}{\omega(IJ)} : \frac{\omega(I'JL)}{\omega(I'J)} - 1 \right| < C\, q^k \tag{2.5}$$

where the constants $C > 0$ and $q \in (0,1)$, depend only on $\underline{A}, \overline{A}$.

Let us give the proof of this statement.

Proof of lemma 2.3. — To begin with we need the following Harnack principle (see also [4], [1]).

LEMMA 2.4. — *(cf. [12], [20]) Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset ... \subset A_n \subset B_n$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω, for $1 \leqslant i \leqslant n$. If the moduli of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$ then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$ and for all $x \in \Omega \setminus B_n$ we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \leqslant K q^n \tag{2.6}$$

where $q<1$ and K are two constants that depend only on the lower bound of the moduli of the annuli.

We use this result to prove the following:

LEMMA 2.5. — *There are constants $K>0$ and $0<q<1$ depending only on $\underline{A},\overline{A}$ such that for all $i,j,k\in\mathbb{N}$ and for all squares $I\in\mathcal{F}_i$, $J\in\mathcal{F}_j, K\in\mathcal{F}_k$ of the construction of $\mathbb{K}$, if $Q=c_0\cdot I$,*

(2.7)
$$\left|\frac{\omega(x,IJK,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}:\frac{\omega(IJK)}{\omega(IJ)}-1\right|<Kq^j \text{ for all } x\in\partial\left\{\frac{1+c_0}{2}\cdot I\right\}.$$

The result applies also to the Cantor set $\mathbb{K}'$.

Proof. — By lemma 2.4,

$$\left|\frac{\omega(x,IJK)}{\omega(x,IJ)}:\frac{\omega(IJK)}{\omega(IJ)}-1\right|<Kq^j \text{ , for } x\notin\frac{1+c_0}{2}\cdot I \tag{2.8}$$

Let $A=\dfrac{\omega(IJK)}{\omega(IJ)}$. We have

$$\omega(x,IJK,Q\setminus\mathbb{K})=\omega(x,IJK)-\int_{\partial Q}\omega(z,IJK)\omega(x,dz,Q\setminus\mathbb{K}),$$

for $x\in\partial\left\{\frac{1+c_0}{2}\cdot I\right\}$.

By the equation (2.8),

$$A\omega(x,IJ)-Kq^jA\omega(x,IJ)\leqslant\omega(x,IJK)\leqslant A\omega(x,IJ)+Kq^jA\omega(x,IJ).$$

We get

$$\begin{aligned}\omega(x,IJK,Q\setminus\mathbb{K}) &\leqslant A\omega(x,IJ)+Kq^jA\omega(x,IJ)-\\ &-\int_{\partial Q}\Big(A\omega(z,IJ)-Kq^jA\omega(z,IJ)\Big)\omega(x,dz,Q\setminus\mathbb{K})\\ &= A\omega(x,IJ)-\int_{\partial Q}A\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})+\\ &+Kq^j\Big(A\omega(x,IJ)+\int_{\partial Q}A\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})\Big)\\ &= A\omega(x,IJ,Q\setminus\mathbb{K})+Kq^j\Big(A\omega(x,IJ)+\\ &+\int_{\partial Q}A\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})\Big)\end{aligned} \tag{2.9}$$

Therefore,

(2.10)
$$\frac{\omega(x,IJK,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}\leqslant A+Kq^jA\frac{\omega(x,IJ)+\int_{\partial Q}\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}$$

It suffices now to show that the quantity

$$\frac{\omega(x,IJ)+\int_{\partial Q}\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}$$

is smaller that a given constant. Take $x_0\in\partial\left\{\frac{1+c_0}{2}\cdot I\right\}$ such that

$$\omega(x_0,IJ)=\max\left\{\omega(x,IJ)\ ;\ x\notin\left\{\frac{1+c_0}{2}\cdot I\right\}\right\}.$$

Using the maximum principle we get

$$\begin{aligned}\omega(x_0,IJ,Q\setminus\mathbb{K}) &= \omega(x_0,IJ)-\int_{\partial Q}\omega(z,IJ)\omega(x_0,dz,Q\setminus\mathbb{K})\\ &\geqslant \omega(x_0,IJ)-\int_{\partial Q}\omega(x_0,IJ)\omega(x_0,dz,Q\setminus\mathbb{K})\\ &= \omega(x_0,IJ)\big(1-\omega(x_0,\partial Q,Q\setminus\mathbb{K})\big)\end{aligned}$$

By standard capacitary techniques one can verify (see [3]) that $1-\omega(x_0,\partial Q,Q\setminus\mathbb{K})$ is greater that a constant $c>0$ depending only on $\underline{A},\overline{A}$.

By using Harnack's principle we get

$$1-\omega(x,\partial Q,Q\setminus\mathbb{K})\geqslant c\text{ , for all }x\in\partial\left\{\frac{1+c_0}{2}\cdot I\right\},$$

for a new constant $c>0$.

Hence, $\dfrac{\omega(x,IJ)+\int_{\partial Q}\omega(z,IJ)\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}\leqslant\dfrac{2}{c}$ and therefore, by relation (2.10),

$$\frac{\omega(x,IJK,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}\leqslant A\Big(1+\frac{2}{c}Kq^j\Big)\tag{2.11}$$

On the other hand $A=\dfrac{\omega(IJK)}{\omega(IJ)}$; we obtain

$$\frac{\omega(x,IJK,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}:\frac{\omega(IJK)}{\omega(IJ)}-1<\frac{2}{c}Kq^j\text{ , for all }x\in\partial\left\{\frac{1+c_0}{2}\cdot I\right\},$$

The left hand inequality and hence the lemma 2.5 is established in the same way. □

It is now evident that

$$\frac{\omega(x,IJK,Q\setminus\mathbb{K})}{\omega(x,IJ,Q\setminus\mathbb{K})}=\frac{\omega(x,I'JK,Q'\setminus\mathbb{K})}{\omega(x,I'J,Q'\setminus\mathbb{K})},$$

for any square $I'\in\mathcal{F}_n$, where $Q'=c_0\cdot I'$. The proof of lemma 2.3 is complete. □

Corollary 2.6. — *There is a constant $\tilde{C} > 1$ such that for any $n, k \in \mathbb{N}$, all $I, I' \in \mathcal{F}_n$ and every $J \in \mathcal{F}_k$ we have*

$$\frac{\omega(IJ)}{\omega(I)} \leqslant \tilde{C}\frac{\omega(I'J)}{\omega(I')} \tag{2.12}$$

where the constant $\tilde{C} > 0$ depends only on $\underline{A}, \overline{A}$.

The proof of the corollary is an easy application of lemma 2.3.

3. Proof of the main result

This section is dedicated to the proof of theorem 1.2. We will make use of the following known version of the theorem of large numbers (see for instance [15]).

Lemma 3.1. — *Let X_n be a sequence of uniformly bounded real random variables on a probability space $(\mathbb{X}, \mathcal{B}, P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-subalgebra of $\mathbb{B}$ such that X_n is measurable with respect to $\mathcal{F}_n$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})) = 0 \quad \textit{P-almost surely} \tag{3.1}$$

The following elementary lemma is also useful; the proof is left to the reader.

Lemma 3.2. — *Let $\alpha_1, ..., \alpha_n$ be real numbers such that $\displaystyle\sum_{i=1}^{n} \alpha_i = 0$. Then, for any choice of real values $h_1, ..., h_n$, we have*

$$\begin{aligned}
&\left|\textstyle\sum_{i=1}^{n} \alpha_i h_i\right| \\
&\qquad \leqslant \max\left\{ \sum_{\{i\ ;\ \alpha_i>0\}} \alpha_i\ ,\ -\sum_{\{i\ ;\ \alpha_i<0\}} \alpha_i \right\} \left(\max_{1\leqslant i\leqslant n} h_i - \min_{1\leqslant i\leqslant n} h_i \right) \\
&\qquad = \sum_{\{i\ ;\ \alpha_i>0\}} \alpha_i \left(\max_{1\leqslant i\leqslant n} h_i - \min_{1\leqslant i\leqslant n} h_i \right).
\end{aligned}$$

Proof of theorem 1.2. — For $p \in \mathbb{N}$ consider the sequence of σ-algebras $(\mathcal{R}_n)_{n\in\mathbb{N}}$ where $\mathcal{R}_n$ is generated by $\mathcal{F}_{np}$.

The hypothesis of lemma 3.1 can be easily verified to hold for the sequence of random variables $(X_n^p)_{n\in\mathbb{N}}$ given by

$$X_n^p(x) = \frac{1}{p}$$

$\left|\log\left(\frac{\omega(I_{np}(x))}{\omega\left(I_{(n-1)p}(x)\right)}\right)\right|$ and the sequence of σ-algebras $(\mathcal{R}_n)_{n\in\mathbb{N}}$.

We get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[X_k^p-\mathbb{E}_\omega(X_k^p|\mathbb{R}_{k-1})\right]=0\ \ \omega\text{-almost everywhere.}\tag{3.2}$$

On the other hand, on $I\in\mathbb{R}_{n-1}$, $n\in\mathbb{N}$,

$$\mathbb{E}_\omega(X_n^p|\mathbb{R}_{n-1})=\frac{1}{p}\sum_{J\in\mathcal{F}_p}\frac{\omega(IJ)}{\omega(I)}\left|\log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right|.$$

We show that this quantity is almost constant on $x\in\mathbb{K}$ if p is taken sufficiently large.

Take $\epsilon>0$, $n\in\mathbb{N}$ and $I\in\mathcal{F}_n$. For $j,k\in\mathbb{N}$ we have

$$\begin{aligned}\sum_{J\in\mathcal{F}_{j+k}}\frac{\omega(IJ)}{\omega(I)}\log\left(\frac{\omega(IJ)}{\omega(I)}\right)&=\sum_{J\in\mathcal{F}_j}\sum_{K\in\mathcal{F}_k}\frac{\omega(IJK)}{\omega(I)}\log\left(\frac{\omega(IJK)}{\omega(I)}\right)\\&=\sum_{J\in\mathcal{F}_j}\sum_{K\in\mathcal{F}_k}\frac{\omega(IJK)}{\omega(IJ)}\log\left(\frac{\omega(IJK)}{\omega(IJ)}\right)\frac{\omega(IJ)}{\omega(I)}\\&\quad+\sum_{J\in\mathcal{F}_j}\frac{\omega(IJ)}{\omega(I)}\log\left(\frac{\omega(IJ)}{\omega(I)}\right)\end{aligned}\tag{3.3}$$

For $L\in\mathcal{F}_n$ and $k,n,j\in\mathbb{N}$ we note

$$h_k(L)=-\frac{1}{k}\sum_{K\in\mathcal{F}_k}\frac{\omega(LK)}{\omega(L)}\log\left(\frac{\omega(LK)}{\omega(L)}\right).$$

In particular, we put

$$h_k(IJ)=-\frac{1}{k}\sum_{K\in\mathcal{F}_k}\frac{\omega(IJK)}{\omega(IJ)}\log\left(\frac{\omega(IJK)}{\omega(IJ)}\right)$$
$$\text{and }\Delta_j^k(I)=\max_{J\in\mathcal{F}_j}h_k(IJ)-\min_{J\in\mathcal{F}_j}h_k(IJ).$$

We will use the following lemma.

Lemma 3.3. — *For all $\epsilon>0$, if $j,k\in\mathbb{N}$ are big enough (depending only on $\underline{A}$ and $\overline{A}$) then for all $n\in\mathbb{N}$ and $I\in\mathcal{F}_n$ we have $\Delta_j^k(I)<\epsilon$.*

We first proceed with the proof of this sub-lemma.

Proof of lemma 3.3. — We can rewrite formula (3.3):

$$(j+k)h_{j+k}(I)=\sum_{J\in\mathcal{F}_j}k\frac{\omega(IJ)}{\omega(I)}h_k(IJ)+jh_j(I)\tag{3.4}$$

By applying formula (3.4) to a cylinder $I = I_1 I_2$ with $I_1 \in \mathcal{F}_{i_1}$ and $I_2 \in \mathcal{F}_{i_2}$ we have

$$(j+k)h_{j+k}(I_1I_2) = \sum_{J\in\mathcal{F}_j} k\frac{\omega(I_1I_2J)}{\omega(I_1I_2)}h_k(I_1I_2J) + jh_j(I_1I_2)$$

Now take j big enough to have $Kq^j < \epsilon$ and afterwards choose k in order to have that $\dfrac{j}{k+j} < \epsilon$. Remark that, by lemma 2.3, $h_k(I_1I_2'J) - h_k(I_1I_2J)$ $< 2\epsilon$. We have

$$\begin{aligned}
\Delta_{i_2}^{k+j}(I_1) &= \max_{I_2\in\mathcal{F}_{i_2}} h_{k+j}(I_1I_2) - \min_{I_2\in\mathcal{F}_{i_2}} h_{k+j}(I_1I_2) \\
&\leqslant 5\epsilon + \max_{I_2,I_2'\in\mathcal{F}_{i_2}} \Bigg[\sum_{J\in\mathcal{F}_j} \frac{\omega(I_1I_2J)}{\omega(I_1I_2)}h_k(I_1I_2J) \\
&\qquad\qquad - \sum_{J\in\mathcal{F}_j} \frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}h_k(I_1I_2'J)\Bigg] \\
&\leqslant 10\epsilon + \max_{I_2,I_2'\in\mathcal{F}_{i_2}} \sum_{J\in\mathcal{F}_j}\left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}\right)h_k(I_1I_2J). \qquad (3.5)
\end{aligned}$$

We can now apply lemma 3.2 to get

$$\Delta_{i_2}^{k+j}(I_1) \leqslant 10\epsilon + \max_{I_2,I_2'\in\mathcal{F}_{i_2}} \sum_{J\in S_{i_1}^j(I_2,I_2')}\left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}\right)\Delta_j^k(I_1I_2) \qquad (3.6)$$

where $\mathcal{S}_{I_1}^j(I_2,I_2')$ is the set of cylinders $J \in \mathcal{F}_j$ such that $\dfrac{\omega(I_1I_2J)}{\omega(I_1I_2)}$ $> \dfrac{\omega(I_1I_2'J)}{\omega(I_1I_2')}$. The following lemma is easy to prove:

Lemma 3.4. — *There is a constant $0 < \zeta < 1$ such that for all i_1, i_2, $j \in \mathbb{N}$, all $I_1 \in \mathcal{F}_{i_1}$ and all I_2,$I_2' \in \mathcal{F}_{i_2}$ we have*

$$\sum_{J\in S_{I_1}^j(I_2,I_2')}\left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}\right) < \zeta$$

where $\mathcal{S}_{I_1}^j(I_2,I_2')$ is the set of cylinders $J \in \mathcal{F}_j$ such that $\dfrac{\omega(I_1I_2J)}{\omega(I_1I_2)}$ $> \dfrac{\omega(I_1I_2'J)}{\omega(I_1I_2')}$.

The proof follows from the translation invariance of ω (corollary 2.6).

Proof of lemma 3.4. — Remark that by lemma 2.3, if $C = \dfrac{K}{1-q}$, we have $\dfrac{\omega(IJ)}{\omega(I'J)} \leqslant C$ for all $I, I' \in \mathcal{F}_n$, $n \in \mathbb{N}$, and all $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Hence,

$$\sum_{J\in S^j_{I_1}(I_2,I'_2)} \left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I'_2J)}{\omega(I_1I'_2)}\right) \leqslant \left(1-\frac{1}{C}\right) \sum_{J\in S^j_{I_1}(I_2,I'_2)} \frac{\omega(I_1I_2J)}{\omega(I_1I_2)} \leqslant 1-\frac{1}{C},$$

which is the lemma conclusion for $\zeta = 1 - \dfrac{1}{C}$. □

By applying lemma 3.4 to relation (3.6) we conclude that there is $0 < \zeta < 1$ depending only on $\underline{A}, \overline{A}$ such that

$$\Delta^{k+j}_{i_2}(I_1) \leqslant 10\epsilon + \zeta\Delta^k_j(I_1I_2) \tag{3.7}$$

By repeating the same reasoning if we write $k = k_1 + k_2$ we can establish the inequalities

$$\Delta^{k_1+k_2}_j(I_1I_2) < 10\epsilon + \zeta\Delta^{k_2}_{k_1}(I_1I_2J). \tag{3.8}$$

Hence,

$$\Delta^{k+j}_{i_2}(I_1) - \frac{10\epsilon}{1-\zeta} \leqslant \zeta\left(\Delta^{k_2}_{k_1}(I_1I_2J) - \frac{10\epsilon}{1-\zeta}\right) \leqslant \zeta^2\left(\Delta^{k_2}_{k_1}(I_1I_2J) - \frac{10\epsilon}{1-\zeta}\right).$$

The sequence $\Delta^m_l(I)$ being uniformly bounded we get, by decomposing again k_2 and repeating the procedure, that if k is big enough, $\Delta^{k+j}_{i_2} < \dfrac{20\epsilon}{1-\zeta}$.

The real constant ζ depending only on $\underline{A}$, $\overline{A}$, the proof is complete. □

We now apply lemma 3.1 to an adapted filtration $\mathcal{R}_n$: By the previous lemma we can choose j, k such that $\Delta^k_j(I) < \epsilon$. By formula (3.4), for this choice of j and k and for all $n \in \mathbb{N}$ there are constants c_n such that

$$\left|\frac{1}{k+j}\mathbb{E}\left\{\sum_{\ell=1}^{k+j} X^1_{n+\ell}\middle|\mathcal{F}_n\right\} - c_n\right| < \epsilon.$$

By lemma 3.1 and the relation (3.2) following it we then deduce (for $p = k + j$)

$$\left|\liminf_{n\to\infty}\frac{1}{n(k+j)}\sum_{\ell=1}^{n(k+j)} X^1_\ell - \liminf_{n\to\infty}\frac{1}{n}\sum_{\ell=1}^{n} c_\ell\right| < \epsilon, \omega \text{ - a.e. on } \mathbb{K}$$

This implies that

$$\left|\liminf_{n\to\infty}\left|\frac{\log\omega(I_{n(k+j)}(x))}{\log\left(\prod_{i=1}^{n(k+j)}a_i\right)}\right|-\liminf_{n\to\infty}\left|\frac{k+j}{\log\left(\prod_{i=1}^{n(k+j)}a_i\right)}\sum_{\ell=1}^{n}c_\ell\right|\right|<\epsilon\ ,$$

ω - a.e. on $\mathbb{K}$.

On the other hand, once we have fixed k,j we can use lemma 2.2 to choose δ in a way that, for all $n\in\mathbb{N}$,

$$|c_n-c'_n|<\epsilon,\ \text{and}\ \frac{1}{n}\left|\log\left(\prod_{i=1}^{n}a_i\right)-\log\left(\prod_{i=1}^{n}a'_i\right)\right|<\epsilon,$$

where c'_n is the same sequence associated to the harmonique measure ω'. We can finally use relation (2.2) to conclude that $|\dim\omega-\dim\omega'|<4\epsilon$. $\square$

4. Consequences and remarks

It is implicitely proved that the harmonic measure of the sets $\mathbb{K}$ studied here satisfy the relationship $\dim_*\omega=h_*^{\mathbb{K}}(\omega)$, where

$$h_*^{\mathbb{K}}(\omega)=\liminf_{n\to\infty}\frac{1}{\log\prod_{i=1}^{n}a_i}\sum_{I\in\mathcal{F}_n}\log\omega(I)\omega(I),$$

and $(a_n)_{n\in\mathbb{N}}$ is the construction sequence associated to $\mathbb{K}$. This fact is a consequence of the space invariance of ω and is a key factor in the proof of our results.

It is natural to ask whether the relation (2.4) suffices to conclude that the dimensions of two measures ω and ω' (not necessarily harmonic) are close. This is not the case. There are counterexemples (see [4]) even when the measures are doubling on $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and exact (cf [6]).

Even if the equality between the Hausdorff dimension and the entropy of the harmonic measure of the complementary of Cantor sets plays a crucial role in the proof of theorem 1.2, we would like to point out that the measures constructed μ and ν in the example of [6] satisfy the relation

$$\left|\frac{\mu(I)}{\mu(\widehat{I})}:\frac{\mu(I)}{\mu(\widehat{I})}-1\right|<\delta,$$

with δ as small as we want, as well as the the equalities $h_*(\mu)=\dim\mu$ and $h_*(\nu)=\dim\nu$. Nevertheless $|\dim\mu-\dim\nu|\geqslant\dfrac{1}{2}$.

To establish the result claimed in remark 1.3 we first need a fine precision of inequalities in theorem 2.2 and secondly we need to quantify the dependance on ϵ of the choice of k. In fact, we can find suitable k's that are bounded by $-C\log(\epsilon)$, where C is a positive constant depending on $\underline{A}$, $\overline{A}$, which is sufficient in order to prove the claim.

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