ANNALES

DE

L'INSTITUT FOURIER

Lucy MOSER-JAUSLIN & Pierre-Marie POLONI

Embeddings of a family of Danielewski hypersurfaces and certain $\mathbf{C}^+$-actions on $\mathbf{C}^3$

Tome 56, n° 5 (2006), p. 1567-1581.

<http://aif.cedram.org/item?id=AIF_2006__56_5_1567_0>

Ann. Inst. Fourier, Grenoble
56, 5 (2006) 1567-1581

EMBEDDINGS OF A FAMILY OF DANIELEWSKI HYPERSURFACES AND CERTAIN $\mathbf{C}^+$-ACTIONS ON $\mathbf{C}^3$

by Lucy MOSER-JAUSLIN & Pierre-Marie POLONI

Abstract. — We consider the family of polynomials in $\mathbf{C}[x,y,z]$ of the form $x^2y - z^2 - xq(x,z)$. Two such polynomials P_1 and P_2 are equivalent if there is an automorphism φ^* of $\mathbf{C}[x,y,z]$ such that $\varphi^*(P_1) = P_2$. We give a complete classification of the equivalence classes of these polynomials in the algebraic and analytic category. As a consequence, we find the following results. There are explicit examples of inequivalent polynomials P_1 and P_2 such that the zero set of $P_1 + c$ is isomorphic to the zero set of $P_2 + c$ for all $c \in \mathbf{C}$. There exist polynomials which are algebraically inequivalent but analytically equivalent. There exist polynomials which are algebraically inequivalent but when considered as polynomials in $\mathbf{C}[x,y,z,w]$ become equivalent. This last result answers a problem posed in [7]. Finally, we get a complete classification of $\mathbf{C}^+$-actions on $\mathbf{C}^3$ which are defined by a triangular locally nilpotent derivation of the form $x^2\partial/\partial z + (2z + xq(x,z))\partial/\partial y$.

Résumé. — Nous considérons la famille de polynômes de $\mathbf{C}[x,y,z]$ de la forme $x^2y - z^2 - xq(x,z)$. Deux polynômes P_1 et P_2 sont dits équivalents s'il existe un automorphisme $\varphi*$ de $\mathbf{C}[x,y,z]$ tel que $\varphi*(P_1) = P_2$. Nous donnons une classification complète des classes d'équivalence de ces polynômes dans les catégories algébrique et analytique. Nous en déduisons les résultats suivants. Il existe des exemples explicites de polynômes non équivalents P_1 et P_2 tels que l'ensemble des zéros de $P_1 + c$ est isomorphe à l'ensemble des zéros de $P_2 + c$ pour tout $c \in \mathbf{C}$. Il existe des polynômes analytiquement équivalents qui ne le sont pas algébriquement. Il existe des polynômes algébriquement non équivalents mais qui, vus comme des polynômes de $\mathbf{C}[x,y,z,w]$, le deviennent. Ce dernier résultat répond à un problème posé dans [7]. Finalement, nous obtenons une classification complète des actions de $\mathbf{C}^+$ sur $\mathbf{C}^3$ définies par une dérivation triangulaire de la forme $x^2\partial/\partial z + (2z + xq(x,z))\partial/\partial y$.

1. Introduction

In this article we will consider polynomials of the form $P = x^2y - z^2 - xq(x,z) + c$. If $c \neq 0$, then the hypersurface $V(P)$ in $\mathbf{C}^3$ defined

Keywords: equivalence of polynomials, stable equivalence, algebraic embeddings, Danielewski surfaces.
Math. classification: 14R10, 14R05, 14L30.

to be the zero set of P is, in the terminology of [2] and [3], an example of a *Danielewski surface*. It has an action of $(\mathbf{C},+)$ defined by the locally nilpotent (triangular) derivation on its coordinate ring given by $\partial = x^2\partial/\partial z + (2z + x\partial q/\partial z)\partial/\partial y$. The quotient map for this action is simply the projection to $\mathbf{C}$ given by x. The fibers over non-zero points are all isomorphic to $\mathbf{C}$, corresponding to an orbit of the action, and the fiber over zero is isomorphic to 2 copies of $\mathbf{C}$, corresponding to two orbits.

The first examples of Danielewski surfaces were studied in [1] where it was shown that if $Y_k = V(x^k y - z^2 - 1)$, then $Y_1 \not\cong Y_2$, but $Y_1 \times \mathbf{C} \cong Y_2 \times \mathbf{C}$. This result was then generalized in [4], where Fieseler showed that $Y_k \cong Y_{k'}$ if and only if $k = k'$, but $Y_k \times \mathbf{C} \cong Y_1 \times \mathbf{C}$ for all k. Since then, these surfaces have been studied by many others. In [4] and in [2] , the abstract notion of Danielewski surfaces was developed, and these surfaces were classified.

Here, we are interested in the embeddings of certain Danielewski hypersurfaces. In [8] and [5], inequivalent embeddings are given. In [8], Shpilrain and Yu use the gradients to distinguish the embeddings. More precisely, they find examples of polynomials p and q such that the zero sets are isomorphic, but the number of zeros of $\mathrm{grad}(p)$ is not equal to the number of zeros of $\mathrm{grad}(q)$. Thus, this gives inequivalent polynomials in the algebraic and analytic category.

The examples of [5] are different, because the authors give analytically equivalent polynomials. But the method used was also to find polynomials p and q such that the zero fibers $\{p = 0\}$ and $\{q = 0\}$ are isomorphic but the fiber $\{p = 1\}$ is not isomorphic to the fiber $\{q = c\}$ for any $c \in \mathbf{C}$. Thus the embeddings of the zero fibers are inequivalent.

Here, we use a different method to show that two hypersufaces have inequivalent embeddings. In fact, as we shall see, there are examples of polynomials Q_k such that $V(Q_k + c) \cong V(Q_{k'} + c)$ and is smooth for all $c \in \mathbf{C}$, but the embedding of $V(Q_k + 1)$ is different for each k. In particular the gradients are never zero for any of the polynomials Q_k.

This article is organized as follows. In section 2, the main results and the consequences are described. In section 3 we give some preliminary lemmas. The difficulty in general to study equivalence of polynomials of three variables is that the group of automorphisms of $\mathbf{C}^3$ is not well understood. We will show that we are actually only concerned with a subgroup of the automorphism group. More precisely, the aim of the preliminary lemmas is two-fold. First of all, we will show that the automorphisms that we must consider all lie in the subgroup of automorphisms of $\mathbf{C}[x, y, z]$ which fix the ideal (x). This is done by using a result of Makar-Limanov (see lemma 3.1).

This lemma is useful when we want to show that certain polynomials are not equivalent. Secondly, we give a method of constructing automorphisms of equivalent polynomials. The idea here is that for each $c \in \mathbf{C}$ there is an isomorphism φ_c of $V(P+c)$ and $V(Q+c)$, and if φ_c varies as a polynomial in c, then these isomorphisms can be put together in an automorphism of $\mathbf{C}^3$ which induces an equivalence of P and Q. In sections 4 and 5 we prove the results given in 2, and finally in section 6, we give another interpretation of several of the results in terms of certain $(\mathbf{C},+)$ actions on $\mathbf{C}^3$.

Notation 1.1. — Throughout this article we denote by $\mathbf{C}^{[n]}$ the polynomial ring on n variables, $\mathbf{C}[x_1,\ldots,x_n]$. Also, if φ is a morphism between two affine varieties $\varphi : X \to Y$, then φ^* denotes the corresponding homomorphism $\varphi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$ where $\mathbf{C}[X]$ is the coordinate ring of X and $\mathbf{C}[Y]$ is the coordinate ring of Y. Also if $P \in \mathbf{C}^{[n]}$, then $V(P)$ denotes the zero set of P in $\mathbf{C}^n$.

2. The main results

DEFINITION 2.1. — *Let P and Q be two polynomials in $\mathbf{C}[x,y,z]$. We say that they are algebraically (resp. analytically) equivalent if there exists an algebraic (resp. analytic) automorphism φ^* of the ring $\mathbf{C}[x,y,z]$ such that $\varphi^*(P) = Q$. We say they are algebraically equivalent in $\mathbf{C}[x,y,z,w]$ if there exists an automorphism ψ^* of $\mathbf{C}[x,y,z,w]$ such that $\psi^*(P) = Q$.*

Geometrically, two polynomials P and Q are equivalent if and only if the fibrations of $\mathbf{C}^3 \to \mathbf{C}$ given by P and Q are equivalent. They are equivalent in $\mathbf{C}[x,y,z,w]$ if and only if the fibrations of $\mathbf{C}^4 \to \mathbf{C}$ given by P and Q are equivalent.

The notion of equivalence in $\mathbf{C}[x,y,z,w]$ is a special case of *stable equivalence*. Two polynomials in $\mathbf{C}[x,y,z]$ are stably equivalent if and only if there exists an integer n such that the polynomials are algebraically equivalent in $\mathbf{C}[x,y,z,w_1,\ldots,w_n]$. For the special examples given in this article, when two polynomials are stably equivalent, then we can choose $n=1$.

DEFINITION 2.2. — *Given two isomorphic hypersurfaces $H_1 = V(P)$ and $H_2 = V(Q)$ in $\mathbf{C}^3$, we say that the embeddings induced by P and Q of H_1 and H_2 are algebraically (resp. analytically) equivalent if there is an automorphism φ of $\mathbf{C}^3$ which sends H_1 to H_2. That is, $\varphi^*(Q) = uP$, where u is an invertible algebraic (resp. analytic) function on $\mathbf{C}^3$. For algebraic equivalences, therefore, u is a non-zero constant.*

If two polynomials P and Q are equivalent, then for any $a \in \mathbf{C}$, the surfaces defined by $P + a$ and $Q + a$ are isomorphic, and the embeddings given by $P + a$ and $Q + a$ are equivalent. One of the consequences of this article is that the converse is not true even if all the fibers are smooth irreducible surfaces (see Corollary 2.8).

The main theorem below gives the classification of the algebraic equivalence classes of the polynomials $x^2y - z^2 - xq(x, z)$.

Notation 2.3. — Given a polynomial $r(t) \in \mathbf{C}[t]$, denote by P_r the polynomial $x^2y - z^2 - xr(z^2)$.

Theorem 2.4 (Main theorem). — *Let $P \in \mathbf{C}[x, y, z]$ be a polynomial of the form $P = x^2y - z^2 - xq(x, z)$. Then the two following conditions hold:*

(i) *P is algebraically equivalent to P_r, where $r \in \mathbf{C}[t]$ is the polynomial defined by $r(z^2) = (q(0, z) + q(0, -z))/2$.*

(ii) *if $r(t)$ and $s(t)$ are polynomials in $\mathbf{C}[t]$, then P_r is algebraically equivalent to P_s if and only if there exists $a \in \mathbf{C}^*$ such that $r(t) = as(t)$.*

The next theorem gives the classification of the analytic and stable algebraic equivalence classes.

Theorem 2.5. — *For any polynomials $r(t)$ and $s(t)$ in $\mathbf{C}[t]$, the following three conditions are equivalent:*

(i) *the polynomials P_r and P_s are algebraically equivalent in $\mathbf{C}[x,y,z,w]$.*

(ii) *either $r(0) = s(0) = 0$ or $r(0)s(0) \neq 0$.*

(iii) *the polynomials P_r and P_s are analytically equivalent (in $\mathbf{C}[x, y, z]$).*

Consider the four following non-isomorphic surfaces in $\mathbf{C}^3$.

- $W_0 = V(x^2y - z^2)$;
- $W_1 = V(x^2y - z^2 - x)$;
- $V_0 = V(x^2y - z^2 + 1)$;
- $V_1 = V(x^2y - z^2 - x + 1)$.

Note that the four surfaces are in fact non-isomorphic. The surface W_0 is singular on the line defined by $x = z = 0$. W_1 is non-singular, and has a projection $\pi : W_1 \to \mathbf{C}$ given by $\pi(x, y, z) = x$ for which all fibers are isomorphic to $\mathbf{C}$ and the zero fiber is non-reduced. The Euler characteristic of W_1 is 1. V_0 and V_1 are non-singular, algebraically non-isomorphic but analytically isomorphic ([5]). The Euler characteristic of these surfaces is 2.

Remark 2.6. — Since W_0 is singular but W_1 is smooth, we have that W_0 is not analytically isomorphic to W_1 and $W_0 \times \mathbf{C} \ncong W_1 \times \mathbf{C}$. This remark will be used in the proof of theorem 2.5.

PROPOSITION 2.7. — *For any $c \in \mathbf{C}$, the surface in $\mathbf{C}^3$ defined by $V(P_r + c)$ is isomorphic to:*

- *W_0 if $r(0) = 0$ and $c = 0$;*
- *W_1 if $r(0) \neq 0$ and $c = 0$;*
- *V_0 if $r(c) = 0$ and $c \neq 0$;*
- *V_1 for all other cases.*

Proof. — First we show that $V(P_r + c) \cong V(P_{r(c)} + c)$. We give a morphism of $\mathbf{C}^3 \to \mathbf{C}^3$ which induces an isomorphism between these surfaces. This morphism depends on c.

For any $c \in \mathbf{C}$ define the polynomial g_c such that $r(z^2) - r(c) = g_c(z^2)(z^2 - c)$. Let φ_c and ψ_c be the morphisms of $\mathbf{C}^3$ defined by

$$\varphi_c(x, y, z) = (x, (1 + xg_c(z^2))y - r(c)g_c(z^2), z)$$

and

$$\psi_c(x, y, z) = (x, (1 - xg_c(z^2))y + r(z^2)g_c(z^2), z).$$

We have $\varphi_c^*(P_r + c) = (1 + xg_c(z^2))(P_{r(c)} + c)$ and $\psi_c^*(P_{r(c)} + c) = (1 - xg_c(z^2))(P_r + c)$. Furthermore, $\varphi_c^* \circ \psi_c^*(y) = y - (g_c(z^2))^2(P_{r(c)} + c)$, and $\psi_c^* \circ \varphi_c^*(y) = y - (g_c(z^2))^2(P_r + c)$. Thus, φ_c and ψ_c induce isomorphisms between $V(P_r + c)$ and $V(P_{r(c)} + c)$.

Now consider the four cases. If $c = r(c) = 0$, then $V(P_{r(c)} + c) = W_0$. If $c = 0$ and $r(c) \neq 0$, then the automorphism ϕ_1 of $\mathbf{C}^3$ defined by $\phi_1(x, y, z) = (x(r(0))^{-1}, (r(0))^2 y, z)$ induces an isomorphism between $V(P_{r(c)} + c)$ and W_1. If $c \neq 0$ and $r(c) = 0$, then the automorphism ϕ_2 of $\mathbf{C}^3$ defined by $\phi_2(x, y, z) = (x, cy, \alpha z)$, where α is a square root of c, defines an isomorphism between $V(P_{r(c)} + c)$ and V_0. Finally, if $c \neq 0$ and $r(c) \neq 0$, then the automorphism ϕ_3 of $\mathbf{C}^3$ defined by $\phi_3(x, y, z) = ((r(c))^{-1} cx, (r(c))^2 c^{-1} y, \alpha z)$, induces an isomorphism between $V(P_{r(c)} + c)$ and V_1. □

From these results, we can find several interesting examples of phenomena concerning polynomials of three variables.

COROLLARY 2.8. — *Suppose $r(t)$ and $s(t)$ are two polynomials with the same zero set but such that $r(t)/s(t)$ is not constant. Then $V(P_r + c) \cong V(P_s + c)$ for all $c \in \mathbf{C}$, however P_r is not equivalent to P_s.*

Remark that $V(P_r+c)$ is always irreducible. By choosing r such that $r(0)\neq 0$, we can even construct examples where $V(P_r+c)$ is smooth and irreducible for all $c\in\mathbf{C}$.

Example 2.9. — For $k\in\mathbf{N}\setminus\{0\}$, the polynomials $Q_k=P_{(t-1)^k}=x^2y-z^2-x(z^2-1)^k$ are all algebraically inequivalent, however the varieties $V(Q_k+c)$ are isomorphic for all $c\in\mathbf{C}$ and $k\in\mathbf{N}\setminus\{0\}$, and they are smooth irreducible surfaces.

The proof of the corollary is evident from proposition 2.7 and the main theorem 2.4.

Corollary 2.10. — *Suppose X is isomorphic to one of the surfaces V_0, V_1, W_0 or W_1. Suppose also that $V(P_r+c)$ and $V(P_s+c')$ are both isomorphic to X. Then the polynomials P_r+c and P_s+c' induce equivalent embeddings of X in $\mathbf{C}^3$ if and only if one of the following two cases holds:*

- (i) *$c=c'=0$ and there exist $\alpha\in\mathbf{C}^*$ and $\beta\in\mathbf{C}^*$ such that $r(t)=\alpha s(\beta t)$; or*
- (ii) *$cc'\neq 0$ and there exists an $\alpha\in\mathbf{C}^*$ such that $r(t)=\alpha s(c't/c)$.*

However, the same polynomials P_r+c and P_s+c' induce equivalent embeddings of $X\times\mathbf{C}$ in $\mathbf{C}^4$ if and only if the following two conditions hold:

- (i) *either $r(0)=s(0)=0$ or $r(0)s(0)\neq 0$; and*
- (ii) *either $c=c'=0$ or $cc'\neq 0$.*

Example 2.11. — For $k\in\mathbf{N}\setminus\{0\}$, the polynomials $R_k=P_{t^k-1}+1=x^2y-z^2-x(z^{2k}-1)+1$ induce inequivalent embeddings of the Danielewski surface V_0 in $\mathbf{C}^3$, however they induce equivalent embeddings of $V_0\times\mathbf{C}$ in $\mathbf{C}^4$.

This corollary answers a question posed in [7], where different embeddings are studied. The authors pose the "stable equivalence problem" which ask if stable equivalence of hypersurfaces in $\mathbf{C}^n$ implies that they have equivalent embeddings. We show here that the answer is negative. Indeed we have proven that there exists polynomials P and Q in $\mathbf{C}^3$ such that the induced embeddings of $V(P)\times\mathbf{C}$ and $V(Q)\times\mathbf{C}$ in $\mathbf{C}^4$ are algebraically equivalent but the induced embeddings of $V(P)$ and $V(Q)$ in $\mathbf{C}^3$ are not. The corresponding question for subvarieties of higher codimension was shown to have a negative answer in [9]. In their example, Shpilrain and Yu use the original Danielewski surfaces $V(xy-z^2-1)$ and $V(x^2y-z^2-1)$, which are not isomorphic but whose cylinders are isomorphic.

Proof. — (Proof of Corollary 2.10) First, note that P_r+c and P_s+c' induce equivalent embeddings if and only if there exists $u\in\mathbf{C}^*$ such that

$P_r + c$ is equivalent to $u(P_s + c')$. By considering an automorphism of $\mathbf{C}^3$ which sends (x, y, z) to $(ux, u^{-1}y, \nu z)$ where ν is a square root of u, one can see that $u(P_s + c')$ is equivalent to $P_{s'} + uc'$ where $s'(t) = s(t/u)$. Now the result follows directly from theorem 2.4. □

3. Preliminary lemmas

One of the principal tools we use to prove the main theorem is a result of Makar-Limanov concerning the automorphism group of any surface of the form $V(P_r + c)$. To explain this result, first recall the definition of the Makar-Limanov invariant ML. Let S be a finitely generated $\mathbf{C}$-algebra. Suppose ∂ is a locally nilpotent derivation on S . It is well-known that ∂ defines a $(\mathbf{C}, +)$-action on S by $t \cdot f = \exp(t\partial f)$ for $f \in S$ and $t \in (\mathbf{C}, +)$. The Makar-Limanov invariant $ML(S)$ is the intersection of the kernels of all locally nilpotent derivations on S. If S is the coordinate ring of an affine variety X, we say that $ML(X) = ML(S)$. If $f \in ML(X)$, then the zero set of f in X is stable under all $(\mathbf{C}, +)$-actions on X.

Lemma 3.1 (Makar-Limanov, [6]). — *Let $X = V(P_r + c)$. Consider the coordinate ring $\mathbf{C}[X]$. We consider generators x, y and z of $\mathbf{C}[X]$ satisfying the relation $P_r + c = 0$. Then $ML(X) = \mathbf{C}[x]$ and a derivation ∂ on $\mathbf{C}[X]$ is locally nilpotent if and only if $\partial = h(x)(x^2\partial/\partial z - (\partial(P_r)/\partial z)\partial/\partial y)$, where $h(x) \in \mathbf{C}[x]$.*

Proof. — Makar-Limanov proved in [6] this result in the case that $X = V(x^n y - q(z))$, where $n \geqslant 2$ and q is a polynomial of degree at least 2. To prove the lemma, by proposition 2.7, it suffices to consider the four cases where $X = W_0$, W_1, V_0 or V_1. If $X = V_0$ or W_0, it follows directly from Makar-Limanov's result. For V_1 and W_1, the proof given by Makar-Limanov still holds. More precisely, the method to prove it is to choose a filtration of $S = \mathbf{C}[X]$ and study the locally nilpotent derivations on $Gr(S)$. The graded ring for $X = V_1$ or W_1 is exactly the same as for W_0. Thus the same argument proves the lemma. □

For the following corollary, we will consider isomorphisms between rings of the form $S_1 = \mathbf{C}[x, y, z]/(P_r + c)$ and $S_2 = \mathbf{C}[x, y, z]/(P_s + c)$. We will denote by x_i, z_i and y_i the class of x, y and z in S_i, $i = 1, 2$.

Corollary 3.2. — *Let $c \in \mathbf{C}$. If ϕ^* is an isomorphism from $S_1 = \mathbf{C}[x, y, z]/(P_r + c)$ to $S_2 = \mathbf{C}[x, y, z]/(P_s + c)$, then there exists $a \in \mathbf{C}^*$ such that $\phi^*(x_1) = ax_2$.*

Proof. — Again we use a proof of Makar-Limanov from [6]. Since ϕ^* induces an isomorphism between $ML(S_1) = \mathbf{C}[x_1]$ and $ML(S_2) = \mathbf{C}[x_2]$, we have that $\phi^*(x_1) = ax_2 + b$ where $a \in \mathbf{C}^*$ and $b \in \mathbf{C}$. Now, $\partial^2(z_2) = 0$ for any locally nilpotent derivation ∂ on S_2. Thus, $\partial^2(\phi^*(z_1)) = 0$ and therefore $\phi^*(z_1) = dz_2 + e$ where $d, e \in \mathbf{C}[x_2]$. Since ϕ^* is inversible, we see that $d \in \mathbf{C}^*$. Let $\epsilon = x_2^2\partial/\partial z_2 - (\partial(P_s)/\partial z_2)\partial/\partial y_2$. ϵ is a locally nilpotent derivation on S_2. Then $(\phi^*)^{-1} \circ \epsilon \circ \phi^*$ is a locally nilpotent derivation on S_1. Now, $(\phi^*)^{-1} \circ \epsilon \circ \phi^*(z_1) = a^{-2}d(x_1 - b)^2$. But, $\partial(z_1)$ is divisible by x_1^2 for any locally derivation ∂ on S_1. This gives that $b = 0$. □

We now prove several lemmas which are used to prove the main theorem.

Lemma 3.3. — *Let P, Q be polynomials in $\mathbf{C}^{[n]}$ and φ be an algebraic or analytic morphism from $\mathbf{C}^n$ to itself such that*

$$\text{for all } c \in \mathbf{C} \qquad \varphi^*(P - c) \in (Q - c).$$

Then $\varphi^*(P) = Q$.

Proof. — For all $c \in \mathbf{C}$, there exists $R_c \in \mathbf{C}^{[n]}$ such that

$$\begin{aligned} R_c(Q - c) &= \varphi^*(P - c) \\ &= \varphi^*(P) - c \\ &= R_0Q - c. \end{aligned}$$

This gives in particular that $c(R_c - 1) = Q(R_c - R_0)$. If we multiply this equation by $(Q - c)$ and we remark that $(R_c - 1)(Q - c) = (R_0 - 1)Q$, we find that $R_0 - 1$ belongs to the ideal $Q - c$ for all $c \in \mathbf{C}^*$. This implies that $R_0 = 1$, and the lemma follows. □

The following lemma concerns algebraic or analytic morphisms of $\mathbf{C}^n$ to itself which depend on a parameter. More specifically, if there exists an algebraic (resp. analytic) morphism $\varphi : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\varphi(c, x_1, \ldots, x_n) = \varphi_c(x_1, \ldots, x_n)$ for all c, we say that the family $(\varphi_c)_{c\in\mathbf{C}}$ of morphisms depends algebraically (or analytically) on the parameter c.

Lemma 3.4. — *Let P, Q be polynomials in $\mathbf{C}^{[n]}$ and $(\varphi_c)_{c\in\mathbf{C}}$ be an algebraic (resp. analytic) family of morphisms of $\mathbf{C}^n$ to itself which depends algebraically (resp. analytically) on a parameter c. Suppose that $\varphi_c^* : \mathbf{C}^{[n]}/(P - c) \to \mathbf{C}^{[n]}/(Q - c)$ is an isomorphism for all $c \in \mathbf{C}$.*

Then P and Q are algebraically (resp. analytically) equivalent.

Proof. — By the definition above, there exists an algebraic or analytic morphism $\varphi : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\varphi_c(x_1, \ldots, x_n) = \varphi(c, x_1, \ldots, x_n)$ for any $c \in \mathbf{C}$. We pose $\Phi : \mathbf{C}^n \to \mathbf{C}^n$ the morphism $\Phi(x_1, \ldots, x_n) = \varphi(Q(x_1, \ldots, x_n), x_1, \ldots, x_n)$. For each $c \in \mathbf{C}$, the restriction of Φ to the hypersurface $V(Q-c)$ is an isomorphism from $V(Q-c)$ to $V(P-c)$. Therefore Φ is an algebraic (resp. analytic) bijective morphism of $\mathbf{C}^n$ to itself. This implies that it is an algebraic (resp. analytic) automorphism. The fact that $\Phi^*(P) = Q$ comes from lemma 3.3. □

4. Algebraic equivalence in $\mathbf{C}^3$.

In this section, we will prove the main theorem.

Proof (of Main Theorem 2.4). — Let $P = x^2y - z^2 - xq(x, z)$. We can rewrite P in the form $x^2y - z^2 - xq(0, z) + x^2q_2(x, z) = x^2(y + q_2(x, z)) - z^2 - xq(0, z)$. By applying the automorphism of $\mathbf{C}^3$ which sends (x, y, z) to $(x, y - q_2(x, z), z)$, we see that we may assume that $q_2 \equiv 0$. Now by lemma 4.1, which is given below, we have that P is equivalent to $x^2y - z^2 - x(q(0, z) + q(0, -z))/2$. This proves the first part of the theorem.

For the second part, suppose that ϕ is an algebraic automorphism of $\mathbf{C}^3$ such that $\phi^*(P_r) = P_s$.

First we show that ϕ^* stabilizes the ideal generated by x. This is a consequence of corollary 3.2. For any $c \in \mathbf{C}$, ϕ^* induces an isomorphism between $S_1 = \mathbf{C}[x, y, z]/(P_r + c)$ and $S_2 = \mathbf{C}[x, y, z]/(P_s + c)$. In particular, by the corollary, we find that for all $c \in \mathbf{C}$, the ideal $\phi^*((x, P_r + c)) = (x, P_s + c) = (x, z^2 - c)$. Thus $\phi^*(x) \in \cap_{c\in\mathbf{C}}(x, z^2 - c) = (x)$. Since ϕ is an automorphism $\phi^*(x)$ is a generator of the ideal (x). Thus $\phi^*(x) = ax$, where $a \in \mathbf{C}^*$.

Now for any $\alpha \in \mathbf{C}$, we have that $\phi^*(P_r + \alpha x) = P_s + a\alpha x$, and therefore for any $\beta \in \mathbf{C}$, ϕ induces an isomorphism between $V(P_s + \alpha x + \beta)$ and $V(P_r + a\alpha x + \beta)$. By proposition 2.7, this means in particular that the zeros of $r + \alpha$ and $s + a\alpha$ are the same for all $\alpha \in \mathbf{C}$. Therefore $s(t) = ar(t)$.

On the other hand, if $s(t) = ar(t)$ for $a \in \mathbf{C}^*$, then one can define ϕ by $\phi(x, y, z) = (ax, a^{-2}y, z)$. □

LEMMA 4.1. — *Let $q(z), \tilde{q}(z) \in \mathbf{C}[z]$ be such that*

$$q(z) + q(-z) = \tilde{q}(z) + \tilde{q}(-z).$$

Then the polynomials $x^2y - z^2 - xq(z)$ and $x^2y - z^2 - x\tilde{q}(z)$ are algebraically equivalent.

Proof. — Note that $\tilde{q} - q$ is a polynomial which has monomials only in odd degrees. In particular, z divides the polynomial $\tilde{q}(z) - q(z)$. Define $\alpha(t) \in \mathbf{C}[t]$ by

$$\alpha(z^2) = \frac{\tilde{q}(z) - q(z)}{2z} \in \mathbf{C}[z^2].$$

Consider the polynomial $f_c = f(z,c) \in \mathbf{C}[z,c]$ such that

$$(z^2 - c)f_c = 2z(\alpha(z^2) - \alpha(c)).$$

To simplify notation, let $\alpha_c = \alpha(c)$. We let

$$\varphi_c(x,y,z) = (x, (1 - xf_c)y + f_c\tilde{q}(z) + (\alpha_c)^2 + \frac{q(z + \alpha_c x) - q(z)}{x}, z + \alpha_c x)$$

and

$$\psi_c(x,y,z) = (x, (1 + xf_c)y - f_c q(z) + (\alpha_c)^2 + \frac{\tilde{q}(z - \alpha_c x) - \tilde{q}(z)}{x}, z - \alpha_c x).$$

For all $c \in \mathbf{C}$, we have

$$\varphi_c^*(x^2y - z^2 - xq(z) + c) = (1 - xf_c)(x^2y - z^2 - x\tilde{q}(z) + c)$$

and

$$\psi_c^*(x^2y - z^2 - x\tilde{q}(z) + c) = (1 + xf_c)(x^2y - z^2 - xq(z) + c).$$

Moreover, we have $\psi_c^* \circ \varphi_c^*(x) = x$, $\psi_c^* \circ \varphi_c^*(z) = z$. Thus, we have

$$\psi_c^* \circ \varphi_c^*(y) = y + \frac{(1 - x\psi_c^*(f_c))(1 + xf_c) - 1}{x^2}(x^2y - z^2 - xq(z) + c).$$

Therefore for all $c \in \mathbf{C}, \varphi_c^*$ is an isomorphism from $\mathbf{C}[x,y,z]/(x^2y - z^2 - xq(z) + c)$ to $\mathbf{C}[x,y,z]/(x^2y - z^2 - x\tilde{q}(z) + c)$.

We can now conclude with lemma 3.4. □

5. Stable equivalence and analytic equivalence

In this section we will prove theorem 2.5. In fact,by the remark 2.6 and proposition 2.7, it suffices to prove the following reformulation of the theorem:

Theorem 2.5′. — *For any polynomial $r(t)$ in $\mathbf{C}[t]$,*

(i) *the polynomials P_r and $P_{r(0)}$ are algebraically equivalent in $\mathbf{C}[x,y,z,w]$.*

(ii) *the polynomials P_r and $P_{r(0)}$ are analytically equivalent (in $\mathbf{C}[x,y,z]$).*

The equivalences in the two parts are quite different, but the idea of the proof is similar. To show that the polynomials P and Q are equivalent, we form an algebraic or analytic family φ_c of automorphism of $\mathbf{C}^4$ or $\mathbf{C}^3$, such that each automorphism φ_c induces an isomorphism between a $V(P + c)$ and $V(Q + c)$. Then by lemma 3.4, we have the given result.

Proof. — The goal is to find equivalences between the polynomials P_r and $P_{r(0)}$.

We start by forming an algebraic family φ_c of morphisms from $\mathbf{C}^3$ to $\mathbf{C}^3$ for which φ_c induces a morphism from $V(P_r+c)$ and $V(P_{r(0)}+c)$. They are not isomorphisms. Afterwards, we will adjust the family to make families of isomorphisms in different ways for each of the two parts of the theorem. Let $\beta(t) = \dfrac{r(0)-r(t)}{2t} \in \mathbf{C}[t]$ and $h(z,t) = \dfrac{r(z^2)+2\beta(t)z^2-r(0)}{z^2-t} \in \mathbf{C}[z,t]$. We set $\beta_c = \beta(c)$ and $h_c = h(z,c)$.

Let φ_c be the morphism of $\mathbf{C}^3$ to itself defined by:

$$\varphi_c(x,y,z) = (x,(1-xh_c)y + h_c r(z^2) + (\beta_c z)^2, (1-\beta_c x)z).$$

For all $c \in \mathbf{C}$, φ_c^* send the polynomial $P_{r(0)}+c$ to the polynomial $(1-xh_c)(P_r+c)$.

Proof of part (ii).

We will change φ_c to an analytic automorphism $\widehat{\varphi}_c$ of $\mathbf{C}^3$ for each c such that $\widehat{\varphi}_c^*(x) = x$, $\widehat{\varphi}_c^*(z) \equiv \varphi_c^*(z) \pmod{x^2}$, and $\widehat{\varphi}_c^*(P_{r(0)}+c) \equiv \varphi_c^*(P_{r(0)}+c) \pmod{x^2}$. One can achieve this by defining $\widehat{\varphi}_c^*(x) = x, \widehat{\varphi}_c^*(z) = e^{-\beta_c x}z$ and $\widehat{\varphi}_c^*(y) = e^{-xh_c}y + z^2(e^{-2\beta_c x} - 1 + 2\beta_c x)/x^2 + h_c r(z^2) + (-z^2+c-xr(z^2))(e^{-xh_c} - 1 + xh_c)/x^2$.

For all $c \in \mathbf{C}$, $\widehat{\varphi}_c^*$ is an analytic automorphism of $\mathbf{C}^3$ such that

$$\widehat{\varphi}_c^*(P_{r(0)}+c) = e^{-xh_c}(P_r+c).$$

By the lemma 3.4, we can now conclude that P_r and $P_{r(0)}$ are analytically equivalent.

Proof of part (i).

This time, we will change φ_c to a morphism $\tilde{\varphi}_c$ of $\mathbf{C}^4$ to $\mathbf{C}^4$ by adding a variable, which we denote by w. We will do this in such a way that for each $c \in \mathbf{C}$: $\tilde{\varphi}_c^*$ is an isomorphism from $\mathbf{C}[x,y,z,w]/(P_{r(0)}+c)$ to $\mathbf{C}[x,y,z,w]/(P_r+c)$. This can be achieved as follows. We choose $\tilde{\varphi}_c$ as follows.

$$\tilde{\varphi}_c^*(x) = x, \tilde{\varphi}_c^*(z) = (1-\beta_c x)z + x^2 w, \tilde{\varphi}_c^*(w) = (1+\beta_c x)w - \beta_c^2 z.$$

Note that $\tilde{\varphi}_c^*(z) = \varphi_c^*(z) \pmod{x^2}$. This allows us to choose $\tilde{\varphi}_c^*(y)$ such that $\tilde{\varphi}_c^*(P_{r(0)}+c) = (1-xh_c)(P_r+c)$. It is easily checked that it suffices to choose

$$\begin{aligned}\tilde{\varphi}_c^*(y) &= (1-xh_c)y + h_c r(z^2) + (\beta_c z)^2 + 2(1-\beta_c x)zw + x^2w^2\\ &= \varphi_c^*(y) + 2(1-\beta_c x)zw + x^2w^2.\end{aligned}$$

We will now construct $\tilde{\psi}_c^*$, the inverse of $\tilde{\varphi}_c^*$ between $\mathbf{C}[x,y,z,w]/(P_{r(0)}+c)$ and $\mathbf{C}[x,y,z,w]/(P_r+c)$. If we pose:

$$\tilde{\psi}_c^*(x)=x, \tilde{\psi}_c^*(z)=(1+\beta_c x)z-x^2w, \tilde{\psi}_c^*(w)=(1-\beta_c x)w+\beta_c^2 z,$$

we have that $\tilde{\psi}_c^*\circ\tilde{\varphi}_c^*(x)=x$, $\tilde{\psi}_c^*\circ\tilde{\varphi}_c^*(z)=z$, and $\tilde{\psi}_c^*\circ\tilde{\varphi}_c^*(w)=w$.

Now we choose $\tilde{\psi}_c^*(y)$ such that $\tilde{\psi}_c^*(P_r+c)=(1+xh_c)(P_{r(0)}+c)$. It is easily checked that it suffices to choose:

$$\begin{aligned}\tilde{\psi}_c^*(y) &= (1+xh_c)y-h_c r(0)+(\beta_c z)^2+[r((\tilde{\psi}_c^*(z))^2)-r(z^2)]/x\\ &\quad -2(1+\beta_c x)zw+x^2w^2.\end{aligned}$$

Therefore, $\tilde{\psi}_c^*\circ\tilde{\varphi}_c^*(P_{r(0)}+c)=(1-x\tilde{\psi}_c^*(h_c))(1+xh_c)(P_{r(0)}+c)$.

Thus we have that

$$\tilde{\psi}_c^*\circ\tilde{\varphi}_c^*(y)=y+\frac{(1-x\tilde{\psi}_c^*(h_c))(1+xh_c)-1}{x^2}(P_{r(0)}+c).$$

Therefore for all $c\in\mathbf{C}, \tilde{\varphi}_c^*$ is an isomorphism from $\mathbf{C}[x,y,z,w]/(P_{r(0)}+c)$ to $\mathbf{C}[x,y,z,w]/(P_r+c)$.

Again by the lemma 3.4, we can now conclude that P_r and $P_{r(0)}$ are algebraically equivalent in $\mathbf{C}[x,y,z,w]$. $\square$

6. $(\mathbf{C},+)$-actions on $\mathbf{C}^3$

The results of the previous section can be interpreted as a classification of a certain family of $\mathbf{C}^+$ actions on $\mathbf{C}^3$.

LEMMA 6.1. — *Let ∂ be a triangular derivation of $\mathbf{C}[x,y,z]$ of the form $\partial=x^2\partial/\partial z+(x\partial q/\partial z+2z)\partial/\partial y$ where $q\in\mathbf{C}[x,z]$, and let $P=x^2y-z^2-xq(x,z)$. Then the kernel of ∂ is the ring $\mathbf{C}[x,P]$.*

Proof. — Note that $\partial=P_y\partial/\partial z-P_z\partial/\partial y$. It is evident that $\mathbf{C}[x,P]\subset\ker\partial$. Now we prove the other inclusion. First, consider the derivation ∂ on the algebra $R=\mathbf{C}[x,x^{-1},y,z]=\mathbf{C}[x,x^{-1},x^{-2}P,x^{-2}z]$. By choosing the coordinate system $u=x$, $v=x^{-2}z$ and $w=x^{-2}P$, one sees that $\partial=\partial/\partial v$, and thus the kernel of ∂ when considered as a derivation of R is $\mathbf{C}[x,x^{-1},P]$. This means that if $g\in\ker\partial$ with $g\in\mathbf{C}[x,y,z]$, then there exists $N\in\mathbf{N}$ such that $x^Ng\in\mathbf{C}[x,P]$. Since $P\equiv z^2(\mod x)$, one can prove that if $N\geqslant 1$, then $x^{N-1}g\in\mathbf{C}[x,P]$. By induction, $g\in\mathbf{C}[x,P]$. $\square$

In particular, if ∂ is of the form given in the lemma, then the induced $(\mathbf{C},+)$-action has the following properties. There is one line of fixed points, defined by the ideal (x,z). All orbits in the open subset defined by $x\neq 0$ can be separated by invariants, however the surface $V(x)$ contains the line of fixed points and pairs of orbits which cannot be separated by invariants.

Now we consider the set of all triangular derivations on $\mathbf{C}[x,y,z]$ of the form $x^2\partial/\partial z+(xp(x,z)+2z)\partial/\partial y$.

Notation 6.2. — If $s(t)\in\mathbf{C}[t]$, we denote by ∂_s the triangular derivation given by $x^2\partial/\partial z+2z(1+xs(z^2))\partial/\partial y$.

Two locally nilpotent derivations ∂_1 and ∂_2 on $\mathbf{C}[x,y,z]$ are equivalent if there exists an automorphism φ^* of $\mathbf{C}[x,y,z]$ such that $\varphi^{\#}\partial_1 = \varphi^*\partial_1(\varphi^*)^{-1}=\partial_2$. That is, they are equivalent if the automorphism φ of $\mathbf{C}^3$ conjugates the action induced by ∂_1 into the action induced by ∂_2. We will denote by $\varphi^{\#}\partial$ the conjugate of ∂ by φ, that is $\varphi^{\#}\partial=\varphi^*\partial(\varphi^*)^{-1}$.

One way of interpreting the results of the previous sections is in terms of $(\mathbf{C},+)$-actions.

Corollary 6.3.

(i) *Let ∂ be the derivation defined by $x^2\partial/\partial z+(xp(x,z)+2z)\partial/\partial y$. Then ∂ is equivalent to ∂_s where $s(z^2)=(p(0,z)-p(0,-z))/2z$;*

(ii) *For all s and $\tilde{s}$ in $\mathbf{C}[t]$, we have that ∂_s is equivalent to a non-zero constant multiple of $\partial_{\tilde{s}}$ if and only if there exist α and β in $\mathbf{C}^*$ such that $\alpha s(\beta t)=\tilde{s}(t)$;*

(iii) *Let $P_0=x^2y-z^2$. For all $s\in\mathbf{C}[t]$, there exists a polynomial f such that ∂_s is analytically equivalent to $\exp(xf(P_0))\partial_0$.*

Proof. — For the first part, let $q(x,z)$ be a polynomial such that $\partial q/\partial z = p$, and choose $r\in\mathbf{C}[t]$ such that $r'(t)=s(t)$ and $r(0)=q(0,0)$. By the main theorem, the polynomial $P=x^2y-z^2-xq(x,z)$ is equivalent to P_r, and by the proof of the main theorem, the equivalence is given by an automorphism φ such that $\varphi^*(x)=x$, $(\varphi^*)^{-1}(z)-z\in\mathbf{C}[x,P]$, and $\varphi^*(P)=P_r$. By checking the images of $\varphi^{\#}\partial$ on x, y and z, we find that $\varphi^{\#}\partial=\partial_s$. For the last part, use the analytic automorphism defined in the proof of 2.5.

This leaves part (ii). Suppose that ∂_s is equivalent to a non-zero constant multiple of $\partial_{\tilde{s}}$. Then the kernels of the two derivations are equivalent. Let r (resp. $\tilde{r}$) be a polynomial such that $r'(t)=s(t)$ (resp. $\tilde{r}'(t)=s'(t)$). By lemma 6.1, there exists φ^* of $\mathbf{C}[x,y,z]$ such that $\mathbf{C}[x,P_r]=\mathbf{C}[\varphi^*(x),\varphi^*(P_{\tilde{r}})]$. Also, $\varphi^*(x)$ must be in the ideal (x), since the surface $V(x)$ is the surface containing all the orbits which cannot be separated by invariants of the $(\mathbf{C},+)-$actions corresponding to the derivations. Thus there is $a \in \mathbf{C}^*$ such that $\varphi^*(x) = ax$. This implies that $\varphi^*(P_{\tilde{r}}) = cP_r + b(x)$, where $c \in \mathbf{C}^*$ and $b(x) \in \mathbf{C}[x]$. First, we show that we can assume that $b(x)$ is of degree at most one. This is done as follows. If $b(x) = b_0 + b_1 x + x^2 \tilde{b}(x)$, we apply the automorphism of $\mathbf{C}^{[3]}$ which fixes x and z and sends y to $y + \tilde{b}(x)$. Thus now we have that, $b(x) = b_1 x + b_0$. By considering the fixed point sets of the two actions, we find that $b_0 = 0$. Thus we find that if ∂_s is equivalent to $\partial_{\tilde{s}}$, then P_r is equivalent to $cP_{\tilde{r}} + b_1 x$. Thus, by Corollary 2.8, there exists α, $\beta \in \mathbf{C}^*$ such that $r(t) = (\alpha/\beta)\tilde{r}(\beta t) + b_1$. In other words, we have that $s(t) = r'(t) = \alpha \tilde{r}'(\beta t) = \alpha \tilde{s}(\beta t)$.

Finally, for the converse, let $\varphi(x,y,z) = (\alpha x, (\beta/\alpha^2)y, \gamma z)$ where $\gamma^2 = \beta$. □

Remark 6.4. — Note that the orbits of the $(\mathbf{C},+)-$actions determined by ∂_0 and by $\exp(xf(P_0))\partial_0$ are identical.

Remark 6.5. — We do not know, in general, for which $s(t)$ the derivation ∂_s is analytically equivalent to ∂_0. However, there is one case which we can treat. If $s(t)$ is of the form $s(t) = \alpha t^k$ with $\alpha \in \mathbf{C}$ and $k \in \mathbf{N}$, we can show that ∂_s is analytically equivalent to ∂_0. Indeed, by Corollary 6.3, ∂_s is analytically equivalent to $\exp(xf(P_0))\partial_0$, where $f(P_0) = \frac{\alpha(-P_0)^k}{2(k+1)}$. Now we construct an automorphism ψ^* of $\mathbf{C}[x,y,z]$ for which $\psi^{\#}[\exp(f(P_0)x)\partial_0] = \partial_0$. Let $a_1, a_2 \in \mathbf{C}$. We pose

$$\psi^*(x) = \exp(a_1 x(-P_0)^k)x, \quad \psi^*(z) = \exp(a_2 x(-P_0)^k)z,$$
$$\text{and} \quad \psi^*(y) = \exp(2(a_2 - a_1)x(-P_0)^k)y.$$

Thus $\psi^*(P_0) = \exp(2a_2 x(-P_0)^k)P_0)$. By choosing $a_1 = \frac{-k\alpha}{(k+1)(4k+1)}$ and $a_2 = \frac{\alpha}{2(k+1)(4k+1)}$, one verifies easily that $x(-P_0)^k$ is fixed by ψ^*, and thus that ψ^* is an automorphism of $\mathbf{C}[x,y,z]$. Also, we have that

$$\psi^{\#}[\exp(f(P_0)x)\partial_0] = \partial_0.$$

Acknowledgement. The authors wish to thank Adrien Dubouloz for pointing out an error in a previous version of Corollary 6.3.

BIBLIOGRAPHY

[1] W. Danielewski, "On the cancellation problem and automorphism groups of affine algebraic varieties", preprint, Warsaw, 1989.

[2] A. Dubouloz, "Sur une classe de schémas avec actions de fibrés en droites", PhD Thesis, Institut Fourier, Grenoble, 2004.

[3] ———, "Danielewski-Fieseler surfaces", *Transformation Groups* **10** (Juin 2005), no. 2, p. 139-162.

[4] K.-H. Fieseler, "On complex affine surfaces with $\mathbf{C}^+$-action", *Comment. Math. Helvetici* **69** (1994), p. 5-27.

[5] G. Freudenburg & L. Moser-Jauslin, "Embeddings of Danielewski surfaces", *Math. Z.* **245** (2003), no. 4, p. 823-834.

[6] L. Makar-Limanov, "On the group of automorphisms of a surface $x^n y = P(z)$", *Israel J. Math.* **121** (2001), p. 113-123.

[7] L. Makar-Limanov, P. van Rossum, V. Shpilrain & J.-T. Yu, "The stable equivalence and cancellation problems", *Comment. Math. Helv.* **79** (2004), p. 341-349.

[8] V. Shpilrain & J.-T. Yu, "Embeddings of hypersurfaces in affine spaces", *J. Alg.* **239** (2001), p. 161-173.

[9] ———, "Affine varieties with equivalent cylinders", *J. Alg.* **251** (2002), p. 295-307.

Manuscrit reçu le 30 juin 2005,
accepté le 12 janvier 2006.

Lucy MOSER-JAUSLIN & Pierre-Marie POLONI
Université de Bourgogne
Institut de Mathématiques de Bourgogne
CNRS–UMR 5584
9, avenue Alain Savary
B.P. 47870
21078 Dijon (France)
moser@u-bourgogne.fr
ppoloni@u-bourgogne.fr