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ON WITTEN MULTIPLE ZETA-FUNCTIONS ASSOCIATED WITH SEMISIMPLE LIE ALGEBRAS I

by Kohji MATSUMOTO & Hirofumi TSUMURA

ABSTRACT. — We define Witten multiple zeta-functions associated with semisimple Lie algebras $\mathfrak{sl}(n)$, $(n = 2, 3, \ldots)$ of several complex variables, and prove the analytic continuation of them. These can be regarded as several variable generalizations of Witten zeta-functions defined by Zagier. In the case $\mathfrak{sl}(4)$, we determine the singularities of this function. Furthermore we prove certain functional relations among this function, the Mordell-Tornheim double zeta-functions and the Riemann zeta-function. Using these relations, we prove new and non-trivial evaluation formulas for special values of this function at positive integers.

RÉSUMÉ. — Nous définissons les fonctions zeta multiples de Witten associées aux algèbres de Lie semi-simples $\mathfrak{sl}(n)$, $(n = 2, 3, \ldots)$, et démontrons leurs continuations analytiques. Elles peuvent être considérées comme des généralisations à plusieurs variables des fonctions zeta de Witten définies par Zagier. Dans le cas $\mathfrak{sl}(4)$, nous déterminons les singularités de la fonction zeta multiple. De plus, nous démontrons plusieurs relations fonctionnelles entre cette fonction, les fonctions zeta doubles de Mordell-Tornheim et la fonction zeta de Riemann. En utilisant ces relations, nous démontrons de nouvelles formules non-triviales pour évaluer des valeurs spécifiques de cette fonction aux points entiers positifs.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

For any semisimple Lie algebra $\mathfrak{g}$, Zagier [26] defined the Witten zeta-function by

$$\zeta_{\mathfrak{g}}(s) = \sum_{\rho} (\dim \rho)^{-s}, \tag{1.1}$$

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where $s \in \mathbb{C}$ and ρ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. The values $\zeta_{\mathfrak{g}}(2k)$ for $k \in \mathbb{N}$ were introduced by Witten [25] in order to calculate the volumes of certain moduli spaces. Indeed it follows from Witten's work that $\zeta_{\mathfrak{g}}(2k) \in \mathbb{Q}\pi^{2kl}$ for $k \in \mathbb{N}$, where l is the number of positive roots of $\mathfrak{g}$ (see [26] Section 7). Zagier showed some explicit forms; $\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s)$, the Riemann zeta-function, and

$$\zeta_{\mathfrak{sl}(3)}(s) = 2^s \sum_{m,n=1}^{\infty} m^{-s} n^{-s} (m+n)^{-s},$$

$$\zeta_{\mathfrak{so}(5)}(s) = 6^s \sum_{m,n=1}^{\infty} m^{-s} n^{-s} (m+n)^{-s} (m+2n)^{-s}.$$

The sum on the right-hand side of the above explicit form for $\zeta_{\mathfrak{sl}(3)}(s)$ was already studied by Mordell [17], who showed that the values $\zeta_{\mathfrak{sl}(3)}(2k)$ ($k \in \mathbb{N}$) can be evaluated by means of $\zeta(2j)$ for $j \in \mathbb{N}$ (see also [19]). Recently Gunnells and Sczech [7] evaluated $\zeta_{\mathfrak{g}}(2k)$ for $k \in \mathbb{N}$ by means of the generalized higher-dimensional Dedekind sums, and gave certain evaluation formulas for $\zeta_{\mathfrak{sl}(3)}(2k)$ and $\zeta_{\mathfrak{sl}(4)}(2k)$ for $k \in \mathbb{N}$ by means of $\zeta(2j)$ for $j \in \mathbb{N}$.

As generalizations of Witten zeta-functions, the first author [14] defined the following complex functions of several variables by

$$\zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3) = \zeta_{MT,2}(s_1, s_2, s_3) = \sum_{m,n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3}, \tag{1.2}$$

$$\zeta_{\mathfrak{so}(5)}(s_1, s_2, s_3, s_4) = \sum_{m,n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3} (m+2n)^{-s_4}, \tag{1.3}$$

and proved the analytic continuation of them by the method using the Mellin-Barnes integral formula (see also [13, 11, 12]). Note that $\zeta_{MT,2}(s_1, s_2, s_3)$ is called the Mordell-Tornheim double zeta-function whose values at positive integers were studied by Tornheim [20] and Mordell [17] (as mentioned above) in the 1950's. As a related result, the second author [23] gave some evaluation formulas for certain values $\zeta_{\mathfrak{so}(5)}(k_1, k_2, k_3, k_4)$ ($k_1, k_2, k_3, k_4 \in \mathbb{N}_0$) by means of $\zeta(j+1)$ for $j \in \mathbb{N}$.

Recently the second author [21] has proved certain functional relations between $\zeta_{MT,2}(s_1, s_2, s_3)$ and $\zeta(s)$, and further proved some related analogues ([24]). These can be regarded as continuous generalizations of the known relations for Mordell-Tornheim and Riemann zeta values at positive integers obtained in [17, 20]. For example,

$$\zeta_{MT,2}(1, s, 3) - \zeta_{MT,2}(1, 3, s) + \zeta_{MT,2}(3, s, 1) = 4\zeta(s+4) - 2\zeta(2)\zeta(s+2). \tag{1.4}$$

Since the first author proved the analytic continuation of (1.2) and determined its possible singularities in [14], we see that the relation (1.4) holds for all $s \in \mathbb{C}$ except for the possible singularities of both sides.

In the present paper, as a generalization of (1.2), we define the Witten multiple zeta-function associated with $\mathfrak{sl}(r+1)$ for $r \in \mathbb{N}$ by

$$\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}}, \tag{1.5}$$

where

$$\mathbf{s} = (s_{jk})_{1\leqslant j \leqslant r;\, 1\leqslant k \leqslant r-j+1} \in \mathbb{C}^{r(r+1)/2} \quad (\Re s_{jk} > 1).$$

In particular when $\mathbf{s} = (s)$, namely $s_{jk} = s$ for all j, k, we can see that $C_{r+1} \cdot \zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ coincides with the ordinary Witten zeta-function associated with $\mathfrak{sl}(r+1)$ defined by Zagier [26], where

$$C_{r+1} = \prod_{1\leqslant j<k\leqslant r+1} (k-j)$$

(see Section 2, Proposition 2.1).

In Section 2, using the Mellin-Barnes method introduced in [13, 11, 12], we prove a certain integral expression of $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ for any $r \in \mathbb{N}$, from which we can show the meromorphic continuation of $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ to the whole complex space $\mathbb{C}^{r(r+1)/2}$. Functional relations similar to (1.4) are expected to hold for any $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$, but the general situation is rather complicated. In the present paper we study the special case $r = 3$ more closely. In Section 3 and Section 4, we determine the singularities of $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$ which are located only on the subset of $\mathbb{C}^6$ defined by several explicit equations, using the method introduced in [14, 15]. In Section 5, we prove certain functional relations between $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$, $\zeta_{\mathfrak{sl}(3)}(\mathbf{s}) = \zeta_{MT,2}(\mathbf{s})$ and $\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s)$, using the method introduced in [21, 24]. From these relations, we prove new and non-trivial relation formulas for the special values of these functions at positive integers. For example, we give an evaluation formula

$$\zeta_{\mathfrak{sl}(4)}(1,1,1,2,1,2) = -\frac{29}{175}\zeta(2)^4 + \zeta(3)\zeta(5) - \frac{1}{2}\zeta(2,6), \tag{1.6}$$

where $\zeta(p,q) = \sum_{1\leqslant m<n} m^{-p}n^{-q}$ is what is called the double zeta value. This formula is a non-trivial analogue of Witten's result which was explicitly calculated by Gunnells and Sczech [7] as follows:

$$\zeta_{\mathfrak{sl}(4)}(2,2,2,2,2,2) = \frac{23}{2554051500}\pi^{12} = \frac{23}{2764}\zeta(12). \tag{1.7}$$

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2. Meromorphic continuation of $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$

First we prove an explicit form of the Witten zeta-function associated with $\mathfrak{sl}(r+1)$ for $r \in \mathbb{N}$ which were defined by Zagier [26]. We quote some notation and results from [10, 18] as follows. For $k \in \mathbb{N}$ with $1 \leqslant k \leqslant r+1$, let e_k be the canonical unit vector which has components 0 except for its kth component equal to 1. Namely $\{e_k\}_{1\leqslant k\leqslant r+1}$ forms the canonical basis of $\mathbb{R}^{r+1}$. Then we see that the set of positive roots for $\mathfrak{sl}(r+1)$ are $\{e_j - e_k \,|\, j < k\}$ (see [10] Chap. IV Example 1). Let δ be half the sum of the positive roots, namely

$$\delta = \frac{1}{2} \sum_{1\leqslant j<k\leqslant r+1} (e_j - e_k) = \frac{1}{2}\sum_{\nu=1}^{r+1}(r-2\nu+2)e_\nu.$$

It follows from the Cartan-Weyl theory of highest weights (see [10] Chapter 4 §7, [18] §3.6) that any highest weight λ for $\mathfrak{sl}(r+1)$ can be parameterized by $\lambda = \sum_{\nu=1}^{r+1} n_\nu e_\nu$ with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_{r+1}$, $n_1 + n_2 + \cdots + n_{r+1} = 0$ and $n_j - n_k \in \mathbb{Z}$ (for any j,k). Let ρ_λ be the finite dimensional irreducible representation corresponding to λ. From the Weyl dimension theorem ([10] Theorem 4.48), we have

$$\begin{aligned}
\dim \rho_\lambda &= \prod_{1\leqslant j<k\leqslant r+1} \frac{(\lambda+\delta, e_j - e_k)}{(\delta, e_j - e_k)} \\
&= \prod_{1\leqslant j<k\leqslant r+1} \frac{\sum_{\nu=1}^{r+1}((n_\nu + (r-2\nu+2)/2)e_\nu, e_j - e_k)}{\sum_{\nu=1}^{r+1}(((r-2\nu+2)/2)e_\nu, e_j - e_k)} \\
&= \prod_{1\leqslant j<k\leqslant r+1} \frac{(n_j - n_k) - (j-k)}{k-j}.
\end{aligned}$$

Hence by (1.1), we obtain

$$\zeta_{\mathfrak{sl}(r+1)}(s) = \sum \left(\prod_{1\leqslant j<k\leqslant r+1} \frac{(n_j - n_k)-(j-k)}{k-j} \right)^{-s}, \tag{2.1}$$

where the sum is taken over all $(n_1, n_2, \ldots, n_{r+1})$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{r+1}$, $n_1 + n_2 + \cdots + n_{r+1} = 0$ and $n_j - n_k \in \mathbb{Z}$ (for any j,k). As mentioned in Section 1, Zagier [26] determined $\zeta_{\mathfrak{sl}(2)}(s) (= \zeta(s))$ and $\zeta_{\mathfrak{sl}(3)}(s)$ explicitly. Furthermore we can check the following result inductively.

Proposition 2.1. — *For* $r \in \mathbb{N}$,

$$\zeta_{\mathfrak{sl}(r+1)}(s) = C_{r+1}^{s} \sum_{m_1,m_2,\dots,m_r=1}^{\infty} \prod_{j=1}^{r} \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s}, \tag{2.2}$$

where

$$C_{r+1} = \prod_{1 \leqslant j < k \leqslant r+1} (k-j).$$

Proof. — In the case $r=1$, we have $1 \leqslant j < k \leqslant 2$, so $j=1, k=2$, $n_j - n_k = n_1 - n_2 \in \mathbb{N}_0$. Hence $C_2 = 1$ and $\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s)$, therefore the assertion holds. Hence we assume that the assertion in the case of r holds and aim to prove the case of $r+1$.

For j,k with $1 \leqslant j < k \leqslant r+1$, we put $m_\nu = n_\nu - n_{\nu+1} + 1$ for $1 \leqslant \nu \leqslant r$. Then we need to prove

$$\prod_{1 \leqslant j < k \leqslant r+1} \{(n_j - n_k) - (j-k)\} = \prod_{p=1}^{r} \prod_{q=1}^{r-p+1} \sum_{\nu=q}^{p+q-1} m_\nu. \tag{2.3}$$

In order to prove this, we divide the left-hand side of (2.3) into two parts as

$$\prod_{1 \leqslant j < k \leqslant r+1} = \prod_{1 \leqslant j < k \leqslant r} \times \prod_{1 \leqslant j < k = r+1}. \tag{2.4}$$

It follows from the induction assumption that the first factor on the left-hand side of (2.4) equals to

$$\prod_{p=1}^{r-1} \prod_{q=1}^{r-p} \sum_{\nu=q}^{p+q-1} m_\nu. \tag{2.5}$$

Furthermore we can see that the second factor on the left-hand side of (2.4) equals to

$$\prod_{1 \leqslant j < r+1} \{(n_j - n_{r+1}) - (j-r-1)\} = \prod_{1 \leqslant j \leqslant r} \sum_{\mu=j}^{r} m_\mu, \tag{2.6}$$

because $\sum_{\mu=j}^{r} (n_\mu - n_{\mu+1} + 1) = n_j - n_{r+1} + r - j + 1$. On the other hand, we divide the right-hand side of (2.3) into two parts as

$$\left(\prod_{p=1}^{r-1} \prod_{q=1}^{r-p+1} \sum_{\nu=q}^{p+q-1} m_\nu \right) \times \sum_{\nu=1}^{r} m_\nu. \tag{2.7}$$

The quantity (2.7) divided by (2.5) is equal to

$$\prod_{p=1}^{r-1} \left(\sum_{\nu=r-p+1}^{r} m_\nu \right) \times \sum_{\nu=1}^{r} m_\nu = \prod_{p=1}^{r} \left(\sum_{\nu=r-p+1}^{r} m_\nu \right),$$

which coincides with the right-hand side of (2.6). Thus we obtain (2.3), which implies the assertion in the case of $r+1$. $\square$

>From the form (2.2), we naturally define Witten multiple zeta-functions of several variables associated with $\mathfrak{sl}(n)$ by (1.5). In the rest of this section, we prove the fact that $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ can be continued meromorphically to the whole complex space $\mathbb{C}^{r(r+1)/2}$ for $r \in \mathbb{N}$. Note that this fact can be obtained from Theorem 3 of [14] (see also Essouabri's work [5, 6]). However, in this section, we give a more explicit result as follows (see Theorem 2.2).

We recall the Mellin-Barnes formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{2.8}$$

where $\Re s > 0$, $|\arg\lambda| < \pi$, $\lambda \neq 0$, $c \in \mathbb{R}$ with $-\Re s < c < 0$, $i=\sqrt{-1}$ and the path (c) of integration is the vertical line $\Re z = c$. From (1.5) in the case of $r+1$, we have

$$\zeta_{\mathfrak{sl}(r+2)}(\mathbf{s}) = \sum_{m_1,\ldots,m_{r+1}=1}^{\infty} \prod_{j=1}^{r+1}\prod_{k=1}^{r-j+2}\left(\sum_{\nu=k}^{j+k-1} m_\nu\right)^{-s_{jk}}, \tag{2.9}$$

where

$$\mathbf{s} = (s_{jk})_{1\leqslant j\leqslant r+1;\, 1\leqslant k\leqslant r-j+2} \in \mathbb{C}^{(r+1)(r+2)/2} \quad (\Re s_{jk} > 1).$$

We consider the terms including m_{r+1} in (2.9). This means $j+k-1 = r+1$, namely $k = r-j+2$ for $1\leqslant j\leqslant r+1$. Put $\underline{s}_j := s_{j,r-j+2}$. Suppose $j\geqslant 2$ and $k = r-j+2$. Applying (2.8) with $\lambda = m_{r+1}/\left(\sum_{\nu=r-j+2}^{r} m_\nu\right)$, we have

$$\begin{aligned}\left(\sum_{\nu=k}^{j+k-1} m_\nu\right)^{-s_{jk}} &= \left(\sum_{\nu=r-j+2}^{r+1} m_\nu\right)^{-\underline{s}_j} \\ &= \left(\sum_{\nu=r-j+2}^{r} m_\nu\right)^{-\underline{s}_j} \times \left(1+\frac{m_{r+1}}{\sum_{\nu=r-j+2}^{r} m_\nu}\right)^{-\underline{s}_j} \\ &= \frac{1}{2\pi i}\int_{(c_j)} \frac{\Gamma(\underline{s}_j+z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)}\left(\sum_{\nu=r-j+2}^{r} m_\nu\right)^{-\underline{s}_j-z_j} m_{r+1}^{z_j} dz_j,\end{aligned} \tag{2.10}$$

where $-\Re \underline{s}_j < c_j < 0$. We divide the right-hand side of (2.9) into two parts corresponding to two cases when $j = 1$ and $j > 1$, that is

(2.11)

$$\begin{aligned}\zeta_{\mathfrak{sl}(r+2)}(\mathbf{s}) = &\sum_{m_1,\ldots,m_{r+1}=1}^{\infty} \prod_{k=1}^{r+1} m_k^{-s_{1k}} \\ &\times \prod_{j=2}^{r+1} \left\{ \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}} \times \left(\sum_{\nu=r-j+2}^{r+1} m_\nu \right)^{-\underline{s}_j} \right\} \\ = &\sum_{m_1,\ldots,m_{r+1}=1}^{\infty} \prod_{k=1}^{r+1} m_k^{-s_{1k}} \\ &\times \prod_{j=2}^{r+1} \left\{ \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}} \frac{1}{2\pi i} \int_{(c_j)} \frac{\Gamma(\underline{s}_j + z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)} \right. \\ &\left. \times \left(\sum_{\nu=r-j+2}^{r} m_\nu \right)^{-\underline{s}_j - z_j} m_{r+1}^{z_j} dz_j \right\}\end{aligned}$$

by using (2.10). Putting

$$A_j = \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}} \qquad (1 \leqslant j \leqslant r+1)$$

(the empty product A_{r+1} is to be regarded as 1) and

$$B_j = \left(\sum_{\nu=r-j+2}^{r} m_\nu \right)^{-\underline{s}_j - z_j} \qquad (2 \leqslant j \leqslant r+1),$$

we have

(2.12)

$$\begin{aligned}\zeta_{\mathfrak{sl}(r+2)}(\mathbf{s}) = &\sum_{m_1,\ldots,m_{r+1}=1}^{\infty} A_1 m_{r+1}^{-s_{1,r+1}} \frac{1}{(2\pi i)^r} \int_{(c_2)} \\ &\cdots \int_{(c_r)} \left(\prod_{j=2}^{r+1} A_j B_j \frac{\Gamma(\underline{s}_j + z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)} m_{r+1}^{z_j} \right) dz_{r+1} \cdots dz_2\end{aligned}$$

$$= \frac{1}{(2\pi i)^r} \int_{(c_2)} \cdots \int_{(c_{r+1})} \left(\prod_{j=2}^{r+1} \frac{\Gamma(\underline{s}_j + z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)} \right)$$
$$\times \sum_{m_1,\ldots,m_{r+1}=1}^{\infty} \left(\prod_{j=1}^{r} A_j B_{j+1} \right) m_{r+1}^{-s_{1,r+1}-z_2-\cdots-z_{r+1}} dz_{r+1} \cdots dz_2.$$

Since

$$A_j B_{j+1} = \prod_{k=1}^{r-j} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}} \times \left(\sum_{\nu=r-j+1}^{r} m_\nu \right)^{-s_{j,r+1}-\underline{s}_{j+1}-z_{j+1}}$$

for $1 \leqslant j \leqslant r$, we obtain

$$\zeta_{\mathfrak{sl}(r+2)}(\mathbf{s}) = \frac{1}{(2\pi i)^r} \int_{(c_2)} \cdots \int_{(c_{r+1})} \prod_{j=2}^{r+1} \frac{\Gamma(\underline{s}_j + z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)}$$
$$\times \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \left\{ \prod_{k=1}^{r-j} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}} \right.$$
$$\left. \times \left(\sum_{\nu=r-j+1}^{r} m_\nu \right)^{-s_{j,r-j+1}-\underline{s}_{j+1}-z_{j+1}} \right\}$$
$$\times \sum_{m_{r+1}=1}^{\infty} m_{r+1}^{-s_{1,r+1}+z_2+\cdots+z_{r+1}} dz_{r+1} dz_r \cdots dz_2. \tag{2.13}$$

Put $\mathbf{z} = (z_2, \ldots, z_{r+1})$ and $\mathbf{s}^* = \mathbf{s}^*(\mathbf{z}) = (s^*_{jk}(\mathbf{z}))_{1 \leqslant j \leqslant r,\ 1 \leqslant k \leqslant r-j+1}$ with

$$s^*_{jk} = \begin{cases} s_{j,r-j+1} + \underline{s}_{j+1} + z_{j+1} & (\text{if } k = r-j+1) \\ s_{jk} & (\text{otherwise}). \end{cases} \tag{2.14}$$

Then, combining (1.5) and (2.13), we obtain the following recursive relations.

Theorem 2.2. — *Let $r \in \mathbb{N}$. Suppose $\mathbf{s} = (s_{jk})_{1 \leqslant j \leqslant r+1;\, 1 \leqslant k \leqslant r-j+2} \in \mathbb{C}^{(r+1)(r+2)/2}$ with $\Re s_{jk} > 1$ for each j, k, and $c_2, \ldots, c_{r+1} \in \mathbb{R}$ with $-\Re \underline{s}_j < c_j < 0$ for each j, where $\underline{s}_j = s_{j,r-j+2}$. Then*

$$\zeta_{\mathfrak{sl}(r+2)}(\mathbf{s}) = \frac{1}{(2\pi i)^r} \int_{(c_2)} \cdots \int_{(c_{r+1})} \prod_{j=2}^{r+1} \frac{\Gamma(\underline{s}_j + z_j)\Gamma(-z_j)}{\Gamma(\underline{s}_j)}$$
$$\times \zeta_{\mathfrak{sl}(r+1)}(\mathbf{s}^*(\mathbf{z}))\zeta(s_{1,r+1} - (z_2 + \cdots + z_{r+1})) dz_{r+1} dz_r \cdots dz_2, \tag{2.15}$$

where $\mathbf{z} = (z_2, \ldots, z_{r+1})$ and $\mathbf{s}^*(\mathbf{z})$ is defined by (2.14).

Remark 2.3. — From (2.15), we can immediately see that $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$ can be continued meromorphically to the whole complex space $\mathbb{C}^{r(r+1)/2}$, by using the induction on r (see [11, 12, 14, 15]). Note that for the case $r=1$, namely the case of $\zeta(s)$, this assertion is well-known. Furthermore we can inductively determine the possible singularities of $\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s})$. Indeed the first author has already determined the possible singularities of $\zeta_{\mathfrak{sl}(3)}(\mathbf{s})$ in [11]. In the next section, we determine the possible singularities of $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$.

3. Possible singularities of $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$

We recall the properties of $\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)=\zeta_{MT,2}(s_1,s_2,s_3)$.

LEMMA 3.1 ([11] Theorem 1). — *$\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)$ can be continued meromorphically to the whole complex space $\mathbb{C}^3$, and all of its singularities are located on the subsets of $\mathbb{C}^3$ defined by one of the equations $s_1+s_3=1-l$, $s_2+s_3=1-l$ ($l\in\mathbb{N}_0$) and $s_1+s_2+s_3=2$.*

Remark 3.2. — The key to the proof of this lemma is the following relation ([11] Equation (5.3)):

(3.1)

$$\begin{aligned}\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3)&=\frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1)\\&+\sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(s_2-k)\\&+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_2+s_3+z)\zeta(s_2-z)dz,\end{aligned}$$

where ε is a small positive number and $M\in\mathbb{N}$ with $M>\Re s_2-1+\varepsilon$. Each singularity in Lemma 3.1 is derived from only one term on the right-hand side of (3.1), hence is not cancelled. Therefore all of them are true singularities (for the details, see [11]).

We aim to prove a kind of generalization of (3.1) corresponding to $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$. >From (2.15) in the case $r=2$, we have

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(\mathbf{s})&=\zeta_{\mathfrak{sl}(4)}(s_{11},s_{12},s_{13},s_{21},s_{22},s_{31})\\&=\frac{1}{(2\pi i)^2}\int_{(c_2)}\int_{(c_3)}\frac{\Gamma(s_{22}+z_2)\Gamma(-z_2)}{\Gamma(s_{22})}\cdot\frac{\Gamma(s_{31}+z_3)\Gamma(-z_3)}{\Gamma(s_{31})}\\&\qquad\times\zeta_{\mathfrak{sl}(3)}(\mathbf{s}^*(\mathbf{z}))\zeta(s_{13}-(z_2+z_3))dz_3dz_2.\end{aligned}$$

For simplicity, we let $\mathbf{s} = (s_1, s_2, \ldots, s_6)$ and replace z_2, z_3, c_2, c_3 with z_5, z_6, c_5, c_6, respectively. Then we have

$$\begin{aligned} (3.2)\quad & \zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) \\ &= \frac{1}{(2\pi i)^2} \int_{(c_5)} \int_{(c_6)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} \cdot \frac{\Gamma(s_6+z_6)\Gamma(-z_6)}{\Gamma(s_6)} \\ &\quad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+z_6)\zeta(s_3-z_5-z_6)dz_6dz_5, \end{aligned}$$

where $\Re s_j > 1$ $(1 \leqslant j \leqslant 6)$, $-\Re s_5 < c_5 < 0$ and $-\Re s_6 < c_6 < 0$. We put

$$\begin{aligned} (3.3)\quad I(s_1, \ldots, s_6; z_5) &= \frac{1}{2\pi i} \int_{(c_6)} \frac{\Gamma(s_6+z_6)\Gamma(-z_6)}{\Gamma(s_6)} \\ &\quad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+z_6)\zeta(s_3-z_5-z_6)dz_6. \end{aligned}$$

Then

$$\begin{aligned} (3.4)\quad & \zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) \\ &= \frac{1}{2\pi i} \int_{(c_5)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} I(s_1, \ldots, s_6; z_5)dz_5. \end{aligned}$$

First we examine $I(s_1, \ldots, s_6; z_5)$. Let $\varepsilon > 0$ be a small number and $M_6 \in \mathbb{N}$ which satisfies

$$(3.5)\qquad \Re s_3 - c_5 - 1 + \varepsilon < M_6.$$

Then we see that $\zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+z_6)$ is convergent absolutely, so is $O(1)$ $(|\Im z_6| \to \infty)$ when $c_6 \leqslant \Re z_6 \leqslant M_6 - \varepsilon$. On the other hand, it is well-known that $|\zeta(\sigma + i\tau)|$ is of at most polynomial order with respect to $|\tau| \gg 1$ (see [9] Theorem 1.9). Furthermore, from the well-known Stirling formula for $\Gamma(s)$, we have

$$|\Gamma(\sigma + i\tau)| = \sqrt{2\pi} e^{-\frac{\pi}{2}|\tau|}(|\tau|+1)^{\sigma-\frac{1}{2}} \left(1 + O\left(\frac{1}{|\tau|+1}\right)\right) \quad (|\tau| \to \infty).$$

Therefore the integrand on the right-hand side of (3.3) tends to zero as $|\Im z_6| \to \infty$ when $c_6 \leqslant \Re z_6 \leqslant M_6 - \varepsilon$. Hence we can shift the path to $\Re z_6 = M_6 - \varepsilon$. We have to check which poles are relevant to this shifting. Poles from $\Gamma(s_6+z_6)$ are $z_6 = -s_6 - l$ $(l \in \mathbb{N}_0)$, but these are irrelevant. Poles from $\Gamma(-z_6)$ are $z_6 = l$ for $l \in \mathbb{N}_0$ with $0 \leqslant l \leqslant M_6 - 1$, and their residues are

$$(3.6)\quad -\binom{-s_6}{l} \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+l)\zeta(s_3-z_5-l) \quad (l \in \mathbb{N}_0),$$

because it follows from $\mathrm{Res}_{z=l}\Gamma(-z) = (-1)^{l+1}/l!$ that

$$\frac{\Gamma(s_6+l)}{\Gamma(s_6)} \cdot \mathrm{Res}_{z_6=l}\Gamma(-z_6) = -\binom{-s_6}{l}.$$

We further check that a pole from $\zeta(s_3 - z_5 - z_6)$ is $z_6 = s_3 - z_5 - 1$ whose residue should be counted because of (3.5). Its residue is

$$(3.7)\quad -\frac{\Gamma(s_3+s_6-z_5-1)\Gamma(-s_3+z_5+1)}{\Gamma(s_6)} \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_3+s_4+s_6-z_5-1),$$

because $\mathrm{Res}_{z_6=s_3-z_5-1}\zeta(s_3 - z_5 - z_6) = -1$.

Now we shift the path on the right-hand side of (3.3) to $\Re z_6 = M_6 - \varepsilon$. From (3.6) and (3.7), we have

(3.8)

$$\begin{aligned} I(s_1,\ldots,s_6;z_5) &= \sum_{k=0}^{M_6-1}\binom{-s_6}{k}\zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+k)\zeta(s_3-z_5-k) \\ &+ \frac{\Gamma(s_3+s_6-z_5-1)\Gamma(-s_3+z_5+1)}{\Gamma(s_6)} \\ &\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_3+s_4+s_6-z_5-1) \\ &+ \frac{1}{2\pi i}\int_{(M_6-\varepsilon)} \frac{\Gamma(s_6+z_6)\Gamma(-z_6)}{\Gamma(s_6)} \\ &\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+z_6)\zeta(s_3-z_5-z_6)dz_6. \end{aligned}$$

For simplicity, we denote the first, the second and the third term on the right-hand side of (3.8) by $\sum_{k=0}^{M_6-1} I_{1k}$, I_2 and I_3, respectively. It follows from Lemma 3.1 and the properties of $\zeta(s)$ and $\Gamma(s)$ that I_{1k} $(0 \leqslant k \leqslant M_6-1)$ and I_2 can be continued meromorphically to the whole complex space $\mathbb{C}^7$, and the singularities of I_{1k} are located on the subsets of $\mathbb{C}^7$ defined by one of the equations

$$(3.9)\qquad s_1+s_4+s_6 = 1-l \quad (l \in \mathbb{N}_0);$$

$$(3.10)\qquad z_5 = -s_2-s_4-s_5-s_6+1-l \quad (l \in \mathbb{N}_0);$$

$$(3.11)\qquad z_5 = -s_1-s_2-s_4-s_5-s_6+2-l \quad (l \in \mathbb{N}_0);$$

$$(3.12)\qquad z_5 = s_3-k-1,$$

while the singularities of I_2 are on the subsets defined by one of the equations

$$z_5 = s_3 + s_6 - 1 + l \quad (l \in \mathbb{N}_0); \tag{3.13}$$

$$z_5 = s_3 - 1 - l \quad (l \in \mathbb{N}_0); \tag{3.14}$$

$$z_5 = s_1 + s_3 + s_4 + s_6 - 2 + l \quad (l \in \mathbb{N}_0); \tag{3.15}$$

$$s_2 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0); \tag{3.16}$$

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3. \tag{3.17}$$

However we can find that the singularities of (3.12) and (3.14) are cancelled to each other. In fact, the residue of I_{1k} at $z_5 = s_3 - k - 1$ equals to

$$-\binom{-s_6}{k}\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_3 + s_5 - k - 1, s_4 + s_6 + k),$$

while, as well as (3.6), we can check that the residue of I_2 at $z_5 = s_3 - k - 1$ equals to

$$\begin{aligned}\frac{\Gamma(s_6 + k)}{\Gamma(s_6)} \cdot \frac{(-1)^k}{k!}\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_3 + s_5 - k - 1, s_4 + s_6 + k)& \\ = \binom{-s_6}{k}\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_3 + s_5 - k - 1, s_4 + s_6 + k)&.\end{aligned}$$

Hence we put

$$\widetilde{I}_{1k} = I_{1k} + \binom{-s_6}{k}\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_3 + s_5 - k - 1, s_4 + s_6 + k)\frac{1}{z_5 - s_3 + k + 1} \tag{3.18}$$

for $0 \leqslant k \leqslant M_6 - 1$, and

$$\widetilde{I}_2 = I_2 - \sum_{l=0}^{M_6-1}\binom{-s_6}{l}\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_3 + s_5 - l - 1, s_4 + s_6 + l)\frac{1}{z_5 - s_3 + l + 1}, \tag{3.19}$$

and rewrite (3.8) as

$$I(s_1, \ldots, s_6; z_5) = \sum_{k=0}^{M_6-1}\widetilde{I}_{1k} + \widetilde{I}_2 + I_3. \tag{3.20}$$

>From the above consideration, $\widetilde{I}_{1k}$ does not have the singularities of the type (3.12). >From Lemma 3.1, the singularities of the second term on the right-hand side of (3.18) are

$$\begin{cases} s_1 + s_4 + s_6 + k = 1 - l \quad (l \in \mathbb{N}_0); \\ s_2 + s_3 + s_4 + s_5 + s_6 - 1 = 1 - l \quad (l \in \mathbb{N}_0); \\ s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3, \end{cases}$$

which coincide with (3.9), (3.16) and (3.17). Similarly we can check that $\widetilde{I}_2$ does not have the singularities of the type (3.14) with $l \leqslant M_6 - 1$ and that the singularities of the second term on the right-hand side of (3.19) are (3.9), (3.16) and (3.17). Thus we can summarize these facts as follows:

(3.21)
$$\begin{cases} \text{Singularities of } \widetilde{I}_{1k} \cdots \underline{(3.9), (3.16), (3.17)}, (3.10), (3.11); \\ \text{Singularities of } \widetilde{I}_2 \cdots \underline{(3.9), (3.16), (3.17)}, (3.13), (3.15), (3.14)_{\{l \geqslant M_6\}}, \end{cases}$$

where the underline parts (3.9), (3.16), (3.17) do not contain z_5.

Next we consider

$$I_3 = \frac{1}{2\pi i} \int_{(M_6-\varepsilon)} \frac{\Gamma(s_6+z_6)\Gamma(-z_6)}{\Gamma(s_6)} \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2+s_5+z_5, s_4+s_6+z_6)\zeta(s_3-z_5-z_6)dz_6.$$

At first we have assumed that $\Re s_j > 1$ $(1 \leqslant j \leqslant 6)$ and $\Re z_5 = c_5$, but the integral I_3 can be continued to a wider region. In fact, as well as I_{1k} and I_2, it follows from Lemma 3.1 and the properties of $\zeta(s)$ and $\Gamma(s)$ that the singularities of the integrand of I_3 are

$$\begin{cases} z_6 = -s_6 - l \quad (l \in \mathbb{N}_0); \\ z_6 = l \quad (l \in \mathbb{N}_0); \\ z_6 = -s_1 - s_4 - s_6 + 1 - l \quad (l \in \mathbb{N}_0); \\ z_6 = -s_2 - s_4 - s_5 - s_6 - z_5 + 1 - l \quad (l \in \mathbb{N}_0); \\ z_6 = -s_1 - s_2 - s_4 - s_5 - s_6 - z_5 + 2; \\ z_6 = s_3 - z_5 - 1. \end{cases} \tag{3.22}$$

Therefore, when $(s_1, \ldots, s_6, z_5)$ varies with satisfying the conditions

$$\begin{cases} -\Re s_6 - l < M_6 - \varepsilon \quad (l \in \mathbb{N}_0); \\ -\Re(s_1+s_4+s_6) + 1 - l < M_6 - \varepsilon \quad (l \in \mathbb{N}_0); \\ -\Re(s_2+s_4+s_5+s_6+z_5) + 1 - l < M_6 - \varepsilon \quad (l \in \mathbb{N}_0); \\ -\Re(s_1+s_2+s_4+s_5+s_6+z_5) + 2 < M_6 - \varepsilon; \\ \Re(s_3 - z_5) - 1 < M_6 - \varepsilon, \end{cases} \tag{3.23}$$

the singularities in (3.22) do not intersect with the path of integration $\Re z_6 = M_6 - \varepsilon$ at all. Note that the last inequality in (3.23) coincides with (3.5) when $\Re z_5 = c_5$. Also, if we choose a small ε then $z_6 = l$ $(l \in \mathbb{N}_0)$ does not intersect with $\Re z_6 = M_6 - \varepsilon$. Moreover, from the proof of Theorem 3 in [14] Section 2 for $\zeta_{MT,2} = \zeta_{\mathfrak{sl}(3)}$, we have obtained the following.

Lemma 3.3. — *$\zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3)$ is of polynomial order with respect to $|\Im s_j| \gg 1$ $(1 \leqslant j \leqslant 3)$.*

Therefore we can continue the integral I_3 holomorphically to all $(s_1, \ldots, s_6, z_5) \in \mathbb{C}^7$ with satisfy

$$
\begin{cases}
\Re s_6 > -M_6 + \varepsilon; \\
\Re(s_1 + s_4 + s_6) > 1 - M_6 + \varepsilon; \\
\Re(s_2 + s_4 + s_5 + s_6) + \Re z_5 > 1 - M_6 + \varepsilon; \\
\Re(s_1 + s_2 + s_4 + s_5 + s_6) + \Re z_5 > 2 - M_6 + \varepsilon; \\
\Re s_3 - \Re z_5 - 1 + \varepsilon < M_6.
\end{cases}
\tag{3.24}
$$

Hence it follows from (3.20) and the above consideration for each $\widetilde{I}_{1k}$, $\widetilde{I}_2$ and I_3 that $I(s_1, \ldots, s_6; z_5)$ can be continued meromorphically to the region determined by (3.24) with the singularities (3.21). Note that (3.21) includes the singularities of the type (3.14) with $l \geqslant M_6$, namely $z_5 = s_3 - l - 1$ with $l \geqslant M_6$. However this means $\Re z_5 \leqslant \Re s_3 - M_6 - 1$ which is contrary to the last inequality in (3.24). Hence the singularities of the type (3.14) with $l \geqslant M_6$ do not occur in the region determined by (3.24). We can easily check that if $M_6 \to \infty$ then the region determined by (3.24) extends to the whole space $\mathbb{C}^7$. Therefore we obtain the following.

Lemma 3.4. — *With the above notation, $I(s_1, \ldots, s_6; z_5)$ can be continued meromorphically to the whole complex space $\mathbb{C}^7$ with the true singularities determined by (3.9), (3.10), (3.11), (3.13), (3.15), (3.16), (3.17).*

Proof. — We have only to check the truth of all singularities mentioned in the statement of Lemma 3.4. We see that singularities (3.13), (3.15), (3.16) and (3.17) are derived from I_2 only, hence are not cancelled. Therefore they determine true singularities. On the other hand, (3.9), (3.10) and (3.11) are derived from I_{1k}, actually from its $\zeta_{\mathfrak{sl}(3)}$-factor. Indeed, from Lemma 3.1, the singularities of $\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_5 + z_5, s_4 + s_6 + k)$ are determined by

$$s_1 + s_4 + s_6 + k = 1 - m \quad (m \in \mathbb{N}_0), \tag{$\star$}$$

$$s_2 + s_4 + s_5 + s_6 + z_5 + k = 1 - m \quad (m \in \mathbb{N}_0), \tag{$\star\star$}$$

$$s_1 + s_2 + s_4 + s_5 + s_6 + z_5 + k = 2. \tag{$\star\star\star$}$$

Putting $k = l$ in $(\star\star\star)$, we obtain (3.11). For each k, a different hyperplane $(\star\star\star)$ is determined. Hence (3.11) is derived from only one equation $(\star\star\star)$, namely is not cancelled. Therefore (3.11) determines a true singularity.

Next, ($\star$) and ($\star\star$) can be rewritten as

$$(*)\qquad s_1 + s_4 + s_6 = 1 - (k+m) \quad (m \in \mathbb{N}_0),$$

$$(**)\qquad s_2 + s_4 + s_5 + s_6 + z_5 = 1 - (k+m) \quad (m \in \mathbb{N}_0).$$

Hence, for any $l \in \mathbb{N}_0$, $(*)$ and $(**)$ for each pair (k,m) with $k+m=l$ give (3.9) and (3.10), respectively. In other words, these singularities are derived from several I_{1k}. Therefore some cancellation between them might occur. In order to check the truth of these singularities, we use the technique of "change of variables" introduced by Akiyama, Egami and Tanigawa ([1]). Put

$$u_4 = s_4 + s_6, \quad u_j = s_j \quad (j \neq 4).$$

Then it follows from the definition of I_{1k} that

$$I_{1k} = \binom{-u_6}{k} \zeta_{\mathfrak{sl}(3)}(u_1, u_2 + u_5 + z_5, u_4 + k)\zeta(s_3 - z_5 - k).$$

Hence $(*)$ and $(**)$ means

$$u_1 + u_4 = 1 - (k+m) \quad (m \in \mathbb{N}_0),$$
$$u_2 + u_4 + u_5 + z_5 = 1 - (k+m) \quad (m \in \mathbb{N}_0),$$

which are independent of u_6. On the other hand, I_{1k} contains the polynomial in u_6 of degree k, that is

$$\binom{-u_6}{k} = \frac{1}{k!}(-u_6)(-u_6-1)\cdots(-u_6-k+1).$$

Hence, for each l, the cancellation between singularities derived from several I_{1k} with $k+m=l$ cannot occur because the degree of I_{1k} with respect to u_6 is different if k is different. This means that (3.9) and (3.10) determine true singularities. Thus we have the assertion. $\square$

>From the proof of Theorem 3 in [14], we can see that I_3 is of polynomial order with respect to $|\Im s_j| \gg 1$ $(1 \leqslant j \leqslant 6)$ and $|\Im z_5| \gg 1$, so is $I(s_1, \ldots, s_6; z_5)$ from Lemma 3.3.

Now we examine the singularities of $\zeta_{\mathfrak{sl}(4)}(s_1, \ldots, s_6)$ based on the above data. We return to the situation $\Re s_j > 1$ $(1 \leqslant j \leqslant 6)$, $-\Re s_5 < c_5 < 0$. Substituting (3.20) into (3.4), we obtain

(3.25)

$$\zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) = \sum_{k=0}^{M_6-1} \frac{1}{2\pi i}\int_{(c_5)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} \widetilde{I}_{1k}\, dz_5$$
$$+ \frac{1}{2\pi i}\int_{(c_5)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} \left\{\widetilde{I}_2 + I_3\right\} dz_5.$$

For simplicity, we denote the right-hand side of (3.25) by $\sum_{k=0}^{M_6-1} J_{1k} + J_2$.

The meromorphic continuation of $J_{1,k}$ can be shown by the same shifting argument as above, but that argument cannot be applied to J_2. Therefore, for both J_{1k} and J_2, here we make use of a different method. That is the same method as in the proof of Theorem 3 in [14] (see also [15] Theorem 5). The singularities (3.9), (3.16) and (3.17) of $\widetilde{I}_{1k}$ are derived from the singularities of $\zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3)$, and can be cancelled by multiplying some linear factors because of (3.1) (see $\Phi(s_1, \ldots, s_6)$ below). Other singularities of $\widetilde{I}_{1k}$ are (3.10) and (3.11). Furthermore J_{1k} has the singularities

$$z_5 = -s_5 - l \quad (l \in \mathbb{N}_0); \tag{3.26}$$

$$z_5 = l \quad (l \in \mathbb{N}_0), \tag{3.27}$$

which are derived from the gamma factors. Since $-\Re s_5 < c_5 < 0$, (3.10), (3.11) and (3.26) are located on the left-hand side of $\Re z_5 = c_5$ and (3.27) is located on the right-hand side of $\Re z_5 = c_5$. We choose an arbitrary $(s_1^0, \ldots, s_6^0) \in \mathbb{C}^6$ and aim to show that J_{1k} can be continued meromorphically to $(s_1^0, \ldots, s_6^0)$. For this aim, we first remove the singularities of the types (3.9), (3.16) and (3.17) from the integrand in J_{1k}. Let L be a sufficiently large positive integer for which

$$\Re(s_1 + s_4 + s_6) > 1 - L,$$
$$\Re(s_2 + s_3 + s_4 + s_5 + s_6) > 2 - L$$

for any $(s_1, \ldots, s_6)$ with $\Re s_j \geqslant \Re s_j^0$ $(1 \leqslant j \leqslant 6)$. Let

$$\Phi(s_1, \ldots, s_6) = \left\{ \prod_{l=0}^{L-1} (s_1 + s_4 + s_6 - 1 + l)(s_2 + s_3 + s_4 + s_5 + s_6 - 2 + l) \right\} \times (s_1 + s_2 + s_3 + s_4 + s_5 + s_6 - 3),$$

and

$$J_{1k}^* = \frac{1}{2\pi i} \int_{(c_5)} \frac{\Gamma(s_5 + z_5)\Gamma(-z_5)}{\Gamma(s_5)} \Phi(s_1, \ldots, s_6) \widetilde{I}_{1k}\, dz_5, \tag{3.28}$$

namely

$$J_{1k} = \Phi(s_1, \ldots, s_6)^{-1} J_{1k}^*. \tag{3.29}$$

>From the choice of L, J_{1k}^* has no singularities of the types (3.9), (3.16) and (3.17) in the region $\Re s_j \geqslant \Re s_j^0$ $(1 \leqslant j \leqslant 6)$ because of the cancellation as noted above.

Since $\Phi(s_1, \ldots, s_6)^{-1}$ is meromorphic in the whole complex space, we have only to consider the meromorphic continuation of J_{1k}^*. >From the above argument, in the region $\Re s_j \geqslant \Re s_j^0$ $(1 \leqslant j \leqslant 6)$, the singularities of the integrand of (3.28) are (3.10), (3.11), (3.26) and (3.27). Corresponding to these types, we let

$$\begin{cases} z^0_{5,1} = -(s^0_2 + s^0_4 + s^0_5 + s^0_6) + 1; \\ z^0_{5,2} = -(s^0_1 + s^0_2 + s^0_4 + s^0_5 + s^0_6) + 2; \\ z^0_{5,3} = -s^0_5; \\ z^0_{5,4} = 0. \end{cases} \tag{3.30}$$

Let N be a sufficiently large positive integer such that $\Re(s^0_j + N) > 1$ $(1 \leqslant j \leqslant 6)$. Put $s^*_j = s^0_j + N$ $(1 \leqslant j \leqslant 6)$ so that each $\Re s^*_j > 1$. Furthermore, for $\nu \in \{1, 2, 3, 4\}$, we define $z^*_{5,\nu}$ by replacing s^0_j with s^*_j $(1 \leqslant j \leqslant 6)$ in (3.30). Note that $\Im s^*_j = \Im s^0_j$ for each j, hence $\Im z^*_{5,\nu} = \Im z^0_{5,\nu}$ for each ν. We will show that J^*_{1k} can be continued meromorphically from $(s^*_1, \ldots, s^*_6)$ to $(s^0_1, \ldots, s^0_6)$.

Case 1. The case that each of $\Im z^0_{5,1}, \Im z^0_{5,2}, \Im z^0_{5,3}$ does not equal to $\Im z^0_{5,4} (= 0)$. We join $z^*_{5,\nu}$ and $z^0_{5,\nu}$ by the segment S_ν which is parallel to the real axis. Since $z^*_{5,1}, z^*_{5,2}, z^*_{5,3}$ are located on the left-hand side of $\Re z_5 = c_5$ and $z^*_{5,4}$ is located on the right-hand side of $\Re z_5 = c_5$, we can deform the path $\Re z_5 = c_5$ to obtain a new oriented path $\mathcal{C}$ from $c_5 - i\infty$ to $c_5 + i\infty$, such that all segments S_ν $(1 \leqslant \nu \leqslant 3)$ are located on the left-hand side of $\mathcal{C}$, while all of the singularities (3.27) are located on the right-hand side of $\mathcal{C}$ (see Figure 3.1). Hence we have

$$J^*_{1k} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s_5 + z_5)\Gamma(-z_5)}{\Gamma(s_5)} \Phi(s_1, \ldots, s_6) \widetilde{I}_{1k} \, dz_5, \tag{3.31}$$

in a sufficiently small neighbourhood of $(s^*_1, \ldots, s^*_6)$. Next, on the right-hand side of (3.31), we move $(s_1, \ldots, s_6)$ from $(s^*_1, \ldots, s^*_6)$ to $(s^0_1, \ldots, s^0_6)$ with keeping the values of imaginary parts of each s_j. This is possible because Lemma 3.3 implies that $\widetilde{I}_{1k}$ is of polynomial order with respect to $|\Im s_j| \gg 1$ $(1 \leqslant j \leqslant 6)$. During this procedure, the path $\mathcal{C}$ does not cross any poles of integrand of (3.31). Therefore (3.31) gives the analytic continuation of J^*_{1k} with no singularities to a small neighbourhood of $(s^0_1, \ldots, s^0_6)$.

Case 2. The case that one of $\Im z^0_{5,1}, \Im z^0_{5,2}, \Im z^0_{5,3}$ equals to $\Im z^0_{5,4} = 0$. For example, assume $\Im z^*_{5,1} = \Im z^*_{5,4} = 0$. First we suppose $\{z^0_{5,1} - l \,|\, l \in \mathbb{N}_0\} \cap \mathbb{N}_0 = \emptyset$. Then, at the neighborhood of the real axis, we can deform the path $\mathcal{C}$ to obtain a new oriented path $\mathcal{C}'$ from $c_5 - i\infty$ to $c_5 + i\infty$, such that $\{z^0_{5,1} - l \,|\, l \in \mathbb{N}_0\}$ are located on the left-hand side of $\mathcal{C}'$, while all of the singularities (3.27) $(= \mathbb{N}_0)$ are located on the right-hand side of $\mathcal{C}'$ (see Figure 3.2). Then we have

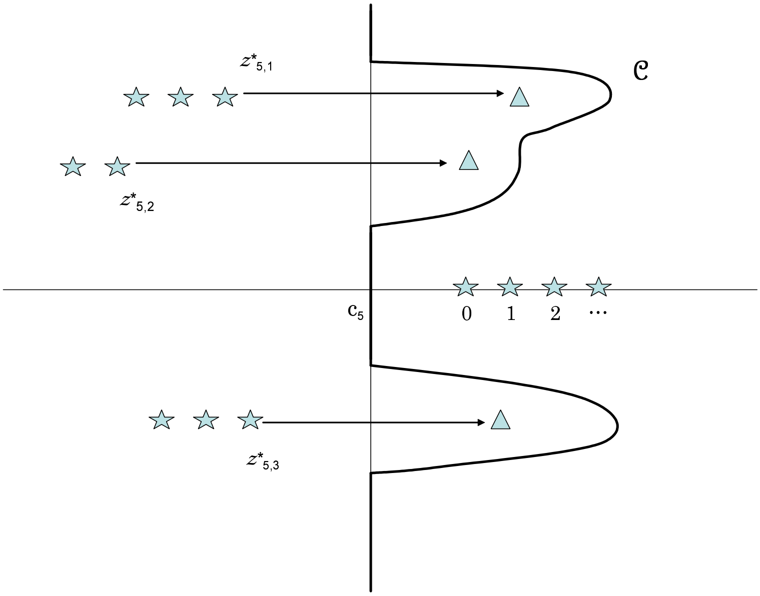

Figure 3.1.

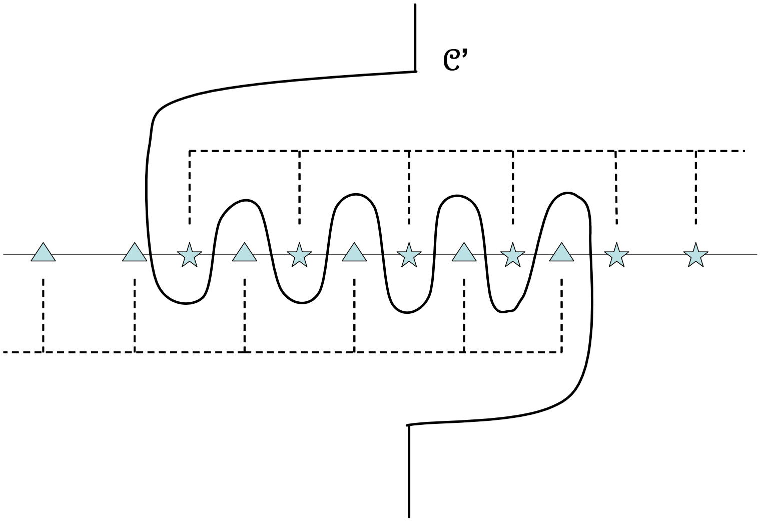

Figure 3.2.

$$J_{1k}^* = \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)\widetilde{I}_{1k}\, dz_5. \tag{3.32}$$

Let η be a small positive number. On the right-hand side of (3.32), we move $(s_1,\ldots,s_6)$ as

$$(s_1^*,\ldots,s_6^*) \to (s_1^*+i\eta,\ldots,s_6^*+i\eta) \to (s_1^0+i\eta,\ldots,s_6^0+i\eta) \to (s_1^0,\ldots,s_6^0).$$

It is possible to define the path $\mathcal{C}'$ such that, during this procedure, $\mathcal{C}'$ does not cross any poles of integrand of (3.32). Therefore (3.32) gives the analytic continuation of J_{1k}^* with no singularities to a small neighbourhood of $(s_1^0,\ldots,s_6^0)$. Secondly we suppose

$$\{z_{5,1}^0 - l \mid l \in \mathbb{N}_0\} \cap \mathbb{N}_0 = \{0,1,\ldots,N\}. \tag{3.33}$$

Then we deform the path $\mathcal{C}$ to obtain a new oriented path $\mathcal{C}''$ from $c_5 - i\infty$ to $c_5 + i\infty$, such that $\{0,1,\ldots,N\}$ are located on the left-hand side of $\mathcal{C}''$, while $\{N+1,N+2,\ldots\}$ are located on the right-hand side of $\mathcal{C}''$ (see Figure 3.3). Then

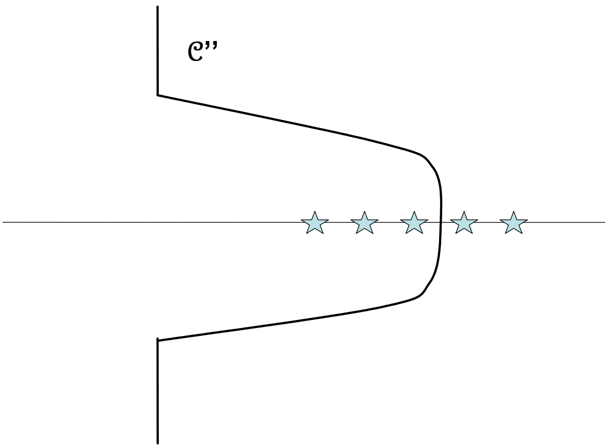

Figure 3.3.

$$J_{1k}^* = -R(s_1,\ldots,s_6) + \frac{1}{2\pi i}\int_{\mathcal{C}''} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)\widetilde{I}_{1k}\, dz_5, \tag{3.34}$$

where $R(s_1,\ldots,s_6)$ consists of the sum of residues at $z_5 = 0,1,\ldots,N$. On the right-hand side of (3.33), we move $(s_1,\ldots,s_6)$ from $(s_1^*,\ldots,s_6^*)$ to $(s_1^0,\ldots,s_6^0)$ with keeping the values of imaginary parts of each s_j. This is also possible. Hence (3.34) gives the meromorphic continuation of J_{1k}^* to a small neighbourhood of $(s_1^0,\dots,s_6^0)$. In this case, the singularities of J_{1k}^* are derived from $R(s_1,\dots,s_6)$. Actually, if (3.10) intersects with (3.27) then we have $-s_2-s_4-s_5-s_6+1-l_1=l_2$ for $l_1,l_2\in\mathbb{N}_0$. Hence the singularities in this case are located on

$$s_2+s_4+s_5+s_6=1-l \quad (l\in\mathbb{N}_0). \tag{3.35}$$

Similarly, if each of (3.11) and (3.26) intersects with (3.27) then we have

$$s_1+s_2+s_4+s_5+s_6=2-l \quad (l\in\mathbb{N}_0); \tag{3.36}$$

$$s_5=-l \quad (l\in\mathbb{N}_0). \tag{3.37}$$

However the type of (3.37) is cancelled by $\Gamma(s_5)$ in the denominator of the integrand of J_{1k}^* (see (3.28)), hence it does not occur. Therefore the remaining possible singularities of J_{1k}^* are (3.35) and (3.36). On the other hand, it is clear that the singularities of $\Phi(s_1,\dots,s_6)^{-1}$ are

$$s_1+s_4+s_6=1-l \quad (l\in\mathbb{N}_0); \tag{3.38}$$

$$s_2+s_3+s_4+s_5+s_6=2-l \quad (l\in\mathbb{N}_0); \tag{3.39}$$

$$s_1+s_2+s_3+s_4+s_5+s_6=3. \tag{3.40}$$

Thus we can conclude that the possible singularities of J_{1k} are (3.35), (3.36), (3.38), (3.39) and (3.40).

Next we consider J_2. As mentioned in the proof of Lemma 3.4, the singularities of $\widetilde{I}_2+I_3$ are (3.9), (3.13), (3.15), (3.16), (3.17). In order to remove the singularities of (3.9), (3.16) and (3.17), we let

$$J_2^*=\frac{1}{2\pi i}\int_{(c_5)}\frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\Phi(s_1,\dots,s_6)(\widetilde{I}_2+I_3)\,dz_5, \tag{3.41}$$

like (3.28), namely

$$J_2=\Phi(s_1,\dots,s_6)^{-1}J_2^*.$$

As well as J_1^*, the singularities of the integrand of J_2^* are (3.13), (3.15), (3.26) and (3.27). When each $s_j=s_j^*$ $(1\leqslant j\leqslant 6)$, we see that (3.13), (3.15), (3.27) are located on the right-hand side of $\Re z_5=c_5$, and (3.26) is located on the left-hand side of $\Re z_5=c_5$. By the same method as that about J_1^* mentioned above, we can modify $\Re z_5=c_5$ as $\mathcal{C},\mathcal{C}',\mathcal{C}''$. The singularities occur only in the following three cases. The first case: (3.13) intersects with (3.26). This means

$$s_3+s_6-1+l_1=-s_5-l_2 \quad (l_1,l_2\in\mathbb{N}_0),$$

namely

$$s_3+s_5+s_6=1-l \quad (l\in\mathbb{N}_0). \tag{3.42}$$

The second case: (3.15) intersects with (3.26). This means

$$s_1 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0). \tag{3.43}$$

The third case: (3.27) intersects with (3.26). This means (3.37) which are not singularities because of the cancellation by the gamma factors as mentioned above. Thus we find that the possible singularities of J_2 are (3.38), (3.39), (3.40), (3.42) and (3.43). By (3.25), we obtain the following.

Theorem 3.5. — *$\zeta_{\mathfrak{sl}(4)}(s_1, \ldots, s_6)$ can be continued meromorphically to the whole complex space $\mathbb{C}^6$, and all of its singularities are located on the subsets of $\mathbb{C}^6$ defined by one of the equations:*

$$s_1 + s_4 + s_6 = 1 - l \quad (l \in \mathbb{N}_0); \tag{3.38}$$

$$s_3 + s_5 + s_6 = 1 - l \quad (l \in \mathbb{N}_0); \tag{3.42}$$

$$s_2 + s_4 + s_5 + s_6 = 1 - l \quad (l \in \mathbb{N}_0); \tag{3.35}$$

$$s_1 + s_2 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0); \tag{3.36}$$

$$s_1 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0); \tag{3.43}$$

$$s_2 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0); \tag{3.39}$$

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3. \tag{3.40}$$

Remark 3.6. — It is important to divide $I(s_1, \ldots, s_6)$ into $\sum_k \widetilde{I}_{1k}$ and $\widetilde{I}_2 + I_3$. In fact, if we do not divide $I(s_1, \ldots, s_6)$ into two parts and apply our method in [14] to it directly, then the fake singularities will occur. The reason why the fake singularities occur is that, if we do not divide, then our method should enumerate the case when the set of singularities of $\sum_k \widetilde{I}_{1k}$ located on the left-hand side of $\Re z_5 = c_5$ intersects with the set of singularities of $\widetilde{I}_2 + I_3$ located on the right-hand side of $\Re z_5 = c_5$. For example, if (3.10) of $\sum_k \widetilde{I}_{1k}$ intersects with (3.15) of $\widetilde{I}_2 + I_3$, then we obtain

$$s_1 + s_2 + s_3 + 2s_4 + s_5 + 2s_6 = 3 - l \quad (l \in \mathbb{N}_0),$$

which is not in the list of Theorem 3.5. Actually this coincides with the sum of (3.38) and (3.39), hence this case is not "fake". However the fake singularities indeed occur as follows. If (3.10) of $\sum_k \widetilde{I}_{1k}$ intersects with (3.13) of $\widetilde{I}_2 + I_3$, then we obtain

$$s_2 + s_3 + s_4 + s_5 + 2s_6 = 2 - l \quad (l \in \mathbb{N}_0). \tag{3.44}$$

This equation cannot be obtained from the combination of equations in Theorem 3.5. Hence (3.44) is likely to be misunderstood as it may be a new one. But there are no singularities of the type (3.44) as checked in the proof of Theorem 3.5. In order to exclude these fake singularities, we need to divide $I(s_1,\ldots,s_6)$ into $\sum_k \widetilde{I}_{1k}$ and $\widetilde{I}_2+I_3$.

This consideration suggests that it is important to check the truth of all singularities in Theorem 3.5. Hence we aim to check this point in the next section.

4. Further consideration of the singularities of $\zeta_{\mathfrak{sl}(4)}(\mathbf{s})$

In this section, we consider the singularities determined by seven equations mentioned in Theorem 3.5 more closely.

We can classify the singularities into the following two types:

(Type I) (3.38), (3.39), (3.40);

(Type II) (3.35), (3.36), (3.42), (3.43).

We see that all of (Type I) are derived from $I(s_1,\ldots,s_6;z_5)$ and all of (Type II) are otherwise.

First we consider (Type II). If we suppose (3.33), namely

$$\{z_{5,1}^0-l\,|\,l\in\mathbb{N}_0\}\cap\mathbb{N}_0=\{0,1,\ldots,N\},$$

then, by the deformation of Figure 3.3, the singularities determined by (3.35) appear. Indeed, let $z_{5,1}^0-l=n$ for $l,n\in\mathbb{N}_0$. Then, by (3.30), we have

$$s_2^0+s_4^0+s_5^0+s_6^0=1-l-n,$$

which means (3.35). In a small neighbourhood of $(s_1^0,\ldots,s_6^0)$, (3.34) holds. Hence we need to examine $R(s_1,\ldots,s_6)$. Since $R(s_1,\ldots,s_6)$ consists of the sum of residues at $z_5=0,1,\ldots,N$, it follows from (3.8) and (3.18) that

(4.1)

$$\begin{aligned}
&R(s_1,\ldots,s_6)\\
&\quad=\sum_{n=0}^{N}\mathrm{Res}_{z_5=n}\left\{\frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)\widetilde{I}_{1k}\right\}\\
&\quad=\sum_{n=0}^{N}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(s_5+n)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)\\
&\qquad\times\left\{\binom{-s_6}{k}\eta_{\mathfrak{sl}(3)}(s_1,s_2+s_5+n,s_4+s_6+k)\zeta(s_3-n-k)\right.\\
&\qquad\left.+\binom{-s_6}{k}\zeta_{\mathfrak{sl}(3)}(s_1,s_2+s_3+s_5-k-1,s_4+s_6+k)\frac{1}{n-s_3+k+1}\right\}.
\end{aligned}$$

We denote the two terms in the curly parentheses on the right-hand side of (4.1) by R_1 and R_2, respectively. Then, by Lemma 3.1 and the property of $\zeta(s)$, we see that the singularities of R_1 are determined by

$$s_1 + s_4 + s_6 = 1 - m - k \quad (m \in \mathbb{N}_0), \tag{4.2}$$

$$s_2 + s_4 + s_5 + s_6 = 1 - m - n - k \quad (m \in \mathbb{N}_0), \tag{4.3}$$

$$s_1 + s_2 + s_4 + s_5 + s_6 = 2 - n - k, \tag{4.4}$$

$$s_3 = 1 + n + k, \tag{4.5}$$

and the singularities of R_2 are determined by

$$s_1 + s_4 + s_6 = 1 - m - k \quad (m \in \mathbb{N}_0), \tag{4.6}$$

$$s_2 + s_3 + s_4 + s_5 + s_6 = 2 - m \quad (m \in \mathbb{N}_0), \tag{4.7}$$

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3, \tag{4.8}$$

$$s_3 = 1 + n + k. \tag{4.9}$$

Hence the singularities determined by (3.35) are derived from R_1 only, that is (4.3). Note that (4.7) in the case $s_3 = 1$ intersects (4.3), but they cannot be cancelled completely.

We see that (4.3) comes from the singularity of $\zeta_{\mathfrak{sl}(3)}(s_1, s_2 + s_5 + n, s_4 + s_6 + k)$. Combining (3.1) and the definition of R_1, we have only to consider

$$\begin{aligned} &\sum_{n=0}^{N} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(s_5 + n)}{\Gamma(s_5)} \Phi(s_1, \ldots, s_6) \\ &\times \binom{-s_6}{k} \frac{\Gamma(s_2 + s_4 + s_5 + s_6 + n + k - 1)\Gamma(1 - s_2 - s_5 - n)}{\Gamma(s_4 + s_6 + k)} \\ &\times \zeta(s_1 + s_2 + s_4 + s_5 + s_6 + n + k - 1)\zeta(s_3 - n - k). \end{aligned} \tag{4.10}$$

Hence, for $l \in \mathbb{N}_0$, we see that (4.10) for any pair (m, n, k) with $m+n+k = l$ gives (3.35). Therefore we need to check the cancellation between them. As well as in the proof of Lemma 3.4, we use the technique of "change of variables" introduced in [1]. Put

$$u_2 = s_2 + s_5, \quad u_4 = s_4 + s_6, \quad u_j = s_j \quad (j \neq 2, 4).$$

Then (4.10) can be rewritten as

$$\begin{aligned} &\sum_{n=0}^{N} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(u_5 + n)}{\Gamma(u_5)} \Phi(u_1, u_2 - u_5, u_3, u_4 - u_6, u_5, u_6) \\ &\times \binom{-u_6}{k} \frac{\Gamma(u_2 + u_4 + n + k - 1)\Gamma(1 - u_2 - n)}{\Gamma(u_4 + k)} \\ &\times \zeta(u_1 + u_2 + u_4 + n + k - 1)\zeta(u_3 - n - k). \end{aligned} \tag{4.11}$$

This contains the polynomial $\binom{-u_6}{k}$ of degree k and

$$\frac{\Gamma(u_5+n-1)}{\Gamma(u_5)}=(u_5+n-1)(u_5+n-2)\cdots(u_5+1)u_5$$

of degree n. Hence, for each l, the cancellation between singularities derived from several terms in (4.11) with $m+n+k=l$ cannot occur. By these considerations, we see that (3.35) determines a true singularity.

The singularities determined by (3.36) are derived from (4.4). By the same method as above, we can see that (3.36) determines a true singularity.

Next we consider (3.42) and (3.43). As well as (3.34), by the deformation of Figure 3.3, it follows from (3.41) that

(4.12)
$$J_2^*=-S(s_1,\ldots,s_6)+\frac{1}{2\pi i}\int_{\mathcal{C}''}\frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)(\widetilde{I}_2+I_3)\,dz_5,$$

where $S(s_1,\ldots,s_6)$ consists of the sum of residues at $z_5=0,1,\ldots,N$. As well as $R(s_1,\ldots,s_6)$, we have

$$S(s_1,\ldots,s_6)=\sum_{n=0}^{N}\frac{(-1)^{n+1}}{n!}\frac{\Gamma(s_5+n)}{\Gamma(s_5)}\Phi(s_1,\ldots,s_6)(S_1-S_2), \tag{4.13}$$

where we put

$$\begin{aligned}S_1=&\frac{\Gamma(s_3+s_5+s_6+n-1)\Gamma(1-s_3-s_5-n)}{\Gamma(s_6)}\\&\times\zeta_{\mathfrak{sl}(3)}(s_1,s_2-n,s_3+s_4+s_5+s_6+n-1),\\S_2=&\sum_{m=0}^{M_6-1}\binom{-s_6}{m}\zeta_{\mathfrak{sl}(3)}(s_1,s_2+s_3+s_5-m-1,s_4+s_6+m)\\&\times\frac{1}{-s_3-s_5-n+m+1}+(\text{integral term}).\end{aligned}$$

Then we can similarly see that (3.42) and (3.43) are derived from the singularities of S_1. By the same argument as in the case of R_1, we can verify that (3.42) and (3.43) determine true singularities. Thus we obtain the assertion that all of the singularities of (Type II) are true.

Finally we examine the singularities of (Type I) which are derived from $\Phi(s_1,\ldots,s_6)$ defined just before (3.28). Combining (3.2) and (3.3), we can write

$$\zeta_{\mathfrak{sl}(4)}(s_1,\ldots,s_6)=\frac{1}{\Phi(s_1,\ldots,s_6)}\widetilde{J}, \tag{4.14}$$

where

$$\widetilde{J} = \frac{1}{2\pi i}\int_{(c_5)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} I(s_1,\ldots,s_6;z_5)\Phi(s_1,\ldots,s_6)dz_5. \tag{4.15}$$

Now we prove that (3.38) determines a true singularity. In order to prove this, we have only to show that $\widetilde{J} \not\equiv 0$ on the hyperplane

$$\mathcal{H}_l :\ s_1+s_4+s_6 = 1-l$$

for any $l \in \mathbb{N}_0$ defined by (3.38). Hence we fix an arbitrary $l \in \mathbb{N}_0$. From (3.8), we see that the singularity determined by (3.38) is derived from I_{1k}. Since $\Phi(s_1,\ldots,s_6) \equiv 0$ on the hyperplane $\mathcal{H}_l$, we have

$$\widetilde{J} = \frac{1}{2\pi i}\int_{(c_5)} \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} \left(\sum_{k=0}^{M_6-1} I_{1k}\right) \Phi(s_1,\ldots,s_6)dz_5. \tag{4.16}$$

Using (3.1), we have

$$\begin{aligned} I_{1k} = \binom{-s_6}{k}\Bigg\{ & \frac{\Gamma(s_2+s_4+s_5+s_6+z_5+k-1)\Gamma(1-s_2-s_5-z_5)}{\Gamma(s_4+s_6+k)} \\ & \quad \times \zeta(s_1+s_2+s_4+s_5+s_6+z_5+k-1) \\ & + \sum_{m=0}^{M-1}\binom{-s_4-s_6-k}{m}\zeta(s_1+s_4+s_6+k+m)\zeta(s_2+s_5+z_5-m) \\ & + (\text{integral term})\Bigg\}\zeta(s_3-z_5-k). \end{aligned}$$

Hence the singularity determined by (3.38) is derived from the factor $\zeta(s_1+s_4+s_6+k+m)$ only. Therefore we have

$$\begin{aligned} \widetilde{J} = \frac{1}{2\pi i}\int_{(c_5)} & \frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)} \sum_{k=0}^{M_6-1}\binom{-s_6}{k} \\ & \times \sum_{m=0}^{M-1}\binom{-s_4-s_6-k}{m}\zeta(s_1+s_4+s_6+k+m)\zeta(s_2+s_5+z_5-m) \\ & \times \zeta(s_3-z_5-k)\Phi(s_1,\ldots,s_6)dz_5. \end{aligned}$$

Since the pole of $\zeta(s_1+s_4+s_6+k+m)$ is $s_1+s_4+s_6=1-k-m$, we have

$$\widetilde{J} = \frac{1}{2\pi i}\cdot\frac{\Phi(s_1,\ldots,s_6)}{s_1+s_4+s_6-1+l} \times\Biggl\{\int_{(c_5)}\frac{\Gamma(s_5+z_5)\Gamma(-z_5)}{\Gamma(s_5)}\sum_{\substack{k,m\geqslant 0\\k+m=l}}\binom{-s_6}{k}\binom{-s_4-s_6-k}{m} \times\zeta(s_2+s_5+z_5-m)\zeta(s_3-z_5-k)\Biggr\} \tag{4.17}$$

on the hyperplane $\mathcal{H}_l$. We denote the part in the curly parentheses on the right-hand side of (4.17) by $\mathcal{A}$. Putting $u_2=s_2+s_5,\ u_4=s_4+s_6$ and $u_j=s_j$ $(j\neq 2,4)$, we have

$$\mathcal{A} = \sum_{\substack{k,m\\k+m=l}}\binom{-u_6}{k}\binom{-u_4-k}{m}\Biggl\{\frac{1}{\Gamma(u_5)}\int_{(c_5)}\Gamma(u_5+z_5)\Gamma(-z_5) \times\zeta(u_2+z_5-m)\zeta(u_3-z_5-k)\,dz_5\Biggr\}. \tag{4.18}$$

We further denote the part in the curly parentheses on the right-hand side of (4.18) by $\mathfrak{J}_{m,k}(u_2,u_3,u_5)$. This $\mathfrak{J}_{m,k}(u_2,u_3,u_5)$ can be defined as a meromorphic function on $\mathbb{C}^3$. Putting $z_5=c_5+it$ $(t\in(-\infty,\infty))$, we have

$$\mathfrak{J}_{m,k}(u_2,u_3,u_5) = \frac{1}{\Gamma(u_5)}\int_{-\infty}^{\infty}\Gamma(u_5+c_5+it)\Gamma(-c_5-it) \times\zeta(u_2+c_5-m+it)\zeta(u_3-c_5-k-it)\,i\,dt.$$

Choose (u_2,u_3,u_5) such as $u_5+c_5=-c_5$ and $u_2+c_5-m=u_3-c_5-k$, namely $(u_2,u_3,u_5)=(m,2c_5+k,-2c_5)$. Then we have

$$\mathfrak{J}_{m,k}(m,2c_5+k,-2c_5) = \frac{i}{\Gamma(-2c_5)}\int_{-\infty}^{\infty}|\Gamma(-c_5+it)|^2|\zeta(c_5+it)|^2dt\neq 0.$$

Thus $\mathfrak{J}_{m,k}(u_2,u_3,u_5)\not\equiv 0$ as a meromorphic function. Furthermore, by (4.18), we have

$$\mathcal{A} = \sum_{\substack{k,m\\k+m=l}}\binom{-u_6}{k}\binom{-u_4-k}{m}\mathfrak{J}_{m,k}(u_2,u_3,u_5).$$

As mentioned before, $\binom{-u_6}{k}$ is a polynomial in u_6 of degree k. Hence we have $\mathcal{A}\not\equiv 0$ as a meromorphic function. By (4.17), we have $\widetilde{J}\not\equiv 0$ on the hyperplane $\mathcal{H}_l$. Thus we see that (3.38) determines a true singularity for any $l\in\mathbb{N}_0$.

Similarly we can verify that $\widetilde{J} \not\equiv 0$ on hyperplanes (3.39) and (3.40). This means that the singularities of (Type I) are true. Thus we obtain the assertion that all of the singularities determined by the seven equations in Theorem 3.5 are true singularities.

Remark 4.1. — Here we guess the location of singularities of $\zeta_{\mathfrak{sl}(r+1)}$ for general r. By Lemma 3.1, we see that the singularities of

$$\zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3) = \sum_{m_1, m_2=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} (m_1 + m_2)^{s_3}}$$

are determined by

$$s_1 + s_3 = 1 - l \quad (l \in \mathbb{N}_0), \tag{4.19}$$

$$s_2 + s_3 = 1 - l \quad (l \in \mathbb{N}_0), \tag{4.20}$$

$$s_1 + s_2 + s_3 = 2. \tag{4.21}$$

We know that the left-hand side of (4.19) and (4.20) are the sum of exponents of the factors containing m_1 and m_2, respectively, and that of (4.21) is the sum of all exponents.

Next, by Theorem 3.5, we see that the singularities of

$$\begin{aligned} &\zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) \\ &= \sum_{m_1, m_2, m_3=1}^{\infty} \frac{1}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}} \end{aligned}$$

are determined by

$$s_1 + s_4 + s_6 = 1 - l \quad (l \in \mathbb{N}_0), \tag{4.22}$$

$$s_2 + s_4 + s_5 + s_6 = 1 - l \quad (l \in \mathbb{N}_0), \tag{4.23}$$

$$s_3 + s_5 + s_6 = 1 - l \quad (l \in \mathbb{N}_0), \tag{4.24}$$

$$s_1 + s_2 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0), \tag{4.25}$$

$$s_1 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0), \tag{4.26}$$

$$s_2 + s_3 + s_4 + s_5 + s_6 = 2 - l \quad (l \in \mathbb{N}_0), \tag{4.27}$$

$$s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 3. \tag{4.28}$$

We know that the left-hand side of (4.22), (4.23) and (4.24) are the sum of exponents of the factors containing m_1, m_2 and m_3, respectively, and that the left-hand side of (4.25), (4.26) and (4.27) are the sum of exponents of the factors containing either m_1 or m_2, either m_1 or m_3, and either m_2 or

m_3, respectively. Furthermore the left-hand side of (4.28) is the sum of all exponents.

By this consideration, we might be able to conjecture, on the singularities of

$$\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{j=1}^{r} \prod_{k=1}^{r-j+1} \left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}},$$

as follows. For any j, k, the general factor can be expressed as

$$\left(\sum_{\nu=k}^{j+k-1} m_\nu \right)^{-s_{jk}}.$$

Hence, if (j,k)-factor contains m_ν then $k \leqslant \nu \leqslant j+k-1$. So we first expect that

$$\sum_{j=1}^{r} \sum_{\nu+1-j \leqslant k \leqslant \nu} s_{jk} = 1 - l \quad (l \in \mathbb{N}_0) \tag{4.29}$$

determine the singularities. Secondly we consider the relations that the sum of exponents of the factors containing either m_ν or m_μ $(\nu \neq \mu)$ equals to $2 - l$ $(l \in \mathbb{N}_0)$ like (4.25) - (4.27). Inductively we consider the relations that the sum of exponents of the factor containing either m_{ν_1} or $\ldots$ or m_{ν_j} equals to $j - l$ $(l \in \mathbb{N}_0)$, where $1 \leqslant j \leqslant r-1$. Additionally we consider the relation that the sum of all exponents equals to r. Then we might be able to conjecture that all of them give the complete list of the true singularities of $\zeta_{\mathfrak{sl}(r+1)}$.

5. Functional relations and evaluation formulas

In this section, we aim to prove certain functional relations for

$$\begin{aligned} &\zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) \\ &\quad = \sum_{l,m,n=1}^{\infty} l^{-s_1} m^{-s_2} n^{-s_3} (l+m)^{-s_4} (m+n)^{-s_5} (l+m+n)^{-s_6}. \end{aligned} \tag{5.1}$$

More strictly, we let

$$\begin{cases} \mathcal{Z}(s_1, s_2, s_3, s_4, s_5) = \zeta_{\mathfrak{sl}(4)}(s_1, 0, s_2, s_3, s_4, s_5), \\ \mathcal{T}(s_1, s_2, s_3, s_4, s_5) = \zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, 0, s_5), \end{cases} \tag{5.2}$$

and prove certain functional relations between $\mathcal{Z}(\mathbf{s})$, $\mathcal{T}(\mathbf{s})$, $\zeta_{\mathfrak{sl}(3)}(\mathbf{s}) = \zeta_{MT,2}(\mathbf{s})$ and $\zeta_{\mathfrak{sl}(2)}(s) = \zeta(s)$ (see Theorem 5.9, Theorem 5.10). Note that these relations can be regarded as triple analogues of those in [21, 24].

We quote some notation and results from [16, 21, 24]. Throughout this section we fix a small $\delta \in \mathbb{R}$ with $\delta > 0$. Let $u \in \mathbb{R}$ with $1 \leqslant u \leqslant 1+\delta$. For $s \in \mathbb{C}$, define

$$\phi(s;u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^s}. \tag{5.3}$$

If $u > 1$ then $\phi(s;u)$ is convergent absolutely for all $s \in \mathbb{C}$. In the case $u = 1$, we have $\phi(s;1) = (2^{1-s}-1)\zeta(s)$. Note that $\phi(-2m;1) = (2^{2m+1}-1)\zeta(-2m) = 0$ for $m \in \mathbb{N}$. Furthermore, for $s \in \mathbb{C}$ with $\Re s > 1$, $t \in \mathbb{C}$ with $\Re t \leqslant 0$ and $u \in [1, 1+\delta]$, we define

$$F_1(t;s;u) = \sum_{m=1}^{\infty} \frac{(-u)^{-m} e^{mt}}{m^s}, \tag{5.4}$$

which can be viewed as a kind of polylogarithm. Applying our previous result in [16] Proposition 2.1 to $F_1(t;s;u)$, we obtain the following.

Lemma 5.1. — *Let $s \in \mathbb{C}$ with $\Re s > 1$ and $u \in [1, 1+\delta]$. Then $F_1(t;s;u)$ can be continued holomorphically to $\mathcal{D}(\pi) := \{t \in \mathbb{C} \mid |t| < \pi\}$ and satisfies*

$$F_1(t;s;u) = \sum_{N=0}^{\infty} \phi(s-N;u) \frac{t^N}{N!} \quad (|t| < \pi). \tag{5.5}$$

Furthermore the right-hand side of (5.5) is uniformly convergent with respect to $u \in [1, 1+\delta]$.

Moreover we can show the following.

Lemma 5.2. — *The function $F_1(t;s;u)$ can be continued holomorphically to all $s \in \mathbb{C}$ when $(t,u) \in \mathcal{D}(\pi) \times [1, 1+\delta]$ and the assertions of Lemma 5.1 hold for all $s \in \mathbb{C}$.*

Proof. — When $\Re s > 1$, Lemma 5.1 states that $F_1(t;s;u)$ is continuous for all $(t,u) \in \mathcal{D}(\pi) \times [1, 1+\delta]$. For an arbitrary $\eta \in (0, \pi) \subset \mathbb{R}$, we put

$$M = M(s,\eta) = \max\left\{ |F_1(t;s;u)| \,\middle|\, t \in \mathbb{C},\ |t| = \eta,\ 1 \leqslant u \leqslant 1+\delta \right\}.$$

Also, let $\gamma := \{z = \eta e^{i\theta} \,|\, 0 \leqslant \theta < 2\pi\}$. Applying well-known Cauchy's theorem to (5.5), we have

$$\frac{\phi(s-n;u)}{n!} = \frac{1}{2\pi i} \int_{\gamma} F_1(z;s;u) z^{-n-1}\, dz \quad (n \in \mathbb{N}_0),$$

hence

$$\frac{|\phi(s-n;u)|}{n!} \leqslant M\eta^{-n} \quad (n \in \mathbb{N}_0). \tag{5.6}$$

Let $s \in \mathbb{C}$ with $\Re s \leqslant 1$, and choose $m \in \mathbb{N}$ such that $\Re(s+m) > 1$. For an arbitrary $\varepsilon \in (0,\pi)$, we choose $\eta \in \mathbb{R}$ with $\varepsilon < \eta < \pi$. From (5.6) with replacing s by $s+m$, we see that

$$\frac{|\phi(s+m-\nu;u)|}{\nu!} \leqslant M\eta^{-\nu} \quad (\nu \in \mathbb{N}_0).$$

When $\nu \geqslant m$, we put $n = \nu - m$. Then we have

(5.7)

$$\begin{aligned}\frac{|\phi(s-n;u)|}{n!} &= \frac{|\phi(s+m-\nu;u)|}{\nu!} \cdot (n+m)(n+m-1)\cdots(n+1) \\ &\leqslant M\varepsilon^{-n-m} \cdot \frac{(n+m)(n+m-1)\cdots(n+1)}{(\eta/\varepsilon)^{n+m}} \quad (n \in \mathbb{N}_0).\end{aligned}$$

Note that the last fraction-factor on the right-hand side of (5.7) is bounded and independent of u because $\varepsilon < \eta$. Hence, for $s \in \mathbb{C}$ with $\Re s \leqslant 1$, we can define $F_1(t;s;u)$ by the right-hand side of (5.5) which is uniformly convergent with respect to $u \in [1, 1+\delta]$ because of (5.7). Moreover it is easy to see that the convergence of the right-hand side of (5.5) is uniform with respect to s in any compact subset of $\mathbb{C}$. Hence $F_1(t;s;u)$ is holomorphic for all $s \in \mathbb{C}$. This completes the proof of Lemma 5.2. □

Let $k \in \mathbb{N}_0$ and $u \in [1, 1+\delta]$. We define

$$\begin{aligned}\mathfrak{F}_1(t;2k+1;u) =& F_1(t;2k+1;u) - F_1(-t;2k+1;u) \\ & - 2\sum_{j=0}^{k} \phi(2k-2j;u)\frac{t^{2j+1}}{(2j+1)!} \quad (|t| < \pi).\end{aligned} \tag{5.8}$$

Let $t = i\theta$ with $\theta \in (-\pi, \pi)$ in (5.8). If $u > 1$ then by the Maclaurin expansion of $\sin x$, we have

$$\begin{aligned}F_1(i\theta;2k+1;u) - F_1(-i\theta;2k+1;u) &= 2i\sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin(m\theta)}{m^{2k+1}} \\ &= 2\sum_{j=0}^{\infty} \phi(2k-2j;u)\frac{(i\theta)^{2j+1}}{(2j+1)!},\end{aligned}$$

hence

$$\mathfrak{F}_1(i\theta;2k+1;u) = 2\sum_{j=k+1}^{\infty} \phi(2k-2j;u)\frac{(i\theta)^{2j+1}}{(2j+1)!}. \tag{5.9}$$

It follows from (5.6) and (5.7) that the right-hand side of (5.9) is uniformly convergent with respect to $u \in [1, 1+\delta]$ when $\theta \in (-\pi,\pi)$. Since $\phi(-2l;1) = 0$ for $l \in \mathbb{N}$, we obtain the following.

Lemma 5.3. — *For $\theta \in (-\pi,\pi)$, $\mathfrak{F}_1(i\theta;2k+1;u) \to 0$ as $u \to 1+0$.*

We fix $s_1, s_2 \in \mathbb{C}$ with $\Re s_j > 1$ $(j = 1, 2)$. For $u \in [1, 1+\delta]$, we define
(5.10)
$$G_2(t; s_1, s_2; u) = F_1(t; s_1; u) F_1(t; s_2; u) = \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n} e^{(m+n)t}}{m^{s_1} n^{s_2}}.$$

By Lemma 5.1, $G_2(t; s_1, s_2; u)$ is holomorphic for $t \in \mathcal{D}(\pi)$, and its Maclaurin expansion is
(5.11)
$$G_2(t; s_1, s_2; u) = \sum_{N=0}^{\infty} \sum_{j=0}^{N} \binom{N}{j} \phi(s_1 - j; u) \phi(s_2 + j - N; u) \frac{t^N}{N!} \quad (|t| < \pi),$$

where the right-hand side is uniformly convergent with respect to $u \in [1, 1+\delta]$. For $u \in [1, 1+\delta]$, we further define

$$R_1(s_1, s_2, s_3; u) = \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n}}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{5.12}$$

This is convergent absolutely for all $(s_1, s_2, s_3) \in \mathbb{C}^3$ if $u > 1$. We can also prove that $R_1(s_1, s_2, s_3; 1)$ can be continued meromorphically to $\mathbb{C}^3$ by the same method as in [11] Theorem 1. However, it is sufficient here to prove the following lemma.

Lemma 5.4. — *Fix $s_1, s_2 \in \mathbb{C}$ with $\Re s_j > 1$ $(j = 1, 2)$. Then $R_1(s_1, s_2, s_3; u)$ can be continued meromorphically for all $s_3 \in \mathbb{C}$. Furthermore*

$$G_2(t; s_1, s_2; u) = \sum_{N=0}^{\infty} R_1(s_1, s_2, -N; u) \frac{t^N}{N!} \quad (|t| < \pi) \tag{5.13}$$

holds, where the right-hand side is uniformly convergent with respect to $u \in [1, 1+\delta]$.

Proof. — We consider the well-known contour integrals (see, for example, [16] Proof of Proposition 2.1) as follows. Let Υ be the path which consists of the positive real axis $[\varepsilon, \infty]$ (top side), a circle C_ε around 0 of radius ε, and the positive real axis $[\varepsilon, \infty]$ (bottom side), where $0 < \varepsilon < \pi$. Note that we interpret t^s as $\exp(s \log t)$, where the imaginary part of $\log t$ varies from 0 (on the top side of the real axis) to 2π (on the bottom side). Let

$$\begin{aligned} H(s; s_1, s_2; u) &= \int_\Upsilon G_2(-t; s_1, s_2; u) t^{s-1} dt \\ &= \left(e^{2\pi i s} - 1\right) \int_\varepsilon^\infty G_2(-t; s_1, s_2; u) t^{s-1} dt + \int_{C_\varepsilon} G_2(-t; s_1, s_2; u) t^{s-1} dt, \end{aligned} \tag{5.14}$$

which is holomorphic for all $s \in \mathbb{C}$. On the other hand, if $\Re s > 1$, then, letting $\varepsilon \to 0$ and using (5.10), we have

$$H(s; s_1, s_2; u) = \left(e^{2\pi i s} - 1\right) \Gamma(s) R_1(s_1, s_2, s; u). \tag{5.15}$$

Using (5.15) we can continue $R_1(s_1, s_2, s_3; u)$ meromorphically to all $s_3 \in \mathbb{C}$. Suppose $u > 1$. Then, by (5.10) and (5.12), we have

$$G_2(i\theta; s_1, s_2; u) = \sum_{N=0}^{\infty} R_1(s_1, s_2, -N; u) \frac{(i\theta)^N}{N!}$$

for $t = i\theta$ with $\theta \in (-\pi, \pi)$. Hence, by (5.11), we see that (5.13) holds for $t \in \mathbb{C}$ with $|t| < \pi$ when $u > 1$. Now let $u \in [1, 1+\delta]$. Using (5.14), (5.15) with $s = -N$ for $N \in \mathbb{N}_0$, we have

$$\begin{aligned} R_1(s_1, s_2, -N; u) \frac{(-1)^N}{N!} &= \frac{1}{2\pi i} H(-N; s_1, s_2; u) \\ &= \frac{1}{2\pi i} \int_{C_\varepsilon} G_2(-t; s_1, s_2; u) t^{-N-1} dt. \end{aligned} \tag{5.16}$$

Let

$$M_1 = M_1(s_1, s_2, \varepsilon) = \max\{|G_2(t; s_1, s_2; u)| \bigm| t \in \mathbb{C},\ |t| = \varepsilon,\ 1 \leqslant u \leqslant 1+\delta\}.$$

Then

$$\frac{|R_1(s_1, s_2, -N; u)|}{N!} \leqslant M_1 \varepsilon^{-N} \quad (u \in [1, 1+\delta]; N \in \mathbb{N}_0), \tag{5.17}$$

where ε is an arbitrary positive number satisfying $\varepsilon < \pi$. This means that the right-hand side of (5.13) is uniformly convergent with respect to $u \in [1, 1+\delta]$ when $|t| < \pi$, hence (5.13) holds for $u \in [1, 1+\delta]$. □

Next we fix $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant 3)$. For $u \in [1, 1+\delta]$, we define

$$F_2(t; \mathbf{s}; u) = \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n} e^{(m+n)t}}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{5.18}$$

This can be regarded as a kind of double analogue of polylogarithm $F_1(t; s; u)$. Note that we have already considered $F_2(t; \mathbf{s}; u)$ in the case when each $s_j \in \mathbb{N}$ in [21]. We can prove the following lemma by the same method as in the proof of Theorem 1.3 and Proposition 2.1 in [16]. This lemma can also be regarded as a double analogue of Lemma 5.1.

Lemma 5.5. — *Let $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant 3)$ and $u \in [1, 1+\delta]$. Then $F_2(t; \mathbf{s}; u)$ can be continued holomorphically to*

$\mathcal{D}(\pi)=\{t\in\mathbb{C}\,|\,|t|<\pi\}$ *and satisfies*

$$F_2(t;\mathbf{s};u)=\sum_{N=0}^{\infty}R_1(s_1,s_2,s_3-N;u)\frac{t^N}{N!}\quad(|t|<\pi). \tag{5.19}$$

Furthermore the right-hand side of (5.19) is uniformly convergent with respect to $u\in[1,1+\delta]$.

Proof. — Using the same notation as in the proof of Lemma 5.4. Let $N\in\mathbb{N}$ with $N\geqslant\Re s_3+1$. We denote by J_1 and J_2, respectively, the first and second term on the right-hand side of (5.14) with $s=s_3-N$. Then we have

$$\begin{aligned}|J_1|&\leqslant\left|e^{2\pi i s_3}-1\right|\cdot\left|\int_{\varepsilon}^{\infty}\sum_{m,n=1}^{\infty}\frac{(-u)^{-m-n}e^{-(m+n)t}}{m^{s_1}n^{s_2}}t^{s_3-N-1}dt\right|\\&\leqslant\varepsilon^{\Re s_3-N-1}\left|e^{2\pi i s_3}-1\right|\sum_{m,n=1}^{\infty}\frac{e^{-(m+n)\varepsilon}}{m^{\Re s_1}n^{\Re s_2}(m+n)}.\end{aligned} \tag{5.20}$$

On the other hand, if $N-s_3\in\mathbb{N}_0$ then it follows from (5.16) that

$$|J_2|\leqslant 2\pi\frac{|R_1(s_1,s_2,s_3-N;u)|}{(N-s_3)!},$$

which with (5.17) implies that

$$|J_2|\leqslant 2\pi M_1\varepsilon^{-N+s_3}. \tag{5.21}$$

Otherwise from (5.13) we have

$$|J_2|\leqslant\varepsilon^{\Re s_3-N}\left|e^{2\pi i s_3}-1\right|\left|\sum_{n=0}^{\infty}R_1(s_1,s_2,-n;u)\frac{(-1)^n\varepsilon^n}{(n+s_3-N)n!}\right|, \tag{5.22}$$

because

$$\int_{C_\varepsilon}t^p dt=\varepsilon^{p+1}\left(\frac{e^{2\pi i p}-1}{p+1}\right)\quad(p\neq-1).$$

Note that the right-hand side of (5.22) is convergent because $\varepsilon<\pi$. From (5.15), we have

$$\begin{aligned}R_1(s_1,s_2,s_3-N;u)&=\frac{H(s_3-N;s_1,s_2;u)}{(e^{2\pi i s_3}-1)\,\Gamma(s_3-N)}\\&=\frac{\Gamma(N+1-s_3)}{2\pi i e^{\pi i s_3}}H(s_3-N;s_1,s_2;u),\end{aligned}$$

by the well known formula $\Gamma(s)\Gamma(1-s)=2\pi i/(e^{\pi i s}-e^{-\pi i s})$. Combining this relation and (5.20)–(5.22), we find that there exists a positive constant

$M_2 = M_2(s_1, s_2, s_3, \varepsilon)$ independent of $u \in [1, 1+\delta]$ such that

$$\frac{|R_1(s_1, s_2, s_3 - N; u)|}{N!} \leqslant M_2 \varepsilon^{-N} \quad (N \geqslant \Re s_3 + 1). \tag{5.23}$$

Finally, we assume $u > 1$ and $t = i\theta$ with $\theta \in (-\pi, \pi)$. Then, substituting the Maclaurin expansion of $e^{(m+n)t}$ into the right-hand side of (5.18), we find that

$$F_2(i\theta; \mathbf{s}; u) = \sum_{N=0}^{\infty} R_1(s_1, s_2, s_3 - N; u) \frac{(i\theta)^N}{N!}. \tag{5.24}$$

By (5.23), we see that the right-hand side of (5.24) is uniformly convergent with respect to $u \in [1, 1+\delta]$. Hence (5.24) holds for $u \in [1, 1+\delta]$. Thus (5.19) holds and its right-hand side is uniformly convergent with respect to $u \in [1, 1+\delta]$. □

For $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $\Re s_j > 1$ $(1 \leqslant j \leqslant 3)$ and $u \in [1, 1+\delta]$, we define

$$G_3(t; 2k+1; \mathbf{s}; u) = \mathfrak{F}_1(t; 2k+1; u) F_2(t; \mathbf{s}; u). \tag{5.25}$$

>From Lemma 5.1, Lemma 5.5 and (5.8), $G_3(t; 2k+1; \mathbf{s}; u)$ is holomorphic with respect to t on $\mathcal{D}(\pi)$. Combining (5.9) and (5.19), we can write $G_3(t; 2k+1; \mathbf{s}; u)$ as

$$G_3(t; 2k+1; \mathbf{s}; u) = \sum_{n=0}^{\infty} \mathfrak{C}_n(u) \frac{t^n}{n!}, \tag{5.26}$$

where $\mathfrak{C}_n(u) = \mathfrak{C}_n(2k+1; \mathbf{s}; u)$ is continuous for $u \in [1, 1+\delta]$, and the right-hand side of (5.26) is uniformly convergent with respect to $u \in [1, 1+\delta]$. Hence, as well as (5.6) and (5.17), for an arbitrary η with $0 < \eta < \pi$, there exists a positive constant M_3 independent of $u \in [1, 1+\delta]$ such that

$$\frac{|\mathfrak{C}_N(u)|}{N!} \leqslant M_3 \eta^{-N} \quad (u \in [1, 1+\delta]; N \in \mathbb{N}_0). \tag{5.27}$$

Substituting (5.8) and (5.18) with $t = i\theta$ into (5.26), we have

(5.28)

$$\begin{aligned} G_3(i\theta; 2k+1; \mathbf{s}; u) &= \mathfrak{F}_1(i\theta; 2k+1; u) F_2(i\theta; \mathbf{s}; u) \\ &= \sum_{l,m,n=1}^{\infty} \frac{(-u)^{-l-m-n} e^{i(l+m+n)\theta}}{l^{2k+1} m^{s_1} n^{s_2} (m+n)^{s_3}} - \sum_{l,m,n=1}^{\infty} \frac{(-u)^{-l-m-n} e^{-i(l-m-n)\theta}}{l^{2k+1} m^{s_1} n^{s_2} (m+n)^{s_3}} \\ &\quad - 2 \left\{ \sum_{j=0}^{k} \phi(2k-2j; u) \frac{(i\theta)^{2j+1}}{(2j+1)!} \right\} F_2(i\theta; \mathbf{s}; u). \end{aligned}$$

For simplicity we denote the first, the second, and the third term on the right-hand side of (5.28) by I_1, I_2 and I_3, respectively. We need to define the following functions:

$$
(5.29)\quad \begin{cases} S_1(s_1,\ldots,s_5;u) = \displaystyle\sum_{l,m,n=1}^{\infty} \frac{(-u)^{-l-m-n}}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(l+m+n)^{s_5}}; \\ S_2(s_1,\ldots,s_5;u) = \displaystyle\sum_{l,m,n=1}^{\infty} \frac{(-u)^{-2l-2m-n}}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(l+m+n)^{s_5}}; \\ S_3(s_1,\ldots,s_5;u) = \displaystyle\sum_{l,m,n=1}^{\infty} \frac{(-u)^{-2l-m-n}}{l^{s_1}m^{s_2}(l+n)^{s_3}(m+n)^{s_4}(l+m+n)^{s_5}}; \\ S_4(s_1,\ldots,s_5;u) = \displaystyle\sum_{l,m,n=1}^{\infty} \frac{(-u)^{-2l-m-2n}}{l^{s_1}m^{s_2}(l+n)^{s_3}(m+n)^{s_4}(l+m+n)^{s_5}}; \\ R_2(s_1,s_2,s_3;u) = \displaystyle\sum_{l,m=1}^{\infty} \frac{(-u)^{-2l-m}}{l^{s_1}m^{s_2}(l+m)^{s_3}} \end{cases}
$$

for $u \in [1, 1+\delta]$. We assume $u > 1$ again. Then the above functions are entire functions of several complex variables. (Note that even if $u = 1$, by the same method as in the proof of [14] Theorem 3, we can prove that the above functions can be continued meromorphically to the whole complex space.)

Since $u > 1$, replacing (l, m, n) with (n, l, m) in I_1, we have

$$
(5.30)\qquad I_1 = \sum_{N=0}^{\infty} S_1(s_1, s_2, 2k+1, s_3, -N; u)\frac{(i\theta)^N}{N!}.
$$

Secondly we divide I_2 into three parts corresponding to $l < m$, $l > m$ and $l = m$ respectively, and put

$$
\begin{cases} \nu = m - l \quad \text{i.e. } m = l + \nu \quad (\text{if } l < m) \\ \mu = l - m \quad \text{i.e. } l = m + \mu \quad (\text{if } l > m). \end{cases}
$$

Then we have

$$
\begin{aligned}
(5.31)\qquad I_2 = &-\sum_{l,n,\nu=1}^{\infty} \frac{(-u)^{-2l-n-\nu}e^{i(n+\nu)\theta}}{l^{2k+1}n^{s_2}(l+\nu)^{s_1}(l+n+\nu)^{s_3}} \\
&-\sum_{m,n,\mu=1}^{\infty} \frac{(-u)^{-2m-n-\mu}e^{i(n-\mu)\theta}}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+\mu)^{2k+1}} \\
&-\sum_{l,n=1}^{\infty} \frac{(-u)^{-2l-n}e^{in\theta}}{l^{2k+1+s_1}n^{s_2}(l+n)^{s_3}}.
\end{aligned}
$$

Furthermore, in the second term on the right-hand of (5.31), we put

$$\begin{cases} \xi = n-\mu \ \ \textit{i.e.}\ n=\xi+\mu \quad (\text{if } \mu<n) \\ \rho=\mu-n \ \ \textit{i.e.}\ \mu=n+\rho \quad (\text{if } \mu>n). \end{cases}$$

Then the second term equals to

$$\begin{aligned} (5.32) \qquad & -\sum_{m,\xi,\mu=1}^{\infty} \frac{(-u)^{-2m-\xi-2\mu}e^{i\xi\theta}}{m^{s_1}(m+\mu)^{2k+1}(\xi+\mu)^{s_2}(m+\xi+\mu)^{s_3}} \\ & -\sum_{m,n,\rho=1}^{\infty} \frac{(-u)^{-2m-2n-\rho}e^{-i\rho\theta}}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+n+\rho)^{2k+1}} \\ & -\sum_{m,n=1}^{\infty} \frac{(-u)^{-2m-2n}}{m^{s_1}n^{s_2}(m+n)^{2k+1+s_3}}. \end{aligned}$$

Combining (5.31) and (5.32), and using (5.29), we have

$$\begin{aligned} (5.33) \qquad I_2 = -\sum_{N=0}^{\infty} \Big\{ & S_3(2k+1,s_2,s_1,-N,s_3;u) \\ & + S_4(s_1,-N,2k+1,s_2,s_3;u) \\ & + (-1)^N S_2(s_1,s_2,-N,s_3,2k+1;u) \\ & + R_2(2k+1+s_1,s_2-N,s_3;u)\Big\} \frac{(i\theta)^N}{N!} \\ & - \sum_{m,n=1}^{\infty} \frac{u^{-2m-2n}}{m^{s_1}n^{s_2}(m+n)^{2k+1+s_3}}. \end{aligned}$$

Thirdly, substituting (5.24) into I_3, we have

(5.34)

$$\begin{aligned} I_3 &= -2\left\{\sum_{j=0}^{k}\phi(2k-2j;u)\frac{(i\theta)^{2j+1}}{(2j+1)!}\right\}\left\{\sum_{N=0}^{\infty}R_1(s_1,s_2,s_3-N;u)\frac{(i\theta)^N}{N!}\right\} \\ &= -2\sum_{N=0}^{\infty}\sum_{j=0}^{k}\binom{N}{2j+1}\phi(2k-2j;u)R_1(s_1,s_2,s_3+2j+1-N;u)\frac{(i\theta)^N}{N!}. \end{aligned}$$

In view of (5.30), (5.33) and (5.34), we define $\widetilde{\mathfrak{C}}_N(u) = \widetilde{\mathfrak{C}}_N(2k+1;\mathbf{s};u)$ for any $N \in \mathbb{N}_0$ and $u \in [1, 1+\delta]$ by

(5.35)

$$\begin{aligned}\widetilde{\mathfrak{C}}_N(u) = {} & S_1(s_1, s_2, 2k+1, s_3, -N; u) \\ & - (-1)^N S_2(s_1, s_2, -N, s_3, 2k+1; u) \\ & - S_3(2k+1, s_2, s_1, -N, s_3; u) \\ & - S_4(s_1, -N, 2k+1, s_2, s_3; u) \\ & - R_2(2k+1+s_1, s_2 - N, s_3; u) \\ & - 2\sum_{j=0}^{k} \binom{N}{2j+1} \phi(2k-2j; u) R_1(s_1, s_2, s_3 + 2j + 1 - N; u).\end{aligned}$$

Then, by combining (5.25), (5.26), (5.30), (5.33) and (5.34), we see that

$$\begin{cases} \mathfrak{C}_N(u) = \widetilde{\mathfrak{C}}_N(u) & (\text{if } N \in \mathbb{N}); \\ \mathfrak{C}_0(u) = \widetilde{\mathfrak{C}}_0(u) - \displaystyle\sum_{m,n=1}^{\infty} \frac{u^{-2m-2n}}{m^{s_1} n^{s_2} (m+n)^{2k+1+s_3}} & (\text{if } N = 0). \end{cases}$$

Combining Lemma 5.3 and (5.25), and using (5.26), we have

$$\lim_{u \to 1+0} \mathfrak{C}_N(u) = 0 \quad (N \in \mathbb{N}_0).$$

Thus we obtain the following.

Lemma 5.6. — *With the above notation,*

$$\lim_{u \to 1+0} \widetilde{\mathfrak{C}}_N(u) = \begin{cases} 0 & (N \in \mathbb{N}); \\ \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2k + 1) & (N = 0). \end{cases}$$

Remark 5.7. — For $N \in \mathbb{Z}$ with $N \leqslant -1$, we define $\widetilde{\mathfrak{C}}_N(u)$ by (5.35) for $u > 1$. Furthermore, if $N \leqslant -1$ then the right-hand side of (5.35) is convergent as $u \to 1+0$, hence we define $\widetilde{\mathfrak{C}}_N(1)$ by this limit.

Now we give explicit relation formulas based on the above data. For $d \in \mathbb{N}$, $\theta \in \mathbb{R}$ and $u \in [1, 1+\delta]$, we define $\mathfrak{L}_j(\theta; d; u) = \mathfrak{L}_j(\theta; 2k+1, \mathbf{s}, d; u)$ $(1 \leqslant j \leqslant 5)$ by

$$(5.36)\quad \begin{cases} \mathfrak{L}_1(\theta;d;u)=\displaystyle\sum_{l,m,n=1}^{\infty}\frac{(-u)^{-l-m-n}\sin\left((l+m+n)\theta\right)}{l^{s_1}m^{s_2}n^{2k+1}(l+m)^{s_3}(l+m+n)^{d+1}}; \\ \mathfrak{L}_2(\theta;d;u)=(-1)^d\displaystyle\sum_{l,m,n=1}^{\infty}\frac{(-u)^{-2l-2m-n}\sin(n\theta)}{l^{s_1}m^{s_2}n^{d+1}(l+m)^{s_3}(l+m+n)^{2k+1}}; \\ \mathfrak{L}_3(\theta;d;u)=\displaystyle\sum_{l,m,n=1}^{\infty}\frac{(-u)^{-2l-m-n}\sin((m+n)\theta)}{l^{2k+1}m^{s_2}(l+n)^{s_1}(m+n)^{d+1}(l+m+n)^{s_3}}; \\ \mathfrak{L}_4(\theta;d;u)=\displaystyle\sum_{l,m,n=1}^{\infty}\frac{(-u)^{-2l-m-2n}\sin(m\theta)}{l^{s_1}m^{d+1}(l+n)^{2k+1}(m+n)^{s_2}(l+m+n)^{s_3}}; \\ \mathfrak{L}_5(\theta;d;u)=\displaystyle\sum_{l,m=1}^{\infty}\frac{(-u)^{-2l-m}\sin(m\theta)}{l^{2k+1+s_1}m^{s_2+d+1}(l+m)^{s_3}}, \end{cases}$$

for k,s_1,s_2,s_3 fixed as above. Assume $u>1$. By (5.29), we have, for example,

$$\mathfrak{L}_1(\theta;d;u)=\sum_{\nu=0}^{\infty}S_1(s_1,s_2,2k+1,s_3,d-2\nu;u)\frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!}.$$

Similarly we can express $\mathfrak{L}_2,\mathfrak{L}_3,\mathfrak{L}_4,\mathfrak{L}_5$ in terms of S_2,S_3,S_4 and R_2. Recall the relation ([22] Equation (3.3)):

$$(5.37)\quad \sum_{\mu=0}^{b}\binom{a-1+b-\mu}{b-\mu}\frac{(-\theta)^{\mu}}{\mu!}\frac{\sin^{(\mu+p)}(\theta x)}{x^{a+b+c-\mu}} = i^{p-1}\sum_{N=0}^{\infty}(-1)^b\binom{N-a}{b}\frac{(i\theta)^N}{N!}\lambda_{p-1+N}\ x^{-a-b-c+N},$$

where we let $\lambda_n=(1+(-1)^n)/2$ and denote the pth derivative of $\sin X$ by $\sin^{(p)}X$ and $\sin^{(p)}X\big|_{X=\alpha}$ by $\sin^{(p)}(\alpha)$ for $\alpha\in\mathbb{R}$. Let $(a,b,c,p)=(d+1,2j+1,s_3,0)$ and $x=l+m$ in (5.37). Then

$$\sum_{\mu=0}^{2j+1}\binom{d+2j+1-\mu}{2j+1-\mu}\frac{(-\theta)^{\mu}}{\mu!}\sum_{l,m=1}^{\infty}\frac{(-u)^{-l-m}\sin^{(\mu)}((l+m)\theta)}{l^{s_1}m^{s_2}(l+m)^{d+2j+2+s_3-\mu}} = i^{-1}\sum_{\nu=0}^{\infty}(-1)^{2j+1}\binom{2\nu-d}{2j+1}R_1(s_1,s_2,s_3+2j+1+d-2\nu;u)\frac{(i\theta)^{2\nu+1}}{(2\nu+1)!},$$

for $u > 1$. Define

$$\mathfrak{L}_6(\theta; d; u) = 2\sum_{j=0}^{k}\phi(2k-2j;u)\sum_{\mu=0}^{2j+1}\binom{d+2j+1-\mu}{2j+1-\mu}\frac{(-\theta)^{\mu}}{\mu!} \times \sum_{l,m=1}^{\infty}\frac{(-u)^{-l-m}\sin^{(\mu)}((l+m)\theta)}{l^{s_1}m^{s_2}(l+m)^{d+2j+2+s_3-\mu}} \tag{5.38}$$

for $\theta \in \mathbb{R}$ and $u \in [1, 1+\delta]$. Then the above calculations show that

$$\mathfrak{L}_6(\theta; d; u) = -2\sum_{\nu=0}^{\infty}\sum_{j=0}^{k}\binom{2\nu-d}{2j+1}\phi(2k-2j;u) \times R_1(s_1, s_2, s_3+2j+1+d-2\nu; u)\frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!},$$

when $u > 1$. It follows from (5.35), (5.36) and the above consideration that

$$\mathfrak{L}_1(\theta; d; u) - \mathfrak{L}_2(\theta; d; u) - \mathfrak{L}_3(\theta; d; u) - \mathfrak{L}_4(\theta; d; u) - \mathfrak{L}_5(\theta; d; u) + \mathfrak{L}_6(\theta; d; u) = \sum_{\nu=0}^{\infty}\widetilde{\mathfrak{C}}_{2\nu-d}(u)\frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!}, \tag{5.39}$$

when $u > 1$. From (5.27), the right-hand side of (5.39) is uniformly convergent with respect to $u \in [1, 1+\delta]$ when $\theta \in (-\pi, \pi)$. Hence (5.39) holds for $u = 1$. By Lemma 5.6, we obtain

$$\mathfrak{L}_1(\theta; d; 1) - \mathfrak{L}_2(\theta; d; 1) - \mathfrak{L}_3(\theta; d; 1) - \mathfrak{L}_4(\theta; d; 1) - \mathfrak{L}_5(\theta; d; 1) + \mathfrak{L}_6(\theta; d; 1) = \sum_{\nu=0}^{[d/2]}\widetilde{\mathfrak{C}}_{2\nu-d}(1)\frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!}, \tag{5.40}$$

when $\theta \in (-\pi, \pi)$, where $[x]$ is the integer part of x. Since $d \in \mathbb{N}$, the both sides of (5.40) are continuous for $\theta \in [-\pi, \pi]$. Hence (5.40) holds for $\theta = \pi$. However we see that $\mathfrak{L}_j(\pi; d; 1) = 0$ $(1 \leqslant j \leqslant 5)$. Thus we obtain

$$\frac{1}{\pi}\mathfrak{L}_6(\pi; d; 1) = \sum_{\nu=0}^{[d/2]}\widetilde{\mathfrak{C}}_{2\nu-d}(1)\frac{(-1)^{\nu}\pi^{2\nu}}{(2\nu+1)!}. \tag{5.41}$$

Furthermore, let $d = 2l + q \geqslant 2$ with $l \in \mathbb{N}$ and $q \in \{0, 1\}$. Then we can differentiate (5.40) with respect to θ and the obtained equation holds for

$\theta = \pi$ because $d \geqslant 2$. We have

(5.42)
$$\mathfrak{L}_1'(\pi; 2l+q; 1) - \mathfrak{L}_2'(\pi; 2l+q; 1) - \mathfrak{L}_3'(\pi; 2l+q; 1) - \mathfrak{L}_4'(\pi; 2l+q; 1) - \mathfrak{L}_5'(\pi; 2l+q; 1) + \mathfrak{L}_6'(\pi; 2l+q; 1) = \sum_{\nu=0}^{l} \widetilde{\mathfrak{C}}_{2\nu-2l-q}(1) \frac{(-1)^\nu \pi^{2\nu}}{(2\nu)!}.$$

Now we make use of the following lemma in [21].

Lemma 5.8 ([21] Lemma 4.4). — *Let $\{\alpha_{2l}\}_{l\in\mathbb{N}_0}$, $\{\beta_{2l}\}_{l\in\mathbb{N}_0}$, $\{\gamma_{2l}\}_{l\in\mathbb{N}_0}$ be sequences such that*

$$\alpha_{2l} = \sum_{\nu=0}^{l} \gamma_{2l-2\nu} \frac{(i\pi)^{2\nu}}{(2\nu)!}, \ \beta_{2l} = \sum_{\nu=0}^{l} \gamma_{2l-2\nu} \frac{(i\pi)^{2\nu}}{(2\nu+1)!}$$

for any $l \in \mathbb{N}_0$. Then

$$\alpha_{2l} = -2 \sum_{\nu=0}^{l} \zeta(2l-2\nu)\, \beta_{2\nu}$$

for any $l \in \mathbb{N}_0$.

In this lemma, we let α_{2l} and β_{2l} be the left-hand side of (5.41) and (5.42) with $d = 2l+q$ for $l \in \mathbb{N}$, respectively, and $\alpha_0 = \beta_0 = \widetilde{\mathfrak{C}}_{-q}(1)$. Note that $\widetilde{\mathfrak{C}}_0(1)$ is determined by Lemma 5.6, and $\widetilde{\mathfrak{C}}_{-1}(1) = (1/\pi)\mathfrak{L}_6(\pi; 1; 1)$ by (5.41). Furthermore, let $\gamma_{2l} = \widetilde{\mathfrak{C}}_{-2l-q}(1)$ for $l \in \mathbb{N}_0$. Then Lemma 5.8 gives that for $l \in \mathbb{N}$ and $q = 0, 1$,

$$\begin{aligned} (5.43) \quad & \mathfrak{L}_1'(\pi; 2l+q; 1) - \mathfrak{L}_2'(\pi; 2l+q; 1) - \mathfrak{L}_3'(\pi; 2l+q; 1) \\ & \quad - \mathfrak{L}_4'(\pi; 2l+q; 1) - \mathfrak{L}_5'(\pi; 2l+q; 1) + \mathfrak{L}_6'(\pi; 2l+q; 1) \\ & = -\frac{2}{\pi} \sum_{\nu=1}^{l} \zeta(2l-2\nu) \mathfrak{L}_6(\pi; 2\nu+q; 1) - 2\zeta(2l)\widetilde{\mathfrak{C}}_{-q}(1). \end{aligned}$$

The rest of our work in this section is to determine each term in (5.43) explicitly. By (5.2) and (5.36), we can easily check that

$$\begin{aligned} \mathfrak{L}_1'(\pi; d; 1) &= \mathcal{T}(s_1, s_2, 2k+1, s_3, d); \\ \mathfrak{L}_2'(\pi; d; 1) &= (-1)^d \mathcal{T}(s_1, s_2, d, s_3, 2k+1); \\ \mathfrak{L}_3'(\pi; d; 1) &= \mathcal{Z}(2k+1, s_2, s_1, d, s_3); \\ \mathfrak{L}_4'(\pi; d; 1) &= \mathcal{Z}(s_1, d, 2k+1, s_2, s_3); \\ \mathfrak{L}_5'(\pi; d; 1) &= \zeta_{\mathfrak{sl}(3)}(2k+1+s_1, s_2+d, s_3). \end{aligned}$$

Also from (5.38) we can see that

$$\mathfrak{L}_6(\pi;d;1) = -2\sum_{j=0}^{k}\phi(2k-2j;1)\sum_{\rho=0}^{j}\binom{d+2j-2\rho}{2j-2\rho}\frac{(-1)^\rho\pi^{2\rho+1}}{(2\rho+1)!} \times \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+d+2j+1-2\rho), \tag{5.44}$$

and

$$\mathfrak{L}'_6(\theta;d;1) = 2\sum_{j=0}^{k}\phi(2k-2j;1)\sum_{\mu=0}^{2j+1}\binom{d+2j-\mu}{2j+1-\mu}\frac{(-\theta)^\mu}{\mu!} \times \sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\sin^{(\mu+1)}((l+m)\theta)}{l^{s_1}m^{s_2}(l+m)^{d+2j+1+s_3-\mu}}, \tag{5.45}$$

because

$$\binom{X+1}{Y+1}-\binom{X}{Y}=\binom{X}{Y+1}.$$

Hence we have

$$\mathfrak{L}'_6(\pi;d;1) = 2\sum_{j=0}^{k}\phi(2k-2j;1)\sum_{\rho=0}^{j}\binom{d+2j-2\rho}{2j+1-2\rho}\frac{(-1)^\rho\pi^{2\rho}}{(2\rho)!} \times \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+d+2j+1-2\rho).$$

Substituting these relations into (5.43), we obtain the following.

Theorem 5.9. — *Let $k\in\mathbb{N}_0$, $l\in\mathbb{N}$ and $q\in\{0,1\}$. Then*

(5.46)

$$\begin{aligned}
&\mathcal{T}(s_1,s_2,2k+1,s_3,2l+q)-(-1)^q\mathcal{T}(s_1,s_2,2l+q,s_3,2k+1)\\
&\qquad -\mathcal{Z}(2k+1,s_2,s_1,2l+q,s_3)-\mathcal{Z}(s_1,2l+q,2k+1,s_2,s_3)\\
&= \zeta_{\mathfrak{sl}(3)}(s_1+2k+1,s_2+2l+q,s_3)\\
&\quad -2\sum_{j=0}^{k}\phi(2k-2j)\sum_{\rho=0}^{j}\binom{2l+q+2j-2\rho}{2j+1-2\rho}\frac{(-1)^\rho\pi^{2\rho}}{(2\rho)!}\\
&\qquad\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+2l+2j+q+1-2\rho)\\
&\quad +4\sum_{\nu=1}^{l}\zeta(2l-2\nu)\sum_{j=0}^{k}\phi(2k-2j)\sum_{\rho=0}^{j}\binom{2\nu+q+2j-2\rho}{2j-2\rho}\frac{(-1)^\rho\pi^{2\rho}}{(2\rho+1)!}\\
&\qquad\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+2\nu+2j+q+1-2\rho)\\
&\quad -2\zeta(2l)\widetilde{\mathfrak{C}}_{-q}(2k+1;\mathbf{s};1)
\end{aligned}$$

holds for all $s_1, s_2, s_3 \in \mathbb{C}$ except for the singular points of each side determined by Lemma 3.1 and Theorem 3.5, where $\phi(s) = (2^{1-s} - 1)\zeta(s)$ and

$$\widetilde{\mathfrak{C}}_{-q}(2k+1; \mathbf{s}; 1) = \begin{cases} \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2k + 1) & (\text{if } q = 0); \\ -2 \displaystyle\sum_{j=0}^{k} \phi(2k-2j) \sum_{\rho=0}^{j} (2j - 2\rho + 1) \frac{(-1)^\rho \pi^{2\rho}}{(2\rho+1)!} & \\ \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2j + 2 - 2\rho) & (\text{if } q = 1). \end{cases}$$

By the same method as in the proof of Theorem 5.9 replacing $2k+1$ with $2k$, we obtain the following.

Theorem 5.10. — *Let $k, l \in \mathbb{N}$ and $q \in \{0, 1\}$. Then*

(5.47)

$$\begin{aligned} &\mathcal{T}(s_1, s_2, 2k, s_3, 2l+q) + (-1)^q \mathcal{T}(s_1, s_2, 2l+q, s_3, 2k) \\ &\qquad + \mathcal{Z}(2k, s_2, s_1, 2l+q, s_3) + \mathcal{Z}(s_1, 2l+q, 2k, s_2, s_3) \\ &= -\zeta_{\mathfrak{sl}(3)}(s_1 + 2k, s_2 + 2l + q, s_3) \\ &\quad + 2 \sum_{j=0}^{k} \phi(2k-2j) \sum_{\rho=0}^{j} \binom{2l+q+2j-2\rho-1}{2j-2\rho} \frac{(-1)^\rho \pi^{2\rho}}{(2\rho)!} \\ &\qquad\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2l + 2j + q - 2\rho) \\ &\quad - 4 \sum_{\nu=1}^{l} \zeta(2l-2\nu) \sum_{j=1}^{k} \phi(2k-2j) \sum_{\rho=0}^{j-1} \binom{2\nu+q+2j-2\rho-1}{2j-2\rho-1} \frac{(-1)^\rho \pi^{2\rho}}{(2\rho+1)!} \\ &\qquad\qquad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2\nu + 2j + q - 2\rho) \\ &\quad - 2\zeta(2l) \widetilde{\mathfrak{C}}^*_{-q}(2k; \mathbf{s}; 1) \end{aligned}$$

holds for all $s_1, s_2, s_3 \in \mathbb{C}$ except for the singular points of each side determined by Lemma 3.1 and Theorem 3.5, where

$$\widetilde{\mathfrak{C}}^*_{-q}(2k; \mathbf{s}; 1) = \begin{cases} -\zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2k) & (\text{if } q = 0); \\ 2 \displaystyle\sum_{j=1}^{k} \phi(2k-2j) \sum_{\rho=0}^{j-1} (2j - 2\rho) \frac{(-1)^\rho \pi^{2\rho}}{(2\rho+1)!} & \\ \qquad \times \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + 2j + 1 - 2\rho) & (\text{if } q = 1). \end{cases}$$

Example 5.11. — As a concrete example, we verify (5.46) in the case $(k,l,q) = (0,1,0)$, namely

(5.48)
$$\begin{aligned}\mathcal{T}(s_1,s_2,1,s_3,2) - \mathcal{T}(s_1,s_2,2,s_3,1) - \mathcal{Z}(1,s_2,s_1,2,s_3) - \mathcal{Z}(s_1,2,1,s_2,s_3)\\ = 3\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+3) + \zeta_{\mathfrak{sl}(3)}(s_1+1,s_2+2,s_3)\\ - 2\zeta(2)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+1).\end{aligned}$$

Indeed, for $\mathbf{s} = (s_1,s_2,s_3) \in \mathbb{C}$, it follows from (5.41) in the case $d = 2$ that

$$\frac{1}{\pi}\mathfrak{L}_6(\pi;2;1) = \widetilde{\mathfrak{C}}_{-2}(1) - \widetilde{\mathfrak{C}}_0(1)\frac{\pi^2}{6}. \tag{5.49}$$

>From (5.44), we have $\mathfrak{L}_6(\pi;2;1) = \pi\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+3)$ because $\phi(0) = -1/2$. Hence, by Lemma 5.6 in the case $k = 0$, we have

$$\widetilde{\mathfrak{C}}_{-2}(1) = \zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+3) + \frac{\pi^2}{6}\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+1). \tag{5.50}$$

Similarly, from (5.42) in the case $(l,q) = (1,0)$, we have

$$\begin{aligned}\mathfrak{L}'_1(\pi;2;1) - \mathfrak{L}'_2(\pi;2;1) - \mathfrak{L}'_3(\pi;2;1) - \mathfrak{L}'_4(\pi;2;1)\\ - \mathfrak{L}'_5(\pi;2;1) + \mathfrak{L}'_6(\pi;2;1) = \widetilde{\mathfrak{C}}_{-2}(1) - \widetilde{\mathfrak{C}}_0(1)\frac{\pi^2}{2}.\end{aligned} \tag{5.51}$$

Considering the derivation of each $\mathfrak{L}_j(\theta;2;1)$ in (5.36) with respect to θ and letting $\theta \to \pi$, we see that the left-hand side of (5.51) equals to

$$\begin{aligned}\mathcal{T}(s_1,s_2,1,s_3,2) - \mathcal{T}(s_1,s_2,2,s_3,1) - \mathcal{Z}(1,s_2,s_1,2,s_3) - \mathcal{Z}(s_1,2,1,s_2,s_3)\\ - \zeta_{\mathfrak{sl}(3)}(s_1+1,s_2+2,s_3) - 2\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+3).\end{aligned}$$

On the other hand, it follows from (5.50) and Lemma 5.6 that the right-hand side of (5.51) equals to

$$\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+3) - \frac{\pi^2}{3}\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+1).$$

Thus we obtain (5.48).

Similarly, applying Theorem 5.9 with $(k,l,q) = (0,1,1)$, we have

$$\begin{aligned}\mathcal{T}(s_1,s_2,1,s_3,3) + \mathcal{T}(s_1,s_2,3,s_3,1) - \mathcal{Z}(1,s_2,s_1,3,s_3) - \mathcal{Z}(s_1,3,1,s_2,s_3)\\ = 4\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+4) + \zeta_{\mathfrak{sl}(3)}(s_1+1,s_2+3,s_3)\\ - 2\zeta(2)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+2).\end{aligned}$$

Also, applying Theorem 5.10 with $(k,l,q)=(1,1,0)$, we have

$$\begin{aligned}(5.52)\qquad 2\mathcal{T}(s_1,s_2,&2,s_3,2)+\mathcal{Z}(2,s_2,s_1,2,s_3)+\mathcal{Z}(s_1,2,2,s_2,s_3)\\ &=-6\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+4)-\zeta_{\mathfrak{sl}(3)}(s_1+2,s_2+2,s_3)\\ &\quad+4\zeta(2)\zeta_{\mathfrak{sl}(3)}(s_1,s_2,s_3+2),\end{aligned}$$

because $\phi(2)=-(1/2)\zeta(2)=-\pi^2/12$.

For example, we numerically check (5.48) in the case $(s_1,s_2,s_3)=(1,3.45,1.98)$. We can see that

$$\begin{aligned}\mathcal{T}(1,3.45,1,1.98,2)&=0.0555888600\cdots\\ \mathcal{T}(1,3.45,2,1.98,1)&=0.1504014027\cdots\\ \mathcal{Z}(1,3.45,1,2,1.98)&=0.0244657460\cdots\\ \mathcal{Z}(1,2,1,3.45,1.98)&=0.0080362256\cdots\\ \zeta_{\mathfrak{sl}(3)}(1,3.45,4.98)&=0.0347323323\cdots\\ \zeta_{\mathfrak{sl}(3)}(2,5.45,1.98)&=0.2977616216\cdots\\ \zeta_{\mathfrak{sl}(3)}(1,3.45,2.98)&=0.1608797409\cdots\end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{T}(1,3.45,1,&1.98,2)-\mathcal{T}(1,3.45,2,1.98,1)-\mathcal{Z}(1,3.45,1,1.98,2)\\ &-\mathcal{Z}(1,2,1,3.45,1.98)-\big\{3\zeta_{\mathfrak{sl}(3)}(1,3.45,4.98)\\ &+\zeta_{\mathfrak{sl}(3)}(2,5.45,1.98)-2\zeta(2)\zeta_{\mathfrak{sl}(3)}(1,3.45,2.98)\big\}\\ &=0.000000000\cdots\end{aligned}$$

which means that (5.48) is correct numerically.

Example 5.12. — >From these functional relations we can further deduce non-trivial relations among the values at integer points. For example, putting $(s_1,s_2,s_3)=(1,1,2)$ in (5.52) and using $\mathcal{Z}(p,q,r,s,t)=\mathcal{Z}(q,p,s,r,t)$, we obtain

$$\begin{aligned}(5.53)\quad \mathcal{T}(1,1,2,2,2)&+\mathcal{Z}(1,2,2,1,2)\\ &=-3\zeta_{\mathfrak{sl}(3)}(1,1,6)-\frac{1}{2}\zeta_{\mathfrak{sl}(3)}(3,3,2)+2\zeta(2)\zeta_{\mathfrak{sl}(3)}(1,1,4).\end{aligned}$$

To proceed further, first we will prove

$$(5.54)\quad \mathcal{T}(1,1,2,2,2)=2\zeta(1,3,4)+2\zeta(2,1,5)+8\zeta(1,2,5)+18\zeta(1,1,6),$$

where $\zeta(p,q,r)=\sum_{1\leqslant l<m<n} l^{-p}m^{-q}n^{-r}$ is the triple zeta value. In order to prove (5.54), we make use of

$$\begin{cases} \dfrac{1}{lm}=\dfrac{1}{l(l+m)}+\dfrac{1}{m(l+m)}; \\ \dfrac{1}{n(l+m)}=\dfrac{1}{n(l+m+n)}+\dfrac{1}{(l+m)(l+m+n)}; \\ \dfrac{1}{lmn}=\dfrac{1}{lm(l+m+n)}+\dfrac{1}{ln(l+m+n)}+\dfrac{1}{mn(l+m+n)}. \end{cases} \tag{5.55}$$

By using these relations, we can easily find that $\mathcal{T}(1,1,0,0,6)=2\zeta(1,1,6)$, $\mathcal{T}(1,0,1,1,5)=3\zeta(1,1,6)$ and $\mathcal{T}(1,2,0,0,5)=2\zeta(1,2,5)+\zeta(2,1,5)$. Hence, by these relations, we have

$$\begin{cases} \mathcal{T}(1,1,0,2,4)=2\zeta(1,3,4); \\ \mathcal{T}(1,0,1,2,4)=3\zeta(1,1,6)+\zeta(1,2,5); \\ \mathcal{T}(1,0,2,2,3)=3\zeta(1,2,5)+\zeta(2,1,5)+6\zeta(1,1,6). \end{cases} \tag{5.56}$$

>From (5.55), we further see that

$$\mathcal{T}(1,1,2,2,2)=\mathcal{T}(1,1,0,2,4)+2\mathcal{T}(1,0,1,2,4)+2\mathcal{T}(1,0,2,2,3). \tag{5.57}$$

Substituting (5.56) into (5.57), we obtain (5.54).

Next, using the method of Borwein and Girgensohn [4] (see also [2, 3]: the table of multiple zeta values), we can evaluate the right-hand side of (5.54) in terms of double and single zeta values, that is

$$\mathcal{T}(1,1,2,2,2)=-\frac{61}{175}\zeta(2)^4+4\zeta(3)\zeta(5)-\zeta(2)\zeta(3)^2-\frac{1}{2}\zeta(2,6). \tag{5.58}$$

On the other hand, using the method in [8], we can evaluate $\zeta_{\mathfrak{sl}(3)}(p,q,r)$ such as

$$\begin{aligned} \zeta_{\mathfrak{sl}(3)}(1,1,6)&=\frac{12}{35}\zeta(2)^4-2\zeta(3)\zeta(5); \\ \zeta_{\mathfrak{sl}(3)}(3,3,2)&=\frac{66}{175}\zeta(2)^4-2\zeta(3)\zeta(5)+\zeta(2,6); \\ \zeta_{\mathfrak{sl}(3)}(1,1,4)&=\frac{12}{35}\zeta(2)^3-\zeta(3)^2. \end{aligned}$$

Hence, by (5.53) and (5.58), we have

$$\begin{aligned} \mathcal{Z}(1,2,2,1,2)&=\zeta_{\mathfrak{sl}(4)}(1,0,2,2,1,2) \\ &=-\frac{32}{175}\zeta(2)^4+3\zeta(3)\zeta(5)-\zeta(2)\zeta(3)^2. \end{aligned} \tag{5.59}$$

Thus we obtain

$$\begin{aligned}(5.60)\qquad \zeta_{\mathfrak{sl}(4)}(1,1,1,2,1,2) &= \mathcal{T}(1,1,2,2,2) - \mathcal{Z}(1,2,2,1,2)\\ &= -\frac{29}{175}\zeta(2)^4 + \zeta(3)\zeta(5) - \frac{1}{2}\zeta(2,6),\end{aligned}$$

which can be regarded as a non-trivial analogue of the Gunnells-Sczech formula (1.7) for $\zeta_{\mathfrak{sl}(4)}(2,2,2,2,2,2)$. Similarly we can obtain, for example,

$$\zeta_{\mathfrak{sl}(4)}(1,1,2,1,2,1) = \frac{2683}{1050}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 16\zeta(3)\zeta(5) + \frac{29}{4}\zeta(2,6);$$

$$\zeta_{\mathfrak{sl}(4)}(1,1,1,2,1,3) = \frac{2}{5}\zeta(2)^2\zeta(5) + 10\zeta(2)\zeta(7) - \frac{53}{3}\zeta(9).$$

Remark 5.13. — It is well-known that $\zeta(1-n) = -B_n/n$ for $n \in \mathbb{N}$, where $\{B_n\}$ are the Bernoulli numbers defined by

$$t/(e^t-1) = \sum_{n\geqslant 0} B_n t^n/n!$$

(see [9]). We can regard $\{\widetilde{\mathfrak{C}}_n(u)\}$ as analogues of $\{B_n\}$. In fact, $\widetilde{\mathfrak{C}}_n(u)$ is the value of a certain finite sum of multiple zeta-functions at nonpositive integers, defined by the right-hand side of (5.35). Furthermore the result in Lemma 5.6 corresponds to the well-known fact that $B_{2n+1}=0$ for $n \in \mathbb{N}$. Additionally, it is important that the radius of convergence of the generating function $G_3(t;2k+1;\mathbf{s};u)$ of $\{\mathfrak{C}_n(u)\}$ (see (5.26)) is π. This corresponds to the fact that the radius of convergence of $t/(e^t-1)$ is 2π. Hence it seems that (5.46) and (5.47) correspond to well-known Euler's formula for $\zeta(2k)$. In the proof of Theorem 5.9, we fixed $s_1, s_2, s_3 \in \mathbb{C}$. However each function on both sides of (5.46) and (5.47) is meromorphic on $\mathbb{C}^3$. Hence we can regard (5.46) and (5.47) as functional relations. If we can find other multiple analogues of Bernoulli numbers having nice properties like $\widetilde{\mathfrak{C}}_n(u)$ as mentioned above, we might be able to prove functional relations among certain multiple zeta-functions corresponding to those numbers, like (5.46) and (5.47). The authors are now studying Witten zeta-functions associated with some other Lie algebras using this method, and will report on these results in the next paper.

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