logoAIF.pdf

ANNALES

DE

L'INSTITUT FOURIER

Adolfo GUILLOT

Semicompleteness of homogeneous quadratic vector fields

Tome 56, nº 5 (2006), p. 1583-1615.

<http://aif.cedram.org/item?id=AIF_2006__56_5_1583_0>

© Association des Annales de l'institut Fourier, 2006, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (http://aif.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://aif.cedram.org/legal/). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

logocedram.pdf

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Ann. Inst. Fourier, Grenoble
56, 5 (2006) 1583-1615

SEMICOMPLETENESS OF HOMOGENEOUS QUADRATIC VECTOR FIELDS

by Adolfo GUILLOT

ABSTRACT. — We investigate the quadratic homogeneous holomorphic vector fields on $\mathbf{C}^n$ that are semicomplete, this is, those whose solutions are single-valued in their maximal definition domain. To a generic quadratic vector field we rationally associate some complex numbers that turn out to be integers in the semicomplete case, thus showing that the linear equivalence classes of semicomplete vector fields are contained in some sort of lattice in the space of linear equivalence classes of quadratic ones. We prove that the foliations of $\mathbf{CP}^{n-1}$ induced by semicomplete quadratic vector fields are linearizable in a neighborhood of their singular points and give some new families of examples in $\mathbf{C}^3$. Finally, we classify the semicomplete isochoric vector fields in $\mathbf{C}^3$ having an isolated singularity at the origin.

RÉSUMÉ. — On étudie les champs de vecteurs holomorphes quadratiques et homogènes de $\mathbf{C}^n$ qui sont semicomplets : ceux dont les solutions sont uniformes dans leurs domaines maximaux de définition. À un champ générique on associe de façon rationnelle quelques nombres complexes qui s'avèrent entiers dans le cas semicomplet. Ceci montre que, dans l'espace des classes d'équivalence linéaire de champs de vecteurs, les semicomplets sont contenus dans une sorte de réseau. On prouve que les feuilletages de $\mathbf{CP}^{n-1}$ induits par des champs quadratiques semicomplets sont linéarisables au voisinage de leurs points singuliers et on donne quelques familles nouvelles d'exemples dans $\mathbf{C}^3$. Finalement, on classifie les champs semicomplets de $\mathbf{C}^3$ qui sont isochores et à singularité isolée.

1. Introduction

A holomorphic action of $\mathbf{C}$ in a complex manifold induces a holomorphic vector field, the infinitesimal generator of this action. However, the converse is only true if the vector field under consideration is complete. Lack of completeness of a vector field is not only a global property but, as

Keywords: Complex differential equation, semicomplete vector field, holomorphic foliation.
Math. classification: 32S65, 34M05, 34M15, 34M35, 37F75.

the work of Rebelo has shed light upon, there exist local obstructions to completeness. Take for example the germ at the origin of the vector field $z^{n+1}\partial/\partial z$ for $n \geqslant 2$. The local solution $z(t)$ that takes the value $z_0 \neq 0$ for $t = 0$ satisfies

$$t = -\frac{1}{n}\left(\frac{1}{z^n(t)} - \frac{1}{z_0^n}\right).$$

In this way, a neighborhood of the origin of the z-plane maps in an n-to-one way to a neighborhood of infinity in the t-plane. Thus, we cannot parametrize *any* neighborhood of $z = 0$ in a single-valued way. The multi-valuedness of the solution is part of the germ of the vector field $z^{n+1}\partial/\partial z$ at the origin. This phenomenon does not arrive in complete vector fields, for if X is a holomorphic vector field on the complex manifold M then for every $p \in M$ there exists a solution of X, $\phi : \mathbf{C} \to M$, that maps 0 to p. Furthermore, if $N \subset M$ is an open submanifold containing p then $\phi|_{\phi^{-1}(N)}$ is the maximal solution with initial condition p, which is still single-valued. This motivates the following definition:

Definition 1.1 (adapted from [19]). — *Let M be a complex manifold and let X be a holomorphic vector field on M. For each point $p \in M$ where X does not vanish, denote by $\mathcal{C}_p$ the integral curve of X that contains p (with the topology given by the local solutions of X). The vector field X is* semicomplete *if for every point $p \in M$ where $X(p) \neq 0$ there exists an open subset $U_p \subset \mathbf{C}$ containing the origin and a covering map $\phi : (U_p, 0) \to (\mathcal{C}_p, p)$ that is a solution of X, this is $\phi_*(\partial/\partial z) = X|_{\mathcal{C}}$.*

Notice that the restriction of a semicomplete vector field to an open submanifold is still semicomplete and that we can thus speak of *germs* of semicomplete vector fields. The obstruction for a germ of singular vector field X to be semicomplete is particularly strong when the first jet of X at the singular point vanishes, this is, when the associated flow is tangent to the identity at a fixed point. We have the following theorem in dimension two:

Theorem 1.2 (Ghys-Rebelo [7]). — *Let X be a semicomplete vector field on the complex surface M. Let $p \in M$ be an isolated singularity of X where the first jet of X vanishes. Then there exists a neighborhood of p in M where the vector field is, up to multiplication by a non-vanishing holomorphic function, holomorphically conjugate to one of the vector fields*

- $x^2\dfrac{\partial}{\partial x} + y(y - nx)\dfrac{\partial}{\partial y}$ *for* $n \in \mathbf{Z}$, $n \geqslant 0$,
- $x(x - 3y)\dfrac{\partial}{\partial x} + y(y - 3x)\dfrac{\partial}{\partial y}$,

- $x(x-2y)\dfrac{\partial}{\partial x}+y(y-2x)\dfrac{\partial}{\partial y}$,
- $x(x-y)\dfrac{\partial}{\partial x}+y(y-2x)\dfrac{\partial}{\partial y}$,

in a neighborhood of the origin.

The proof of this result starts with the following observation [19]: if X is a semicomplete vector field in neighborhood of $0 \in \mathbf{C}^n$ and has a vanishing first jet, then the quadratic homogeneous vector field X_2 given by its second jet is a semicomplete vector field in $\mathbf{C}^n$. The authors then classify all semicomplete quadratic homogeneous vector fields in $\mathbf{C}^2$ and prove that every semicomplete vector field with an isolated singularity and vanishing first jet is holomorphically orbitally equivalent to the homogeneous one given by its second jet, proving the theorem.

In the list of the above theorem we have every semicomplete quadratic homogeneous vector field with an isolated singularity at the origin in $\mathbf{C}^2$ (up to linear equivalence). All the items in the first family of this list can be integrated by rational functions and have a rational first integral. Moreover, the projection onto the first coordinate maps these vector fields onto the semicomplete vector field $z^2\partial/\partial z$ on $\mathbf{C}$. The other three vector fields have holomorphic first integrals and can be explicitly solved by elliptic functions. In all these, the solutions are defined in the complement of a discrete set of points in $\mathbf{C}$.

Already at the level of quadratic homogeneous vector fields, the situation is completely different in dimension three. To begin with, we have *Halphen's vector fields*, those given by

$$
\begin{aligned}
\mathcal{H}(\alpha_1,\alpha_2,\alpha_3) &= [\alpha_1 z_1^2+(\alpha_1-1)(z_1z_2-z_2z_3+z_3z_1)]\frac{\partial}{\partial z_1}\\
&\quad+[\alpha_2 z_2^2+(\alpha_2-1)(z_1z_2+z_2z_3-z_3z_1)]\frac{\partial}{\partial z_2}\\
&\quad+[\alpha_2 z_2^2+(\alpha_3-1)(-z_1z_2+z_2z_3+z_3z_1)]\frac{\partial}{\partial z_3}.
\end{aligned}
\tag{1.1}
$$

They are semicomplete as soon as the quantities $m_i=(\alpha_1+\alpha_2+\alpha_3-2)/\alpha_i$ are positive integers different from unity. When $\sum 1/m_i<1$, these fields display some fascinating dynamical properties that are not present in lower dimensions. To begin with, the definition domain of the general solution has as natural boundary a circle or a straight line. Moreover, these vector fields do not have *any* meromorphic first integral though they have a real-valued continuous first integral. These vector fields were introduced and studied by Halphen [14]; a dynamical study of their properties can be found in [12].

Another phenomenon is portrayed by *Lins Neto's vector fields*, the ones in the linear span of

$$\begin{aligned}X_\infty &= 2z_2(-z_1+z_3)\frac{\partial}{\partial z_1}+(3z_1^2-z_2^2)\frac{\partial}{\partial z_2}+2z_3z_2\frac{\partial}{\partial z_3},\\ X_0 &= (-3z_1^2+z_2^2+2z_1z_3)\frac{\partial}{\partial z_1}+2z_2(-3z_1+2z_3)\frac{\partial}{\partial z_2}\\ &\quad +2z_3(3z_1-z_3)\frac{\partial}{\partial z_3}.\end{aligned}$$

Every element of this family is semicomplete and a generic one has an isolated singularity at the origin. This family contains an *uncountable* number of (linearly inequivalent) semicomplete vector fields. These were introduced in [9], where an account of their semicompleteness is given (the foliations that these vector fields induce in $\mathbf{CP}^2$ were introduced and studied by Lins Neto in his research on the Poincaré Problem [17]).

The richness of these examples motivated our research towards a general understanding of semicompleteness in the context of singular vector fields with vanishing first jets. Our results concern mainly quadratic homogeneous vector fields in dimension three.

The main tool in our study is to exploit the homogeneity of the semicomplete vector field X in order to build a maximal local action of the affine group that is naturally associated to X. The locus of points where this action fails to be locally free will be given generically by the union of 2^n-1 *radial orbits*, lines through the origin of $\mathbf{C}^n$ that are invariant by X. We will assign to each radial orbit an unordered set of $n-1$ of complex numbers –its *eigenvalues*– that are intrinsically attached to this orbit and that give a first order approximation of the dynamics of X around it. Our results in this direction can be stated as follows:

Theorem A. — *Let V_n denote the vector space of quadratic homogeneous vector fields in $\mathbf{C}^n$ and let $W_n\subset V_n$ be the Zariski open cone of fields having an isolated singularity at the origin. The function that to each field in W_n associates the eigenvalues of its radial orbits is the restriction of the $PGL_n(\mathbf{C})$-invariant rational map*

$$\mathsf{Spectrum}:\mathbf{P}V_n \dashrightarrow \mathrm{Sym}^{2^n-1}\mathbf{CP}^{n-1},$$

where $\mathbf{CP}^{n-1}$ is taken as a compactification of $\mathrm{Sym}^{n-1}\mathbf{C}\approx\mathbf{C}^{n-1}$. When restricted to the set of semicomplete vector fields within W_n, it takes values in $\mathrm{Sym}^{2^n-1}\mathrm{Sym}^{n-1}\mathbf{Z}$ and, for $n=3$, the preimage of a point under this map is generically given by a finite number of orbits of the action of $PGL_3(\mathbf{C})$ on V_3.

Analysis of the structure of semicomplete vector fields in the neighborhood of the radial orbits yields the following result:

THEOREM B. — *Let X be a quadratic homogeneous vector field in $\mathbf{C}^{n+1}$ having an isolated singularity at the origin. Then the foliation that X induces in $\mathbf{CP}^n$ has simple singularities. For each one of these the eigenvalues are commensurable and the foliation is locally linearizable.*

The eigenvalues of a quadratic homogeneous vector field are far from being arbitrary. They are bound by some relations, of which the most interesting one is the following: if X is a quadratic homogeneous vector field in $\mathbf{C}^n$ with an isolated singularity and radial orbits $\rho_1, \ldots, \rho_{2^n-1}$ and if we denote by ξ_i the product of the eigenvalues of ρ_i, then

$$\sum_{i=1}^{2^n-1} \frac{1}{\xi_i} = (-1)^{n+1}. \tag{1.2}$$

In this way, every semicomplete quadratic homogeneous vector field furnishes a solution to this Diophantine equation. For example, for $n = 3$, semicomplete fields within Halphen's family (1.1) yield the infinite three-parameter family of solutions

$$\{(1, m_1, -m_1, m_2, -m_2, m_3, -m_3)\}$$

and a generic element in Lins Neto's family gives the solution $(2, 2, 6, 6, 6, -3, -6)$. The study of this "Egyptian fractions" problem led to the discovery of some countable families of semicomplete vector fields in $\mathbf{C}^3$. These families are closely related to Halphen's equations and are described in Subsection 3.1. Finally, in Subsection 3.4, the particularity of the solutions of (1.2) arising from isochoric (divergence-free) semicomplete vector fields allows us to prove the following:

THEOREM C. — *Let X be an isochoric semicomplete quadratic vector field in $\mathbf{C}^3$ having the origin as an isolated singularity. Then X is linearly equivalent to one of the semicomplete isochoric vector fields*

$$z_1^2\frac{\partial}{\partial z_1} + z_2^2\frac{\partial}{\partial z_2} + z_3^2\frac{\partial}{\partial z_3} - (z_1z_2 + z_2z_3 + z_3z_1)\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\right), \tag{1.3a}$$

$$z_1^2\frac{\partial}{\partial z_1} + z_2^2\frac{\partial}{\partial z_2} + 3z_3^2\frac{\partial}{\partial z_3} + (z_1z_2 - z_2z_3 - z_1z_3)\left(3\frac{\partial}{\partial z_1} + 3\frac{\partial}{\partial z_2} + 5\frac{\partial}{\partial z_3}\right), \tag{1.3b}$$

$$(2z_1^2 - 4z_1z_2 + 4z_2z_3 - z_1z_3)\frac{\partial}{\partial z_1} + (2z_2^2 - 4z_1z_2 + 4z_1z_3 - z_2z_3)\frac{\partial}{\partial z_2} + z_3^2\frac{\partial}{\partial z_3}. \tag{1.3c}$$

Without the hypothesis on the isolated nature of the singularity, there exists a twofold of (linear equivalence classes of) isochoric semicomplete vector fields. This family is described in Example 3.9.

Our gratitude goes to Étienne Ghys for his enlightenment and support. We wish to thank Dominique Cerveau, Bertrand Deroin and Bruno Sévennec for several enlightening conversations. The last issues of this article were settled while enjoying the hospitality of IMPA at Rio.

2. On quadratic vector fields

Let V_n be the vector space of homogeneous quadratic vector fields on $\mathbf{C}^n$. A basis for V_n is given by the vector fields of the form $z_i z_j \partial/\partial z_k$ for i, j, k in $\{1, \ldots, n\}$, with $i \leqslant j$. The dimension of V_n is thus $n^2(n+1)/2$. The group $\mathrm{GL}(n, \mathbf{C})$ acts upon this vector space via the linear changes of coordinates of the underlying space $\mathbf{C}^n$. The property of semicompleteness is invariant under the action of the group. As a representation space for $\mathrm{GL}_n(\mathbf{C})$, this space is isomorphic to $\mathrm{Sym}^2 W^* \otimes W$, where W is the standard representation of $\mathrm{GL}_n(\mathbf{C})$ on $\mathbf{C}^n$. We have the equivariant contraction $\nabla : \mathrm{Sym}^2 W^* \otimes W \to W^*$ given by

$$\ell_1 \ell_2 \otimes w \mapsto \langle \ell_1, w \rangle \ell_2 + \langle \ell_2, w \rangle \ell_1.$$

Its kernel is the irreducible representation $\Gamma_{1,0,\ldots,0,2}$ and the above representation decomposes as the direct sum $\Gamma_{1,0,\ldots,0,2} \oplus W^*$ [6]. In the context of vector fields, these constructions behave as follows. Let E denote Euler's vector field $\sum_{i=1}^n z_i \partial/\partial z_i$. Products of a linear form $\ell \in (\mathbf{C}^n)^*$ with E, quadratic vector fields of the form $\ell \cdot E$, form an irreducible representation, isomorphic to $(\mathbf{C}^n)^*$. These fields are always collinear to Euler's vector field and will be henceforth called *radial*. The contraction $\nabla : V_n \to (\mathbf{C}^n)^*$, $\nabla(\sum_i P_i \partial/\partial z_i) = \sum_i \partial P_i/\partial z_i$, is given by divergence with respect to the volume form $dz_1 \wedge \cdots \wedge dz_n$. The kernel of ∇ is the subspace of *isochoric* vector fields, stable under the action of $\mathrm{GL}(n, \mathbf{C})$. The space V_n is the direct sum of radial and isochoric vector fields; the divergence of the radial vector field $\ell \cdot E$ is $(n+1)\ell$.

Let X be a quadratic homogeneous vector field on $\mathbf{C}^n$. The homogeneity of X can be infinitesimally expressed as $[E, X] = X$. Recall that $\mathrm{Aff}(\mathbf{C})$, the complex affine group, is the group of matrices of the form

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} ; a \in \mathbf{C}^*, b \in \mathbf{C} \right\}$$

and that a basis for its Lie algebra $\mathfrak{aff}(\mathbf{C})$ is given by

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

These elements satisfy the commutator relation $[y,x]=x$. The corresponding *right*-invariant vector fields $\widetilde{x}$ and $\widetilde{y}$ on $\mathrm{Aff}(\mathbf{C})$ satisfy the bracket relation $[\widetilde{y},\widetilde{x}]=-\widetilde{x}$. Let $\Psi_X:\mathfrak{aff}(\mathbf{C})\to\mathfrak{X}(\mathbf{C}^n)$ be the Lie algebra morphism, from the Lie algebra of right invariant vector fields on $\mathrm{Aff}(\mathbf{C})$ to the Lie algebra of holomorphic vector fields in $\mathbf{C}^n$, determined by

$$\Psi_X(\widetilde{x})=X,\qquad \Psi_X(\widetilde{y})=-E.$$

We would like to construct some kind of action of the affine group associated with this representation:

Definition 2.1 (Palais [18]). — *A holomorphic* maximal local action to the left *of the complex Lie group G on the manifold M is a holomorphic mapping $\Phi:\mathcal{U}\longrightarrow M$ defined on an open set $\mathcal{U}\subset G\times M$ with $\{e\}\times M\subset\mathcal{U}$, satisfying the following conditions:*

1. *$\Phi(e,p)=p$ pour tout $p\in M$,*
2. *$\Phi(g_2,\Phi(g_1,p))=\Phi(g_2g_1,p)$ if both members of this equation are defined and*
3. *for every $p\in M$ and every sequence $\{g_i\}\subset\mathcal{U}_p=\{g\in G;\ (g,p)\in\mathcal{U}\}$ such that $\lim_{i=\infty}g_i\in\partial\mathcal{U}_p$, the sequence $\Phi(g_i,p)$ escapes from every compact subset of M.*

Such a local action induces a Lie algebra morphism $\Phi_*:\mathfrak{g}\to\mathfrak{X}(M)$. When $G=\mathbf{C}$, this definition becomes that of Rebelo's *semiglobal flow*. The vector field on M associated to a semiglobal flow is semicomplete and, conversely, such a vector field has an associated semiglobal flow.

Proposition 2.2. — *The vector field $X\in V_n$ is semicomplete if and only there exists a maximal local action Φ of $\mathrm{Aff}(\mathbf{C})$ on $\mathbf{C}^n$ such that $\Phi_*=\Psi_X$.*

Proof. — Assume X to be semicomplete. For every $p\in\mathbf{C}^n$ let $\phi^\tau(p):\mathcal{U}_p\to\mathbf{C}^n$ be the maximal solution of X having p as initial condition evaluated at time τ. The relation $[-E,X]=-X$ can be restated as the fact that, for every $q\in\mathbf{C}^n$, $\alpha\in\mathbf{C}^*$ and every $\tau\in\mathcal{U}_q$, we have

$$\alpha\phi^{\alpha\tau}(\alpha^{-1}q)=\phi^\tau(q). \tag{2.1}$$

Let $\mathcal{U}$ be the open set given by

$$\mathcal{U}=\{((a,b),q)\in\mathrm{Aff}(\mathbf{C})\times\mathbf{C}^n;\ a^{-1}b\in\mathcal{U}_q\}.$$

Let $\Phi:\mathcal{U}\to\mathbf{C}^n$ be defined by

$$\Phi((a,b),q)=\phi^b(a^{-1}q). \tag{2.2}$$

It satisfies trivially the condition (1) in the definition of a maximal local action. For condition (2), notice that

$$\begin{aligned}\Phi((a_1,b_1),\Phi((a_0,b_0),q)) &= \phi^{b_1}(a_1^{-1}\phi^{b_0}(a_0^{-1}q))\\ &= \phi^{b_1}\circ\phi^{a_1b_0}(a_1^{-1}a_0^{-1}q)\\ &= \Phi((a_1,b_1)\cdot(a_0,b_0),q),\end{aligned}$$

where defined. For condition (3), let $\{((a_i,b_i),q)\}\subset\mathcal{U}$ be a sequence that accumulates to a point in $\partial(\mathcal{U}\cap\mathrm{Aff}(\mathbf{C})\times\{q\})$ and such that the sequence $\{\Phi((a_i,b_i),q)\}\subset\mathbf{C}^n$ converges. Because $\Phi((a_i,b_i),q)=a_i^{-1}\phi^{a_i^{-1}b_i}(q)$ and because $\{a_i^{-1}b_i\}\longrightarrow\partial\mathcal{U}_q$, we must have $a_i\longrightarrow\infty$, but this contradicts the fact that the original sequence accumulates the boundary of $\mathcal{U}\cap(\mathrm{Aff}(\mathbf{C})\times\{q\})$. Finally, when restricted to the one-parameter groups generated by x and y, this action gives the semi-global flow of X and the flow of $-E$. For the converse statement, it suffices to restrict the maximal local action to a suitable one-parameter group. □

The above machinery works well beyond the scope of this article, being, as it is, purely based of the lie algebraic relation of X and E and on the *global* nature of the solutions of E. This has been exploited in [11], where we investigate the existence of isochronous settings for Calogero's many-body "goldfish" problem.

2.1. Radial orbits and eigenvalues

The main tool in our analysis of semicompleteness of quadratic homogeneous vector fields will be the study of their dynamics of such a vector field in the neighborhood of points belonging to some special orbits:

Definition 2.3. — *Let X be a quadratic homogeneous vector field in $\mathbf{C}^n$. A* radial orbit *of X is a one dimensional subspace of $\mathbf{C}^n$ (a line through the origin) where X and E are collinear. A radial orbit is said to be* degenerate *if X vanishes identically on it.*

A generic quadratic homogeneous vector field in $\mathbf{C}^n$ has 2^n-1 non-degenerate radial orbits [10]. To such a radial orbit we will associate an unordered collection of $(n-1)$ complex numbers that will be called its *eigenvalues*. This is done in the following way. Let ρ be a non-degenerate radial orbit of X and let $q\in\rho$. Let $\alpha_q\in\mathbf{C}$ be the *unique* complex number such that the vector field $L=\alpha_qX-E$ vanishes at q. The projection of L onto the (local) space of orbits of X is well-defined and has a singularity at the point corresponding to ρ. It will be denoted by $L^\flat$.

DEFINITION 2.4. — *The* extended linear type of ρ *is the conjugacy class of the linear term of* L *at* q. *The* reduced linear type of ρ *is the conjugacy class of the linear term of* $L^\flat$ *around* ρ. *The eigenvalues of the latter are said to be the* eigenvalues of X along ρ.

It is not difficult to see that these notions are intrinsically attached to the radial orbit and do not depend on the point q. Consider now the following lemma:

LEMMA 2.5. — *Let* L *and* X *be vector fields in a neighborhood of the origin in* $\mathbf{C}^n$ *such that* $L(0) = 0$, $X(0) \neq 0$ *and* $[L, X] = -X$. *If the germ of* L *at the origin is semicomplete and all its solutions are* $2i\pi$*-periodic then there exists a change of coordinates that simultaneously redresses* X *onto* $\partial/\partial z_1$ *and* L *onto a vector field of the form* $z_1\partial/\partial z_1 + \sum_{i=2}^n \lambda_i z_i \partial/\partial z_i$ *with* $\lambda_i \in \mathbf{Z}$.

Proof. — Suppose, without loss of generality, that $X = \partial/\partial z_1$. The field L is necessarily of the form $z_1\partial/\partial z_1 + \sum_{i=1}^n f_i(z_2, \ldots, z_n)\partial/\partial z_i$, where, for every i, f_i is a function that vanishes at the origin. The field $L^\flat = \sum_{i=2}^n f_i \partial/\partial z_i$, image of L under the projection unto the hyperplane $\{z_1 = 0\}$, has only periodic solutions, whose periods are multiples of $2i\pi$. The restriction of the maximal local action of $\mathbf{C}/2i\pi$ induced by $L^\flat$ to the real compact Lie subgroup $i\mathbf{R}/2i\pi\mathbf{Z}$ gives a local maximal action by biholomorphisms in a neighborhood of the origin of $\mathbf{C}^{n-1}$. According to the Bochner-Cartan Theorem [2], this action is holomorphically linearizable and there exists thus a biholomorphism $F = (F_2, \ldots, F_n)$, fixing the origin of $\mathbf{C}^{n-1}$, that maps $L^\flat$ onto a holomorphic vector field whose imaginary flow is linear and 2π-periodic, this is, a vector field of the form $\sum \lambda_i z_i \partial/\partial z_i$ with $\lambda_i \in \mathbf{Z}$. The biholomorphism $(z_1, F_2, \ldots, F_n)$ preserves $\partial/\partial z_1$ and maps L onto a vector field of the form $[z_1 + h(z_2, \ldots, z_n)]\partial/\partial z_1 + \sum_{i=2}^n \lambda_i z_i \partial/\partial z_i$, for some holomorphic function h that vanishes at the origin. Let $\sum_{i_2,\ldots,i_n=0}^\infty a_{i_2\cdots i_n} z_2^{i_2} \cdots z_n^{i_n}$ be the Taylor development of h at the origin. Set

$$b_{i_2\cdots i_n} = \begin{cases} 1, & \text{if } \sum_j \lambda_j i_j = 1; \\ 1 - \sum_j \lambda_j i_j, & \text{if } \sum_j \lambda_j i_j \neq 1. \end{cases}$$

Let $g(z_2, \ldots, z_n)$ be the function defined in a neighborhood of $0 \in \mathbf{C}^{n-1}$ by the series

$$\sum_{i_2,\ldots,i_n=0}^{\infty} \frac{a_{i_2\cdots i_n}}{b_{i_2\cdots i_n}} z_2^{i_1} \cdots z_n^{i_n}.$$

Its coefficients are smaller in absolute value than those of h and this guarantees its convergence. The change of coordinates $(z_1, z_2, \ldots, z_n) \mapsto$

$(z_1 + g, z_2, \ldots, z_n)$ preserves $\partial/\partial z_1$ and maps L onto a vector field

$$\widetilde{L} = [z_1 + \widetilde{h}(z_2, \ldots, z_n)]\partial/\partial z_1 + \sum_{i=2}^{n} \lambda_i z_i \frac{\partial}{\partial z_i},$$

where the Taylor series of $\widetilde{h}$ has only monomials $z_2^{i_2} \cdots z_n^{i_n}$ such that $\sum_{j=2}^{n} \lambda_j i_j = 1$ (resonant monomials). Let $\phi : U \to \mathbf{C}^n$ be a solution of $\widetilde{L}$ with the initial condition $(y_1, y_2, \ldots, y_n)$. For $i \geqslant 2$, $z_i \circ \phi(t) = y_i e^{\lambda_i t}$ and thus $\widetilde{h}(z_2 \circ \phi, \ldots, z_n \circ \phi) = ke^t$ for some $k \in \mathbf{C}$ depending on $(y_2, \ldots, y_n)$. The function $\zeta(t) = z_1 \circ \phi(t)$ solves the differential equation $d\zeta/dt = \zeta + ke^t$, whose general solution is $\zeta(t) = (kt + \zeta_0)e^t$. The latter cannot be a periodic function of time unless k vanishes. In this way, the solutions of $\widetilde{L}$ coincide with those of $z_1\partial/\partial z_1 + \sum_{i=2}^{n} \lambda_i z_i \partial/\partial z_i$. □

Corollary 2.6 (Integrality of eigenvalues). — *If X is a semicomplete quadratic homogeneous vector field and ρ a non-degenerate radial direction of X then the extended and the reduced linear type of X around ρ are diagonalizable and have integral eigenvalues.*

Proof. — Suppose X is semicomplete. The restriction of the maximal local action Φ of equation (2.2) to the one parameter subgroup generated by $\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$ is the flow of $\alpha X - E$, and this vector field is thus semicomplete. Because the one parameter subgroup associated to $\alpha x + y \in \mathfrak{aff}(\mathbf{C})$,

$$\exp \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} t = \begin{pmatrix} e^t & \alpha[e^t - 1] \\ 0 & 1 \end{pmatrix},$$

has $2i\pi\mathbf{Z}$ in its kernel, the solutions of $\alpha X - E$ have $2i\pi$ among their periods. In the coordinates guaranteed by the Lemma the projection onto the leaf space of X is given by $(z_1, z_2, \ldots, z_n) \mapsto (z_2, \ldots, z_n)$ and $L^\flat$ is thus given by $\sum_{i=2}^{n} \lambda_i z_i \partial/\partial z_i$. This proves the Corollary. □

Thus, in the semicomplete case, the eigenvalues (that are, a priori, a first order approximation) determine completely, up to a holomorphic change of coordinates, the dynamics of a semicomplete vector field in a neighborhood of the points that belong to a non-degenerate radial orbit.

Corollary 2.7. — *Let $X \in V_n$ be a semicomplete vector field.*

(1) *If the origin is an isolated singularity of X then the radial orbits are isolated (the locus of points belonging to radial orbits is one-dimensional).*
(2) *Every isolated non-degenerate radial orbit is simple.*

(3) *If ρ is a non-degenerate radial orbit having no zero eigenvalues then the foliation that X induces in $\mathbf{CP}^{n-1}$ is linearizable at the corresponding singular point.*

Proof. — (1) Denote by $\mathcal{V} \subset \mathbf{C}^n$ the locus of collinearity of X and E, $\mathcal{V} = \{X \wedge E = 0\}$. Define on $\mathcal{V}$ the E-invariant function given by $z_1(p)E(p)/X(p)$. This defines a rational function on $\mathbf{P}\mathcal{V}$, a variety whose dimension is strictly greater than zero if the radial orbits are not isolated. Any $p \in \mathcal{V}$ projecting unto the polar locus of this function lies on a degenerate radial orbit and thus, if the origin is an isolated singularity of X, the dimension of $\mathbf{P}\mathcal{V}$ is zero: the radial orbits are isolated.

(2) Let $q \in \rho$ and choose the coordinates around q guaranteed by Lemma 2.5. The locus of radial orbits is locally given by $\left\{\bigcap_{i>1} \lambda_i w_i = 0\right\}$. Because ρ is supposed to be isolated, $\lambda_i \neq 0$ for all i. In this case, ρ is given by the intersection of $n-1$ transverse hyperplanes and is thus simple.

(3) Let $q \in \mathbf{C}^n$ be a point lying on ρ. Let $[q] \in \mathbf{CP}^{n-1}$ be the corresponding singularity of the induced foliation. Take the coordinates around q guaranteed by Lemma 2.5. The hypersurface $\{w_1 = 0\}$ is transverse to E at q and cuts once every orbit of E. The vector field

$$(\alpha - w_1)X - E = \sum_{i=2}^{n} \lambda_i w_i \partial/\partial w_i$$

is tangent to $\{w_1 = 0\}$ and, by definition, is in the linear span of E and X. It gives thus the expression of a vector field tangent to the induced foliation in $\mathbf{CP}^{n-1}$. This finishes the proof of the corollary and of Theorem B. □

Consider the vector field $X = \sum a_{ij}^k z_i z_k \partial/\partial z_k$. Suppose that X has a non-degenerate radial orbit ρ at the direction $[1 : 0 : \ldots : 0]$, so that $a_{11}^j = 0$ if $j \geqslant 2$. Suppose, without loss of generality, that $a_{11}^1 = 1$. We will calculate the eigenvalues of this radial orbit in three different ways. The naturality of the following calculations as well as their invariance with respect to the action of the linear group show that they are all equivalent.

From the definition. Make the change of coordinates $z_1 = \epsilon + \widetilde{z}_1$ for $\epsilon \in \mathbf{C}$. In the coordinates $(\widetilde{z}_1, z_2, \ldots, z_n)$ the vector field $\frac{1}{\epsilon}X - E$ vanishes at the origin and the linear part of its Taylor development is given by

$$\left(\widetilde{z}_1 + \sum_{j=2}^{n} a_{1j}^1 z_j\right) \frac{\partial}{\partial \widetilde{z}_1} + \sum_{j=2}^{n} \left(\sum_{k=2}^{n} a_{1k}^j z_k - z_j\right) \frac{\partial}{\partial z_j}.$$

From a Lie bracket. Consider the constant vector field (homogeneous of degree zero) that has the chosen radial direction as an orbit (this vector field is unique up to multiplication by a constant). In our case all admissible vector fields are of the form $\lambda\partial/\partial z_1$. Consider the Lie bracket $\ell = [\lambda\partial/\partial z_1, X]$. This is a linear vector that has an orbit along the radial direction under consideration, for it is a common orbit of X and $\partial/\partial z_1$. Normalize the vector field so that the eigenvalue of ℓ corresponding to this direction is 2 (in our case, $\lambda = 1$) and then substract Euler's vector field. In the present setting this yields:

$$\begin{aligned}\left[\frac{\partial}{\partial z_1}, X\right] - E = & \sum_{i=1}^{n}\left(\frac{\partial P_i}{\partial z_1} - z_i\right)\frac{\partial}{\partial z_i} \\ = & \left(z_1 + \sum_{j=2}^{n} a_{1j}^1 z_j\right)\frac{\partial}{\partial z_1} + \sum_{j=2}^{n}\left(\sum_{k=2}^{n} a_{1k}^j z_k - z_j\right)\frac{\partial}{\partial z_j}.\end{aligned}$$

The resulting linear vector field represents the extended linear type.

From the foliation. The eigenvalues can be also calculated directly from the foliation of the vector field in $\mathbf{C}^n$, after blowing up the origin. Consider once again the vector field X. Blow up the origin and consider the chart of this blow up given by $z_1 = u_1$ and $z_i = u_i z_1$ for $i = 2, \dots, n$. In the coordinates $(u_1, u_2, \dots, u_n)$ we have, for the strict transform of the vector field,

$$\begin{aligned}\frac{1}{u_1}\widetilde{X} = & u_1 P_1(1, u_2, \dots, u_n)\frac{\partial}{\partial u_1} \\ & + \sum_{i\geqslant 2}\left[P_i(1, u_2, \dots, u_n) - u_i P_1(1, u_2, \dots, u_n)\right]\frac{\partial}{\partial u_i}.\end{aligned}$$

The linear part of the right-hand side of this expression is

$$u_1\frac{\partial}{\partial u_1} + \sum_{j=2}^{n}\left(\sum_{k=2}^{n} a_{1k}^j u_k - u_j\right)\frac{\partial}{\partial u_j}.$$

The restriction of this field to the exceptional divisor $\{u_1 = 0\}$ matches the reduced linear type. In this way, the eigenvalues of the vector field X with respect to to the radial direction ρ can be calculated from the eigenvalues of a vector field tangent to the foliation induced by X after blowing up the origin: the eigenvalues of X with respect to ρ are the eigenvalues of the linear part of the restriction of the vector field to the exceptional divisor when the vector field has been normalized in such a way that the eigenvalue corresponding to the invariant curve given by ρ is 1. This is the definition of the eigenvalues of a quadratic homogeneous vector field along a radial orbit given in [10]. A detailed proof of the rational character of the function Spectrum that we refer to in Theorem A is found in this article, where the reader will also find the following results:

- Let B be a homogeneous invariant polynomial in $\mathrm{Hom}(\mathbf{C}^{n-1}, \mathbf{C}^{n-1})$ of degree $d \leqslant n-1$. Let $X \in V_n$ be a quadratic homogeneous vector field with simple radial directions and an isolated singularity at the origin. Let $\rho_1, \ldots, \rho_{2^n-1}$ be the radial orbits and let A_i be the reduced linear type of ρ_i. Then the value of the expression
$$\sum_{i=1}^{2^n-1} \frac{B(A_i)}{\det(A_i)}$$
depends only on n and B.
- Let $X \in V_n$ be an isochoric vector field having non-degenerate radial orbits ρ_1, …, ρ_{2^n-1}. For each $\ell \in (\mathbf{C}^n)^*$, let $\tau_i(\ell)$ denote the sum of the eigenvalues of the radial orbit ρ_i of the vector field $X + \ell \cdot E$. Then the function $\Xi_X : (\mathbf{C}^n)^* \to \mathbf{C}^{2^n-1}$ given by
$$\Xi_X(\ell) = \left(\frac{1}{\tau_1(\ell)} + \frac{1}{n+1}, \ldots, \frac{1}{\tau_{2^n-1}(\ell)} + \frac{1}{n+1}\right)$$
is injective and linear (*op. cit.*, Lemma 5).

Example 2.8. — The simplest family of homogeneous quadratic semi-complete vector fields is
$$Q_n = \sum z_i^2 \frac{\partial}{\partial z_i}.$$
This vector field is symmetric under the full permutation group in n symbols acting by permuting the variables. Up to these permutations, all the radial orbits are of the form $[\underbrace{1 : \cdots : 1}_{m} : 0 : \cdots : 0]$. For the eigenvalues of this last orbit we have:
$$\left[\sum_{i=1}^{m} \frac{\partial}{\partial z_i}, Q_n\right] = \sum_{i=1}^{m} 2z_i \frac{\partial}{\partial z_i},$$
and thus this radial orbit has the eigenvalue 1 with multiplicity $m-1$ and the eigenvalue -1 with multiplicity $n-m$.

Remark 2.9. — The eigenvalues of isochoric vector fields are very particular. In the above setting, the divergence of X evaluated at p is given by $2 + \sum_{j=2}^{n} a_{1j}^{j}$. The sum of the eigenvalues of ρ is $\left(\sum_{j=2}^{n} a_{1j}^{j}\right) - (n-1)$ and thus a non-degenerate radial orbit lies in the locus of zero divergence if and only if the sum of its eigenvalues is $-(n+1)$. In particular, the sum of the eigenvalues of a non-degenerate radial orbit of an isochoric vector field is never zero.

2.2. The leafwise affine structure on the induced foliation

Being homogeneous, the vector field $X \in V_n$ induces a foliation in $\mathbf{CP}^{n-1}$, for its orbits get permuted under the action of homothecies. We will denote this foliation by $\mathcal{F}_X$. The natural parametrization of the orbits of X gives an extra structure to the leaves of this foliation.

Let Σ be a holomorphic curve. An *affine structure* on Σ is given by an atlas taking values in $\mathbf{C}$ whose coordinate changes lie in $\mathrm{Aff}(\mathbf{C})$. A *Euclidean structure* is an affine structure whose coordinate changes are in $\mathbf{C}$, the commutator subgroup of $\mathrm{Aff}(\mathbf{C})$. Let $p \in \Sigma$. An affine structure on Σ induces a *developing map* $\mathcal{D} : (\widetilde{\Sigma}, p) \to \mathbf{C}$ and a *monodromy morphism* $\mu : \pi_1(\Sigma, p) \to \mathrm{Aff}(\mathbf{C})$ that satisfy the relation

$$\mathcal{D}(\alpha \cdot q) = \mu(\alpha) \cdot \mathcal{D}(q) \tag{2.3}$$

for every $\alpha \in \pi_1(\Sigma)$ [20]. The affine structures on a curve form an affine space over the vector space of differential forms on Σ. Let $\phi_1 : U_1 \to \mathbf{C}$ and $\phi_2 : U_2 \to \mathbf{C}$ be coordinate charts of two different affine structures around the point $p \in \Sigma$. Let $f = \phi_2 \circ \phi_1^{-1}$. Then, the differential form

$$\frac{f''(t)}{f'(t)} dt$$

does not depend on the charts chosen and vanishes if and only if ϕ_1 and ϕ_2 define the same affine structure [13]. In our context, let $X \in V_n$ be a (not necessarily semicomplete) quadratic vector field and let p^* a point that does not belong to a radial orbit. Let $\mathcal{O}^*$ be the orbit of X through p^* (with the topology given by the local solutions of X). Let $\Pi : \mathbf{C}^n \backslash \{0\} \to \mathbf{CP}^{n-1}$ be the standard projection and let $\mathcal{O}$ be the image of $\mathcal{O}^*$ under this map (as a curve). Let $q \in \mathcal{O}^*$ and let $\phi_q : U_q \to \mathcal{O}^*$ be a local solution of X having initial condition q. Relation (2.1) shows that the maps $\{(\Pi \circ \phi_q)^{-1}\}$ give the atlas of an affine structure on $\mathcal{O}$.

Definition 2.10. — *An affine structure on the curve Σ is said to be* uniformizable *if there exists an open set $U \in \mathbf{C}$ and a subgroup Γ of the affine group leaving U invariant and acting properly discontinuously on U in such a way that Σ is affinely isomorphic to U/Γ.*

The reader will recognize in Definition 1.1 that of uniformizability of the Euclidean structure induced by a vector field. Relation (2.3) shows that the developing map is well-defined in the covering $\Sigma_\mu \longrightarrow \Sigma$ corresponding to the kernel of the monodromy, this is, we have a map $\mathcal{D}_\mu : \Sigma_\mu \to \mathbf{C}$. It is not difficult to see that the uniformizability is equivalent to the injectivity of $\mathcal{D}_\mu$.

Proposition 2.11. — *A vector field $X \in V_n$ is semicomplete if and only if the affine structure of every leaf of $\mathcal{F}_X$ is uniformizable.*

Proof. — Let X be a quadratic homogeneous vector field. The map $\Pi|_{\mathcal{O}^*} : \mathcal{O}^* \to \mathcal{O}$ is a normal covering, whose group of deck transformations is the group of homothecies of $\mathbf{C}^n$ preserving $\mathcal{O}^*$. Let $\widetilde{\mathcal{O}}$ denote the universal covering of both $\mathcal{O}^*$ and $\mathcal{O}$. Let $\mathcal{D} : \widetilde{\mathcal{O}} \to \mathbf{C}$ be the developing map of both the Euclidean structure on $\mathcal{O}^*$ and the affine one on $\mathcal{O}$. Let $\mu^* : \pi_1(\mathcal{O}^*) \to \mathbf{C}$ and $\mu : \pi_1(\mathcal{O}) \to \mathrm{Aff}(\mathbf{C})$ denote the corresponding monodromy morphisms. Let $\mu' : \pi_1(\mathcal{O}) \to \mathbf{C}^*$ be the composition of μ with the abelianization of $\mathrm{Aff}(\mathbf{C})$. Associated to the groups

$$\{e\} \subset \ker(\mu) \subset \ker(\mu') \subset \pi_1(\mathcal{O}),$$

we have the covering maps

$$\widetilde{\mathcal{O}} \longrightarrow \mathcal{O}_\mu \longrightarrow \mathcal{O}'_\mu \longrightarrow \mathcal{O}.$$

We claim that $\mathcal{O}^* = \mathcal{O}'$ and that $\mu^* = \mu'$. Let $\alpha \in \pi_1(\mathcal{O})$. Let α^* denote the class of α in the group of deck transformations of the cover $\Pi : \mathcal{O}^* \to \mathcal{O}$. According to the relation (2.1), this action changes the natural parametrization of $\mathcal{O}^*$ (given by X) if and only if $\mu'(\alpha) \neq 1$. Hence, the action of α^* on $\mathcal{O}^*$ fixes p^* if and only if $\mu'(\alpha) = 1$ and hence the group of deck transformations of Π is naturally identified with $\ker(\mu')/\ker(\mu)$. The cover $\mathcal{O}_\mu$ is, in this way, the covering of $\mathcal{O}^*$ corresponding to the kernel of μ^*. The injectivity of the developing map $\mathcal{D}_\mu : \mathcal{O}_\mu \to \mathbf{C}$ accounts thus for the uniformizability of both structures. □

Example 2.12 (In dimension two). — The most general quadratic homogeneous vector field in $\mathbf{C}^2$ having non-degenerate and simple radial orbits in the directions $[1:0]$, $[0:1]$ and $[1:1]$ is a multiple of the vector field

$$z_1(\mu_2 z_1 + \mu_1[\mu_2+1]z_2)\frac{\partial}{\partial z_1} + z_2(\mu_1 z_2 + \mu_2[\mu_1+1]z_1)\frac{\partial}{\partial z_2},$$

with $\mu_1\mu_2 \neq 0$. The eigenvalues of the radial directions $[1:0]$ and $[0:1]$ are, respectively, μ_1 and μ_2. The eigenvalue of the third radial orbit, is determined by the relation $\sum_{i=1}^{3} 1/\mu_i = -1$. The only integral solutions to these equations are those of the form $\{-1,-m,m\}$ for $m \in \mathbf{Z}$ and the exceptional solutions $\{-2,-4,-4\}$, $\{-2,-3,-6\}$, $\{-3,-3,-3\}$. The first family gives the vector fields in item (1) of Theorem 1.2; the other three yield the vector fields in items (2)–(4). As to the induced affine structure, the inverse of the developing map $\mathcal{D}$ is given by $f(t) = [z_1(t) : z_2(t)]$.

Let $[s:1]$ be an affine coordinate in the target space of f. Then

$$\frac{\mathcal{D}''(s)}{\mathcal{D}'(s)}ds = -\frac{f''}{(f')^2}ds = \frac{\mu_1(\mu_2+1)-\mu_2(\mu_1-1)\,s}{\mu_1\mu_2 s(s-1)}ds.$$

In this way, the developing map of the affine structure is given, as a multivalued function on $\mathbf{CP}^1\backslash\{0,1,\infty\}$, by any non-constant solution of the differential equation

$$g''(s) = \frac{\left(1+\frac{1}{\mu_2}\right)-\left(1-\frac{1}{\mu_1}\right)s}{s(s-1)}g'(s).$$

For example, if $\mu_3=-1$ then $\mu_2=-\mu_1$ and this equation becomes $sg''=(\mu_1^{-1}-1)g'$. It is solved by $g^{\mu_1}(s)=s$. In the other cases, the restriction of a non-constant solution to the upper-half plane gives the uniformization of the Euclidean triangle of internal angles $-\pi/\mu_i$ [15]. The quotient of the plane under the associated triangle group gives (when restricted to the set of points with trivial stabilizer) the uniformization of the affine structure.

Example 2.13 (Euler's top). — The time evolution of the angular momentum of a rigid body moving freely in space is given by a system of quadratic homogeneous differential equations. The complexification of these equations gives, after a linear change of coordinates [16], the vector field

$$z_2z_3\frac{\partial}{\partial z_1}+z_1z_3\frac{\partial}{\partial z_2}+z_1z_2\frac{\partial}{\partial z_3}. \tag{2.4}$$

This field has the quadratic first integrals given by $\sum a_iz_i^2$ for $\sum a_i=0$. The leaves of the induced foliation belong thus to a pencil of conics. In the affine coordinates $[u+1:v+1:1]$, this foliation is given by the kernel of the form

$$v(u+1)(v+2)\,du-u(v+1)(u+2)\,dv.$$

After blowing up the origin by $u=sv$ and dividing by v^2 this form becomes

$$s(s-1)\,dv+(v+2)(vs+1)\,ds.$$

The general integral curve of the foliation is given by

$$v(s) = -2\frac{(s-1)\lambda+1}{(s^2-1)\lambda+1},$$

for $\lambda\in\mathbf{CP}^1$. For the values $\lambda\in\{0,1,\infty\}$, we get the invariant lines $\{z_i+z_j=0\}$. If we set $s(t)=(z_1-z_3)/(z_2-z_3)$ then we find, after substituting the above value of v,

$$-\frac{s''}{(s')^2}ds = -\frac{1}{2}\frac{3\lambda s^2-4\lambda s+2s+\lambda-1}{s(s-1)(\lambda s-\lambda+1)}ds = d\log P(s)^{-1/2},$$

for $P(s) = s(s-1)(\lambda s - \lambda + 1)$. This affine structure is uniformizable for the reasons that follow. An elliptic curve Σ is endowed with a canonical affine structure, induced by any holomorphic vector field. This structure is invariant under an elliptic involution. Thus, the quotient of Σ under the elliptic involution is —outside the ramification values— endowed with a uniformizable affine structure. If $\Sigma = \mathbf{C}/\Lambda$ and if $f : \Sigma \to \mathbf{CP}^1$ is the quotient under this involution, then f satisfies a differential equation of the form $(f')^2 = P(f)$, for P a polynomial of degree three or four. The invariant of the affine structure in $\mathbf{CP}^1$ is given by the form

$$-\frac{f''}{(f')^2}ds = -\frac{1}{2}\frac{P'(s)}{P(s)}ds.$$

Hence, the leafwise affine structures induced by equation (2.4) are uniformizable and the vector field is semicomplete.

3. In dimension three

3.1. Vector fields of Halphen type

The most interesting family of quadratic semicomplete vector fields in $\mathbf{C}^3$ is undoubtedly that of Halphen's equations (1.1). The classification of semicomplete vector fields within this family is due to Halphen [14]. We have explored the geometric and dynamical features of this family in [12] and we will refer to this article for some properties of these fields that will be used throughout this section. We will obtain other semicomplete quadratic homogeneous vector fields from the family (1.1) by considering quotients of the symmetric cases and rational changes of coordinates.

Definition 3.1. — *A quadratic homogeneous vector field X in $\mathbf{C}^3$ is of* Halphen type *if there exists a rational vector field Z, homogeneous of degree zero, such that $[Z, X] = 2E$.*

The homogeneity of the various vector fields involved in this definition guarantees the relations $[E, X] = X$ and $[E, Z] = -Z$. In this way, the vector fields E, X and Z generate a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. Because the vector fields X and Z are homogeneous, they both induce foliations in $\mathbf{CP}^2$. In all the examples that follow, the solutions of Z will be rational, so the foliation induced by X will actually be a Riccati foliation. Besides from the three-parameter family of vector fields of Halphen type given by Halphen's vector fields (1.1), for which Z is the holomorphic vector field $\sum_i \partial/\partial z_i$, we will describe one two-parameter family and five one-parameter families of these vector fields.

If in the vector fields (1.1) we have $\alpha_1=\alpha$ and $\alpha_2=\alpha_3=\beta$ then the vector field is invariant under the linear involution σ that exchanges z_2 and z_3. The σ-invariant polynomial map

$$(z_1,z_2,z_3)\mapsto(z_1,z_2+z_3,z_2z_3)=(\eta_1,\eta_2,\eta_3)$$

maps the vector field $\mathcal{H}(\alpha,\beta,\beta)$ onto the vector field

$$\begin{aligned}H_2(\alpha,\beta) &= (\alpha\eta_1^2+[1-\alpha][\eta_1\eta_2-\eta_3])\frac{\partial}{\partial\eta_1}\\ &\quad +(\beta\eta_2^2+[2-4\beta]\eta_3)\frac{\partial}{\partial\eta_2}+(\eta_3\eta_2+[\beta-1][\eta_2^2-4\eta_3]\eta_1)\frac{\partial}{\partial\eta_3}.\end{aligned}$$

This vector field is semicomplete if and only if we are in one of the following cases [12]:

- $2(\alpha+2\beta-2)/\alpha\in\mathbf{Z}\backslash\{-1,0,1\}$ and $(\alpha+2\beta-2)/\beta\in\mathbf{Z}\backslash\{-1,0,1\}$.
- $2(\alpha+2\beta-2)/\alpha\in\{-2,2\}$ and $(\alpha+2\beta-2)/\beta\in\{-1,1\}$.
- $(\alpha+2\beta-2)/\beta\in\{-2,2\}$ and $2(\alpha+2\beta-2)\alpha\in\{-1,1\}$.

The rational change of coordinates $(\zeta_1,\zeta_2,\zeta_3)=(\eta_1,\eta_2,\eta_3/\eta_2)$ maps the above vector field to the restriction to the image of the vector field

(3.1)
$$\begin{aligned}\mathcal{H}_2(\alpha,\beta) &= (\alpha\zeta_1^2+[1-\alpha][\zeta_1-\zeta_3]\zeta_2)\frac{\partial}{\partial\zeta_1}+\zeta_2(\beta\zeta_2+[2-4\beta]\zeta_3)\frac{\partial}{\partial\zeta_2}\\ &\quad +([4\beta-2]\zeta_3^2+[\beta-1][\zeta_1\zeta_2-\zeta_2\zeta_3-4\zeta_1\zeta_3])\frac{\partial}{\partial\zeta_3}.\end{aligned}$$

Because the composition of these maps is a homogeneous and open one, the radial vector field is mapped unto itself. The image of the σ-invariant vector field $\sum_i \partial/\partial z_i$ is given by

$$\frac{\partial}{\partial\zeta_1}+2\frac{\partial}{\partial\zeta_2}+\frac{\zeta_2^2+\zeta_3^2}{(\zeta_2+\zeta_3)^2}\frac{\partial}{\partial\zeta_3},$$

and this vector field ensures the Halphen character of the family $\mathcal{H}_2$. Semi-completeness within this family is given by the same conditions as before, for these conditions guarantee the semicompleteness of $\mathcal{H}_2$ in restriction to the invariant and unattained plane $\{\zeta_2=0\}$.

The vector field $\mathcal{H}(\alpha,\alpha,\alpha)$ is invariant under the linear action of S_3 (the group of permutations on three symbols), acting upon $\mathbf{C}^3$ by permuting the variables z_1,z_2 and z_3. The S_3-invariant mapping

$$(z_1,z_2,z_3)\mapsto(z_1+z_2+z_3,z_1z_2+z_2z_3+z_3z_1,z_1z_2z_3)$$

maps $\mathcal{H}(\alpha,\alpha,\alpha)$ onto the holomorphic vector field

$$\begin{aligned}(3.2)\quad H_3(\alpha) = {}& [\alpha\eta_1^2 + (1-3\alpha)\eta_2]\frac{\partial}{\partial\eta_1}\\ &+ [\alpha\eta_1\eta_2 + (6-9\alpha)\eta_3]\frac{\partial}{\partial\eta_2} + [(4-3\alpha)\eta_1\eta_3 + (\alpha-1)\eta_2^2]\frac{\partial}{\partial\eta_3}.\end{aligned}$$

It maps the invariant vector field $\sum_i \partial/\partial z_i$ to the vector field

$$3\frac{\partial}{\partial\eta_1} + 2\eta_1\frac{\partial}{\partial\eta_2} + \eta_2\frac{\partial}{\partial\eta_3}, \tag{3.3}$$

while the vector field E is mapped to

$$\eta_1\frac{\partial}{\partial\eta_1} + 2\eta_2\frac{\partial}{\partial\eta_2} + 3\eta_3\frac{\partial}{\partial\eta_3}. \tag{3.4}$$

The vector field $H_3(\alpha)$ is semicomplete if and only if

$$2(3\alpha-2)/\alpha \in \mathbf{Z}\backslash\{-1,0,1\}$$

[12]. Consider now the following five families of vector fields:

$$\begin{aligned}\mathcal{H}_{3a}(\gamma) = {}& \zeta_1\left(2\gamma\zeta_1 - [6\gamma+1]\zeta_3\right)\frac{\partial}{\partial\zeta_1}\\ &+ \left(\zeta_2^2 + 2[2\gamma-1]\zeta_1\zeta_2 - 3[2\gamma-1]\zeta_1\zeta_3\right)\frac{\partial}{\partial\zeta_2}\\ &+ \zeta_3\left([6\gamma+1]\zeta_3 - \zeta_2\right)\frac{\partial}{\partial\zeta_3},\end{aligned} \tag{3.5a}$$

$$\begin{aligned}\mathcal{H}_{3b}(\gamma) = {}& \left(\gamma\zeta_1^2 - 2[6\gamma+1]\zeta_2\zeta_3\right)\frac{\partial}{\partial\zeta_1}\\ &+ \zeta_2\left(\zeta_2 + [2\gamma+1]\zeta_1 - 3[2\gamma-1]\zeta_3\right)\frac{\partial}{\partial\zeta_2}\\ &+ \zeta_3\left(3[2\gamma-1]\zeta_3 - [\gamma-1]\zeta_1 - 2\zeta_2\right)\frac{\partial}{\partial\zeta_3},\end{aligned} \tag{3.5b}$$

$$\begin{aligned}\mathcal{H}_{3c}(\gamma) = {}& \zeta_1\left(8\gamma\zeta_1 + 12\zeta_2 - [6\gamma-5]\zeta_3\right)\frac{\partial}{\partial\zeta_1}\\ &+ \left([6\gamma-1]\zeta_2^2 + [2\gamma-1][8\zeta_2+3\zeta_3]\zeta_1\right)\frac{\partial}{\partial\zeta_2}\\ &+ \left(2[6\gamma-5]\zeta_3^2 + 8[3\gamma-2][3\zeta_2-2\zeta_3]\zeta_2\right)\frac{\partial}{\partial\zeta_3},\end{aligned} \tag{3.5c}$$

$$\begin{aligned}\mathcal{H}_{3d}(\gamma) = \mathcal{H}_{3b}(\gamma) + (\gamma-1)\Big\{& -2\zeta_2(\zeta_1 + [6\gamma+1]\zeta_3)\frac{\partial}{\partial\zeta_2}\\ &+\zeta_3(3\zeta_1 + 2[6\gamma+1]\zeta_3)\frac{\partial}{\partial\zeta_3}\Big\},\end{aligned} \tag{3.5d}$$

$$\begin{aligned}\mathcal{H}_{3e}(\gamma) = \mathcal{H}_{3c}(\gamma) + \frac{1}{2}(6\gamma-1)\Big\{\zeta_1(12\zeta_2+[6\gamma+7]\zeta_3)\frac{\partial}{\partial\zeta_1}\\ +2([6\gamma-1]\zeta_2^2-\zeta_1\zeta_3)\frac{\partial}{\partial\zeta_2}+\zeta_3(4[6\gamma-7]\zeta_2 r\\ -[6\gamma+7]\gamma_2-[6\gamma+7]\zeta_3)\frac{\partial}{\partial\zeta_3}\Big\},\end{aligned} \tag{3.5e}$$

Under the polynomial maps

$$\Phi_a = \left(\frac{1}{2}\zeta_1, \frac{1}{2}\zeta_1\zeta_3, \frac{1}{12}\zeta_1\zeta_2\zeta_3\right), \tag{3.6a}$$

$$\Phi_b = \left(\frac{1}{4}\zeta_1, \frac{1}{2}\zeta_2\zeta_3, \frac{1}{12}\zeta_2^2\zeta_3\right), \tag{3.6b}$$

$$\Phi_c = \left(2\zeta_1+3\zeta_2, 3\zeta_2^2-\zeta_1\zeta_3, \frac{1}{3}\zeta_2[3\zeta_2^2-\zeta_1\zeta_3]\right), \tag{3.6c}$$

$$\Phi_d = \Phi_b + \left(0, 0, -\frac{\gamma-1}{12}\zeta_1\zeta_2\zeta_3\right), \tag{3.6d}$$

$$\begin{aligned}\Phi_e = \Phi_c + (6\gamma-1)\\ \left(\frac{3}{2}\zeta_2, 3\zeta_2^2-\zeta_1\zeta_3, -\frac{1}{3}[18\gamma^2-6\gamma-1][3\zeta_2^2-\zeta_1\zeta_3]\zeta_2\right),\end{aligned} \tag{3.6e}$$

the vector fields (3.5a)–(3.5e) get mapped to the vector field $(6\gamma-1)H_3(\alpha)$ of equation (3.2) for $\alpha = 4\gamma/(6\gamma-1)$. These maps are invertible (within the class of rational maps) and their Jacobian determinants are given, up to a constant factor, by

$$\Delta_a = \zeta_1^2\zeta_3, \tag{3.7a}$$

$$\Delta_b = \zeta_2^2\zeta_3, \tag{3.7b}$$

$$\Delta_c = \zeta_1(3\zeta_2^2-\zeta_1\zeta_3), \tag{3.7c}$$

$$\Delta_d = \zeta_2^2\zeta_3, \tag{3.7d}$$

$$\Delta_e = \gamma^3(2\gamma-1)\zeta_1(3\zeta_2^2-\zeta_1\zeta_3). \tag{3.7e}$$

Notice, furthermore, that the zero locus of these polynomials gives an invariant (homogeneous) surface for the corresponding vector field. Hence, in restriction to the complement of this zero locus, the vector fields (3.5a)–(3.5e) are semicomplete either if $\gamma = 0$ or γ is the inverse of an integer (but $\gamma^2 \neq 1$). It is not difficult to verify that these conditions imply the semicompleteness of the vector fields in restriction to $\{\Delta = 0\}$. The image of the vector field E under these maps is —by homogeneity considerations— the vector field (3.4). Finally, the pull-back of the vector field (3.3) gives a rational vector field that ensures the Halphen character of these examples.

3.2. The eigenvalues as local coordinates

Proposition 3.2. — *The set of eigenvalues characterizes locally the vector fields in V_3, this is, for a generic quadratic homogeneous vector field on $\mathbf{C}^3$, any deformation that preserves the eigenvalues comes from a linear change of coordinates.*

Proof. — If the radial orbits of a vector field are $\rho_1, \dots, \rho_7$ and if the eigenvalues of ρ_i are u_i and v_i then an eventual deformation of the vector field preserves, for each radial orbit, the function of the eigenvalues

$$\frac{(u_i + v_i)^2}{u_i v_i},$$

that is intrinsically attached to the singular points of the induced foliation for it does not depend on the vector field inducing it. These numbers are usually called the *Baum-Bott indexes* of the singular points of the foliation [1]. If the deformation of a vector field is trivial (if it is given by a linear change of coordinates) then the deformation at the level of the foliations is also trivial. Thus, in order to prove the proposition, it suffices to exhibit a foliation such that every deformation that preserves its Baum-Bott indexes is a trivial one. Start with the vector field

$$X_0 = z_1(2z_2 - z_3)\frac{\partial}{\partial z_1} + z_2(2z_3 - z_1)\frac{\partial}{\partial z_2} + z_3(2z_1 - z_2)\frac{\partial}{\partial z_3}.$$

The vector field is invariant under cyclic permutations of the coordinates. The induced foliation has singularities at the points $[0:1:0]$, $[0:0:1]$, $[0:-2:1]$, $[1:0:0]$, $[-2:1:0]$, $[1:0:-2]$ and $[1:1:1]$. These will be noted $\rho_1, \dots, \rho_7$ (respectively). The ratios of the eigenvalues of ρ_1 and ρ_2 are both $[-2:1]$ and the ratio of the eigenvalues of ρ_3 is $[2:7]$. Consider the vector fields $Y_1, \dots, Y_6$ given, respectively, by $z_2(z_1 - z_3)\partial/\partial z_1$, $z_3(z_2 - z_1)\partial/\partial z_2$, $z_1(z_3 - z_2)\partial/\partial z_3$, $z_1(z_2 - z_3)\partial/\partial z_1$, $z_2(z_3 - z_1)\partial/\partial z_2$, $z_3(z_1 - z_2)\partial/\partial z_3$. We will consider the foliation induced by the vector field $X_\epsilon = X_0 + \sum_i \epsilon_i Y_i$, for $\epsilon = (\epsilon_1, \dots, \epsilon_6)$ in some neighborhood U of the origin of $\mathbf{C}^6$. The corresponding foliations give a six dimensional deformation of $\mathcal{F}_X$ that is transverse to the orbits of the action of $\mathrm{PGL}(n, \mathbf{C})$ in the space of foliations, because the foliations under consideration have (fixed) singularities at the points ρ_1, ρ_2, ρ_4 and ρ_7. The divergence has been uniquely chosen so that the vector field vanishes at the radial orbits given by ρ_1, ρ_2 and ρ_4 and the vector field uniquely normalized in such a way that $\langle dz_1, X_0\rangle(1,1,1) = 1$. The singularities of the foliation ρ_1, ρ_2 and ρ_3 lie in the line $\{z_1 = 0\}$ and this line is invariant by the foliation $\mathcal{F}_{X_0}$. Consequently, the Camacho-Sad relation –relative to this invariant line–

holds [3]. Let $\rho_i(\epsilon)$ be the singularity of the foliation induced by the vector field X_ϵ that corresponds to a deformation of ρ_i. For $i=1,\dots,3$, let $r_i(\epsilon)$ be the ratio of the eigenvalues of $\rho_i(\epsilon)$ taken in such a way that $r_i(0)$ gives the contribution of ρ_i to the Camacho-Sad relation relative to the invariant line $\{z_1=0\}$ of X_0. Consider the function $\mathsf{CS}: U\to\mathbf{C}$ given by

$$\mathsf{CS}(\epsilon)=r_1(\epsilon)+r_2(\epsilon)+r_3(\epsilon).$$

We claim that this continuous function is holomorphic in a neighborhood of $\epsilon=0$. We need only verify that all its directional derivatives exist. This function takes the constant value 1 in the zero locus of ϵ_1, because in this case the line $\{z_1=0\}$ is invariant by the foliation and the Camacho-Sad relation holds: the directional derivatives along $Y_2,\dots,Y_6$ vanish and thus they certainly exist. For Y_1, we have that the vector field $X+\epsilon_1 z_2(z_1-z_3)\partial/\partial z_1$ has still the directions ρ_1 and ρ_2 as radial orbits. The ratios of the eigenvalues are $r_1=-2-\epsilon_1$ and $r_2=-1/2$. The deformed vector field has a radial orbit in the direction $[2\epsilon_1,-14,7+3\epsilon_1]$, which corresponds to a deformation of ρ_3. The eigenvalues of the latter are the roots of the polynomial

$$x^2+(25\epsilon_1+63)x+14(\epsilon_1+7)(3\epsilon_1+7).$$

Consequently, we have

$$\begin{aligned}\mathsf{CS}(\epsilon_1,0,\dots,0)&=-2-\epsilon_1-\frac{1}{2}+\frac{-(25\epsilon_1+63)-\sqrt{1225+1582\epsilon_1+457\epsilon_1^2}}{-(25\epsilon_1+63)+\sqrt{1225+1582\epsilon_1+457\epsilon_1^2}}\\&=-\epsilon_1-\frac{5}{2}+\frac{(25\epsilon_1+63)+(35+\frac{113}{5}\epsilon_1-\frac{96}{125}\epsilon_1^2+\cdots)}{(25\epsilon_1+63)-(35+\frac{113}{5}\epsilon_1-\frac{96}{125}\epsilon_1^2+\cdots)}\\&=1+\frac{2}{5}\epsilon_1-\frac{213}{875}\epsilon_1^2+\cdots\end{aligned}$$

Thus, CS is a regular function at the origin of U and the level surface $\{\mathsf{CS}=1\}$ coincides with the set of foliations where the line $\{z_1=0\}$ is invariant. Thus, *every deformation of $\mathcal{F}_{X_0}$ preserving the Camacho-Sad relation among the deformations of the singular points lying at the line $\{z_1=0\}$ must preserve the invariance of the latter*. By the symmetry of the equation, any deformation of $\mathcal{F}_{X_0}$ that preserves the Baum-Bott indexes must preserve the invariance of the three lines $\{z_i=0\}$ and, in consequence, is induced by a vector field of the form

$$X_0+\delta_1 z_1(z_2-z_3)\frac{\partial}{\partial z_1}+\delta_2 z_2(z_3-z_1)\frac{\partial}{\partial z_2}+\delta_3 z_3(z_1-z_2)\frac{\partial}{\partial z_3},$$

with δ_i close to 0. The Baum-Bott index of the induced foliation at the singular point $[1:0:0]$ is

$$-\frac{(1-\delta_2+\delta_3)^2}{(2+\delta_3)(1+\delta_2)} = -\frac{(2\delta_3-\delta_2+3)(\delta_3-2\delta_2)}{2(2+\delta_3)(1+\delta_2)} - \frac{1}{2}.$$

If the Baum-Bott index is preserved then this expression equals $-1/2$ for small values of the δ_i and we thus have $\delta_3 = 2\delta_2$. By the symmetry of the foliation, $\delta_3 = 2^3\delta_3$ and all the δ_i must vanish. This proves the proposition and completes the proof of Theorem A. □

Corollary 3.3. — *The image of the Baum-Bott map, the map*

$$BB : \{\textit{Degree two foliations of } \mathbf{CP}^2\} \dashrightarrow \mathrm{Sym}^7 \mathbf{C},$$

that associates to a generic foliation the Baum-Bott indexes of its seven singularities has a dominant image in the hyperplane $\{\sum x_i = (2+2)^2\}$ *defined by Baum-Bott's theorem.*

In other words, the algebraic relation given by Baum-Bott's theorem is the only one relating the Baum-Bott indexes of a degree two foliation of the plane. A. Lins Neto and J. V. Pereira have recently shown that this is also true for foliations of the plane of arbitrary degree.

Definition 3.4. — *The vector field $X \in V_n$ is said to* admit an isospectral deformation *if there exists a nontrivial deformation within V_n that preserves the eigenvalues of X.*

Proposition 3.2 can be rephrased in the following way: *a generic quadratic homogenous vector field in $\mathbf{C}^3$ does not admit an isospectral deformation.* The family of Lins Neto's vector fields is the only example we know of an isospectral deformation of semicomplete vector fields (with an isolated singularity at the origin).

3.3. Towards a classification

>From the results in the last part, we know that given a set of seven couples of complex numbers (u_i, v_i) there exists, generically, a finite number of quadratic homogeneous vector fields in $\mathbf{C}^3$ having them as eigenvalues. Furthermore, these eigenvalues are tied by some algebraic relations. An elementary calculation shows that there must be at least five (algebraically

independent) relations in the present case. From the Main Theorem in [12] we find the three relations

$$\sum_{i=1}^{7} \frac{(u_i+v_i)^j}{u_i v_i} = (-4)^j, \tag{3.8}$$

for $j \in \{0,1,2\}$. A semicomplete vector field with an isolated singularity gives seven couples of integers satisfying these equations. Trying to classify semicomplete vector fields with an isolated singularity by solving this system of Diophantine equations is of course very tempting. It seems, however, that a reasonable classification following these lines will have to wait until the discovery of more relations binding the eigenvalues of a quadratic vector field. In what follows, we will indicate how one may approach the study of the solutions of the system (3.8).

For each i, let $\xi_i = u_i v_i$. The first one of the equations (3.8) reads

$$\sum_{i=1}^{7} \frac{1}{\xi_i} = 1. \tag{3.9}$$

Integer solutions to the system (3.8) are subordinate to the solutions of this equation in the sense that every solution to (3.9) belongs —at most— to a *finite* number of solutions of the system (3.8). This equation belongs to the theory of "Egyptian fractions" [5], that considers the problem of expressing a rational number as a sum of aliquot parts of unity (though in general only positive ones are considered).

A solution $(\xi_1,\ldots,\xi_7) \in \mathbf{Z}^7$ to equation (3.9) is said to be *ordered* if it satisfies the following three conditions:

(1) ξ_1 is positive.
(2) If ξ_i and ξ_j are positive and $i<j$ then $\xi_i \leqslant \xi_j$; if ξ_i and ξ_j are negative and $i<j$ then $\xi_i \geqslant \xi_j$.
(3) If $\sum_{i=1}^{j} 1/\xi_i > 1$ then ξ_{j+1} is negative ; if $\sum_{i=1}^{j} 1/\xi_i \leqslant 1$ then ξ_{j+1} is positive.

It is not difficult to see that every solution admits such an order and that it is *unique* up to symmetries of the solution. This order allows us to partition the set of solutions of (3.9) in six families. The ordered solution $(\xi_1,\ldots,\xi_7)$ of equation (3.9) is said to *belong to the n^{th} family* if n is the smallest natural number such that $\sum_{i=1}^{n} 1/\xi_i = 1$ (the sixth family is empty). We say that $(\xi_1,\ldots,\xi_n)$ is the *principal part* of the ordered solution $(\xi_1,\ldots,\xi_7)$.

PROPOSITION 3.5. — *The seventh family is finite. There exists an algorithm to find the solutions belonging to this family.*

Proof. — Remark that if $(\xi_1, \xi_2, \dots, \xi_7)$ is an ordered solution of (3.9), the integer ξ_1 belongs to the interval $[1, 7] \subset \mathbf{Q}$. Let $j \in \{1, 2, 3, \dots, 6\}$ and denote by s_j the partial sum $\sum_{i=1}^{j} 1/\xi_i$. Let ν_+ be the greatest positive number in $\{\xi_1, \dots, \xi_j\}$ and ν_- the smallest negative one in $\{-1, \xi_1, \xi_2, \dots, \xi_j\}$. Then,

- If $s_j > 1$, the integer ξ_{j+1} belongs to the interval $\left[\dfrac{1 - s_j}{7 - j}, \nu_-\right]$.
- If $s_j < 1$, the integer ξ_{j+1} belongs to the interval $\left[\nu_+, \dfrac{1 - s_j}{7 - j}\right]$.
- The case $s_j = 1$ cannot arrive in the seventh family.

In this way, $(\xi_1, \dots, \xi_j)$ determines ξ_{j+1} up to a finite choice. □

The same argument shows that the set of principal parts belonging to each one of the first five families is finite. At this point, the reader should have the following picture in mind:

- There exist an uncountable number of inequivalent semicomplete quadratic homogeneous vector fields having an isolated singularity. Vector fields belonging to an isospectral deformation are the sole reason for this uncountability: vector fields that do not belong to an isospectral deformation are, up to linear equivalence, countable in number.
- Each principal part defines a $\mathrm{GL}_3(\mathbf{C})$-invariant variety in V_3. Semicomplete vector fields that are neither part of an isospectral deformation nor belong to one of the five families are finite in number.

The combinatorial approach here explained should be carried on considering also the very strong conditions imposed by the fact that the singular points of the foliation induced by a semicomplete vector field are all linearizable. For example, when the two eigenvalues of a radial orbit of a semicomplete vector field coincide, we have the following:

Lemma 3.6. — *A degree two foliation of the plane having a singularity that is locally given by the kernel of the form $x\,dx - y\,dy$ is a Riccati foliation with respect to the pencil of lines passing through the singular point.*

Proof. — In an affine chart $[x : y : 1]$, the most general degree two foliation of $\mathbf{CP}^2$ having such a singularity and at the origin is given —up to a linear transformation fixing the origin— by the kernel of the form

$$[y + a_1x^2 + a_2xy + a_3y^2 + y(b_1x^2 + b_2xy + b_3y^2)]dx \\ - [x + a_4x^2 + a_5xy + a_6y^2 + x(b_1x^2 + b_2xy + b_3y^2)]dy.$$

Blowing up the origin by $y = sx$ and then dividing by x^2, the above one-form becomes

$$[(a_1 + (a_2 - a_4)s + (a_3 - a_5)s^2 - a_6s^3]dx \\ - [1 + x(a_4 + a_5s + a_6s^2) + x^2(b_1 + b_2s + b_3s^2)]ds.$$

This form can be rewritten as

$$\frac{dx}{ds} = \frac{1 + x(a_4 + a_5s + a_6s^2) + x^2(b_1 + b_2s + b_3s^2)}{a_1 + (a_2 - a_4)s + (a_3 - a_5)s^2 - a_6s^3},$$

giving thus a Riccati differential equation. The function y/x becomes the function s. The three roots of the denominator of the right-hand side of the previous equation give invariant lines for the original one. □

For singularities with resonant linear parts in the Poincaré domain, linearization is given by a single condition. The local theory of linearization of singularities in Siegel's domain is an achieved one, but the global obstructions imposed by the linearization of such a singularity in a foliation of the plane seem far from begin understood (see, for example, [4], where the authors show that in a degree two foliation the existence of a linearizable center implies the existence of an invariant line).

These things said, we pose the following

Conjecture 3.7. — *Let X be a semicomplete quadratic homogeneous vector field having the origin as its sole singularity. Then, with the exception of a finite number of linear equivalence classes, at least one of the following conditions is satisfied:*

(1) *X admits an isospectral deformation.*
(2) *There exists a polynomial dominant map $\pi\colon \mathbf{C}^3 \to \mathbf{C}^{3-j}$, $0<j<3$, and a vector field Y on $\mathbf{C}^{3-j}$ such that $\pi_*(X) = Y$ (X is an imprimitive vector field).*
(3) *X is a vector field of Halphen type and, except for a finite number of linear equivalence classes, X belongs to one of the families $\mathcal{H}$, $\mathcal{H}_2$, $\mathcal{H}_{3a}$, $\mathcal{H}_{3b}$, $\mathcal{H}_{3c}$, $\mathcal{H}_{3d}$, $\mathcal{H}_{3e}$.*

Because we are willing to exclude a finite number of conjugacy classes from our understanding, we need not worry about vector fields giving solutions to the system (3.8) that belong to the seventh family, and we need only prove that this conjecture holds for vector fields whose principal part belongs to one of the five others. The conjecture is indeed satisfied for the first one:

Proposition 3.8. — *A semicomplete quadratic vector field in* $\mathbf{C}^3$ *having an isolated singularity at the origin and belonging to the first family is either imprimitive or is linearly equivalent to one of Halphen's vector fields* $\mathcal{H}(\alpha_1, \alpha_2, \alpha_3)$.

Proof. — The first family is characterized by the existence of a radial orbit such that the product of its eigenvalues is 1. There are thus two cases, for the eigenvalues can be either $(-1,-1)$ or $(1,1)$. In the first case, suppose that the vector field is of the form $\sum a_{ij}^k z_i z_k \partial/\partial z_k$ and suppose that the radial orbit under consideration is in the direction $[1:0:0]$, so a_{11}^2 and a_{11}^3 are both zero. We will also assume that $a_{11}^1 = 1$. Calculating the eigenvalues through the Lie bracket with the vector field $\partial/\partial z_1$, we have that the linear vector field $[\partial/\partial z_1, X]$ is diagonalizable and that it has two vanishing eigenvalues. If we assume this vector field to be already diagonalized, we have that $[\partial/\partial z_1, X] = 2z_1\partial/\partial z_1$ and thus $X = z_1^2 + \sum_i P_i(z_2,z_3)\partial/\partial z_i$. The linear projection $\pi(z_1,z_2,z_3) = (z_2,z_3)$ maps this vector field onto a quadratic two-dimensional one. In the second case, fix the radial orbit under consideration at the direction $[1:1:1]$. We have that $[\sum \partial/\partial z_i, X] = 2E$. Transforming this vector field by linear change of coordinates preserving the vector field $\sum_i \partial/\partial z_i$, we can suppose that it has radial orbits at the directions $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. In this way we find one of Halphen's fields (1.1). □

A proof of the Conjecture for the second family can be found in [8].

3.4. Classification of semicomplete isochoric vector fields

As we previously singled out in Remark 2.9, isochoric vector fields in V_3 are characterized by the fact that the sum of the eigenvalues of every non-degenerate radial orbit is -4. Thus, the eigenvalues of a non-degenerate isochoric vector field are of the form $\lambda_i - 2$ and $-\lambda_i - 2$ for a unique $\lambda \in \mathbf{Z}$, $\lambda \geqslant 0$. These numbers are bound by the relation

$$\sum_{i=1}^{7} \frac{1}{4-\lambda_i^2} = 1.$$

The values taken by $4-\lambda_i^2$ are those in $\{4, 3, -5, -12, -21, -32, -45, \ldots\}$. If, in an isochoric semicomplete vector field we have two radial orbits with $\lambda_i = 0$ then, according to Lemma 3.6, the plane containing them is an invariant one. In restriction to this plane the eigenvalues of these radial orbits are both -2 and thus the third one cannot be but degenerate.

The only solution to the above equation that does not display twice the summand 1/4 is

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}-\frac{1}{12}-\frac{1}{12}-\frac{1}{12}=1.$$

Because the singularity with $\lambda_i = 0$ is a dicritical one, we have a Riccati equation (Lemma 3.6) and three invariant planes. The only two-dimensional semicomplete vector fields with an isolated singularity that have -2 as an eigenvalue are those whose eigenvalues are in

$$\{(-2,2,-1),(-2,-3,-6),(-2,-4,-4)\}.$$

Hence, the invariant planes of an isochoric vector field with an isolated singularity are of one of the following two forms:

(1) In restriction to the invariant plane the eigenvalues are $(-2,-3,-6)$; the other eigenvalues are, respectively, $(-2,-1,2)$. In the semicomplete case the holonomy of the Riccati equation around this invariant line is trivial

(2) In restriction to the invariant plane the eigenvalues are $(-2,2,-1)$. The order of the corresponding element in the group of the Riccati equation –in the semicomplete case– is three.

We will proceed to construct every possible vector field having this data. Assume that the singularity having eigenvalues $(-2,-2)$ is placed at the direction $[1:1:1]$. The induced foliations will be Riccati ones with respect to the pencil of lines that pass through this point. We will suppose that the three invariant lines are of the form $\{z_i = z_j\}$ and that the vector fields have singularities at the points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. If two of the three invariant planes are of the form (1) then the group of the Riccati equation is trivial, for it is generated by two trivial elements. The third invariant plane is necessarily of the form (1) too. If we impose in such a field that the eigenvalues at the three fixed radial orbits are $(-1,-3)$ then the vector field is necessarily linearly equivalent to the vector field (1.3a). If the eigenvalues of the radial orbits at the points $[1:0:0]$ and $[0:1:0]$ have the eigenvalues $(-1,-3)$ and the invariant lines $\{z_i = z_3\}$ are of type (2) then the vector field is a multiple of the field

$$\begin{aligned}
&\left([2-\lambda]z_1^2-3z_1z_3+3\lambda[z_1-z_3]z_2\right)\frac{\partial}{\partial z_1}\\
&+\left([2-\lambda]z_2^2-3z_2z_3+3\lambda[z_2-z_3]z_1\right)\frac{\partial}{\partial z_2}\qquad (3.10)\\
&+\left(3z_3^2+[\lambda+4][z_1z_2-z_2z_3-z_1z_3]\right),\frac{\partial}{\partial z_3}
\end{aligned}$$

with $\lambda \neq -1$. The eigenvalues of the point $[0:0:1]$ are $-2+\lambda$ and $-2-\lambda$. The latter is the eigenvalue tangent to the invariant plane $\{z_1 = z_2\}$. Thus, if the third invariant plane is of type (2), the only possibility is that $\lambda = -4$. This gives a multiple of the vector field (1.3c). If the third plane is of type (1), we can choose either $\lambda = 1$ or $\lambda = 4$, but the resulting vector fields are easily seen to be linearly equivalent. Choosing $\lambda = 1$ yields the vector field (1.3b). It remains to show that these three vector fields are actually semicomplete. The group of the corresponding equation is, assuming semicompleteness,

- the trivial group,
- a cyclic group of order three or
- the Euclidean triangular group $T(3,3,3)$.

We will now show that these vector fields are indeed semicomplete.

The vector field (1.3a) has the polynomial first integrals

$$R = \frac{(z_2 - z_3)^3(3z_1^2 - z_1z_2 - z_2z_3 - z_1z_3)}{(z_1 - z_3)^3(3z_2^2 - z_1z_2 - z_2z_3 - z_1z_3)},$$

$$Q = (3z_1^2 - z_1z_2 - z_2z_3 - z_3z_1)(3z_2^2 - z_1z_2 - z_2z_3 - z_3z_1)(3z_3^2 - z_1z_2 - z_2z_3 - z_3z_1).$$

The associated foliation has thus three invariant conics. All of them pass through the point $[1:1:1]$. This foliation is, in the affine chart $[u+1 : v+1 : 1] = [z_1 : z_2 : z_3]$, given by the kernel of the form

$$(4v + 3v^2 + 2uv + uv^2)du - (4u + 3u^2 + 2uv + u^2v)dv.$$

If we blow the origin by setting $u = sv$ and then divide by v^2, the above form becomes

$$(sv^2 + [2s+3]v + 4)\,ds - (s^2 - s)\,dv,$$

and this explicits its form of a Riccati equation. Because any four solutions of a Riccati equation have a constant cross-ratio [15] and we have three explicit ones given by the invariant conics, a straightforward calculation shows that the integral curve $\{R = \lambda\}$ is parametrized by

$$v_\lambda(s) = -2\frac{\lambda s^4 - 2\lambda s^3 + 2s - 1}{s(\lambda s^3 - 3\lambda s^2 + 3s - 1)}.$$

Coming back to the vector field, if we set $s(t) = [z_1(t) - z_3(t)]/[z_2(t) - z_3(t)]$ then, taking derivatives with respect to t, then substituting the values $z_2 = (v_\lambda(s)+1)z_3$, $z_1 = (sv_\lambda(s)+1)z_3$, we find that

$$-\frac{s''}{(s')^2}d\tau = -\frac{2\lambda s^3 - 3\lambda s^2 + 1}{\lambda s^4 - 2\lambda s^3 + 2s - 1}ds = d\log P^{-1/2},$$

for $P(\tau) = \lambda s^4 - 2\lambda s^3 + 2s - 1$. By the same arguments of Example 2.13, this vector field is semicomplete. Its general solution is thus an elliptic curve that projects unto a rational quintic.

The vector field (1.3b), the one whose foliation has cyclic holonomy, has the polynomial first integral given by

$$\begin{aligned}&(z_1 - z_3)(z_2 - z_3)\\ &\quad(9[z_1 + z_2]^2 z_3^2 - [z_1 + z_2][z_1^2 + 26 z_1 z_2 + z_2^2] z_3 + z_1 z_2 [z_1^2 + 18 z_1 z_2 + z_2^2]).\end{aligned}$$

Furthermore, it has a homogeneous one given by

$$\sigma = -\frac{(z_2 - z_3)(z_1^2 + 5 z_1 z_2 - 3 z_1 z_3 - 3 z_2 z_3)^3}{(z_1 - z_3)(z_2^2 + 5 z_1 z_2 - 3 z_1 z_3 - 3 z_2 z_3)^3}.$$

In this way, the vector field is completely integrable. In the affine chart $[u+1 : v+1 : 1] = [z_1 : z_2 : z_3]$ the induced foliation is given by the kernel of the form

$$(4v + v^2 - 2uv - 5uv^2)\, du - (4u + u^2 - 2uv - 5u^2 v)\, dv.$$

Blowing up by $u = sv$ we obtain the Riccati foliation given by

$$3(s^2 - s)\, dv - (4 + v - 2sv - 5sv^2)\, ds.$$

>From the first integrals we get three invariant curves that can be parametrized by

$$\begin{aligned} v_0(\tau) &= -2\frac{2\tau^3 + 1}{\tau^3(5 + \tau^3)},\\ v_\infty(\tau) &= -2\frac{\tau^3 + 2}{5\tau^3 + 1},\\ v_1(\tau) &= 2\frac{\tau + 1}{\tau(\tau^2 + 3\tau + 1)}, \end{aligned}$$

for $\tau^3 = s$. Once again, because the cross-ratio of any four solutions to a Riccati equation is constant, the remaining integral curves are parametrized by

$$v_\lambda(\tau) = 2\frac{\tau^4 - 2\lambda\tau^3 + 2\tau - \lambda}{\tau(\lambda\tau^5 - 5\tau^3 + 5\lambda\tau^2 - 1)}.$$

In order to calculate the affine structure induced in these curves, set $\tau^3(t) = [z_1(t) - z_3(t)]/[z_2(t) - z_3(t)]$. Developing the expression then substituting $\lambda^3 = \sigma$, $z_1 = (\tau^3 v_\lambda + 1) z_3$ and $z_2 = (v_\lambda + 1) z_3$ we obtain

$$-\frac{\tau''}{(\tau')^2} = -\frac{2\tau^3 - 3\lambda\tau^2 + 1}{\tau^4 - 2\lambda\tau^3 + 2\tau - \lambda} = d \log P^{-1/2}$$

for $P(\tau) = \tau^4 - 2\lambda\tau^3 + 2\tau - \lambda$, and the vector field is thus semicomplete.

The vector field (1.3c) has the polynomial first integral

$$Q = 2z_3^3(z_1 - z_3)(z_2 - z_1)(z_3 - z_2).$$

It is in the linear span of Euler's vector field and of the field $\mathcal{H}(-1,-1,1)$, given by

(3.11)

$$(-z_1^2+2[z_1z_2-z_2z_3+z_1z_3])\frac{\partial}{\partial z_1}+(-z_2^2+2[z_1z_2+z_2z_3-z_1z_3])\frac{\partial}{\partial z_1}+z_3^2\frac{\partial}{\partial z_3},$$

belonging to Halphen's family (1.1). The induced foliations in $\mathbf{CP}^2$ are thus the same. Let f be the elliptic function satisfying the differential equation $(f')^2 = f^4 + 2f$ and set $g = \frac{1}{2}(f'/f + f)$. Notice that g satisfies the differential equation $(g')^2 = g^4 - 2g$. From the analysis of Halphen's vector fields found in [12], we learn that for every $c \in \mathbf{C}$, the curve

$$\phi(t) =$$

$$\left[\frac{1}{(ct+1)^2} f\left(\frac{t}{ct+1}\right) - \frac{c}{ct+1}, \frac{1}{(ct+1)^2} g\left(\frac{t}{ct+1}\right) - \frac{c}{ct+1}, -\frac{c}{ct+1}\right]$$

gives a solution to (3.11) and that, in this way, we parametrize *every* curve in the foliation $\mathcal{F}_{\mathcal{H}}$ with the exception of the three invariant lines $\{z_i - z_j = 0\}$. If we set

$$t(s) = \frac{1}{c^2s^2} - \frac{1}{c}$$

and multiply this solution by a suitable factor in order for it to lie on the surface $\{Q = 1\}$, we get the curve

$$\phi(s) = \left[sf\left(\frac{1}{c} - s^2\right) - \frac{1}{s}, sg\left(\frac{1}{c} - s^2\right) - \frac{1}{s}, -\frac{1}{s}\right],$$

which is, for every $c \in \mathbf{C}^*$, a solution of (1.3c). As c varies in $\mathbf{C}^*$ (actually, in an arbitrary neighborhood of the origin), we get every solution lying on the surface $\{Q = 1\}$. The homogeneity of the first integral and the semicompleteness of the field in restriction to the four invariant planes contained in $\{Q = 0\}$ guarantees the semicompleteness of this vector field. This finishes the proof of Theorem C.

The situation is very different if we do not impose the condition that the origin is an isolated singularity of the vector field:

Example 3.9. — Consider the family of isochoric vector fields given by:

$$X = (z_2^2 - 2z_2z_3 + z_1z_2 + b_3z_1z_3)\frac{\partial}{\partial z_2} + (z_3^2 - 2z_2z_3 - b_2z_1z_2 - z_2z_3)\frac{\partial}{\partial z_3}$$

These vector fields give a two-dimensional family of vector fields that are not linearly equivalent. The above vector field has the homogeneous polynomial first integrals given by z_1 and by

$$\Lambda = 2z_2z_3(z_2 - z_3) + z_1(b_2z_2^2 + 2z_2z_3 + b_3z_3^2).$$

Thus, the induced foliation is the pencil of cubics generated by z_1^3 and Λ. These fields have three non-degenerate radial orbits in the hyperplane $\{z_1 = 0\}$ having eigenvalues $(-1, -3)$. The other four radial orbits are degenerate and, because the vector field is isochoric, they give linearizable centers in the induced foliation. From the equations we obtain that

$$\begin{aligned}\left(\frac{dz_2}{dt}\right)^2 &= z_2^4 + 2z_1(1+b_2)z_2^3 + z_1^2(1-b_2b_3)z_2^2 - 2\Lambda z_2 + b_3\Lambda z_1,\\ \left(\frac{dz_3}{dt}\right)^2 &= z_3^4 - 2z_1(1+b_3)z_3^3 + z_1^2(1-b_2b_3)z_3^2 + 2\Lambda z_3 + b_2\Lambda z_1.\end{aligned}$$

And thus, because z_1 is constant, the solutions of the vector field are given by elliptic functions. These fields are semicomplete (we thank J. V. Pereira for this example).

BIBLIOGRAPHY

[1] P. Baum & R. Bott, "Singularities of holomorphic foliations", *J. Differential Geometry* **7** (1972), p. 279-342.

[2] G. Cairns & É. Ghys, "The local linearization problem for smooth SL(n)-actions", *Enseign. Math. (2)* **43** (1997), no. 1-2, p. 133-171.

[3] C. Camacho & P. Sad, "Invariant varieties through singularities of holomorphic vector fields", *Ann. of Math. (2)* **115** (1982), no. 3, p. 579-595.

[4] D. Cerveau & A. Lins Neto, "Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{CP}(n)$, $n \geqslant 3$", *Ann. of Math. (2)* **143** (1996), no. 3, p. 577-612.

[5] P. Erdös & R. L. Graham, "Old and new problems and results in combinatorial number theory", in *L'Enseignement Mathématique*, Université de Genève, Geneva, 1980, p. 128.

[6] W. Fulton & J. Harris, *Representation theory*, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics, xvi+551 pages.

[7] É. Ghys & J.-C. Rebelo, "Singularités des flots holomorphes II", *Ann. Inst. Fourier (Grenoble)* **47** (1997), no. 4, p. 1117-1174.

[8] A. Guillot, "Champs quadratiques uniformisables", PhD Thesis, ÉNS-Lyon, 2001.

[9] ———, "Sur les exemples de Lins Neto de feuilletages algébriques", *C. R. Math. Acad. Sci. Paris* **334** (2002), no. 9, p. 747-750.

[10] ———, "Un théorème de point fixe pour les endomorphismes de l'espace projectif avec des applications aux feuilletages algébriques", *Bull. Braz. Math. Soc. (N.S.)* **35** (2004), no. 3, p. 345-362.

[11] ———, "The Painlevé property for quasihomogenous systems and a many-body problem in the plane", *Comm. Math. Phys.* **256** (2005), no. 1, p. 181-194.

[12] ———, "Sur les équations d'Halphen et les actions de $SL_2(\mathbf{C})$", in preparation, 2006.

[13] R. C. Gunning, *Lectures on Riemann surfaces*, Princeton University Press, Princeton, N.J., 1966, Princeton Mathematical Notes, iv+254 pages.

[14] G.-H. Halphen, "Sur certains systèmes d'équations différentielles", *Comptes Rendus Hebdomadaires de l'Académie des Sciences* **XCII** (1881), no. 24, p. 1404-1406.

[15] E. Hille, *Ordinary differential equations in the complex domain*, Dover Publications Inc., Mineola, NY, 1997, Reprint of the 1976 original, xii+484 pages.

[16] L. Landau & E. Lifchitz, *Physique théorique. Tome I. Mécanique*, Éditions Mir, Moscou, 1982, Quatrième édition revue et complétée, 227 pages.

[17] A. Lins Neto, "Some examples for the Poincaré and Painlevé problems", *Ann. Sci. École Norm. Sup. (4)* **35** (2002), no. 2, p. 231-266.

[18] R. S. Palais, "A global formulation of the Lie theory of transformation groups", *Mem. Amer. Math. Soc. No.* **22** (1957), p. iii+123.

[19] J. C. Rebelo, "Singularités des flots holomorphes", *Ann. Inst. Fourier (Grenoble)* **46** (1996), no. 2, p. 411-428.

[20] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy, x+311 pages.

Manuscrit reçu le 6 décembre 2004,
accepté le 21 mars 2005.

Adolfo GUILLOT
Unidad Cuernavaca
Instituto de Matemáticas UNAM
Av. Universidad s/n, col. Lomas de Chamilpa
C.P. 62210, Cuernavaca, Morelos (Mexico)
adolfo@matcuer.unam.mx