



# ANNALES

DE

# L'INSTITUT FOURIER

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**Puiseux series polynomial dynamics and iteration of complex cubic polynomials**

Tome 56, n° 5 (2006), p. 1337-1404.

[http://aif.cedram.org/item?id=AIF\\_2006\\_\\_56\\_5\\_1337\\_0](http://aif.cedram.org/item?id=AIF_2006__56_5_1337_0)

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# PUISEUX SERIES POLYNOMIAL DYNAMICS AND ITERATION OF COMPLEX CUBIC POLYNOMIALS

by Jan KIWI (\*)

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ABSTRACT. — We let  $\mathbb{L}$  be the completion of the field of formal Puiseux series and study polynomials with coefficients in  $\mathbb{L}$  as dynamical systems. We give a complete description of the dynamical and parameter space of cubic polynomials in  $\mathbb{L}[\zeta]$ . We show that cubic polynomial dynamics over  $\mathbb{L}$  and  $\mathbb{C}$  are intimately related. More precisely, we establish that some elements of  $\mathbb{L}$  naturally correspond to the Fourier series of analytic almost periodic functions (in the sense of Bohr) which parametrize (near infinity) the quasiconformal classes of non-renormalizable complex cubic polynomials. Our techniques are based on the ideas introduced by Branner and Hubbard to study complex cubic polynomials.

RÉSUMÉ. — Nous considérons la complétion  $\mathbb{L}$  du corps des séries formelles de Puiseux et nous étudions les polynômes à coefficients dans  $\mathbb{L}$  en tant que systèmes dynamiques. Nous donnons une description complète de l'espace dynamique et l'espace des paramètres des polynômes cubiques à coefficients dans  $\mathbb{L}$ . Nous démontrons que la dynamique cubique sur  $\mathbb{L}$  et sur  $\mathbb{C}$  sont intimement liées. Plus précisément, nous montrons que certains éléments de  $\mathbb{L}$  correspondent de manière naturelle à des séries de Fourier de fonctions analytiques presque périodiques (au sens de Bohr) qui paramétrisent (à l'infini) les classes quasi-conformes des polynômes complexes cubiques non renormalisables. Nos techniques s'appuient sur des idées introduites par Branner et Hubbard pour l'étude des polynômes cubiques complexes.

## 1. Introduction

The aim of this paper is to study the dynamics of polynomials over the completion  $\mathbb{L}$  of the field of formal Puiseux series with coefficients in an algebraic closure of  $\mathbb{Q}$  (see Subsection 2.1). Our interest arises from the extensive research on the dynamics of rational functions over  $\mathbb{C}$  and the

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*Keywords:* Puiseux series, Julia sets.

*Math. classification:* 37F45, 12J25, 32S99.

(\*) Supported by "Proyecto Fondecyt #1020711".

recent one over  $\mathbb{C}_p$ . Non-Archimedean fields such as  $\mathbb{L}$  seem to be a natural dynamical space to explore the interplay between non-Archimedean and complex dynamics. The focus of this paper is on cubic polynomials. We will show that the techniques developed by Branner and Hubbard [10, 11] to study complex cubic polynomials merge with some basic ideas from  $p$ -adic dynamics to give a complete picture of the dynamical behavior and the parameter space of cubic polynomials with coefficients in  $\mathbb{L}$ . Although for simplicity we restrict to  $\mathbb{L}$ , our description extends to cubic polynomials with coefficients in any complete algebraically closed non-Archimedean field with residual field of characteristic different than 2 and 3, and dense valuation group.

In this paper we show that the dynamics of a family of cubic polynomials acting on  $\mathbb{L}$  is intimately related to the structure of the parameter space of complex cubic polynomials near infinity. In particular, we show that some elements of  $\mathbb{L}$  naturally correspond to the Fourier series of analytic almost periodic functions (in the sense of Bohr) which parametrize (near infinity) the quasiconformal classes of non-renormalizable complex cubic polynomials.

Let us now summarize our results regarding dynamics over  $\mathbb{L}$ . Given a degree  $d \geq 2$  polynomial  $\varphi$  with coefficients in  $\mathbb{L}$ , in analogy with complex polynomial dynamics (see Section 18 in [23]), the set of non-escaping points is the filled Julia set  $K(\varphi)$  and its boundary  $J(\varphi)$  is called the Julia set of  $\varphi$  (see Chapter 6 in [29]). In complex dynamics it is useful to study the connected components of the filled Julia set. Non-Archimedean fields are totally disconnected and, following Rivera [29], the analogue discussion requires to replace the definition of connected components by the weaker notion of infraconnected components (see Subsection 2.5). As in complex dynamics, the behavior of the critical points under iterations also plays a central rôle in the study of dynamics over  $\mathbb{L}$ . Our first result, which is the analogue of one by Branner and Hubbard for complex cubic polynomials, describes the geometry of the filled Julia set according to the behavior of the critical points.

**THEOREM 1.1.** — *Let  $\varphi \in \mathbb{L}[\zeta]$  be a cubic polynomial. Then one of the following three (exclusive) possibilities hold:*

- (i) *Both critical points have bounded orbits. Then  $\varphi$  is simple,  $K(\varphi)$  is a closed ball, and  $J(\varphi)$  is empty.*
- (ii) *Both critical points escape to infinity. Then  $K(\varphi) = J(\varphi)$  is a compact set (and thus a Cantor set). The dynamics over  $J(\varphi)$  is topologically equivalent to the one-sided shift on three symbols.*

- (iii) *One critical point escapes and the other one has bounded orbit. Two sub-cases appear. Let  $U$  be the infraconnected component of  $K(\varphi)$  that contains the critical point with bounded orbit.*
  - (a) *If  $U$  is not periodic, then  $K(\varphi) = J(\varphi)$  is a non-empty compact (Cantor) set.*
  - (b) *If  $U$  is periodic, then it is a closed ball. In this case, an infraconnected component  $V$  of  $K(\varphi)$  is either a closed ball or a point according to whether  $\varphi^n(V) = U$  for some  $n \geq 0$  or not.*

Recent results in  $p$ -adic dynamics of rational maps reveal the convenience of studying the action of rational maps over the “Berkovich projective line” (e.g., see [2, 17]). For simplicity, we will restrict to polynomial dynamics over  $\mathbb{L}$  since on one hand our techniques do not require to pass to the Berkovich space and on the other hand the expert reader will immediately visualize the Berkovich space dynamical consequences of our results.

Although the precise definition of  $\varphi$  being “simple” is given in Subsection 2.5, intuitively this means that the study of the dynamics of  $\varphi$  reduces to that of a cubic polynomial  $\tilde{\varphi} : \mathbb{Q}^a \rightarrow \mathbb{Q}^a$ , where  $\mathbb{Q}^a$  is an algebraic closure of  $\mathbb{Q}$ . Case (ii) holds in greater generality. More precisely, in Section 3 we show that a polynomial  $\varphi$  of degree  $d \geq 2$  with all its critical points escaping has a Cantor set as Julia set  $J(\varphi)$  and the dynamics over  $J(\varphi)$  is topologically equivalent to the one-sided shift on  $d$  symbols. Note that only in case (iii a) the Julia set contains a critical point.

As an immediate consequence of the theorem above we obtain the following result.

**COROLLARY 1.2.** — *For any cubic polynomial  $\varphi \in \mathbb{L}[\zeta]$ , every infraconnected component of the filled Julia set is either a singleton or an eventually periodic closed ball.*

According to Benedetto [5], Sullivan’s no wandering domain Theorem (e.g., see Appendix F in [24]) does not hold for  $p$ -adic polynomials. That is, there exist polynomials in  $\mathbb{C}_p[\zeta]$  with non-trivial wandering infraconnected components of their filled Julia set. Moreover, according to Fernandez [18], this phenomenon occurs in a rather large subset of parameter space. It is reasonable to conjecture that over fields with residual characteristic zero, such as  $\mathbb{L}$ , every non-trivial infraconnected component of the filled Julia set is eventually periodic (compare with [3]).

In parameter space we work in the space  $\mathcal{P}_{\mathbb{L}} \equiv \mathbb{L}^2$  where to each pair  $(\alpha, \nu) \in \mathbb{L}^2$  we associate the polynomial

$$\varphi_{\alpha, \nu}(\zeta) = \zeta^3 - 3\alpha^2\zeta + 2\alpha^3 + \nu.$$

Note that the critical points of  $\varphi_{\alpha,\nu}$  are  $\pm\alpha$  and that  $\nu$  is the critical value  $\varphi_{\alpha,\nu}(\alpha)$ . Every cubic polynomial is affinely conjugate to at least one in this family. Moreover, after identification of  $(\alpha, \nu)$  with  $(-\alpha, -\nu)$  the space  $\mathcal{P}_{\mathbb{L}}$  becomes the moduli space of cubic polynomials with marked critical points. The above parameter space is the analogous of the one used in [23] to study complex cubic polynomials.

Parameter space is subdivided according to dynamics as follows. We say that the set of parameters  $(\alpha, \nu)$  so that the associated polynomial has an infraconnected filled Julia set is the *infraconnectedness locus*  $\mathcal{C}_{\mathbb{L}}$ . From the previous theorem it is easy to conclude that  $\mathcal{C}_{\mathbb{L}} = \{(\alpha, \nu) \in \mathcal{P}_{\mathbb{L}} \mid |\alpha|_o \leq 1, |\nu|_o \leq 1\}$ . The *shift locus*  $\mathcal{S}_{\mathbb{L}}$  is the set of parameters  $(\alpha, \nu)$  such that both critical points escape under iterations of the corresponding polynomial  $\varphi_{\alpha,\nu}$ . Our parameter space description will supply us with the location of the polynomials that fall into case (iii a) of the theorem above. More precisely, let  $\mathcal{A}_{\mathbb{L}}$  be the set of all  $(\alpha, \nu)$  such that under iterations of  $\varphi_{\alpha,\nu}$  one critical point escapes, the other one has bounded orbit, and the infraconnected component of the non-escaping critical point is not periodic.

**THEOREM 1.3.** — *The boundary of the shift locus is  $\mathcal{A}_{\mathbb{L}}$ .*

In particular, cubic polynomials with critical point free Julia sets are dense in parameter space. According to Benedetto [4] such polynomials exhibit some sort of hyperbolicity.

We also obtain the following characterization of cubic polynomials having compact Julia sets. Related results for polynomials over  $\mathbb{C}_p$  were obtained by Bezinin (see [7]).

**COROLLARY 1.4.** — *Let  $\varphi = \varphi_{\alpha,\nu}$  be a cubic polynomial. Then the following are equivalent:*

- (i) *The Julia set  $J(\varphi)$  is a compact non-empty set.*
- (ii)  *$J(\varphi) = K(\varphi)$ .*
- (iii) *All the cycles of  $\varphi$  are repelling.*
- (iv)  *$(\alpha, \nu)$  is in the closure of the shift locus  $\mathcal{S}_{\mathbb{L}}$ .*

In fact, our description of both, dynamical and parameter space, is far more detailed. A complete discussion is given in Sections 4–6. From this detailed description we are able to establish the existence of cubic polynomials with coefficients in  $\mathbb{Q}^a((t))$  which have a recurrent critical point (see Corollary 5.17 and compare with [28]).

Let us now outline our results regarding complex cubic polynomials. Following Milnor [23], we work in the parameter space  $\mathcal{P}_{\mathbb{C}} \equiv \mathbb{C}^2$  where the

complex cubic polynomial associated to  $(a, v)$  is

$$f_{a,v}(z) = z^3 - 3a^2z + 2a^3 + v.$$

The critical points of  $f_{a,v}$  are  $\pm a$  and  $v = f_{a,v}(a)$  is a critical value. Every complex cubic polynomial is affinely conjugate to at least one of the above form, and after identification of  $(a, v)$  with  $(-a, -v)$  the parameter space  $\mathcal{P}_{\mathbb{C}}$  becomes the moduli space of complex cubic polynomials with marked critical points up to affine conjugation.

Following Branner and Hubbard,  $\mathcal{P}_{\mathbb{C}}$  is subdivided according to how many critical points escape to  $\infty$ . The *connectedness locus*  $\mathcal{C}_{\mathbb{C}}$  is the compact subset of  $\mathcal{P}_{\mathbb{C}}$  formed by all  $(a, v)$  such that the polynomial  $f_{a,v}$  has connected Julia set (i.e., no critical point escapes to  $\infty$ ). The *shift locus*  $\mathcal{S}_{\mathbb{C}}$  consists of all parameters  $(a, v)$  so that the corresponding polynomial  $f_{a,v}$  has all its critical points in the basin of infinity. This set is open and unbounded. The rest of parameter space is the set  $\mathcal{E}$  formed by the parameters of polynomials  $f_{a,v}$  such that exactly one critical point escapes to  $\infty$  and the other one has bounded orbit.

Branner and Hubbard [10, 11] gave a fairly complete and beautiful description of the complement of the connectedness locus. Here we revisit the structure of parameter space near infinity. Our emphasis will be on the geometry of  $\mathcal{E}$ . This set is naturally subdivided into  $\mathcal{E}^{\pm}$  according to whether  $+a$  or  $-a$  is the escaping critical point. That is,

$$\mathcal{E}^{\pm} := \{(a, v) \in \mathcal{P}_{\mathbb{C}} \mid \pm a \notin K(f_{a,v}) \ni \mp a\}.$$

We may restrict to the study of  $\mathcal{E}^{-}$ , since parameter space is endowed with the (polynomial) involution  $(a, v) \rightarrow (-a, v + 4a^3)$  that switches the marking of the critical points, and therefore interchanges  $\mathcal{E}^{-}$  with  $\mathcal{E}^{+}$ .

Polynomials in  $\mathcal{E}^{-}$  can be either renormalizable or non-renormalizable. More precisely, given  $(a, v) \in \mathcal{E}^{-}$  we say that  $f_{a,v}$  is *renormalizable* if the connected component of  $K(f_{a,v})$  which contains the critical point  $+a$  is periodic (see [22]). Otherwise, we say that  $f_{a,v}$  is *non-renormalizable*. Therefore,  $\mathcal{E}^{-}$  splits into two sets:

$$\begin{aligned} \mathcal{R}_{\mathbb{C}} &:= \{(a, v) \in \mathcal{E}^{-} \mid f_{a,v} \text{ is renormalizable}\}, \\ \mathcal{NR}_{\mathbb{C}} &:= \mathcal{E}^{-} \setminus \mathcal{R}_{\mathbb{C}}. \end{aligned}$$

Branner and Hubbard reduced the description of  $\mathcal{R}_{\mathbb{C}}$  to that of the Mandelbrot set and gave a *complete* description of  $\mathcal{NR}_{\mathbb{C}}$ . The aim of Section 7 of this paper is to revisit Branner and Hubbard’s description for  $\mathcal{NR}_{\mathbb{C}}$  from a different, but not independent, perspective. More precisely, we will show that non-Archimedean dynamics naturally produces a model for  $\mathcal{NR}_{\mathbb{C}}$  and

that there is a natural homeomorphism from this model onto  $\mathcal{NR}_{\mathbb{C}}$  (in a neighborhood of infinity).

To study  $\mathcal{P}_{\mathbb{C}}$  near infinity it is convenient to compactify parameter space by adding a line  $\mathcal{L}_{\infty}$  at infinity and identify the resulting space with  $\mathbb{CP}^2 \equiv \mathcal{P}_{\mathbb{C}} \cup \mathcal{L}_{\infty}$ . Thus,  $\mathcal{P}_{\mathbb{C}} \equiv \{[a : v : 1] \mid (a, v) \in \mathbb{C}^2\}$  and  $\mathcal{L}_{\infty} \equiv \{[a : v : 0] \mid (a, v) \in \mathbb{C}^2\}$ . The closure of  $\mathcal{E}^-$  in  $\mathbb{CP}^2$  intersects the line  $\mathcal{L}_{\infty}$  at  $\{[1 : 1 : 0], [1 : -2 : 0]\}$  (see Corollary 7.5). Thus we will be interested on describing  $\mathcal{NR}_{\mathbb{C}}$  in a neighborhood of  $\{[1 : 1 : 0], [1 : -2 : 0]\}$ . More precisely, for  $\varepsilon > 0$ , we will consider the neighborhood

$$V_{\varepsilon} = \{[1 : \bar{v} : \bar{a}] \in \mathcal{P}_{\mathbb{C}} \mid 0 < |\bar{a}| < \varepsilon\}$$

and denote by  $\Pi_{\bar{v}} : V_{\varepsilon} \rightarrow \mathbb{C}$  the projection to the  $\bar{v}$ -coordinate.

In order to properly state our results we need to consider the *one-parameter* family of cubic polynomials with coefficients in  $\mathbb{L}$  given by:

$$\psi_{\nu}(\zeta) = t^{-2}(\zeta - 1)^2(\zeta + 2) + \nu, \quad \nu \in \mathbb{L}.$$

Note that  $\psi_{\nu}$  has critical points  $\omega^{\pm} = \pm 1$ . It is not difficult to check that, for all  $\nu$ , at most one critical point is in the filled Julia set  $K(\psi_{\nu})$ . The analogue of  $\mathcal{NR}_{\mathbb{C}}$  for this family is  $\mathcal{NR}_{\mathbb{L}}$ . By definition  $\mathcal{NR}_{\mathbb{L}}$  consists of all parameters  $\nu \in \mathbb{L}$  such that  $\omega^+ = +1 \in K(\psi_{\nu})$  and the infraconnected component of  $K(\psi_{\nu})$  containing  $\omega^+$  is not periodic. The choice of this family  $\psi_{\nu}$  will be justified after stating our main results.

For  $\varepsilon > 0$ , let

$$\mathbb{H}_{\varepsilon} = \left\{ h \in \mathbb{C} \mid \text{Im}(h) > -\frac{\log \varepsilon}{2\pi} \right\}.$$

**THEOREM 1.5.** — *There exists  $\epsilon > 0$  and a map  $\Phi : \mathbb{H}_{\epsilon} \times \mathcal{NR}_{\mathbb{L}} \rightarrow \mathcal{NR}_{\mathbb{C}} \cap V_{\epsilon}$  such that:*

- (i)  $\Phi(\cdot, \nu)$  is holomorphic for all  $\nu$ . Moreover,  $\Pi_{\bar{v}} \circ \Phi(\cdot, \nu)$  is an analytic almost periodic function in the sense of Bohr, for all  $\nu$ .
- (ii)  $\Phi(h, \cdot)$  is continuous for all  $h \in \mathbb{H}_{\epsilon}$ . Furthermore,  $\nu \mapsto \Pi_{\bar{v}} \circ \Phi(\cdot, \nu)$  is a continuous map from the topology of  $\mathbb{L}$  to the sup-norm topology on functions.
- (iii)  $\Phi$  is surjective.
- (iv) For all  $\nu \in \mathcal{NR}_{\mathbb{L}}$ , the critical marked grid (see Subsection 4.1) of  $\psi_{\nu}$  is the same as the critical marked grid of  $f_{\Phi(h, \nu)}$  for any  $h \in \mathbb{H}_{\epsilon}$ .

Every  $\nu \in \mathcal{NR}_{\mathbb{L}}$  is represented by a series of the form

$$\nu = \sum_{\lambda \in \Lambda} a_{\lambda} t^{\lambda}$$

where  $\Lambda \subset [0, +\infty) \cap \mathbb{Q}$  is a discrete subset of  $[0, +\infty)$  and  $a_\lambda \in \mathbb{Q}^a \subset \mathbb{C}$  (see Section 2). In a sense to be precised in Section 7, for  $\varepsilon > 0$  small,

$$\nu(e^{2\pi ih}) = \sum_{\lambda \in \Lambda} a_\lambda e^{2\pi i \lambda h}$$

converges for all  $h \in \mathbb{H}_\varepsilon$  to an analytic almost periodic function. The map  $\Phi$  of the previous theorem is simply given by

$$\Phi(h, \nu) = [1 : \nu(e^{2\pi ih}) : e^{2\pi ih}].$$

This map is not injective. In fact, let  $\sigma : \mathbb{L} \rightarrow \mathbb{L}$  be the unique Galois automorphism of  $\mathbb{L}$  over  $\mathbb{Q}^a((t))$  such that  $\sigma(t^{1/m}) = e^{2\pi i/m} t^{1/m}$  for all  $m \in \mathbb{N}$ . It will easily follow that  $\mathcal{NR}_{\mathbb{L}}$  is invariant under  $\sigma$  and

$$\Phi(h - 1, \sigma(\nu)) = \Phi(h, \nu).$$

Passing to the quotient we obtain a complete description of  $\mathcal{NR}_{\mathbb{C}}$  near infinity.

**THEOREM 1.6.** — *Define  $\Sigma$  to be the quotient of  $\mathbb{H}_\varepsilon \times \mathcal{NR}_{\mathbb{L}}$  by the identification  $(h - 1, \sigma(\nu)) \cong (h, \nu)$ , and let  $\varpi : \mathbb{H}_\varepsilon \times \mathcal{NR}_{\mathbb{L}} \rightarrow \Sigma$  be the natural projection. Then the map*

$$\begin{aligned} \Phi_\Sigma : \quad \Sigma &\rightarrow \mathcal{NR}_{\mathbb{C}} \cap V_\varepsilon \\ \varpi((h, \nu)) &\mapsto \Phi(h, \nu) \end{aligned}$$

*is well defined. Moreover,  $\Phi_\Sigma$  is a homeomorphism.*

From the complex dynamics viewpoint, the previous results do not add any extra information about the structure of  $\mathcal{NR}_{\mathbb{C}}$  to that already given in Branner and Hubbard’s work. The novelty resides on the construction of a model for  $\mathcal{NR}_{\mathbb{C}}$  which has a non-Archimedean dynamical nature and in the technique used to prove the properties of  $\Phi$  and to show that  $\Phi_\Sigma$  is a homeomorphism. We would like to stress that we do not only use Branner and Hubbard’s insight of cubic parameter space but also our proofs use two of the main ideas in [10] and [11]: marked grids and the wringing construction. Moreover, the picture given by Branner and Hubbard of the complement of the connectedness locus is far more complete than the one presented in this paper. Rather than summarizing this picture here we refer the reader to the excellent exposition given by Branner in [9].

Let us discuss the main ideas involved in the proofs of theorems 1.5 and 1.6. Periodic curves play a key rôle. For each  $n \in \mathbb{N}$ , the periodic curve  $\text{Per}(n) \subset \mathbb{C}^2 \equiv \mathcal{P}_{\mathbb{C}}$  is the algebraic set formed by all  $(a, v)$  such that the critical point  $+a$  has period exactly  $n$  under  $f_{a,v}$ . Periodic curves of cubic polynomials were studied by Milnor in [23]. The analogous curves for



quadratic rational functions have been extensively studied by Rees (e.g. see [26]). From general complex dynamics results,  $\mathcal{NR}_{\mathbb{C}} \setminus \text{int } \mathcal{NR}_{\mathbb{C}}$  is contained in the closure of  $\cup \text{Per}(n)$ . Although Branner and Hubbard proved that  $\text{int } \mathcal{NR}_{\mathbb{C}} = \emptyset$  we will give a different (but not independent) proof of this result. Thus, our strategy to describe  $\mathcal{NR}_{\mathbb{C}}$  near  $\mathcal{L}_{\infty}$  will be to study the branches of periodic curves at  $\mathcal{L}_{\infty}$  and then “pass to limit”. More precisely, Puiseux series of the branches of periodic curves at  $\mathcal{L}_{\infty}$  constitute the bridge which will allow us to move between dynamics over  $\mathbb{L}$  and dynamics over  $\mathbb{C}$ . The main reason being that  $\nu \in \mathbb{L}$  is the Puiseux series of a branch of  $\text{Per}(n)$  if and only if the critical point  $\omega^+$  has period exactly  $n$  under  $\psi_{\nu}$  (Corollary 7.15). So we let  $\text{Per}_{\mathbb{L}}$  be the set of all  $\nu \in \mathbb{L}$  which are the Puiseux series of a branch of some periodic curve and use our results about dynamics over  $\mathbb{L}$  to show that the closure  $\overline{\text{Per}_{\mathbb{L}}}$  of  $\text{Per}_{\mathbb{L}}$  in  $\mathbb{L}$  is the set  $\text{Per}_{\mathbb{L}} \cup \mathcal{NR}_{\mathbb{L}}$  (Corollary 7.16). We will show that the position of the branches of  $\text{Per}(n)$  at  $\mathcal{L}_{\infty}$  is quite special. In certain sense, they are “uniformly transversal” to  $\mathcal{L}_{\infty}$ . More precisely, the projection of any branch on a fixed small curve  $C$  transversal to  $\mathcal{L}_{\infty}$  is an unramified covering (Proposition 7.6). This gives a uniform “parametrization” of all branches of  $\cup \text{Per}(n)$  in a neighborhood of infinity by  $\mathbb{H}_{\varepsilon} \times \text{Per}_{\mathbb{L}}$  (Corollary 7.7) which can be pushed to a “parametrization” of  $\mathcal{NR}_{\mathbb{C}}$  by  $\mathbb{H}_{\varepsilon} \times \mathcal{NR}_{\mathbb{L}}$ . The detailed order in which these arguments are organized is described in the introduction to Section 7.

Let us now outline the structure of the paper:

Section 2 consists of some preliminaries. After giving a short discussion about the field  $\mathbb{L}$  we summarize the basic properties of the action of polynomials on  $\mathbb{L}$ . Then we introduce “affine partitions” of a closed ball (which in the language of [16] are the “classes” of a closed ball) and show that polynomials act on affine partitions. We continue with some dynamical aspects of polynomials in  $\mathbb{L}$  such as their Fatou and Julia sets, and infraconnected components of their filled Julia set. Simultaneously we discuss the basic combinatorial structure of the dynamical space of polynomials in  $\mathbb{L}$  given by balls and annuli of level  $n$ .

Section 3 is devoted to the proof of Theorem 3.1 which describes the Julia set of polynomials with all their critical points escaping.

Section 4 contains a detailed study of the geometry of the filled Julia set of cubic polynomial with one critical point non-escaping and the other one in the basin of infinity. This study is based on Branner and Hubbard’s ideas for organizing the relevant combinatorial information by introducing

marked grids. This section concludes with the proof of a stronger version of Corollary 1.2, a result which establishes the equivalence of the first three statements of Corollary 1.4 and Proposition 4.9 regarding the topological entropy of cubic polynomials.

In Section 5 for any  $\alpha \in \mathbb{L}$  outside the closed unit ball, we consider the one-parameter family of cubic polynomials of the form  $\zeta \mapsto \alpha^{-2}(\zeta - 1)^2(\zeta + 2) + \nu$  where  $\nu \in \mathbb{L}$  and give a detailed description of the corresponding parameter space. These families will be fundamental to obtain a description of the parameter space  $\mathcal{P}_{\mathbb{L}}$  and, in the case  $\alpha = t$ , to study the parameter space  $\mathcal{P}_{\mathbb{C}}$  close to infinity.

In Section 6 we give a detailed description of the parameter space  $\mathcal{P}_{\mathbb{L}}$ . In particular, we use the results of Section 5 to prove Theorem 1.3 and we finish the proof of Corollary 1.4.

In Section 7 we prove theorems 1.5 and 1.6 following the ideas explained above. We summarize the organization of this section in its introduction.

**Acknowledgments.** I would like to thank Juan Rivera Letelier for introducing me into non-Archimedean dynamics. The influence of conversations with him are scattered all over this work. I am grateful to John Milnor since an important motivation for this paper was my interest on understanding the Puiseux-Laurent series of some ends of periodic curves computed by him in a short note that he kindly gave to me some years ago. I thank Manuel Elgueta and Alejandro Ramirez for useful conversations regarding almost periodic functions. I would like to thank the Royal Society for funding my trip to England during April 2004. The conversations held with Adam Epstein and Mary Rees during this trip were very helpful to organize the exposition of these results. I am extremely grateful to the referee that read an earlier version of this paper and made detailed suggestions to improve the statements, organization and exposition of the results.

## 2. Preliminaries

### 2.1. The completion of formal Puiseux series

Let  $\mathbb{Q}^a((t))$  be the field of formal Laurent series in  $t$  with coefficients in  $\mathbb{Q}^a \subset \mathbb{C}$  where  $\mathbb{Q}^a$  is the algebraic closure of  $\mathbb{Q}$  contained in  $\mathbb{C}$ . Given a non-zero Laurent series

$$\zeta = \sum_{j \geq j_0} a_j t^j \in \mathbb{Q}^a((t))$$

define the order of  $\zeta$  by

$$\text{ord}(\zeta) = \min\{j \mid a_j \neq 0\}$$

and consider the non-Archimedean valuation in  $\mathbb{Q}^a((t))$  given by

$$|\zeta|_o = e^{-\text{ord}(\zeta)}.$$

The field of formal Puiseux series with coefficients in  $\mathbb{Q}^a$ , denoted  $\mathbb{Q}^a\langle\langle t \rangle\rangle$ , is the algebraic closure of  $\mathbb{Q}^a((t))$  (e.g., see page 17 in [13]). Each element of  $\mathbb{Q}^a\langle\langle t \rangle\rangle$  may be identified with a Laurent series in  $t^{1/m}$  for some  $m \in \mathbb{N}$ . That is, for any  $\zeta \in \mathbb{Q}^a\langle\langle t \rangle\rangle$  there exists  $m \in \mathbb{N}$  such that

$$\zeta = \sum_{j \geq j_0} a_j t^{j/m} \in \mathbb{Q}^a((t^{1/m})).$$

The unique extension of  $|\cdot|_o$  from  $\mathbb{Q}^a((t))$  to  $\mathbb{Q}^a\langle\langle t \rangle\rangle$  is given by

$$|\zeta|_o = e^{-\text{ord}(\zeta)}$$

where

$$\text{ord}(\zeta) = \frac{\min\{j \mid a_j \neq 0\}}{m}$$

provided that  $\zeta \neq 0$ . The valuation group of  $\mathbb{Q}^a\langle\langle t \rangle\rangle$  is  $e^{\mathbb{Q}}$ .

We denote by  $\mathbb{L}$  the completion of  $\mathbb{Q}^a\langle\langle t \rangle\rangle$ . The elements of  $\mathbb{L}$  may be identified with the series

$$\zeta = \sum_{\lambda \in \mathbb{Q}} a_\lambda t^\lambda$$

where  $a_\lambda \in \mathbb{Q}^a$  and the set  $\{\lambda \mid a_\lambda \neq 0\}$  is discrete and bounded below in  $\mathbb{R}$  (i.e., an increasing sequence of rationals tending to  $\infty$ ). Moreover,  $|\zeta|_o = e^{-\text{ord}(\zeta)}$  where  $\text{ord}(\zeta) = \min\{\lambda \in \mathbb{Q} \mid a_\lambda \neq 0\}$  if  $\zeta \neq 0$ . Therefore, the valuation group  $|\mathbb{L}^*|_o$  is also  $e^{\mathbb{Q}}$ . Since  $\mathbb{L}$  is the completion of an algebraically closed field we have that  $\mathbb{L}$  is also algebraically closed (e.g., see [14]).

The ring of integers  $\mathcal{O}_{\mathbb{L}} = \{\zeta \in \mathbb{L} \mid |\zeta|_o \leq 1\}$  contains as unique maximal ideal  $\mathcal{M}_{\mathbb{L}} = \{z \in \mathbb{L} \mid |\zeta|_o < 1\}$ . The residual field  $\tilde{\mathbb{L}}$  is by definition  $\mathcal{O}_{\mathbb{L}}/\mathcal{M}_{\mathbb{L}}$  which is canonically isomorphic to  $\mathbb{Q}^a$ .

It is worth to mention that any algebraically closed, complete non-Archimedean field with valuation group dense in  $[0, +\infty) \subset \mathbb{R}$  and characteristic 0 residual field contains a subfield isomorphic to  $\mathbb{L}$ .

### 2.2. Polynomial maps in $\mathbb{L}$

In this subsection we summarize some basic properties of polynomial maps in  $\mathbb{L}$ . Although most of these properties also hold for the larger class of holomorphic maps we only state them for polynomials in order to keep the exposition as simple as possible. For general background in non-Archimedean dynamics we refer the reader to [30] and Chapter 6 in [29] which is not contained in [30].

For  $r \in |\mathbb{L}^*|_o$  and  $\zeta_0 \in \mathbb{L}$  we say that

$$B_r^+(\zeta_0) = \{\zeta \in \mathbb{L} \mid |\zeta - \zeta_0|_o \leq r\}$$

is a *closed ball* and

$$B_r(\zeta_0) = \{\zeta \in \mathbb{L} \mid |\zeta - \zeta_0|_o < r\}$$

is an *open ball*. If  $r \notin |\mathbb{L}^*|_o$ , then  $B_r^+(\zeta_0) = B_r(\zeta_0)$  is an *irrational ball*. The reader should be aware that, despite these names, topologically speaking every ball is open and closed.

Consider  $\varphi(\zeta) \in \mathbb{L}[\zeta]$  and  $\zeta_0 \in \mathbb{L}$ . The largest integer  $d_0$  such that  $(\zeta - \zeta_0)^{d_0}$  divides  $\varphi(\zeta) - \varphi(\zeta_0)$  is called *the degree of  $\varphi$  at  $\zeta_0$*  and denoted by  $\deg_{\zeta_0}(\varphi)$ . If the degree of  $\varphi$  at  $\zeta_0$  exceeds 1, we say that  $\zeta_0$  is a *critical point of multiplicity*  $\text{mult}_{\varphi}(\zeta_0) = \deg_{\zeta_0}(\varphi) - 1$ .

Suppose that  $\varphi(B) = B'$  where  $B$  is some subset of  $\mathbb{L}$ . If there exists an integer  $d_B \geq 1$  such that

$$d_B = \sum_{\{\zeta \in B \mid \varphi(\zeta) = \zeta'\}} \deg_{\zeta}(\varphi)$$

for all  $\zeta' \in B'$ , then we say that  $\varphi : B \rightarrow B'$  has degree  $d_B = \deg_B(\varphi)$ .

Balls map onto balls under polynomial maps (see [30] page 167).

PROPOSITION 2.1. — *Let  $\varphi(\zeta) \in \mathbb{L}[\zeta]$  be a polynomial of degree  $\deg(\varphi)$ . Consider a closed (resp. open, irrational) ball  $B \subset \mathbb{L}$ . Then the following hold:*

- (i)  $\varphi(B)$  is a closed (resp. open, irrational) ball.
- (ii)  $\varphi : B \rightarrow \varphi(B)$  has a well defined degree  $\deg_B(\varphi)$ .
- (iii)  $\varphi^{-1}(B)$  is a disjoint union of closed (resp. open, irrational) balls  $B_1, \dots, B_k$  such that

$$\sum \deg_{B_i}(\varphi) = \deg(\varphi).$$

(iv)

$$\deg_B(\varphi) - 1 = \sum_{\zeta \in B} (\deg_{\zeta}(\varphi) - 1) = \sum_{\zeta \in \text{Crit}(\varphi) \cap B} \text{mult}(\zeta)$$

where  $\text{Crit}(\varphi)$  is the set formed by the critical points of  $\varphi$ .

Statement (iv) makes a substantial difference between dynamics over fields with characteristic zero residual fields (e.g.,  $\mathbb{L}$ ) and dynamics over fields with residual fields with non-vanishing characteristic (e.g.,  $\mathbb{C}_p$ ).

*Sketch of the Proof.* — Statements (i)–(iii) follow by inspection of the Newton polygon of  $\varphi$ . We refer the reader to [14] for background on Newton polygons and [30] for a proof of (i)–(iii) in the context of  $p$ -adic holomorphic functions that applies without modifications to our context. Statement (iv) follows from a simple observation. Without loss of generality we may assume that:  $B$  and  $\varphi(B)$  are balls which contain the origin,  $\varphi(0) \neq 0$ , and  $\varphi'(0) \neq 0$ . Since natural numbers have valuation 1, the Newton polygon of  $\varphi$  translated to the left by 1 and restricted to the right half plane is the Newton polygon of  $\varphi'$ . Therefore the number of zeros of  $\varphi$  in  $B$  minus 1 coincides with the number of zeros of  $\varphi'$  in  $B$ .  $\square$

We say that  $A \subset \mathbb{L}$  is an *annulus* if

$$A = \{\zeta \in \mathbb{L} \mid \log |\zeta - \zeta_0|_o \in I\}$$

for some  $\zeta_0 \in \mathbb{L}$  and some interval  $I \subset (-\infty, \infty)$ . We say that  $A$  is an open (resp. closed) annulus if  $I$  is open (resp. closed) interval. The length of  $I$  is by definition the *modulus of  $A$* , denoted  $\text{mod } A$ . The next proposition describes how the modulus of an annulus changes under the action of a polynomial  $\varphi$ .

**PROPOSITION 2.2.** — *If  $A, A'$  are annuli and  $\varphi(\zeta) \in \mathbb{L}[\zeta]$  is such that  $\varphi(A) = A'$ , then  $\varphi : A \rightarrow A'$  has a well defined degree  $\deg_A(\varphi)$  and*

$$\deg_A(\varphi) \cdot \text{mod } A = \text{mod } A'.$$

The statement of Lemma 5.3 in [27] is the same than the one of the previous proposition but in the context of holomorphic functions in  $\mathbb{C}_p$ . Rivera's proof applies to our setting as well.

We will also need the following version of Schwarz's Lemma (see [30])

LEMMA 2.3. — Consider  $\varphi(\zeta) \in \mathbb{L}[\zeta]$ . Assume that  $\varphi(B_0) \subset B_1$  where  $B_i$  is a ball of radius  $r_i$  for  $i = 0, 1$ . Then, for all  $\zeta_1, \zeta_2 \in B_0$ :

$$(2.1) \quad |\varphi(\zeta_1) - \varphi(\zeta_2)|_o \leq \frac{r_1}{r_0} |\zeta_1 - \zeta_2|_o$$

$$(2.2) \quad |\varphi'(\zeta_1)|_o \leq \frac{r_1}{r_0}.$$

Moreover, equality holds at some  $\zeta_1, \zeta_2$  in (2.1) or at some  $\zeta_1$  in (2.2) if and only if equality holds for all  $\zeta_1, \zeta_2$  in (2.1) and all  $\zeta_1$  in (2.2).

The next lemma will be useful to count the number of fixed points inside a given closed ball.

LEMMA 2.4. — Let  $\varphi \in \mathbb{L}[\zeta]$ . Let  $B$  and  $B'$  be closed balls such that  $B' = \varphi(B) \supset B$ . Denote by  $|\text{Fix}_B(\varphi)|$  the number of fixed points of  $\varphi$  in  $B$  counting multiplicities. If  $\deg_B(\varphi) > 1$  or  $B \subsetneq B'$ , then

$$|\text{Fix}_B(\varphi)| = \deg_B(\varphi).$$

*Proof.* — Without loss of generality  $B = B_1^+(0)$ .

In the case that there exists  $\zeta_0 \in B$  such that  $|\varphi'(\zeta_0)|_o > 1$ , after conjugation by  $\zeta \mapsto \zeta - \zeta_0$ , we may assume that  $\zeta_0 = 0$ . It follows that the Newton polygons for  $\varphi(\zeta)$  and  $\varphi(\zeta) - \zeta$  coincide and therefore  $|\text{Fix}_B(\varphi)| = \deg_B(\varphi)$ .

For the case in which  $|\varphi'(\zeta)|_o \leq 1$  for all  $\zeta \in B$  we write

$$\varphi(\zeta) = \alpha_0 + \alpha_1\zeta + \dots + \alpha_n\zeta^n$$

and observe that  $\varphi(B) = B$  and that  $|\alpha_k|_o \leq 1$  for all  $k$ . Also, the number of zeros of  $\varphi$  in  $B$  is  $\deg_B(\varphi)$  and coincides with the maximal index  $k$  for which  $|\alpha_k|_o = 1$ . Since the coefficient of  $\zeta^k$  in  $\varphi(\zeta) - \zeta$  coincides with  $\alpha_k$  for all  $k \neq 1$ , if  $\deg_B(\varphi) > 1$ , then  $\varphi(\zeta) - \zeta$  has exactly  $\deg_B(\varphi)$  zeros in  $B$  (counting multiplicities). □

### 2.3. Affine Partitions

In the study of iterations of rational functions on  $p$ -adic fields it is useful to consider their action on projective systems (see [30]). For polynomials the situation is simpler and we will just need to consider affine partitions (compare with the “classes” of a ball in [16]).

By definition, the *canonical affine partition*

$$\mathcal{P}_c = \{B_1(c) \mid c \in \widetilde{\mathbb{L}}\}$$

is the collection of equivalence classes of the ring  $\mathcal{O}_{\mathbb{L}} = B_1^+(0)$  modulo the ideal  $\mathcal{M}_{\mathbb{L}} = B_1(0)$ . The *affine partition*  $\mathcal{P}_{B_0}$  associated to a closed ball  $B_0$  is:

$$\{h^{-1}(B) \mid B \in \mathcal{P}_c\}$$

where  $h : \mathbb{L} \rightarrow \mathbb{L}$  is an affine map such that  $h(B_0) = B_1^+(0)$ . Affine partitions are parametrized by the residual field  $\tilde{\mathbb{L}}$  and the parametrization is unique up to  $\tilde{\mathbb{L}}$ -affine maps. Therefore, affine partitions inherit the affine structure of  $\mathbb{A}^1(\tilde{\mathbb{L}})$ .

**PROPOSITION 2.5.** — *Let  $\varphi : \mathbb{L} \rightarrow \mathbb{L}$  be a polynomial. Given a closed ball  $B_0 \subset \mathbb{L}$ , let  $B_1 = \varphi(B_0)$ . Denote by  $\mathcal{P}_0$  and  $\mathcal{P}_1$  the associated affine partitions. Then:*

- (i) *There is a well defined induced action on the affine partitions given by:*

$$\begin{aligned} \varphi_* : \mathcal{P}_0 &\rightarrow \mathcal{P}_1 \\ B &\mapsto \varphi(B) \end{aligned}$$

Moreover,  $\varphi_*$  is a polynomial from the affine structure of  $\mathcal{P}_0$  to that of  $\mathcal{P}_1$ .

- (ii)  $\deg(\varphi_*) = \deg_{B_0}(\varphi)$ .
- (iii)  $\deg_B(\varphi_*) = \deg_B(\varphi)$  for all  $B \in \mathcal{P}_0$ .

*Proof.* — We first apply an affine change of coordinates in the domain and the range so that  $B_0 = B_1 = B_1^+(0)$ . Hence  $\varphi(\zeta) = \alpha_0 + \dots + \alpha_n \zeta^n$  with  $|\alpha_k|_o \leq 1$  for all  $k = 0, \dots, n$ . Now let  $\pi : B_1^+(0) \rightarrow \tilde{\mathbb{L}}$  be the quotient map and for  $\zeta \in B_1^+(0)$  let  $\tilde{\zeta} = \pi(\zeta)$ . It follows that  $\tilde{\varphi}(\tilde{\zeta}) = \tilde{\alpha}_0 + \dots + \tilde{\alpha}_d \tilde{\zeta}^d$  is such that  $\pi \circ \varphi = \tilde{\varphi} \circ \pi$  where  $d = \deg_{B_0}(\varphi)$ . Thus  $\varphi_*(\pi^{-1}(\tilde{\zeta})) = \pi^{-1}(\tilde{\varphi}(\tilde{\zeta}))$  and  $\varphi_*$ , in these coordinates, becomes  $\tilde{\varphi}$ . From where (i) and (ii) easily follow.

For (iii), without loss of generality we may assume that  $B = \varphi(B) = \pi^{-1}(0)$ . Under this assumption  $\tilde{\varphi}(\tilde{\zeta}) = \tilde{\alpha}_j \tilde{\zeta}^j + O(\tilde{\zeta}^{j+1})$ . It follows that  $j$  is the smallest index such that  $|\alpha_j|_o = 1$ . Looking at the Newton polygon of  $\varphi$  we conclude that  $j$  is the degree of  $\varphi : B \rightarrow B$  and (iii) follows.  $\square$

### 2.4. Fatou and Julia Sets

Given  $\varphi \in \mathbb{L}[\zeta]$ , in analogy with complex polynomial dynamics, the *filled Julia set* is defined by

$$K(\varphi) := \{\zeta \in \mathbb{L} \mid |\varphi^n(\zeta)|_o \not\rightarrow \infty\}.$$

That is, the filled Julia set is the complement of the basin of  $\infty$ . The Julia set  $J(\varphi)$  is the boundary of  $K(\varphi)$  and the Fatou set  $F(\varphi)$  is  $\mathbb{L} \setminus J(\varphi)$ .

Although  $J(\varphi)$  might be empty (e.g.,  $J(\zeta^2) = \emptyset$ ), the filled Julia set  $K(\varphi)$  is always non-empty since it contains the periodic points of  $\varphi$ . According to Proposition 6.2 in [29] a polynomial Julia set can be characterized as follows:

$$\begin{aligned} J(\varphi) &= \partial K(\varphi) \\ &= \{ \zeta \in \mathbb{L} \mid \bigcup_{n \geq 1} \varphi^n(U) = \mathbb{L} \text{ for all open sets } U \text{ with } \zeta \in U \}. \end{aligned}$$

### 2.5. Dynamical balls, dynamical ends, and infraconnected components of a filled Julia set

Throughout this subsection, let  $\varphi$  be a degree  $d > 1$  polynomial of the form:

$$\varphi(\zeta) = \alpha_0 + \alpha_1\zeta + \dots + \alpha_d\zeta^d \in \mathbb{L}[\zeta]$$

where  $\alpha_d \neq 0$ . Following Section 6.1 of [29], let

$$R_\varphi := \max \left( \left| \frac{\alpha_i}{\alpha_d} \right|_o^{\frac{1}{d-i}}, \left| \frac{1}{\alpha_d} \right|_o^{\frac{1}{d-1}} \right).$$

Then it is easy to check that  $K(\varphi) \subset \varphi^{-1}(\{|\zeta|_o \leq R_\varphi\}) \subset \{|\zeta|_o \leq R_\varphi\}$  and

$$K(\varphi) = \{ \zeta \in \mathbb{L} \mid |\varphi^n(\zeta)|_o \leq R_\varphi \text{ for all } n \geq 1 \}.$$

LEMMA 2.6. — *Given a polynomial  $\psi \in \mathbb{L}[\zeta]$  there exists another polynomial  $\varphi \in \mathbb{L}[\zeta]$  affinely conjugate to  $\psi$  such that*

$$R_\varphi = \text{diam } K(\varphi) = \sup\{|\zeta_1 - \zeta_2|_o \mid \zeta_1, \zeta_2 \in K(\varphi)\}.$$

*Proof.* — After an affine conjugacy  $\psi$  becomes  $\varphi(\zeta) = \alpha_1\zeta + \dots + \alpha_{d-1}\zeta^{d-1} + \zeta^d$ . Note that

$$R_\varphi = \max(\{|\alpha_j|_o^{\frac{1}{d-j}} \mid 1 \leq j < d\} \cup \{1\}).$$

Hence, if  $R_\varphi = 1$ , then  $K(\varphi) = B_1^+(0)$ . Otherwise,  $R_\varphi > 1$  and from the Newton polygon of  $\varphi$  we deduce that there exists  $\zeta_0$  such that  $|\zeta_0|_o = R_\varphi$  and  $\varphi(\zeta_0) = 0 \in K(\varphi)$ . □

DEFINITION 2.7. — *We say that  $D_0 = B_{R_\varphi}^+(0)$  is the dynamical ball of level 0 of  $\varphi$ . The set  $\varphi^{-n}(D_0)$  is the union of finitely many pairwise disjoint closed balls which we call level  $n$  dynamical balls.*



Later we will introduce “parameter” balls of level  $n$ . Often, when clear from the context, a dynamical ball will be simply called a *ball*.

Observe that each ball of level  $n > 0$  is contained in exactly one of level  $n - 1$  and maps onto a level  $n - 1$  ball.

DEFINITION 2.8. — *A dynamical end  $E$  is a sequence  $\{D_n\}_{n \geq 0}$  such that  $D_n$  is a ball of level  $n$  and  $D_{n+1} \subset D_n$  for all  $n$ .*

The map  $\varphi$  acts on ends. In fact, given an end  $E = \{D_n(E)\}$  let  $D_n(\varphi(E)) = \varphi(D_{n+1}(E))$  for all  $n \geq 0$ . It follows that  $\varphi(E) = \{D_n(\varphi(E))\}$  is an end which we call *the image of  $E$  under  $\varphi$* .

Following Escassut [15] a bounded subset  $X$  of  $\mathbb{L}$  is called *infraconnected* if for all disjoint closed balls  $B_0, B_1$  such that  $X \subset B_0 \cup B_1$  we have that either  $X \subset B_0$  or  $X \subset B_1$ . An infraconnected component of  $Y \subset \mathbb{L}$  is an equivalence class of the relation that identifies two points  $\zeta_0, \zeta_1$  if there exists an infraconnected subset of  $Y$  containing both  $\zeta_0$  and  $\zeta_1$ .

Proposition 6.8 in [29] reads as follows:

LEMMA 2.9. —

- (i) *If  $E = \{D_n\}$  is an end, then  $\cap D_n$  is empty, or a singleton, or a closed ball, or an irrational ball.*
- (ii) *If  $\zeta \in K(\varphi)$ , then there exists a unique end  $E(\zeta) = \{D_n(\zeta)\}$  such that  $\zeta \in \cap D_n(\zeta)$ . Moreover, the infraconnected component of  $K(\varphi)$  which contains  $\zeta$  is  $\cap D_n(\zeta)$ .*
- (iii) *For any  $\zeta \in K(\varphi)$ , the infraconnected component of  $K(\varphi)$  which contains  $\zeta$  is a singleton if and only if  $\zeta \in J(\varphi)$ .*

If the filled Julia set of a polynomial  $\varphi$  is a closed ball, then the main dynamical features of  $\varphi$  are described by the action of  $\varphi_*$  in the affine partition associated to  $K(\varphi)$ . Following Rivera (see [30] Definition 4.31) we have the following definition.

DEFINITION 2.10. — *We say that a polynomial  $\varphi \in \mathbb{L}[\zeta]$  is simple if there exists a closed ball  $B$  such that  $\varphi(B) = B$  and  $\deg_B(\varphi) = \deg(\varphi)$ . When  $B = B_1^+(0)$  we say that  $\varphi$  has good reduction.*

A well known result in complex polynomial dynamics states that the filled Julia set of a polynomial  $f$  is connected if and only if all the critical points of  $f$  have bounded orbit (e.g., see Theorem 9.5 in [24]). A similar result is also valid for polynomial dynamics over  $\mathbb{L}$ :

**COROLLARY 2.11.** — *Let  $\varphi \in \mathbb{L}[\zeta]$  and denote by  $\text{Crit}(\varphi)$  the set of critical points of  $\varphi$ . Then  $K(\varphi)$  is infraconnected if and only if  $\text{Crit}(\varphi) \subset K(\varphi)$ . In this case,  $K(\varphi)$  is a closed ball and  $\varphi$  is simple.*

*Proof.* — First suppose that  $\text{Crit}(\varphi) \subset K(\varphi)$ . In view of Lemma 2.6 we may assume the  $R_\varphi = \text{diam } K(\varphi)$ . From Proposition 2.1 (iv) it follows that there exists a unique level 1 dynamical ball  $D_1$  and that  $\deg_{D_1} \varphi = \deg \varphi$ . Hence,  $D_1$  must coincide with the level 0 ball  $D_0$  since  $R_\varphi = \text{diam } K(\varphi)$  and  $K(\varphi) \subset D_1$ . Therefore  $K(\varphi) = D_0$  and  $\varphi$  is simple.

If  $\text{Crit}(\varphi) \not\subset K(\varphi)$ , then there exist a level with at least two disjoint balls, say  $B_1$  and  $B_2$ . Each one of these balls  $B_i$  contains a periodic point  $\zeta_i$  because there exists  $k$  such that  $\varphi^k(B_i) \supsetneq B_i$  (Lemma 2.4). It follows that the infraconnected components  $C(\zeta_0), C(\zeta_1)$  of  $K(\varphi)$  containing  $\zeta_0, \zeta_1$  (respectively) are distinct and therefore  $K(\varphi)$  is not infraconnected.  $\square$

Regarding compactness of  $J(\varphi)$  we have the following result. (Compare with [7].)

**COROLLARY 2.12.** — *Given  $\varphi \in \mathbb{L}[\zeta]$  the following hold:*

- (i) *If  $J(\varphi)$  is compact and non-empty, then every infraconnected component of  $K(\varphi)$  is a singleton.*
- (ii) *If every infraconnected component of  $K(\varphi)$  is a singleton, then all the cycles of  $\varphi$  are repelling.*

*Proof.* — For (i) we proceed by contradiction and suppose that  $J(\varphi)$  is compact and non-empty and that there exists and end  $E = \{D_n\}$  such that  $C = \bigcap D_n$  is a ball or empty. Now let  $\zeta_0 \in J(\varphi)$ . For all  $n \geq 0$  there exists  $\zeta_n \in D_n$  such that  $\varphi^n(\zeta_n) = \zeta_0$  since  $\varphi^n(D_n) = D_0 \supset J(\varphi) \ni \zeta_0$ . Therefore, after passing to a convergent subsequence we obtain a limit point  $\zeta \in J(\varphi) \cap C = \emptyset$  which is a contradiction.

For (ii), suppose that  $\zeta_0$  is a period  $p$  periodic point. Then  $\{\zeta_0\} = \bigcap D_n(\zeta_0)$  where  $D_n(\zeta_0)$  is the level  $n$  ball containing  $\zeta_0$ . The orbit of  $\zeta_0$  does not contain critical points, for otherwise the infraconnected component of  $\zeta_0$  in  $K(\varphi)$  would contain points that are attracted to the cycle of  $\zeta_0$ . Hence, for  $n$  large,  $\varphi^p : D_n(\zeta_0) \rightarrow D_{n-p}(\zeta_0)$  has degree 1. By Schwarz Lemma 2.3,  $|(\varphi^p)'(\zeta_0)|_o > 1$  and  $\zeta_0$  is repelling.  $\square$

### 2.6. Points and Annuli of level $n$

Consider a polynomial  $\varphi \in \mathbb{L}[\zeta]$ , an integer  $n \in \mathbb{N}$  and a point  $\zeta \in \varphi^{-n}(D_0)$  where  $D_0 = B_{R_\varphi}^+(0)$  is the level 0 ball of  $\varphi$ . In this case we say

that  $\zeta$  is a *level  $n$  point*. Note that  $\zeta$  is a level  $k$  point for all  $k \leq n$ . Also,  $\zeta$  is contained in a unique level  $n$  ball denoted  $D_n(\zeta)$ . The radius of  $D_n(\zeta)$  will be denoted by  $r_n(\zeta)$ .

Now let  $\hat{r}$  denote the radius of  $\varphi(D_0)$ . We say that  $A_0 = B_{\hat{r}}(0) \setminus D_0$  is the *level 0 annulus* of  $\varphi$ . For  $n \in \mathbb{N}$ , we say that the *annulus of level  $n$  around  $\zeta$*  is  $A_n(\zeta) = B_{r_{n-1}(\zeta)}(\zeta) \setminus D_n(\zeta)$  where  $\zeta$  is a level  $n$  point. Note that:

$$\log \hat{r} - \log r_n(\zeta) = \sum_{\ell=0}^n \text{mod } A_\ell(\zeta).$$

Similarly if  $E = \{D_n\}$  is an end, then we denote by  $D_n(E)$  the ball of level  $n$  participating in  $E$  and its radius by  $r_n(E)$ . The *level  $n \geq 1$  annulus* of  $E$  is  $A_n(E) = B_{r_{n-1}(E)}(\zeta) \setminus D_n(E)$  where  $\zeta$  is any point of  $D_n(E)$ . Also,

$$\log \hat{r} - \log r_n(E) = \sum_{\ell=0}^n \text{mod } A_\ell(E).$$

We omit the straightforward proof of the following result which shows the importance of studying the convergence of the sum of the moduli of annuli.

LEMMA 2.13. — *Let  $\zeta \in K(\varphi)$  and let  $E$  be an end. Then the following are equivalent:*

- (i)  $r_n(\zeta) \rightarrow 0$  (resp.  $r_n(E) \rightarrow 0$ ).
- (ii)  $\sum_{\ell=0}^\infty \text{mod } A_\ell(\zeta) = +\infty$  (resp.  $\sum_{\ell=0}^\infty \text{mod } A_\ell(E) = +\infty$ ).
- (iii)  $\{\zeta\} = \cap D_n(\zeta)$  (resp.  $\cap D_n(E)$  is a singleton).

### 3. Polynomials with all critical points escaping

The Julia set of a degree  $d > 1$  polynomial  $f : \mathbb{C} \rightarrow \mathbb{C}$  with all its critical points escaping is a Cantor set. Moreover, the dynamics over its Julia set  $J(f)$  is topologically equivalent to the one-sided shift on  $d$  symbols, and  $f$  is uniformly expanding in a neighborhood of  $J(f)$  (e.g., see Theorem 9.9 in [8]). The aim of this section is to prove the analogous result for polynomials acting on  $\mathbb{L}$ .

THEOREM 3.1. — *Let  $\varphi : \mathbb{L} \rightarrow \mathbb{L}$  be a degree  $d \geq 2$  polynomial with all critical points escaping (i.e.,  $\omega \notin K(\varphi)$  for all critical points  $\omega$ ). Then  $\varphi : K(\varphi) \rightarrow K(\varphi)$  is topologically equivalent to the one-sided shift on  $d$  symbols. Moreover,  $\varphi$  is uniformly expanding in a neighborhood of  $J(\varphi)$ . In particular,  $K(\varphi)$  is a Cantor set and  $J(\varphi) = K(\varphi)$ . Furthermore, the intersection of every end is a singleton.*

Before proving the theorem let us be more precise about the definitions involved in the statement. (Compare with Definition 3.1. in [4] and Definition 3 in [7]).

DEFINITION 3.2. — We say that  $\varphi$  is uniformly expanding on a neighborhood  $V$  of  $J(\varphi)$  if there exist real numbers  $0 < c_1 < c_2$ , a bounded continuous function  $\tau : V \rightarrow [c_1, c_2]$  and  $\lambda > 1$  such that

$$\tau(\varphi(\zeta))|\varphi'|_o \geq \lambda\tau(\zeta)$$

for all  $\zeta \in V$ .

*Proof of Theorem 3.1.* — Let  $N \geq 1$  be such that  $\varphi^N(\omega) \notin D_0$  for all critical points  $\omega$ . That is, the level  $N$  balls are critical point free and the level  $N - 1$  balls are critical value free. Therefore each level  $N$  ball maps bijectively onto one of level  $N - 1$ .

We first show that the intersection of every end  $E = \{D_n\}$  is a singleton. For this we consider the metric on  $\varphi^{-(N-1)}(D_0)$  defined by:

$$\rho(\zeta, \zeta') = \begin{cases} |\zeta - \zeta'|_o \cdot r_{N-1}(\zeta)^{-1} & \text{if } D_{N-1}(\zeta) = D'_{N-1}(\zeta'), \\ |\zeta - \zeta'|_o & \text{otherwise,} \end{cases}$$

where  $r_N(\zeta)$  denotes the radius of the level  $N$  ball which contains  $\zeta$ . Let

$$\lambda = \min \left\{ \frac{r_{N-1}(\zeta)}{r_N(\zeta)} \mid \zeta \in \varphi^{-N}(D_0) \right\} > 1.$$

By Schwarz Lemma,

$$(3.1) \quad \rho(\varphi(\zeta), \varphi(\zeta')) \geq \lambda\rho(\zeta, \zeta')$$

if  $D_N(\zeta) = D_N(\zeta')$ . Moreover,

$$r_{N-1}(\varphi(\zeta))^{-1}|\varphi'(\zeta)|_o = r_N(\zeta)^{-1} \geq \lambda r_{N-1}(\zeta)^{-1}$$

for all  $\zeta \in \varphi^{-N}(D_0)$ . In particular,  $\varphi$  is uniformly expanding on the neighborhood  $\varphi^{-N}(D_0)$  of  $K(\varphi)$  taking  $\tau(\zeta) = r_{N-1}(\zeta)^{-1}$  in Definition 3.2.

For  $n \geq N - 1$ , let

$$R_n = \max\{\rho(D_n) \mid D_n \text{ ball of level } n\}$$

where  $\rho(D_n)$  is the  $\rho$ -diameter of  $D_n$ . By (3.1),  $R_{N-1+k}\lambda^k \leq R_{N-1}$ .

It follows that if  $E = \{D_n\}$  is an end, then  $\rho(D_n) \rightarrow 0$  as  $n \rightarrow \infty$ . From the completeness of  $\mathbb{L}$  we conclude that the intersection of  $E$  is a point. By Lemma 2.9, every infraconnected component of  $K(\varphi)$  is a point and  $J(\varphi) = K(\varphi)$ .

The preimage of each level  $N - 1$  ball consists of  $d$  level  $N$  balls. We label each level  $N$  ball  $D_N$  with an integer  $L(D_N) \in \{1, \dots, d\}$  so that if the

image of two distinct level  $N$  balls coincide, then their corresponding labels are distinct. This labeling determines an itinerary for each end. Namely, let **Ends** denote the collection of all ends and

$$\begin{aligned} it : \mathbf{Ends} &\rightarrow \{1, \dots, d\}^{\mathbb{N} \cup \{0\}} \\ \{D_n\} &\rightarrow (j_0, j_1, \dots) \text{ if } j_k = L(\varphi^k(D_{N+k})). \end{aligned}$$

It follows that the itinerary function is bijective. Moreover, for  $\zeta \in K(\varphi)$ , let  $E(\zeta) = \{D_n(\zeta)\}$  be the end with intersection  $\{\zeta\}$ . Then the map  $\zeta \mapsto it(E(\zeta))$  gives the desired topological conjugacy between  $\varphi : K(\varphi) \rightarrow K(\varphi)$  and the one-sided shift on  $d$  symbols. □

### 4. Cubic polynomials: the dynamical space

From Corollary 2.11 and Theorem 3.1 we conclude that the filled Julia set of quadratic polynomials is either a closed ball or a Cantor set according to whether the unique critical point belongs to the filled Julia set or escapes to infinity. For a cubic polynomial  $\varphi \in \mathbb{L}[\zeta]$  we have three possibilities:

- (i) Both critical points have bounded orbits. Then  $\varphi$  is simple,  $K(\varphi)$  is a closed ball, and  $J(\varphi)$  is empty. (See Corollary 2.11).
- (ii) Both critical points escape to infinity. Then  $K(\varphi) = J(\varphi)$  is a compact set (and thus a Cantor set). The dynamics over  $J(\varphi)$  is topologically conjugate to the one-side shift on three symbols. (Theorem 3.1).
- (iii) One critical point escapes to infinity and the other belongs to  $K(\varphi)$ .

The aim of this section is to describe  $K(\varphi)$  for polynomials as in (iii) and finish the proof of Theorem 1.1.

Given a cubic polynomial  $\varphi$  with two distinct critical points  $\omega^\pm$  and exactly one of them escaping after an affine conjugacy, if necessary, we may assume that  $\omega^-$  is the escaping critical point and that  $R_\varphi = \text{diam } K(\varphi)$  (see Lemma 2.6). It follows that  $\varphi(\omega^+) \in D_0 = B_{R_\varphi}^+(0)$ .

#### 4.1. Branner–Hubbard Marked Grids

*Our standing assumption for this subsection is that  $\varphi$  is a cubic polynomial with two distinct critical points  $\omega^\pm$  such that  $\omega^- \notin K(\varphi)$ ,  $\varphi(\omega^+) \in D_0$  and  $R_\varphi = \text{diam } K(\varphi)$  where  $D_0$  is the level 0 ball of  $\varphi$ . The level 0 annulus of  $\varphi$  will be denoted  $A_0$ . (See Lemma 2.6 and Definition 2.7).*

In view of Lemma 2.13 to study the geometry of  $K(\varphi)$  it is convenient to compute the moduli of the annuli of level  $n$ , for all  $n$ . The next pair of lemmas describe the behavior of level  $n$  annuli under iterations:

LEMMA 4.1. — *Let  $\varphi$  be a cubic polynomial with critical points  $\omega^\pm$  such that  $R_\varphi = \text{diam } K(\varphi)$ . Suppose that  $\varphi(\omega^+) \in D_0$  and  $\omega^- \notin K(\varphi)$ . Then the following hold:*

- (i)  $\varphi(\omega^-) \notin D_0$ .
- (ii) *There are exactly two level 1 balls:  $D_1(\omega^+)$  and  $D_1(\gamma^+)$  where  $\varphi(\gamma^+) = \varphi(\omega^+)$  and  $\gamma^+ \neq \omega^+$ .*
- (iii) *The degree of  $\varphi : D_1(\omega^+) \rightarrow D_0$  is 2 and the degree of  $\varphi : D_1(\gamma^+) \rightarrow D_0$  is 1.*
- (iv)  *$\varphi(A_1(\omega^+)) = A_0$  and  $\varphi : A_1(\omega^+) \rightarrow A_0$  is a degree 2 map. Also,  $\varphi(A_1(\gamma^+)) = A_0$  and  $\varphi : A_1(\gamma^+) \rightarrow A_0$  is a degree 1 map.*

Following Branner and Hubbard we say that the point  $\gamma^+$  as in part (ii) of the lemma is the cocritical point of  $\omega^+$ .

*Proof.* — For (i) we proceed by contradiction, if  $\varphi(\omega^-) \in D_0$ , then both critical points must be in the same level 1 ball. Hence there would be only one level 1 ball. This level 1 ball would contain  $K(\varphi)$  and have radius strictly smaller than  $R_\varphi$ . This is a contradiction since  $R_\varphi = \text{diam } K(\varphi)$ .

To prove statements (ii) and (iii) just observe that from (i) it follows that  $\text{deg}_{D_1(\omega^+)}(\varphi) = 2$ . Thus there exists another level 1 ball which bijectively maps onto  $D_0$  under  $\varphi$ .

For (iv), note that  $\varphi^{-1}(D_0) \cap B_{R_\varphi}(\omega^+) = D_1(\omega^+)$  for otherwise  $K(\varphi) \subset \varphi^{-1}(D_0) \subset B_{R_\varphi}(\omega^+)$  and  $\text{diam } K(\varphi) < R_\varphi$ . Now since  $\varphi(B_{R_\varphi}(\omega^+)) = B_{\hat{r}}(0)$  where  $\hat{r}$  is the radius of  $\varphi(D_0)$ , it follows that  $\varphi(A_1(\omega^+)) = A_0$  and the degree of  $\varphi : A_1(\omega^+) \rightarrow A_0$  is 2. The rest of (iv) follows along the same lines. □

LEMMA 4.2. — *Let  $\varphi$  be a cubic polynomial with critical points  $\omega^\pm$  such that  $R_\varphi = \text{diam } K(\varphi)$ . Suppose that  $\omega^- \notin K(\varphi)$ . Consider  $n \geq 1$  and assume that  $\varphi^n(\omega^+) \in D_0$ . Let  $\zeta_0$  be a level  $n$  point and let  $E$  be an end. Then the following hold:*

- (i) *For any element  $P$  of the affine partition associated to  $D_{n-1}(\zeta_0)$  or to  $D_{n-1}(E)$  there exists at most one ball of level  $n$  contained in  $P$  (see Subsection 2.3).*
- (ii)  $\varphi(A_n(\zeta_0)) = A_{n-1}(\varphi(\zeta_0))$  and  $A_n(\zeta_0) \subset \mathbb{L} \setminus K(\varphi)$ .
- (ii')  $\varphi(A_n(E)) = A_{n-1}(\varphi(E))$  and  $A_n(E) \subset \mathbb{L} \setminus K(\varphi)$ .

(iii)

$$\deg_{A_n(\zeta_0)}(\varphi) = \begin{cases} 1 & \text{if } \omega^+ \notin D_n(\zeta_0), \\ 2 & \text{if } \omega^+ \in D_n(\zeta_0). \end{cases}$$

(iii')

$$\deg_{A_n(E)}(\varphi) = \begin{cases} 1 & \text{if } \omega^+ \notin D_n(E), \\ 2 & \text{if } \omega^+ \in D_n(E). \end{cases}$$

*Proof.* — We proceed by induction. For  $n = 1$  the previous lemma implies (i)–(iii). Consider  $n \geq 2$  and suppose that (i)–(iii) hold for  $1, \dots, n - 1$ . Now we show that (i)–(iii) hold for  $n$ :

Note that  $P \setminus D_n(\zeta_0) = A_n(\zeta_0)$ .

To prove (i) we proceed by contradiction and suppose that  $P$  contains  $D_n(\zeta_0)$  and another level  $n$  ball  $D_n(\zeta_1)$ . By the inductive hypothesis, the unique ball inside  $\varphi(P)$  is  $D_{n-1}(\varphi(\zeta_0))$ . Therefore,  $\deg_P(\varphi) = 2$  and  $P$  contains the critical point  $\omega^+$  which has to be outside  $\varphi^{-1}(D_{n-1}(\varphi(\zeta_0)))$ . Hence,  $\varphi(\omega^+) \in A_{n-1}(\varphi(\zeta_0))$  which contradicts the hypothesis of the lemma.

From (i) we have that  $\varphi^{-1}(D_{n-1}(\varphi(\zeta_0))) \cap P = D_n(\zeta_0)$ . Hence  $\varphi(A_n(\zeta_0)) = \varphi(P \setminus D_n(\zeta_0)) = \varphi(P) \setminus D_{n-1}(\varphi(\zeta_0)) = A_{n-1}(\varphi(\zeta_0))$  and (ii) follows.

For (iii) since  $\varphi^{-1}(A_{n-1}(\varphi(\zeta_0))) \cap P = A_n(\zeta_0)$ , the degree of  $\varphi$  in  $A_n(\zeta_0)$  coincides with that of  $\varphi$  in  $P$ . The degree of  $\varphi$  in  $P$  is 1 or 2 according to whether  $\omega^+ \notin P$  or  $\omega^+ \in P$ . From (i),  $\omega^+ \in P$  if and only if  $\omega^+ \in D_n(\zeta_0)$ . Thus (iii) holds.

Choosing  $\zeta_0 \in D_n(E)$  parts (ii') and (iii') follow as well. □

Following Branner and Hubbard [11] we will introduce marked grids in order to keep track of the moduli of annuli.

*Notation 4.3.* — Let  $\ell, k \geq 0$  be integers. Given  $\zeta \in \mathbb{L}$  such that  $\zeta_k = \varphi^k(\zeta)$  is a level  $\ell$  point we denote by  $A_{\ell,k}(\zeta)$  the level  $\ell$  annulus  $A_\ell(\zeta_k)$  around  $\zeta_k$ . Similarly, if  $E$  is an end, we denote by  $A_{\ell,k}(E)$  the level  $\ell$  annulus of  $\varphi^k(E)$ .

**DEFINITION 4.4.** — Let  $\zeta$  be a level  $n \geq 1$  point. The level  $n$  marked grid, denoted  $\mathbf{M}_n(\zeta)$  or sometimes simply  $\mathbf{M}(\zeta)$  is the two dimensional array  $(M_{\ell,k}(\zeta))$  where  $0 \leq \ell, 0 \leq k, \ell + k \leq n$ , and

$$M_{\ell,k}(\zeta) = \begin{cases} 1 & \text{if } A_{\ell,k}(\zeta) = A_\ell(\omega^+), \\ 0 & \text{otherwise.} \end{cases}$$

If  $\zeta \in K(\varphi)$ , then the marked grid of  $\zeta$  is the infinite array  $\mathbf{M}(\zeta) = (M_{\ell,k}(\zeta))$  where  $\ell, k \geq 0$ . Similarly, given an end  $E$  we define the corresponding marked grid  $\mathbf{M}(E) = (M_{\ell,k}(E))$  where  $M_{\ell,k}(E)$  is 0 or 1 according to whether  $A_{\ell,k}(E)(\omega^+) \neq A_{\ell,k}(E)$  or  $A_{\ell,k}(E)(\omega^+) = A_{\ell,k}(E)$ .

Also the marked grid of  $\omega^+$  is called the *critical marked grid* of  $\varphi$ . A position  $M_{\ell,k}$  is said to be *marked* if  $M_{\ell,k} = 1$ .

Marked grids are useful to compute the moduli of the annuli of level  $n$ . In fact, from Lemma 4.2, if  $\omega^+$  and  $\zeta$  are level  $n$  points, and

$$S_\ell = \sum_{i=0}^{\ell-1} M_{\ell-i,i}(\zeta),$$

then

$$\text{mod } A_\ell(\zeta) = 2^{-S_\ell} \text{ mod } A_0$$

for all  $\ell \leq n$ .

Marked grids satisfy four simple rules:

PROPOSITION 4.5. — Suppose that  $\omega^+$  is a level  $n$  point. Given a level  $n$  point  $\zeta$  (resp. an end  $E$ ) let  $M_{\ell,k} = M_{\ell,k}(\zeta)$  (resp.  $M_{\ell,k} = M_{\ell,k}(E)$ ) for  $\ell + k \leq n$ . Then the following hold:

- (Ma) If  $M_{\ell,k}$  is marked, then  $M_{j,k}$  is marked for all  $j \leq \ell$ .
- (Mb) If  $M_{\ell,k}$  is marked, then  $M_{\ell-i,k+i} = M_{\ell-i,i}(\omega^+)$  for  $0 \leq i \leq \ell$ .
- (Mc) If  $\ell + m + 1 \leq n$ ,  $M_{\ell-i,i}(\omega^+)$  is not marked for all  $0 < i < k$ ,  $M_{\ell+1-k,k}(\omega^+)$  is marked,  $M_{\ell,m}$  is marked, and  $M_{\ell+1,m}$  is not marked, then  $M_{\ell+1-k,m+k}$  is not marked.
- (Md) If  $\ell + k + 1 \leq n$ ,  $M_{1,\ell}(\omega^+)$  is not marked,  $M_{\ell,k}$  is marked,  $M_{\ell+1,k}$  is not marked and  $M_{\ell-i,k+i}$  is not marked for  $0 < i < \ell$ , then  $M_{1,k+\ell}$  is marked.

Remark 4.6. — The rule (Md) is not explicitly stated in Branner and Hubbard’s work. The necessity of adding a rule to the original ones first appears in the literature in [20]. Although the rule (Md) stated above is not the same as the fourth rule in [20], it agrees with the one which was recently and independently found by De Marco and McMullen. In fact, (Md) implies the fourth rule in [20] but not conversely. For example, let  $\mathbf{M} = (M_{\ell,k})$  be such that  $1 = M_{5,1} = M_{5,2} = M_{5,3}$ ,  $1 = M_{\ell,0} = M_{0,k}$  for all  $\ell, k \geq 0$ , and with all the other positions unmarked (i.e., 0). Then  $\mathbf{M}$  satisfies (Ma)-(Mc) and the fourth rule in [20] but not (Md). Such a grid is not the critical marked grid of a cubic polynomial.



*Proof.* — We may assume that  $M_{\ell,k} = M_{\ell,k}(\zeta)$  for some  $\zeta \in \varphi^{-n}(D_0)$ . As usual, let  $\zeta_k = \varphi^k(\zeta)$ .

(Ma) follows directly from the definitions.

For (Mb) note that if  $M_{\ell,k}$  is marked, then  $A_{\ell,k}(\zeta) = A_{\ell,0}(\omega^+)$ . Therefore,  $A_{\ell-i,k+i} = \varphi^i(A_{\ell}(\zeta_k)) = \varphi^i(A_{\ell}(\omega^+)) = A_{\ell-i,i}(\omega^+)$ .

Under the hypothesis of (Mc) it follows that  $\zeta_m \in D_{\ell}(\omega^+) \setminus D_{\ell+1}(\omega^+)$ . Since  $D_{\ell+1}(\omega^+)$  is the only preimage of  $D_{\ell}(\varphi(\omega^+))$  inside  $D_{\ell}(\omega^+)$ , we conclude that  $\zeta_{m+1} \in D_{\ell-1}(\varphi(\omega^+)) \setminus D_{\ell}(\varphi(\omega^+))$ . Now  $\varphi^{k-1}$  is one-to-one on  $D_{\ell-1}(\varphi(\omega^+))$ , therefore  $\zeta_{m+k} \in D_{\ell-k}(\varphi^k(\omega^+)) \setminus D_{\ell-k+1}(\varphi^k(\omega^+))$ . By assumption  $M_{\ell-k+1,k}(\omega^+)$  is marked, thus  $D_{\ell-k+1}(\varphi^k(\omega^+)) = D_{\ell-k+1}(\omega^+)$ . Hence,  $D_{\ell-k+1}(\zeta_{m+k}) \neq D_{\ell-k+1}(\omega^+)$  and  $M_{\ell-k+1,m+k}$  is unmarked.

Now under the hypothesis of (Md) we have that  $\zeta_k \in D_{\ell}(\omega^+) \setminus D_{\ell+1}(\omega^+)$ . It follows that  $\zeta_{k+\ell} \in D_0 \setminus D_1(\varphi^{\ell}(\omega^+))$ . By hypothesis,  $D_1(\varphi^{\ell}(\omega^+)) = D_1(\gamma^+) \neq D_1(\omega^+)$  where  $\gamma^+$  is the cocritical point of  $\omega^+$ . Therefore,  $\zeta_{k+\ell} \in D_1(\omega^+)$  because there are only two level 1 balls. Hence  $M_{1,k+\ell}$  is marked. □

The marked grid of the critical point plays a central role. If  $\omega^+ \in K(\varphi)$ , then the critical marked grid  $(M_{\ell,k}(\omega^+))$  is defined for all  $\ell, k \geq 0$ . In this case, the critical marked grid is said to be *periodic* of period  $p > 0$  if the  $p$ -th column is marked. More precisely,  $M_{\ell,p}(\omega^+) = 1$  for all  $\ell \geq 0$  and  $p > 0$  is minimal with this property.

For annuli such that the corresponding grids satisfy (Ma)–(Mc) of Proposition 4.5 and that satisfy parts (ii)–(iii) of Lemma 4.2, Branner and Hubbard (see Theorem 4.3 in [11]) established the following:

**THEOREM 4.7** (Branner and Hubbard). — *Suppose that  $\varphi$  is a cubic polynomial such that  $R_{\varphi} = \text{diam } K(\varphi)$ ,  $\omega^+ \in K(\varphi)$  and  $\omega^- \notin K(\varphi)$ . Then the following hold:*

- (i) *If the critical marked grid  $\mathbf{M}(\omega^+)$  is not periodic, then*

$$\sum_{\ell \geq 0} \text{mod } A_{\ell}(E)$$

*is divergent for all ends.*

- (ii) *If the critical marked grid  $\mathbf{M}(\omega^+)$  is periodic, then*

$$\sum_{\ell \geq 0} \text{mod } A_{\ell}(E)$$

*is convergent if and only if there exists  $k \geq 0$  such that  $A_{\ell,k}(E) = A_{\ell}(\omega^+)$  for all  $\ell \geq 0$ .*

*Proof of Theorem 1.1.* — By Corollary 2.11 and Theorem 3.1, we may assume that  $\varphi$  has one critical point  $\omega^-$  escaping to  $\infty$  and another one  $\omega^+$  in  $K(\varphi)$ . Moreover, we may also assume that  $\varphi$  is normalized so that  $R_\varphi = \text{diam } K(\varphi)$  (Lemma 2.6). Let  $U = \cap D_n(\omega^+)$  be the infraconnected component of  $K(\varphi)$  containing  $\omega^+$ . Note that  $U$  is periodic if and only if  $\mathbf{M}(\omega^+)$  is periodic. By Lemma 2.13 and the previous theorem, provided that  $U$  is not periodic we have that  $\cap D_n(E)$  is a singleton for all ends  $E$ . It follows that  $J(\varphi) = K(\varphi)$  is compact and non-empty. Now if  $U$  is periodic and  $V$  is an infraconnected component of  $K(\varphi)$  which is not a singleton, then  $V = \cap D_n(E)$  where  $E$  is an end such that  $\varphi^k(E) = \{D_n(\omega^+)\}$  for some  $k \geq 0$ , by part (ii) of the previous theorem. Hence  $\varphi^k(V) = U$ .  $\square$

Now we are ready to prove a slightly stronger version of Corollary 1.2.

**COROLLARY 4.8.** — *Let  $\varphi \in \mathbb{L}[\zeta]$  be a cubic polynomial. Then the following hold:*

- (i) *Every end of  $\varphi$  has non-empty intersection.*
- (ii) *Every infraconnected component of  $K(\varphi)$  is either a closed ball or a point.*
- (iii) *An infraconnected component  $C$  of  $K(\varphi)$  is a closed ball if and only if  $C$  eventually maps onto a periodic infraconnected component containing a critical point.*

*Proof.* — By Theorem 3.1 and Corollary 2.11, we may assume that  $\varphi$  has one critical point  $\omega^-$  escaping to  $\infty$  and another one  $\omega^+$  in  $K(\varphi)$ . Moreover, we may also assume that  $\varphi$  is normalized so that  $R_\varphi = \text{diam } K(\varphi)$  (Lemma 2.6). Therefore the definitions and results of this subsection apply to  $\varphi$ .

Let  $E$  be an end. If  $r_n(E) \rightarrow 0$  then the intersection of  $E$  is a point. If  $r_n(E) \not\rightarrow 0$ , then for some  $k \geq 0$  we have that  $\omega^+ \in \varphi^k(D_n(E))$  for all  $n \geq 0$ , by Theorem 4.7. In particular  $\cap D_n(E) \neq \emptyset$  and (i) follows.

Also note that  $K(\varphi)$  has a non-trivial infraconnected component if and only if the critical marked grid of  $\varphi$  is periodic. In this case, every non-trivial infraconnected component eventually maps onto the periodic infraconnected component  $U = \cap D_n(\omega^+)$ . Therefore, to finish the proof of the corollary, it suffices to show that  $U$  is a closed ball when the critical marked grid is periodic. In fact, if  $\mathbf{M}(\omega^+)$  is periodic, say of period  $p$ , then there exists  $\ell_0$  such that

$$\text{mod } A_{\ell+p}(\omega^+) = \frac{1}{2} \text{mod } A_\ell(\omega^+)$$

for all  $\ell \geq \ell_0$ . It follows that

$$a = \frac{1}{\text{mod } A_0} \sum_{\ell=0}^{\infty} \text{mod } A_{\ell}(\omega^+)$$

is rational and therefore the radius of  $U$  is  $\hat{r}^{a-1} R_{\varphi}^a \in |\mathbb{L}^*|_o$  where  $\hat{r}$  is the radius of  $\varphi(D_0)$ . □

### 4.2. Topological entropy

We end this section with a basic result about the topological entropy of cubic polynomials.

**PROPOSITION 4.9.** — *Let  $\varphi : \mathbb{L} \rightarrow \mathbb{L}$  be a cubic polynomial. For a compact invariant subset  $X$  of  $J(\varphi)$  we denote by  $h_{top}(\varphi, X)$  the topological entropy of  $\varphi : X \rightarrow X$ . If  $J(\varphi) \neq \emptyset$ , then*

$$h_{top}(\varphi) := \sup_X h_{top}(\varphi, X) = \log 3.$$

*Proof.* — Suppose that  $\varphi$  is normalized so that  $R_{\varphi} = \text{diam } K(\varphi)$ . We may assume that  $\varphi$  has exactly one critical point  $\omega^+$  in  $K(\varphi)$ , for otherwise  $J(\varphi) = \emptyset$  or  $\varphi : J(\varphi) \rightarrow J(\varphi)$  is topologically conjugated to the one-sided shift on 3 symbols. In the latter case the topological entropy is clearly  $\log 3$ .

Let **Ends** be the set of all ends of  $\varphi$  endowed with the metric defined by  $\rho(\{D_n\}, \{D'_n\}) = 1/(k+1)$  if  $k$  is the largest integer such that  $D_k = D'_k$ . Denote by  $\varphi_{\#}$  the action induced by  $\varphi$  on **Ends**. For  $\zeta \in K(\varphi)$ , the map  $\pi : \zeta \mapsto \{D_n(\zeta)\}$  is a semiconjugacy between  $\varphi : K(\varphi) \rightarrow K(\varphi)$  and  $\varphi_{\#} : \mathbf{Ends} \rightarrow \mathbf{Ends}$ . Since the number of dynamical balls of level  $n$  is  $(3^n + 1)/2$ , it follows that the topological entropy of  $\varphi_{\#}$  is exactly  $\log 3$ .

If the marked grid of  $\omega^+$  is not periodic, then  $J(\varphi)$  is compact and  $\pi : J(\varphi) \rightarrow \mathbf{Ends}$  is a topological conjugacy. Hence, the claim of the proposition follows in this case.

In the case that the marked grid of  $\omega^+$  is periodic we label each level  $n \geq 1$  ball  $D$  by  $L(D) \in \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$  so that  $\text{deg}_D \varphi$  coincides with the cardinality of  $L(D)$ . We may choose a labeling such that if  $D' \subset D$ , then  $L(D') \subset L(D)$ ; and if  $\varphi(D) = \varphi(D')$ , then  $D = D'$  or  $L(D) \cap L(D') = \emptyset$ . Necessarily  $L(D_1(\omega^+)) = \{1, 2\}$  and the label of the other level 1 ball  $D_1(\gamma^+)$  is  $\{3\}$ . It is easy to recursively label all dynamical balls of level  $n \geq 2$  so that the labeling complies with the above conditions. Nevertheless, several choices of labelings are allowed. Now consider the itinerary function

$$it : \mathbf{Ends} \rightarrow \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}^{\mathbb{N} \cup \{0\}}$$

defined by

$$it(E) = (i_k(E))_{k \geq 0} = (\lim_{n \rightarrow \infty} L(D_n(\varphi_{\#}^k(E))))_{k \geq 0}.$$

Denote the critical end  $\{D_n(\omega^+)\}$  by  $E^*$ . Observe that  $i_k(E) = \{1, 2\}$  if and only if  $\varphi_{\#}^k(E)$  is the critical end  $E^*$ .

The itinerary function is injective. In fact, the stronger assertion holds, if  $E \neq E'$ , then  $i_k(E) \cap i_k(E') = \emptyset$  for some  $k \geq 0$ . To see this let  $n$  be the smallest integer such that  $D_n(E) \neq D_n(E')$  and consider  $\ell \leq n$  such that  $\varphi^\ell(D_n(E)) = \varphi^\ell(D_n(E'))$ . From the properties of the labeling, it follows that  $i_{\ell-1}(E) \cap i_{\ell-1}(E') = \emptyset$ .

To characterize the image of  $it$ , given  $(i_k) \in \{\{1\}, \{2\}, \{3\}\}^{\mathbb{N} \cup \{0\}}$ , consider the nested sequence of non-empty closed subsets of **Ends** formed by

$$C_{i_0 \dots i_n} = \{E \in \mathbf{Ends} \mid i_k \subset i_k(E) \text{ for all } 0 \leq k \leq n\}.$$

It follows that  $C_{(i_k)} = \cap C_{i_0 \dots i_n}$  is non-empty and, by the strong form of injectivity proved above, we have that  $C_{(i_k)}$  is a singleton.

It follows that the image of **Ends** can be characterized as the set of sequences  $(i_k) \in \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}^{\mathbb{N} \cup \{0\}}$  such that:

- (a) if  $i_k = \{1, 2\}$  for some  $k \geq 0$ , then  $i_{k+\ell} = i_\ell(E^*)$  and;
- (b) if for some  $k \geq 0$  and all  $\ell \geq 0$  we have that  $i_{k+\ell} \subset i_\ell(E^*)$ , then  $i_k = \{1, 2\}$ .

Also,  $it \circ \pi$  is injective over the Julia set and  $it \circ \pi(J(\varphi))$  is the set of all itineraries (in the image of  $it$ ) with no symbol equal to  $\{1, 2\}$ . So it is sufficient to construct compact subsets of  $it \circ \pi(J(\varphi))$  invariant under the one-sided shift  $\sigma$  with topological entropy arbitrarily close to  $\log 3$ . For this, let  $p$  denote the period of  $E^*$  and for each  $N > p$ , consider the set  $Y_N$  of all symbol sequences  $(i_k)$  with  $i_k \neq \{1, 2\}$  for all  $k \geq 0$  and such that for some  $0 \leq j < N$  and all  $\ell \geq 0$

$$i_{\ell N+j} = i_{\ell N+1+j} = \dots = i_{\ell N+p-1+j} = \{3\}.$$

Since the topological entropy of  $\sigma : Y_N \rightarrow Y_N$  is  $(1 - pN^{-1}) \log 3$ , the proposition follows. □

It is worth to mention that  $\varphi_{\#} : \mathbf{Ends} \rightarrow \mathbf{Ends}$  is topologically conjugated to the dynamics of  $\varphi$  over the Julia set of  $\varphi$  in the Berkovich analytic space induced by  $\mathbb{L}$  (compare with [27, 31]). Favre and Rivera’s construction in [17] (compare with [2]) produces an equilibrium measure  $\mu$  supported in the Berkovich space Julia set of  $\varphi$ . Combining the basic properties of the equilibrium measure with our results it is not difficult to check that  $J(\varphi) \subset \mathbb{L}$  has full measure whenever  $J(\varphi) \neq \emptyset$ . Moreover, from

the symbolic dynamics described above one can also describe  $\mu$  and check that the measure theoretical entropy of  $\mu$  is  $\log 3$  (for  $J(\varphi) \neq \emptyset$ ). That is, for cubic polynomial dynamics  $\mu$  is a measure of maximal entropy.

### 5. Some one parameter families

As a preparation for the study of parameter space we fix  $\alpha \in \mathbb{L}$  such that

$$|\alpha|_o > 1$$

and consider the one parameter family

$$\psi_\nu(\zeta) = \alpha^2(\zeta - 1)^2(\zeta + 2) + \nu$$

where  $\nu \in \mathbb{L}$ . We will identify the cubic polynomial  $\psi_\nu \in \mathbb{L}[\zeta]$  with  $\nu \in \mathbb{L}$ . Note that the critical points of  $\psi_\nu$  are  $\omega^\pm = \pm 1$ ,  $\nu = \varphi(\omega^+)$  is a critical value, and  $\nu = \varphi(\gamma^+)$  where  $\gamma^+ = -2$  is the cocritical value of  $\omega^+$ .

The aim of this section is to study the parameter space structure of  $\psi_\nu$  around the values of  $\nu$  for which  $\omega^+ \in K(\psi_\nu)$ . In the next section we will see that every cubic polynomial with one critical point in the filled Julia set and the other in the basin of infinity is affinely conjugate to at least one in a family of this type (i.e., for some  $\alpha$ ).

#### *The level zero set.*

Let  $D'_0$  denote the level 0 dynamical ball of  $\psi_\nu$ . We say that

$$\mathcal{L}_0 := \{\nu \in \mathbb{L} \mid \psi_\nu(\omega^+) \in D'_0\}$$

is the level 0 set of the family  $\psi_\nu$ . It consists of all parameter such that the critical value  $\nu$  lies in the level 0 dynamical ball  $D'_0$ .

LEMMA 5.1. — *The level 0 set of  $\psi_\nu$  is  $B_1^+(0)$ . If  $\psi_\nu \in \mathcal{L}_0$ , then the following hold:*

- (i)  $R_{\psi_\nu} = 1 = \text{diam } K(\psi_\nu)$ .
- (ii)  $\psi_\nu(\omega^-) \notin D'_0 \ni \nu$  where  $D'_0 = B_1^+(0)$  is the level 0 ball of  $\psi_\nu$ .

*Proof.* — According to the formula for  $R_\varphi$  given in Subsection 2.5, if  $|\nu|_o > 1$ , then  $R_\varphi < |\nu|_o$  and  $\nu = \varphi(\omega^+) \notin K(\varphi)$ . Moreover, if  $|\nu|_o \leq 1$ , then  $R_\varphi = 1$  and  $\nu \in D'_0$ . Therefore,  $\mathcal{L}_0 = B_1^+(0)$ .

Now, if  $\psi_\nu \in \mathcal{L}_0$ , then  $|\psi_\nu(\omega^-)|_o = |4\alpha^3 + \nu|_o = |\alpha^3|_o > 1$ . Hence,  $\psi_\nu(\omega^-) \notin D'_0$ . It follows that  $\omega^+$  and  $\gamma^+ = -2$  are in different level 1 balls. Since both level 1 balls contain fixed points (see Lemma 2.4), we have that  $\text{diam } K(\varphi) = 1$  and the lemma follows. □

From the previous lemma, for all  $\psi_\nu \in \mathcal{L}_0$  we have that  $R_{\psi_\nu} = 1 = \text{diam } K(\psi_\nu)$ ,  $\omega^- \notin K(\psi_\nu)$  and  $\psi_\nu(\omega^+) \in D'_0$ . Thus, the assumptions and therefore the definitions and results contained in Subsection 4.1 apply to  $\psi_\nu \in \mathcal{L}_0$ .

*Results.*

We now state the main results concerning the structure of the parameter space of  $\psi_\nu$ .

In this section we will prove the analogue of Theorem 1.3 for this family (see (iii) of Theorem 5.3 below). In the next section this analogue will be used to prove Theorem 1.3. More precisely we will show that  $\nu \in \partial\{\nu \in \mathcal{L}_0 \mid \{\omega^\pm\} \subset \mathbb{L} \setminus K(\varphi)\}$  if and only if  $\omega^+ \in K(\psi_\nu)$  and  $\mathbf{M}^\nu(\omega^+)$  is not periodic. This fact will be a consequence of our parameter space description for the family  $\psi_\nu$ . In particular, we will characterize “where” a polynomial with a given critical marked grid can be found. In order to be precise we will need the following definition.

DEFINITION 5.2. — *A two dimensional array  $\mathbf{M} = (M_{\ell,k})_{\ell,k \geq 0}$  such that  $M_{\ell,k} \in \{0,1\}$  is called an admissible marked grid if (Ma)–(Md) of Proposition 4.5 hold for all  $n$ . If moreover  $M_{\ell,0}$  is marked for all  $\ell$ , then we say that  $\mathbf{M}$  is an admissible critical marked grid. Similarly, an array  $\mathbf{M}_n = (M_{\ell,k})$  where  $\ell + k \leq n$  for which (Ma)–(Md) hold is called an admissible marked grid of level  $n$ .*

THEOREM 5.3. — *Consider an admissible critical marked grid  $\mathbf{M}$  and let*

$$C_{\mathbf{M}} = \{\psi_\nu \in \mathcal{L}_0 \mid \omega^+ \in K(\psi_\nu) \text{ and } \mathbf{M} = \mathbf{M}^\nu(\omega^+)\}$$

where  $\mathbf{M}^\nu(\omega^+)$  is the marked grid of the critical point  $\omega^+$  under iterations of  $\psi_\nu$ . Then the following hold:

- (i) *If  $\mathbf{M}$  is periodic of period  $p$ , then  $C_{\mathbf{M}}$  is a non-empty union of finitely many pairwise disjoint closed balls.*
- (ii) *If  $\mathbf{M}$  is not periodic, then  $C_{\mathbf{M}}$  is a non-empty compact set which is either finite or a Cantor set.*
- (iii)  *$\psi_\nu \in \partial\{\psi_\nu \in \mathcal{L}_0 \mid \{\omega^\pm\} \subset \mathbb{L} \setminus K(\varphi)\}$  if and only if  $\omega^+ \in K(\psi_\nu)$  and  $\mathbf{M}^\nu(\omega^+)$  is aperiodic.*

We prove this theorem in Subsection 5.3. The proof relies on describing how polynomials are organized in  $\mathcal{L}_0$ . To describe  $\mathcal{L}_0$ , for  $n \geq 0$  we introduce the level  $n$  sets

$$\mathcal{L}_n = \{\psi_\nu \in \mathcal{L}_0 \mid \psi_\nu^{n+1}(\omega^+) \in D'_0\}.$$

Note that  $\mathcal{L}_n$  is a finite disjoint union of closed balls. Each of these balls is called a *parameter ball of level  $n$* . Also observe that  $\mathcal{L}_{n+1} \subset \mathcal{L}_n$  for all  $n \geq 0$ .

We denote the level  $n$  dynamical ball of  $\psi_\nu \in \mathcal{L}_0$  containing  $\zeta$  by  $D_n^\nu(\zeta)$ , the radius of  $D_n^\nu(\zeta)$  by  $r_n^\nu(\zeta)$ , the level  $n$  annulus around  $\zeta$  by  $A_n^\nu(\zeta)$  and the level  $n$  marked grid of  $\zeta$  by  $\mathbf{M}_n^\nu(\zeta)$  with entries  $M_{\ell,k}^\nu(\zeta)$ .

DEFINITION 5.4. — *Let  $n \in \mathbb{N}$ . We say that  $\psi_\nu \in \mathcal{L}_0$  is a center of level  $n$  if*

- (i)  $\psi_\nu^p(\omega^+) = \omega^+$  and
- (ii)  $\omega^+ \notin \psi_\nu^k(D_{n+1}^\nu(\omega^+))$  for  $k = 1, \dots, p - 1$ .

The correspondence between level  $n$  dynamical and parameter balls is stated in the next proposition.

PROPOSITION 5.5. — *Let  $\mathcal{D}_n$  be a parameter ball of level  $n$ . Then the following hold:*

- (i)  $\mathcal{D}_n = D_n^\nu(\nu)$  for all  $\nu \in \mathcal{D}_n$ .
- (ii)  $\mathbf{M}_{n+1}^\nu(\omega^+) = \mathbf{M}_{n+1}^{\nu'}(\omega^+)$  for all  $\nu, \nu' \in \mathcal{D}_n$ .
- (iii) *There exists a unique center of level  $n$  in  $\mathcal{D}_n$ . The period of this center is  $\min\{k \geq 1 \mid M_{n+1-k,k}^\nu(\omega^+) = 1\}$  for any  $\nu \in \mathcal{D}_n$ .*

The proof of this proposition is contained in Subsection 5.2.

In particular, the above proposition shows that the radius of  $\mathcal{D}_n$  is easily computed from  $\mathbf{M}_{n+1}^\nu(\omega^+)$  for any  $\nu \in \mathcal{D}_n$  and coincides with the radius of the level  $n$  dynamical ball around the critical value  $\nu = \psi_\nu(\omega^+)$ . The proposition also says that if  $\nu \in \mathcal{D}_n$  and the critical point  $\omega^+$  is periodic of period  $q$  under  $\psi_\nu$ , then  $q \geq p$  where  $p$  is the period of the center of  $\mathcal{D}_n$ . Moreover,  $p = q$  if and only if  $\psi_\nu$  is the unique level  $n$  center in  $\mathcal{D}_n$ .

Also in Subsection 5.2 we will obtain the following consequence of the previous proposition.

COROLLARY 5.6. — *Let  $\text{Per}$  be the set of parameters  $\nu \in \mathbb{L}$  such that  $\omega^+$  is periodic under  $\psi_\nu$ . Then the set of accumulation points of  $\text{Per}$  coincides with the set of parameters  $\nu \in \mathbb{L}$  such that  $\omega^+ \in K(\psi_\nu)$  and the critical marked grid of  $\psi_\nu$  is not periodic.*

The next proposition describes the correspondence between level  $n + 1$  parameter and dynamical balls.

PROPOSITION 5.7. — *Consider a level  $n$  parameter ball  $\mathcal{D}_n$  and let  $P$  be an element of the affine partition associated to  $\mathcal{D}_n$ . For any  $\nu \in \mathcal{D}_n$  we have the following:*

There exists a level  $n + 1$  parameter ball contained in  $P$  if and only if there exists a level  $n + 1$  dynamical ball  $D_{n+1}^\nu(\zeta)$  contained in  $P$ . In this case, the level  $n + 1$  parameter ball  $\mathcal{D}_{n+1}$  is unique and

$$\mathbf{M}_{n+1}^\nu(\zeta) = \mathbf{M}_{n+1}^{\nu'}(\nu')$$

for all  $\nu' \in \mathcal{D}_{n+1}$ . In particular, the radii of  $D_{n+1}^\nu(\zeta)$  and  $\mathcal{D}_{n+1}$  coincide.

The proof of this proposition is also contained in Subsection 5.2.

### 5.1. Thurston map

Our next result is the key to prove propositions 5.5 and 5.7. It shows that given a polynomial  $\psi_\nu$  and a level  $n + 1$  dynamical ball  $D_{n+1}$  inside the critical value ball of level  $n$  there exists a parameter  $\nu'$  close to the level  $n + 1$  ball  $D_{n+1}$  such that the critical point of  $\psi_{\nu'}$  is periodic with orbit close to that of the points in  $D_{n+1}$ . The precise statement is as follows:

PROPOSITION 5.8. — Consider a parameter  $\hat{\nu} \in \mathcal{L}_n$  and let  $\hat{\zeta}_1$  be a level  $n + 1$  point such that  $D_{n+1}^{\hat{\nu}}(\hat{\zeta}_1) \subset D_n^{\hat{\nu}}(\hat{\nu})$ . For  $k \geq 0$ , let  $\hat{\zeta}_{k+1} = \psi_{\hat{\nu}}^k(\hat{\zeta}_1)$  and

$$\hat{p} = \min\{k \geq 1 \mid \omega^+ \in D_{n+2-k}^{\hat{\nu}}(\hat{\zeta}_k)\}.$$

Then there exists a unique  $\nu'$  such that:

- (i)  $|\nu' - \hat{\zeta}_1|_o < r_n^{\hat{\nu}}(\hat{\nu})$ .
- (ii)  $\psi_{\nu'}^{\hat{p}}(\omega^+) = \omega^+$ .

This subsection is devoted to the proof of the previous proposition so throughout we consider  $\hat{\nu}$ ,  $\hat{\zeta}_k$  and  $\hat{p}$  as above. The parameter  $\nu'$  is obtained as the first coordinate of the fixed point of an appropriate “Thurston map” which acts on:

$$\mathbb{B} := \{(\zeta_1, \dots, \zeta_{\hat{p}} = \omega^+) \mid |\zeta_k - \hat{\zeta}_k|_o < \rho_k\}$$

where  $\rho_k = r_{n+1-k}^{\hat{\nu}}(\hat{\zeta}_k)$ .

We start with two lemmas which apply to an arbitrary  $\nu \in \mathcal{L}_n$ .

LEMMA 5.9. — Let  $n \geq 0$  be an integer and consider  $\nu \in \mathcal{L}_n$ . For  $k \geq 1$ , let  $\nu_k = \psi_\nu^k(\omega^+)$ . Then

- (i)  $r_n^\nu(\nu) < r_{n+1-k}^\nu(\nu_k)$  for  $k = 2, \dots, n$ .



(ii) Assume that  $\zeta_1$  is a level  $n + 1$  point in  $D_n^\nu(\nu)$ . For  $k \geq 1$ , let  $\zeta_{k+1} = \psi_\nu^k(\zeta_1)$  and

$$p = \min\{k \geq 1 \mid \omega^+ \in D_{n+2-k}^\nu(\zeta_k)\}.$$

Then

$$r_{n+2-(k+1)}^\nu(\zeta_{k+1}) = |\alpha|_o^2 \cdot |\zeta_k - \omega^+|_o \cdot r_{n+2-k}^\nu(\zeta_k)$$

for all  $k = 1, \dots, p - 1$ .

*Proof.* — For each  $\ell$  such that  $0 \leq \ell \leq n + 1 - k$  let  $\delta(\ell) \geq 1$  be the integer such that  $A_{\ell+\delta(\ell)}^\nu(\nu_{k-\delta(\ell)})$  is critical but  $A_{\ell+i}^\nu(\nu_{k-i})$  is not critical for all  $0 < i < \delta(\ell)$ . To find such an integer  $\delta(\ell)$  start at  $M_{\ell,k}(\omega^+)$  in the critical marked grid and follow the southwest diagonal until you hit a critical position. The number of columns that you moved to the left is  $\delta(\ell)$ . Note that  $\ell + \delta(\ell)$  is a strictly increasing function of  $\ell$  and  $0 < \ell + \delta(\ell) \leq n + 1$ . Moreover,

$$2 \bmod A_{\ell+\delta(\ell)}^\nu(\omega^+) = 2 \bmod A_{\ell+\delta(\ell)}^\nu(\nu_{k-\delta(\ell)}) = \bmod A_\ell^\nu(\nu_k).$$

Since  $\psi_\nu(D_0^\nu) = B_{|\alpha|_o^2}^+(0)$  and  $\bmod A_0^\nu = \log |\alpha|_o^2$ ,

$$\begin{aligned} \log |\alpha|_o^2 - \log r_n^\nu(\nu) &= \sum_{\ell=0}^n \bmod A_\ell^\nu(\nu) = 2 \cdot \sum_{\ell=1}^{n+1} \bmod A_\ell^\nu(\omega^+) \\ &> 2 \cdot \sum_{\ell=0}^{n+1-k} \bmod A_{\ell+\delta(\ell)}^\nu(\omega^+) = \sum_{\ell=0}^{n+1-k} \bmod A_\ell^\nu(\nu_k) \\ &= \log |\alpha|_o^2 - \log r_{n+1-k}^\nu(\nu_k). \end{aligned}$$

Hence (i) follows.

Now we prove (ii). Fix  $k \geq 1$  and let  $\ell$  be such that  $A_\ell^\nu(\zeta_k)$  is critical (i.e.,  $= A_\ell^\nu(\omega^+)$ ) but  $A_{\ell+1}^\nu(\zeta_k)$  is not. It follows that  $D_\ell^\nu(\omega^+) = D_\ell^\nu(\zeta_k)$  and  $D_{\ell+1}^\nu(\omega^+) \neq D_{\ell+1}^\nu(\zeta_k)$ . By Lemma 4.2 (i),

$$r_\ell^\nu(\zeta_k) = |\zeta_k - \omega^+|_o.$$

Note that:

$$2 \bmod A_j^\nu(\zeta_k) = \begin{cases} \log |\alpha|_o^4 & \text{if } j = 0, \\ \bmod A_{j-1}^\nu(\zeta_{k+1}) & \text{if } 1 \leq j \leq \ell. \end{cases}$$

Also,  $\text{mod } A_j^\nu(\zeta_k) = \text{mod } A_{j-1}^\nu(\zeta_{k+1})$  if  $j \geq \ell + 1$ . Hence:

$$\begin{aligned} \log |\alpha|_o^6 - \log r_{n+2-(k+1)}^\nu(\zeta_{k+1}) &= \log |\alpha|_o^4 + \sum_{j=0}^{n+2-(k+1)} \text{mod } A_j^\nu(\zeta_{k+1}) \\ &= 2 \cdot \sum_{j=0}^{\ell} \text{mod } A_j^\nu(\zeta_k) + \sum_{j=\ell+1}^{n+2-k} \text{mod } A_j^\nu(\zeta_k) \\ &= \log |\alpha|_o^2 - \log r_\ell^\nu(\zeta_k) + \log |\alpha|_o^2 \\ &\quad - \log r_{n+2-k}^\nu(\zeta_k). \end{aligned}$$

Statement (ii) follows after replacing  $r_\ell^\nu(\zeta_k)$  by  $|\zeta_k - \omega^+|_o$ . □

LEMMA 5.10. — Consider  $n \geq 1$  and  $\nu \in \mathcal{L}_n$ . Let  $\nu' \in D_n^\nu(\nu)$  and, for all  $k \geq 0$ , let  $\nu_k = \psi_\nu^k(\omega^+)$  and  $\nu'_k = \psi_{\nu'}^k(\omega^+)$ . Then for all  $k$  such that  $0 \leq k \leq n$  the following hold:

- (i)  $D_{n+1-k}^\nu(\nu_k) = D_{n+1-k}^{\nu'}(\nu'_k)$ . In particular,  $\mathbf{M}_{n+1}^\nu(\omega^+) = \mathbf{M}_{n+1}^{\nu'}(\omega^+)$ .
- (ii) Let  $\mathcal{P}_{n+1-k}$  be the affine partition associated to  $D_{n+1-k}^\nu(\nu_k)$ , then:

$$\psi_{\nu*} = \psi_{\nu'*} : \mathcal{P}_{n+1-k} \rightarrow \mathcal{P}_{n+1-(k+1)}.$$

Proof. — Let  $k$  be such that  $0 \leq k \leq n$  and  $\zeta' \in D_{n+1-k}^\nu(\nu_k)$ . Then

$$\begin{aligned} |\psi_{\nu'}(\zeta') - \nu_{k+1}|_o &= |\psi_{\nu'}(\zeta') - \psi_\nu(\zeta') + \psi_\nu(\zeta') - \nu_{k+1}|_o \\ &\leq \max\{r_n^\nu(\nu), |\psi_\nu(\zeta') - \nu_{k+1}|_o\} \\ &< r_{n+1-(k+1)}^\nu(\nu_{k+1}). \end{aligned}$$

Therefore,  $\psi_{\nu'}(\zeta') \in D_{n+1-(k+1)}^\nu(\nu_{k+1})$ . Hence  $\psi_{\nu'}^{n+1-k}(D_{n+1-k}^\nu(\nu_k)) \subset D_0$  and  $\nu'_k \in D_{n+1-k}^{\nu'}(\nu'_k) \subset D_{n+1-k}^{\nu'}(\nu_k)$ . In particular,  $r_n^{\nu'}(\nu') \geq r_n^\nu(\nu)$  and  $\nu \in D_n^{\nu'}(\nu')$ . After switching  $\nu$  for  $\nu'$  and repeating the above argument it follows that  $D_{n+1-k}^{\nu'}(\nu'_k) \subset D_{n+1-k}^\nu(\nu_k)$  and (i) follows.

For (ii), let  $P$  be an element of the partition  $\mathcal{P}_{n+1-k}$  and choose  $\zeta \in P$ . Then

$$|\psi_\nu(\zeta) - \psi_{\nu'}(\zeta)|_o \leq r_n^\nu(\nu) < r_{n+1-(k+1)}^\nu.$$

Therefore,  $\psi_{\nu'}(P) = \psi_\nu(P) \in \mathcal{P}_{n+1-(k+1)}$ . □

LEMMA 5.11. — Let

$$\mathbb{B} := \{(\zeta_1, \dots, \zeta_{\hat{p}} = \omega^+) \mid |\zeta_k - \hat{\zeta}_k|_o < \rho_k\}$$

where  $\rho_k = r_{n+1-k}^{\hat{\zeta}_k}$ . Then for each  $(\zeta_1, \dots, \zeta_{\hat{p}}) \in \mathbb{B}$  there exists a unique  $(\zeta'_1, \dots, \zeta'_{\hat{p}}) \in \mathbb{B}$  such that  $\psi_{\zeta_1}(\zeta'_k) = \zeta_{k+1}$  for  $1 \leq k < \hat{p}$ .

*Proof.* — Let  $\mathcal{P}_{n+1-k}$  be the partition associated to  $D_{n+1-k}^{\hat{\zeta}_k}$  and denote by  $P(\zeta) = B_{\rho_k}(\zeta)$  the element of  $\mathcal{P}_{n+1-k}$  that contains  $\zeta$ . Since  $\psi_{\zeta_{1*}} = \psi_{\hat{\nu}^*} : \mathcal{P}_{n+1-k} \rightarrow \mathcal{P}_{n+1-(k+1)}$  and  $\psi_{\hat{\nu}} : P(\hat{\zeta}_k) \rightarrow P(\hat{\zeta}_{k+1})$  is one-to-one for  $1 \leq k < \hat{p}$  we have that  $\psi_{\zeta_1} : P(\hat{\zeta}_k) \rightarrow P(\hat{\zeta}_{k+1})$  is also one-to-one. Hence, there exists a unique  $\zeta'_k \in P(\hat{\zeta}_k)$  such that  $\psi_{\zeta_1}(\zeta'_k) = \zeta_{k+1} \in P(\hat{\zeta}_{k+1})$ .  $\square$

We may now define a Thurston map as

$$T : \mathbb{B} \rightarrow \mathbb{B} \\ (\zeta_1, \dots, \zeta_{\hat{p}}) \mapsto (\zeta'_1, \dots, \zeta'_{\hat{p}})$$

if  $\psi_{\zeta_1}(\zeta'_k) = \zeta_{k+1}$  for all  $k = 1, \dots, \hat{p} - 1$ .

LEMMA 5.12. — *A parameter  $\nu'$  is such that (i) and (ii) of Proposition 5.8 hold if and only if  $(\nu', \psi_{\nu'}^2(\omega^+), \dots, \psi_{\nu'}^{\hat{p}-1}(\omega^+), \omega^+)$  is a fixed point of  $T$ .*

*Proof.* — Given  $\nu'$  such that (i) of Proposition 5.8 holds, by Lemma 5.10 (ii), for  $k = 1, \dots, \hat{p} - 1$ , we have that  $|\psi_{\nu'}^k(\omega^+) - \hat{\zeta}_k|_o < \rho_k$ . Therefore,  $(\nu', \psi_{\nu'}^2(\omega^+), \dots, \psi_{\nu'}^{\hat{p}-1}(\omega^+), \omega^+)$  belongs to  $\mathbb{B}$  and clearly is a fixed point of  $T$ . The converse is straightforward.  $\square$

It follows that to prove the proposition is sufficient to show that  $T$  has a unique fixed point.

In  $\mathbb{L}^{\hat{p}}$  we consider the sup-norm:

$$\|\vec{\zeta} = (\zeta_1, \dots, \zeta_{\hat{p}})\|_{\infty} = \max\{|\zeta_1|_o, \dots, |\zeta_{\hat{p}}|_o\}.$$

LEMMA 5.13. — *For all  $\vec{\zeta} = (\zeta_1, \dots, \zeta_{\hat{p}}) \in \mathbb{B}$  we have that  $T^n(\vec{\zeta})$  converges to a fixed point of  $T$ .*

*Proof.* — Consider  $(\zeta_1^{(0)}, \dots, \zeta_{\hat{p}}^{(0)}) \in \mathbb{B}$  and let

$$(\zeta_1^{(n)}, \dots, \zeta_{\hat{p}}^{(n)}) = T^n(\zeta_1^{(0)}, \dots, \zeta_{\hat{p}}^{(0)}).$$

For  $k = 1, \dots, \hat{p} - 1$ ,

$$|\zeta_k^{(n+1)} - \zeta_k^{(n+2)}|_o \leq \frac{1}{|\alpha|_o^2 |\hat{\zeta}_k - \omega^+|_o} \cdot \max\{|\zeta_{k+1}^{(n)} - \zeta_{k+1}^{(n+1)}|_o, |\zeta_1^{(n)} - \zeta_1^{(n+1)}|_o\} \\ = \frac{\rho_k}{\rho_{k+1}} \max\{|\zeta_{k+1}^{(n)} - \zeta_{k+1}^{(n+1)}|_o, |\zeta_1^{(n)} - \zeta_1^{(n+1)}|_o\}$$

Hence, for all  $n$ , there exist  $k_1, \dots, k_j$  such that  $1 \leq k_i \leq \hat{p} - 1$ ,  $k_1 + \dots + k_j = n$ , and

$$|\zeta_1^{(n+1)} - \zeta_1^{(n+2)}|_o \leq \frac{\rho_1}{\rho_{k_1+1}} \dots \frac{\rho_1}{\rho_{k_j+1}} \max\{|\zeta_{k_j+1}^{(0)} - \zeta_{k_j+1}^{(1)}|_o, |\zeta_1^{(0)} - \zeta_1^{(1)}|_o\}.$$

Now let

$$\lambda = \max \left\{ \frac{\rho_1}{\rho_k} \mid k = 2, \dots, \hat{p} - 1 \right\} < 1.$$

Since,  $j \geq \frac{n}{\hat{p}-1}$ , it follows that

$$|\zeta_1^{(n+1)} - \zeta_1^{(n+2)}|_o \leq \lambda^{\frac{n}{\hat{p}-1}} \|\vec{\zeta}^{(0)} - \vec{\zeta}^{(1)}\|_\infty.$$

Therefore,  $\zeta_1^{(n)}$  converges to some  $\zeta_1^{(\infty)}$  as  $n \rightarrow \infty$  and  $\zeta_{k+1}^{(n)} \rightarrow \zeta_{k+1}^{(\infty)} = \psi_{\zeta_1^{(\infty)}}^k(\zeta_1^{(\infty)})$  for  $k = 1, \dots, \hat{p} - 1$ . It follows that  $(\zeta_1^{(\infty)}, \dots, \zeta_{\hat{p}}^{(\infty)})$  is a fixed point for  $T$ . □

LEMMA 5.14. —  $T$  has a unique fixed point in  $\mathbb{B}$ .

*Proof.* — Suppose that  $\vec{\zeta} = (\zeta_1, \dots, \zeta_{\hat{p}})$  and  $\vec{\eta} = (\eta_1, \dots, \eta_{\hat{p}})$  are fixed points of  $T$ . For  $k = 1, \dots, \hat{p} - 1$  the polynomial  $h : \zeta \mapsto \alpha^2(\zeta - 1)^2(\zeta + 2)$  maps  $B_{\rho_k}(\zeta_k) = B_{\rho_k}(\eta_k)$  isomorphically onto its image and

$$\left| \frac{dh}{d\zeta}(\zeta_k) \right|_o = |\alpha|_o^2 |\zeta_k - \omega^+|_o.$$

Therefore,

$$\begin{aligned} |\zeta_{k+1} - \eta_{k+1} + \eta_1 - \zeta_1|_o &= |h(\zeta_k) - h(\eta_k)|_o \\ &= |\zeta_k - \eta_k|_o |\alpha|_o^2 |\zeta_k - \omega^+|_o. \end{aligned}$$

Hence,

$$\begin{aligned} |\zeta_k - \eta_k|_o &\leq \frac{1}{|\alpha|_o^2 |\zeta_k - \omega^+|_o} \max\{|\zeta_{k+1} - \eta_{k+1}|_o, |\eta_1 - \zeta_1|_o\} \\ &= \frac{\rho_k}{\rho_{k+1}} \max\{|\zeta_{k+1} - \eta_{k+1}|_o, |\eta_1 - \zeta_1|_o\}. \end{aligned}$$

It follows that for some  $k$  such that  $2 \leq k < \hat{p}$ :

$$|\zeta_1 - \eta_1|_o \leq \frac{\rho_1}{\rho_k} |\eta_1 - \zeta_1|_o.$$

Since  $\rho_1 < \rho_k$  we have that  $\zeta_1 = \eta_1$  and  $\vec{\zeta} = \vec{\eta}$ . □

### 5.2. Parameter balls

Here we prove propositions 5.5 and 5.7 together with Corollary 5.6.

*Proof of Proposition 5.5.* — By Lemma 5.10 (i) we have that  $D_n^\nu(\nu) \subset \mathcal{D}_n$  for all  $\nu \in \mathcal{D}_n$ . Moreover, for all  $\nu, \nu' \in \mathcal{D}_n$ , the dynamical balls  $D_n^\nu(\nu)$  and  $D_n^{\nu'}(\nu')$  are equal or disjoint. By Proposition 5.8, for each  $\nu \in \mathcal{D}_n$  the ball  $D_n^\nu(\nu)$  contains at least one element of the finite set

$$\{\nu \mid \psi_\nu^k(\omega^+) = \omega^+ \text{ for some } 1 \leq k \leq n + 2\}.$$

It follows that  $\mathcal{D}_n = \cup_{\nu \in \mathcal{D}_n} D_n^\nu(\nu)$  is a finite union of closed and pairwise disjoint balls. This is only possible if  $\mathcal{D}_n = D_n^\nu(\nu)$  for all  $\nu \in \mathcal{D}_n$ . Hence we have proven statement (i). Statement (ii) now follows from Lemma 5.10 (i).

For (iii), let  $\hat{\nu} \in \mathcal{D}_n$  and note that  $\hat{\nu} \in D_n^{\hat{\nu}}(\hat{\nu}) \subset D_{n-1}^{\hat{\nu}}(\hat{\nu})$ . Let  $\hat{p} = \min\{k \geq 1 \mid \omega^+ \in D_{n+1-k}^{\hat{\nu}}(\psi_{\hat{\nu}}^k(\omega^+))\}$ . Proposition 5.8 says that there exists a unique  $\nu'$  such that  $\psi_{\nu'}^{\hat{p}}(\omega^+) = \omega^+$  and  $|\nu' - \hat{\nu}|_o < r_{n-1}^{\hat{\nu}}(\hat{\nu})$ . We must show that  $\nu' \in \mathcal{D}_n$ . In fact, since  $\omega^+ \in \psi_{\hat{\nu}}^{\hat{p}-1}(D_n^{\hat{\nu}}(\hat{\nu}))$  there exist a level  $n + 1$  ball  $D_{n+1}^{\hat{\nu}}(\hat{\zeta}_1)$  in  $D_n^{\hat{\nu}}(\hat{\nu})$  that maps onto  $D_1^{\hat{\nu}}(\omega^+)$  under  $\psi_{\hat{\nu}}^{\hat{p}-1}$ . By Proposition 5.8 we have that  $|\nu' - \hat{\zeta}_1|_o < r_n^{\hat{\nu}}(\hat{\nu})$ . Therefore  $\nu' \in \mathcal{D}_n$ .  $\square$

We leave record of a straightforward consequence of Proposition 5.5 in the following statement (see Theorem 4.7 and Lemma 2.13).

**COROLLARY 5.15.** — *If  $\omega^+ \in K(\psi_\nu)$ ,  $\mathbf{M}^\nu(\omega^+)$  is not periodic and  $\mathcal{D}_n$  denotes the level  $n$  parameter ball containing  $\nu$ , then  $\{\nu\} = \cap \mathcal{D}_n$ .*

Now we can show how to deduce Corollary 5.6 from the already proven Proposition 5.5.

*Proof of Corollary 5.6.* — Provided that  $\psi_\nu$  has aperiodic critical marked grid, the radius of the level  $n$  dynamical disk the radii of the level  $n$  parameter balls  $\mathcal{D}_n$  containing  $\nu$  converge to zero. Hence, the centers  $\nu_n$  of  $\mathcal{D}_n$  converge to  $\nu$ . Thus, parameters with aperiodic critical marked grid are accumulation points of Per.

We consider a parameter  $\hat{\nu}$  with periodic critical marked grid, say of period  $q$ , and show that  $\hat{\nu}$  is not an accumulation point of Per. For this let  $\mathcal{D}_\infty$  be the intersection of all parameter balls containing  $\hat{\nu}$  and let  $D_\infty$  be the infraconnected component of  $K(\psi_{\hat{\nu}})$  containing  $\omega^+$ . It follows that  $D_\infty$  is an infraconnected component of  $K(\psi_\nu)$  which is periodic of period  $q$  for all  $\nu \in \mathcal{D}_\infty$ . The return map  $\psi_\nu^q : D_\infty \rightarrow D_\infty$  has degree 2. Now we consider a parametrization of the affine partitions of  $D_\infty$  (resp.  $\mathcal{D}_\infty$ ) by  $\mathbb{Q}^a$  so that the natural projection  $\pi : D_\infty \rightarrow \mathbb{Q}^a$  (resp.  $\Pi : \mathcal{D}_\infty \rightarrow \mathbb{Q}^a$ ) map  $\omega^+$  to 0 (resp. the unique element  $\nu_{center}$  of  $\mathcal{D}_\infty$  for which  $\omega^+$  has period  $q$  to 0).

As  $\nu$  varies over  $\mathcal{D}_\infty$ , we have that  $c = \Pi(\nu)$  takes all values in  $\mathbb{Q}^a$ . The projection of the map  $\psi_\nu^g : D_\infty \rightarrow D_\infty$  only depends on  $c \in \mathbb{Q}^a$  so is a polynomial of the form  $z \mapsto \alpha(c)z^2 + \beta(c)$  where  $\alpha(c)$  and  $\beta(c)$  are polynomials in  $c$  with coefficients in  $\mathbb{Q}^a$ . Since this map has degree 2 for all  $c = \Pi(\nu)$ , we have that  $\alpha(c)$  must be a non-zero constant. Moreover, the unique solution  $\nu \in \mathcal{D}_\infty$  of  $\psi_\nu^g(\omega^+) - \omega^+ = 0$  is  $\nu_{center}$ , say of multiplicity  $m$  (it can be shown that  $m = 1$ ). Now  $m$  is the degree of  $\psi_\nu^g(\omega^+)$  as a function from  $\mathcal{D}_\infty$  onto  $D_\infty$ . Hence,  $m$  is also the degree of  $\beta(c)$ . But since  $c^m$  divides  $\beta(c)$  we have that  $\beta(c)$  is a constant times  $c^m$ . After an affine change of coordinates in the dynamical  $z$ -plane and a rescaling in the parameter  $c$ -plane, we may assume that the projection of  $\psi_\nu^g : D_\infty \rightarrow D_\infty$  is the family  $f_c(z) = z^2 + c^m$  where  $c \in \mathbb{Q}^a$ .

For the standard quadratic family  $Q_c(z) = z^2 + c$  it is well known that  $Q_c^n(0) = 0$  has only simple solutions for all  $n \geq 1$ . It follows that for our family, all solutions of  $f_c^n(0) = 0$  are simple provided that  $n \geq 2$ . Hence two distinct parameters of Per do not lie in the same element of the affine partition of  $\mathcal{D}_\infty$ . Therefore  $\hat{\nu}$  is not an accumulation point of Per.  $\square$

*Proof of Proposition 5.7.* — By Proposition 5.8, if an element  $P$  of the partition  $\mathcal{P}_n$  associated to  $\mathcal{D}_n$  contains a level  $n + 1$  dynamical ball of some  $\psi_\nu$  with  $\nu \in \mathcal{D}_n = D_n^\nu(\nu)$ , then  $P$  contains a level  $n + 1$  parameter ball. Conversely, if  $P$  contains a parameter ball  $\mathcal{D}_{n+1}$  of level  $n + 1$  and  $\nu' \in \mathcal{D}_{n+1}$ , then  $\psi_{\nu'}^n(P) = \psi_\nu^n(P)$  for all  $\nu \in \mathcal{D}_n$ . Moreover, since  $\psi_{\nu'}^n(\nu')$  is a level 1 point we have that  $\psi_{\nu'}^n(P) = B_1(\omega^+)$  or  $B_1(\gamma^+)$ . In either case the preimage of  $D_1(\omega^+)$  or  $D_1(\gamma^+)$  under  $\psi_\nu : P \rightarrow B_1(\omega^+)$  or  $\psi_\nu : P \rightarrow B_1(\gamma^+)$  is a level  $n + 1$  dynamical ball contained in  $P$ .

We must show that  $P$  contains at most one parameter ball of level  $n + 1$ . Suppose that  $\mathcal{D}_{n+1}$  and  $\mathcal{D}'_{n+1}$  are level  $n + 1$  parameter balls contained in  $P$ . From Proposition 5.8, it is sufficient to show that the periods  $p$  and  $p'$  of their centers  $\mu$  and  $\mu'$  coincide. By Lemma 5.10 we have that  $\psi_\mu^k(P) = \psi_{\mu'}^k(P)$  for  $k = 1, \dots, n$ . Moreover, since there is at most one dynamical ball inside  $\psi_\mu^{k-1}(P)$  and  $D_{n+1}^\mu(\mu) = \mathcal{D}_{n+1} \subset P$  we have that:  $\omega^+ \in \psi_\mu^k(D_{n+2}^\mu(\omega^+)) = \psi_\mu^{k-1}(D_{n+1}^\mu(\mu)) \subset \psi_\mu^{k-1}(P)$  if and only if  $\omega^+ \in \psi_\mu^{k-1}(P)$ . Similarly,  $\omega^+ \in \psi_{\mu'}^k(D_{n+2}^{\mu'}(\omega^+))$  if and only if  $\omega^+ \in \psi_{\mu'}^{k-1}(P)$ . Therefore,  $p = p'$ .

Now let  $P$  be an element of the affine partition of  $\mathcal{D}_n$  and let  $\nu \in \mathcal{D}_n$ . Suppose that there exists a level  $n + 1$  parameter ball  $\mathcal{D}_{n+1} \subset P$  and a level  $n + 1$  dynamical ball  $D_{n+1}^\nu(\zeta) \subset P$ . To complete the proof of the

proposition, given  $\nu' \in \mathcal{D}_{n+1}$  we must show that

$$\mathbf{M}_{n+1}^\nu(\zeta) = \mathbf{M}_{n+1}^{\nu'}(\nu').$$

Since  $\zeta \in D_n^\nu(\nu)$  we have that  $\mathbf{M}_n^\nu(\zeta) = \mathbf{M}_n^\nu(\nu) = \mathbf{M}_n^{\nu'}(\nu')$ . Thus we just need to prove that  $\omega^+ \in D_{n+1-k}^\nu(\psi_\nu^k(\zeta))$  if and only if  $\omega^+ \in D_{n+1-k}^{\nu'}(\psi_{\nu'}^k(\nu'))$ , for all  $k = 0, \dots, n + 1$ . Since  $\nu' \in D_n^\nu(\nu)$ , Lemma 5.10 implies that  $\psi_{\nu^*} = \psi_{\nu'^*} : \mathcal{P}_{n+1-k} \rightarrow \mathcal{P}_{n+1-(k+1)}$  for all  $k$  such that  $1 \leq k \leq n$ , where  $\mathcal{P}_{n+1-k}$  is the affine partition associated to  $D_{n+1-k}^\nu(\psi_\nu^k(\omega^+)) = D_{n+1-k}^{\nu'}(\psi_{\nu'}^k(\omega^+))$ . Now  $\omega^+ \in D_{n+1-k}^\nu(\psi_\nu^k(\zeta))$  if and only if  $\omega^+ \in \psi_\nu^k(P)$  and  $\omega^+ \in \psi_{\nu'}^k(P)$  if and only if  $\omega^+ \in D_{n+1-k}^{\nu'}(\psi_{\nu'}^k(\nu'))$ . Therefore the proposition follows from the fact that  $\psi_\nu^k(P) = \psi_{\nu'}^k(P)$ . □

### 5.3. Realization

In order to prove Theorem 5.3 we first have to show that every admissible critical marked grid of level  $n$  is realized by a cubic polynomial.

**PROPOSITION 5.16.** — *Let  $n \geq 0$  and  $\mathbf{M}_{n+1}$  be an admissible critical marked grid of level  $n + 1$ . Then there exists  $\nu \in \mathcal{L}_n$  such that  $\mathbf{M}_{n+1}^\nu(\omega^+) = \mathbf{M}_{n+1}$ .*

*Proof.* — Since the proposition is clearly true for  $n = 0$  we proceed by induction. That is we suppose that  $\mathbf{M}_{n+2}$  is an admissible critical marked grid of level  $n + 2$  and  $\nu \in \mathcal{L}_n$  is such that  $\mathbf{M}_{n+1}^\nu(\omega^+)$  coincides with  $\mathbf{M}_{n+2}$  in all the positions  $(\ell, k)$  with  $\ell + k \leq n + 1$ . By Proposition 5.8, it suffices to show that there exists a level  $n + 1$  point  $\zeta$  contained in  $D_n^\nu(\nu)$  such that  $\mathbf{M}_{n+1}^\nu(\zeta)$  coincides with the grid obtained from  $\mathbf{M}_{n+2}$  after erasing its first column. For this purpose let  $\nu_j = \psi_\nu^j(\omega^+)$ ,  $p$  be the minimal  $k \geq 1$  such that the  $(n + 2 - k, k)$  position of  $\mathbf{M}_{n+2}$  is marked and  $k_1, \dots, k_m$  be such that  $0 < k_1 < \dots < k_m < p$  and  $(n + 1 - j, j)$  is marked in  $\mathbf{M}_{n+2}$  if and only if  $j = k_i$  for some  $i$ .

Our task boils down to find a dynamical ball  $D_{n+1}$  of level  $n + 1$  contained in  $D_n^\nu(\nu)$  such that  $\omega^+ \notin D_{n+1} \cup \dots \cup \psi_\nu^{p-2}(D_{n+1})$  and  $\psi_\nu^{p-1}(D_{n+1}) = D_{n+2-p}(\omega^+)$ .

We first claim that there exist 2 level  $n + 2 - k_m$  dynamical balls  $B_m^0$  and  $B_m^1$  contained in  $D_{n+1-k_m}^\nu(\nu_{k_m})$  such that for  $i = 0, 1$  we have that  $\omega^+ \notin B_m^i \cup \dots \cup \psi^{p-1-k_m}(B_m^i)$  and  $\omega^+ \in \psi^{p-k_m}(B_m^i)$ . There are two cases according to whether  $p < n + 2$  or  $p = n + 2$ .

In the case that  $p < n + 2$ , by (Mc) of Proposition 4.5,  $g_m = \psi_\nu^{p-k_m} : D_{n+1-k_m}^\nu(\nu_{k_m}) \rightarrow D_{n+1-p}^\nu(\nu_p)$  has degree 2 and  $g_m(\omega^+) \notin D_{n+2-p}^\nu(\omega^+)$ .

Hence, in this case we let  $B_m^0$  and  $B_m^1$  be the two preimages of  $D_{n+2-p}^\nu(\omega^+)$  under  $g_m$ .

In the case that  $p = n + 2$ , by (Md) of Proposition 4.5, the position  $(1, n + 1 - k_m)$  in  $\mathbf{M}_{n+2}$  is marked (otherwise, taking  $\ell = n + 1 - k_m$  and  $k = k_m$  in (Md) we would have that  $M_{1,n+1}$  would be marked and therefore  $p = n + 1$ ). Hence,  $g_m = \psi_\nu^{n+1-k_m} : D_{n+1-k_m}^\nu(\nu_{k_m}) \rightarrow D_0$  has degree 2 and  $g_m(\omega^+) \in D_1(\omega^+)$ . Therefore, in this case, we let  $B_m^0$  and  $B_m^1$  be the two preimages of  $D_1(-2)$  under  $g_m$ .

We now claim that for all  $j = 1, \dots, m$  there exist at least 2 level  $n+2-k_j$  dynamical balls  $B_j^0$  and  $B_j^1$  contained in  $D_{n+1-k_j}^\nu(\nu_{k_j})$  such that for  $i = 0, 1$  we have that  $\omega^+ \notin B_j^i \cup \dots \cup \psi_\nu^{p-1-k_j}(B_j^i)$  and  $\omega^+ \in \psi_\nu^{p-k_j}(B_j^i)$ . In fact, for  $j = m$  we have already established this, so we may assume the above true for  $j + 1$  and prove it for  $j$ . Since  $g_j = \psi_\nu^{k_{j+1}-k_j} : D_{n+1-k_j}^\nu(\nu_{k_j}) \rightarrow D_{n+1-k_{j+1}}^\nu(\nu_{k_{j+1}})$  has degree 2, it follows that at least one of the two balls  $B_{j+1}^0, B_{j+1}^1$  does not contain  $g_j(\omega^+)$ , say  $B_{j+1}^0$ , and we may let  $B_j^0$  and  $B_j^1$  be the preimages of  $B_{j+1}^0$  under  $g_j$ .

Finally, note that  $g_0 = \psi_\nu^{k_1-1} : D_n^\nu(\nu) \rightarrow D_{n+1-k_1}(\nu_{k_1})$  is one-to-one. The preimage of  $B_1^0$  under  $g_0$  is a level  $n+1$  dynamical ball  $D_{n+1}$  contained in  $D_n^\nu(\nu)$  such that  $\omega^+ \notin D_{n+1} \cup \dots \cup \psi_\nu^{p-2}(D_{n+1})$  and  $\omega^+ \in \psi_\nu^{p-1}(D_{n+1})$ . □

*Proof of Theorem 5.3.* — Let  $M_{\ell,k}$  denote the  $(\ell, k)$  entry of  $\mathbf{M}$ . For  $n \geq 0$ , let  $\mathbf{M}_n$  denote the level  $n$  grid with entries  $M_{\ell,k}$  where  $\ell + k \leq n$ .

(i) Suppose that  $\mathbf{M}$  is periodic of period  $p$ . Let  $n_0$  be such that  $\mathbf{M}_{n,n-k}$  is unmarked for all  $1 < k < p$  and all  $n > n_0$ . Denote by  $\nu_1, \dots, \nu_m$  the level  $n_0$  centers with critical marked grid of level  $n_0 + 1$  that coincides with  $\mathbf{M}_{n_0+1}$ . It follows that  $\mathbf{M} = \mathbf{M}^{\nu_i}(\omega^+)$  for all  $i = 1, \dots, m$ . For each  $i$ , let  $X_i$  be the infraconnected component of  $K(\psi_{\nu_i})$  that contains  $\nu_i$ . Recall that  $X_i$  is a closed ball (Corollary 4.8). Moreover, if  $\mathcal{D}_n(\nu_i)$  denotes the level  $n$  parameter ball which contains  $\nu_i$ , then  $X_i = \bigcap \mathcal{D}_n(\nu_i)$ . Hence,  $\mathbf{M}^\nu(\omega^+) = \mathbf{M}^{\nu_i}(\omega^+) = \mathbf{M}$  for all  $\nu \in X_i$ . That is,  $X_1 \cup \dots \cup X_m \subset C_{\mathbf{M}}$ . Given  $\nu \in C_{\mathbf{M}}$ , to complete the proof of (i), it is sufficient to show that  $\nu \in X_i$  for some  $i$ . In fact, for  $n > n_0$ , the center  $\nu_c$  of the parameter ball  $\mathcal{D}_n(\nu)$  of level  $n$  containing  $\nu$  is such that  $\mathbf{M}_{n+1}^{\nu_c}(\omega^+) = \mathbf{M}_{n+1}^\nu(\omega^+) = \mathbf{M}_{n+1}$ . Therefore  $\nu_c = \nu_i$  for some  $i$  and  $\nu \in X_i$ .

(ii) Suppose that  $\mathbf{M}$  is an aperiodic admissible marked grid. Let  $Y_n$  be the union of all level  $n$  parameter balls  $\mathcal{D}_n$  such that  $\mathbf{M}_{n+1} = \mathbf{M}_{n+1}^\nu(\omega^+)$  for all  $\nu \in \mathcal{D}_n$ . It follows that  $C_{\mathbf{M}} = \bigcap_{n \geq 0} Y_n$ . The radius  $r_n$  of the parameter balls that participate in  $Y_n$  coincide since it only depends on  $\mathbf{M}_{n+1}$ . Moreover



$r_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $C_{\mathbf{M}}$  is a compact non-empty set. The number  $b_n$  of parameter balls of level  $n$  in  $Y_n$  is a non-decreasing function of  $n$ . Moreover, the number of level  $n+1$  balls in  $Y_{n+1}$  which lie inside a ball  $\mathcal{D}_n$  of level  $n$  contained in  $Y_n$  is independent of  $\mathcal{D}_n$  (Proposition 5.7). Hence,  $C_{\mathbf{M}}$  is a Cantor set if  $b_n \rightarrow \infty$  as  $n \rightarrow \infty$  and a finite set otherwise.

(iii) Given an aperiodic critical marked grid, for all  $\nu' \in C_{\mathbf{M}}$  the level  $n$  parameter ball  $\mathcal{D}_n(\nu')$  which contains  $\nu'$  has non-empty intersection with  $\mathcal{L}_n \setminus \mathcal{L}_{n+1}$ . If  $\nu \in \mathcal{L}_n \setminus \mathcal{L}_{n+1}$ , then both critical points of  $\psi_\nu$  escape to infinity. It follows that  $\nu' \in \partial\{\nu \in \mathcal{L}_0 \mid \{\omega^\pm = \pm 1\} \subset \mathbb{L} \setminus K(\psi_\nu)\}$ , since  $\{\nu'\} = \cap \mathcal{D}_n(\nu')$ .

Conversely, if  $\nu \in \partial\{\nu \in \mathcal{L}_0 \mid \{\omega^\pm = \pm 1\} \subset \mathbb{L} \setminus K(\psi_\nu)\}$ , then  $\omega^+ \in K(\psi_\nu)$ , since the parameters for which both critical points escape form an open set. By (i), the critical marked grid  $\mathbf{M}^\nu$  must be aperiodic. □

### 5.4. Dynamics over finite extensions of $\mathbb{Q}^a((t))$

The aim of this subsection is to discuss the dynamical behavior of cubic polynomials with coefficients in a finite extension of  $\mathbb{Q}^a((t))$ . In particular, we show that some cubic polynomials with coefficients in a finite extension of  $\mathbb{Q}^a((t))$  have a non-periodic recurrent critical point. That is, a critical point which is an accumulation point of its orbit but it is not periodic. Examples of non-Archimedean dynamical systems over finite extensions of  $\mathbb{Q}_p$  with wild recurrent critical points were recently given by Rivera in [28]. We emphasize that Rivera shows the existence of a *wild* recurrent critical point (i.e., a critical point where the local degree is a multiple of  $p$ ).

If  $\psi_\nu$  is such that  $\omega^+ \in K(\psi_\nu)$  and  $\omega^-$  escapes to infinity, then  $\omega^+$  is recurrent if and only if the critical marked grid of  $\psi_\nu$  is not periodic and has marked columns of arbitrarily long depth. There is a stronger notion of critical recurrence associated to marked grids called *persistent recurrence* (e.g., see [20, 25]).

As a corollary of our description of the parameter space of cubic polynomials we will be able to use the examples of recurrent and persistently recurrent critical marked grids due to Harris [20] to prove the following:

**COROLLARY 5.17.** — *Let  $\psi_\nu(\zeta) = t^{-2}(\zeta - 1)^2(\zeta + 2) + \nu$ . Then the following hold:*

- (i) *There exists  $\nu_a \in \mathbb{Q}^a((t))$  such that the critical point  $\omega^+ = +1$  is recurrent and not periodic under iterations of  $\psi_{\nu_a}$ .*

- (ii) *There exists  $\nu_b \in \mathbb{Q}^a((t^{1/2}))$  such that critical point  $\omega^+ = +1$  is persistently recurrent and not periodic under iterations of  $\psi_{\nu_b}$ .*

The corollary will follow from two lemmas and the work of Harris cited above.

In order to simplify notation, for  $\zeta \in \mathbb{Q}^a\langle\langle t \rangle\rangle$  we say that the *algebraic degree of  $\zeta$*  is

$$\delta(\zeta) = [\mathbb{Q}^a((t))(\zeta) : \mathbb{Q}^a((t))].$$

When  $\mathbb{Q}^a\langle\langle t \rangle\rangle$  is regarded as the inductive limit of  $\{\mathbb{Q}^a((t^{1/m})); m \in \mathbb{N}\}$  the algebraic degree of  $\zeta$  coincides with the smallest  $m$  such that  $\zeta \in \mathbb{Q}^a((t^{1/m}))$ .

Let us fix  $\alpha \in \mathbb{Q}^a\langle\langle t \rangle\rangle$  and consider as before  $\psi_\nu(\zeta) = \alpha^{-2}(\zeta - 1)^2(\zeta + 2) + \nu$  with  $\nu \in \mathbb{L}$ . We define the *algebraic degree of  $\psi_\nu$*  as

$$\delta(\psi_\nu) = \max\{\delta(\alpha), \delta(\nu)\} \quad \text{if } \nu \in \mathbb{Q}^a\langle\langle t \rangle\rangle,$$

and  $\infty$  otherwise. The *algebraic degree of a ball  $B \subset \mathbb{L}$*  is defined as:

$$\delta(B) = \min\{\delta(\psi_\nu) \mid \nu \in B\}$$

We will be interested in computing the algebraic degree of parameter balls. Clearly  $\delta(\mathcal{D}_0) = \delta(\alpha)$ . Our next result shows that the center of a parameter ball minimizes the algebraic degree of the elements of the ball. More precisely:

LEMMA 5.18. — *Let  $\mathcal{D}_{n+1} \subset \mathcal{D}_n$  be parameter balls of levels  $n + 1$  and  $n$  respectively with  $n \geq 0$ . Denote by  $\nu_{n+1}$  the center of  $\mathcal{D}_{n+1}$  and by  $P_n$  the element of the affine partition of  $\mathcal{D}_n$  that contains  $\mathcal{D}_{n+1}$ . Then:*

$$\delta(\psi_{\nu_{n+1}}) = \delta(\mathcal{D}_{n+1}) = \delta(P_n).$$

Before proving the lemma let us remark that if  $\varphi$  is a polynomial with coefficients in  $\mathbb{Q}^a((t^{1/m}))$ , and  $B, B' \subset \mathbb{L}$  are balls such that  $\varphi : B \rightarrow B'$  is bijective, then  $\delta(\zeta) \leq \max\{m, \delta(\varphi(\zeta)), \delta(\nu)\}$  for all  $\nu \in B$ . This easily follows from the Newton polygon of  $\varphi(\cdot - \nu) - \varphi(\zeta)$ .

*Proof.* — Let  $\hat{\nu} \in P_n$  be such that  $\delta(\psi_{\hat{\nu}}) = \delta(P_n)$ . Let  $\hat{p}$  be the period of the center  $\psi_{\nu_{n+1}}$ . By Proposition 5.7 there exists a level  $n + 1$  dynamical ball  $D_{n+1}^{\hat{\nu}}$  contained in  $P_n$  such that  $\psi_{\hat{\nu}}^{\hat{p}-1} : D_{n+1}^{\hat{\nu}} \rightarrow D_{n+2-\hat{p}}^{\hat{\nu}}(\omega^+)$  is one-to-one. Hence there exists a unique  $\hat{\zeta}_1 \in D_{n+1}^{\hat{\nu}}$  such that  $\psi_{\hat{\nu}}^{\hat{p}-1}(\hat{\zeta}_1) = \omega^+$ . It follows that  $\delta(\hat{\zeta}_1) \leq \delta(\psi_{\hat{\nu}})$ . Let  $\hat{\zeta}_k = \psi_{\hat{\nu}}^{k-1}(\hat{\zeta}_1)$  and consider  $\mathbb{B}$  as in Lemma 5.11. From this lemma we conclude that there is a well defined Thurston map  $T : \mathbb{B} \rightarrow \mathbb{B}$ . If  $T(\zeta_1, \dots, \zeta_{\hat{p}}) = (\zeta'_1, \dots, \zeta'_{\hat{p}})$ , then  $\delta(\zeta'_i) \leq \max\{\delta(\zeta'_{i+1}), \delta(\psi_{\zeta_1})\}$  since  $\zeta'_i$  is obtained as a preimage of  $\zeta_{i+1}$  under the

restriction of  $\psi_{\hat{\zeta}_1}$  to a ball where this polynomial is injective. The first coordinates of  $T^k(\hat{\zeta}_1, \dots, \hat{\zeta}_p)$  converge to  $\nu_{n+1}$  as  $k \rightarrow \infty$ . Therefore,

$$\delta(\psi_{\nu_{n+1}}) \leq \delta(\psi_{\hat{\zeta}_1}) \leq \delta(\psi_{\hat{\nu}}) = \delta(P_n).$$

The lemma easily follows. □

Next we show that  $\delta(\mathcal{D}_n)$  is in fact computable from the information contained in the level  $n + 1$  critical marked grid of the parameters in  $\mathcal{D}_n$ .

LEMMA 5.19. — *Let  $\mathcal{D}_{n+1} \subset \mathcal{D}_n$  be parameter balls of levels  $n + 1$  and  $n$  with centers  $\nu_{n+1}$  and  $\nu_n$ , respectively. Denote by  $r_n$  the radius of  $\mathcal{D}_n$  and let  $s_n = \min\{s \in \mathbb{N} \mid s \log r_n \in \mathbb{N}\}$ . Then the following hold:*

- (i) *If  $\nu_{n+1} = \nu_n$ , then  $\delta(\mathcal{D}_{n+1}) = \delta(\mathcal{D}_n)$ .*
- (ii) *If  $\nu_{n+1} \neq \nu_n$ , then  $\delta(\mathcal{D}_{n+1}) = \max\{s_n, \delta(\mathcal{D}_n)\}$ .*

*Proof.* — Part a) follows immediately from the previous lemma. Suppose that  $\nu_{n+1} \neq \nu_n$  and observe that, by Proposition 5.7,  $|\nu_{n+1} - \nu_n|_0 = r_n$ . Hence  $\text{ord}(\nu_{n+1} - \nu_n) = -\log r_n$ . It follows that

$$s_n \leq \max\{\delta(\nu_{n+1}), \delta(\nu_n)\} \leq \max\{\delta(\psi_{\nu_{n+1}}), \delta(\psi_{\nu_n})\} = \delta(\psi_{\nu_{n+1}}).$$

Therefore,  $\max\{s_n, \delta(\psi_{\nu_n})\} \leq \delta(\psi_{\nu_{n+1}})$ .

Now let  $P_n$  be the ball of the affine partition of  $\mathcal{D}_n$  that contains  $\mathcal{D}_{n+1}$ . It follows that  $P_n$  contains a series of the form  $\nu_n + at^{q/s_n}$  for some  $0 \neq a \in \mathbb{Q}^a$  and some  $q \in \mathbb{N}$  relatively prime with  $s_n$ . So

$$\delta(\psi_{\nu_{n+1}}) = \delta(P_n) \leq \max\{\delta(\psi_{\nu_n}), s_n\}$$

which, in view of the previous lemma, finishes the proof. □

Let us now illustrate how the above lemmas may be used to compute the algebraic degree of some parameters. For simplicity we restrict to the case in which  $\alpha = t^{-1}$  and let  $\psi_\nu$  be a polynomial with critical marked grid  $\mathbf{M} = (M_{\ell,k})$ . For all  $n$ , we denote by  $\mathcal{D}_n$  the parameter ball of level  $n$  containing  $\nu$ , by  $r_n$  its radius and by  $\nu_n$  its center. Furthermore we suppose that  $\omega^+$  is periodic of period  $p$  under  $\psi_\nu$  if the critical marked grid is periodic of period  $p$ . Now the smallest integer  $s_k$  such that

$$s_k \log r_n = s_k \cdot 2 \cdot \sum_{\ell=1}^{k+1} 2^{\sum_{i=0}^{\ell-1} M_{\ell-i,i}} \in \mathbb{N}$$

is clearly computable from  $\mathbf{M}$ . The previous lemma implies that

$$\delta(\mathcal{D}_{n+1}) = \max\{s_k \mid \nu_{k+1} \neq \nu_k, 0 \leq k \leq n\}.$$

Moreover,  $\nu$  is algebraic over  $\mathbb{Q}^a((t))$  if and only if

$$\delta(\mathbf{M}) = \sup\{s_k \mid \nu_{k+1} \neq \nu_k\} < \infty$$

and in this case the algebraic degree of  $\psi_\nu$  coincides with  $\delta(\mathbf{M})$ .

The above formula for  $\delta(\psi_\nu)$  coincides with Branner and Hubbard’s formula for the “length” of a “turning curve” passing through a complex cubic polynomial with critical marked grid  $\mathbf{M}$ . In [20], Harris shows the existence of critical marked grids satisfying rules (Ma) through (Md) which are (resp. persistently) critically recurrent and aperiodic such that  $\delta(\mathbf{M}) = 1$  (resp.  $\delta(\mathbf{M}) = 2$ ). Corollary 5.17 now follows.

### 6. Parameter space of cubic polynomials

As explained in the introduction we work in the parameter space  $\mathcal{P}_{\mathbb{L}} \equiv \mathbb{L}^2$  where the cubic polynomial corresponding to  $(\alpha, \nu) \in \mathbb{L}^2$  is

$$\varphi_{\alpha,\nu}(\zeta) = \zeta^3 - 3\alpha^2\zeta + 2\alpha^3 + \nu.$$

Note that the critical points of  $\varphi_{\alpha,\nu}$  are  $\pm\alpha$  and that  $\nu = \varphi_{\alpha,\nu}(\alpha)$  is a critical value. The aim of this section is to prove Theorem 1.3 and Corollary 1.4.

The infraconnectedness locus  $\mathcal{C}_{\mathbb{L}}$  is the subset of  $\mathcal{P}_{\mathbb{L}}$  formed by all  $(\alpha, \nu)$  such that  $K(\varphi_{\alpha,\nu})$  is infraconnected. According to Lemma 6.1 below

$$\mathcal{C}_{\mathbb{L}} = B_1^+(0) \times B_1^+(0).$$

In particular,  $\mathcal{C}_{\mathbb{L}}$  is both closed and open in  $\mathcal{P}_{\mathbb{L}} \equiv \mathbb{L}^2$ . The shift locus  $\mathcal{S}_{\mathbb{L}} \subset \mathcal{P}_{\mathbb{L}}$  consists of all parameters corresponding to cubic polynomials  $\varphi_{\alpha,\nu}$  for which both critical points escape to infinity. It follows that  $\mathcal{S}_{\mathbb{L}}$  is open. The rest of parameter space  $\mathcal{E}_{\mathbb{L}}$  consists of the parameters of cubic polynomials  $\varphi_{\alpha,\nu}$  with one critical point in  $K(\varphi_{\alpha,\nu})$  and the other one escaping. It is subdivided into  $\mathcal{E}_{\mathbb{L}}^+$  and  $\mathcal{E}_{\mathbb{L}}^-$  according to whether  $+\alpha$  escape to infinity or  $-\alpha$  escapes to infinity.

Note that  $\varphi_{\alpha,\nu}(\zeta) = \varphi_{-\alpha,\nu+4\alpha^3}(\zeta)$ . Therefore, the parameter space involution  $(\alpha, \nu) \rightarrow (-\alpha, \nu + 4\alpha^3)$  fixes the infraconnectedness locus as well as the shift locus, and interchanges  $\mathcal{E}_{\mathbb{L}}^-$  with  $\mathcal{E}_{\mathbb{L}}^+$ .

LEMMA 6.1. — *The following hold:*

$$\begin{aligned} \mathcal{C}_{\mathbb{L}} &= B_1^+(0) \times B_1^+(0), \\ \mathcal{E}_{\mathbb{L}}^- &\subset \{(\alpha, \nu) \in \mathbb{L}^2 \mid |\alpha|_o = |\nu|_o > 1\}, \\ \mathcal{E}_{\mathbb{L}}^+ &\subset \{(\alpha, \nu) \in \mathbb{L}^2 \mid |\alpha|_o^3 = |\nu|_o > 1\}. \end{aligned}$$

*Proof.* — If  $(\alpha, \nu) \in B_1^+(0) \times B_1^+(0)$ , then  $|\varphi_{\alpha, \nu}^n(\pm\alpha)|_o \leq 1$ , for all  $n \geq 0$ . Therefore,  $(\alpha, \nu) \in \mathcal{C}_{\mathbb{L}}$ .

If  $|\alpha|_o \leq 1$  and  $|\nu|_o > 1$ , then  $|\varphi_{\alpha, \nu}(-\alpha)|_o = |4\alpha^3 + \nu|_o = |\nu|_o > R_{\varphi_{\alpha, \nu}}$  and  $|\varphi_{\alpha, \nu}(\alpha) = \nu|_o > R_{\varphi_{\alpha, \nu}}$  since  $R_{\varphi_{\alpha, \nu}} = |\nu|_o^{1/3}$ . Therefore,  $B_1^+(0) \times (\mathbb{L} \setminus B_1^+(0)) \subset \mathcal{S}_{\mathbb{L}}$ .

If  $|\alpha|_o > 1$  and  $|\alpha|_o < |\nu|_o$ , then  $R_{\varphi_{\alpha, \nu}} = |\nu|_o^{1/3} < |\nu|_o$ . Therefore,  $\alpha \notin K(\varphi_{\alpha, \nu})$ .

If  $|\alpha|_o > 1$  and  $|\alpha|_o > |\nu|_o$ , then  $R_{\varphi_{\alpha, \nu}} = |\alpha|_o < |\alpha|_o^3 = |\varphi_{\alpha, \nu}(\nu)|_o$ . Therefore,  $\alpha \notin K(\varphi_{\alpha, \nu})$ .

We conclude that  $\mathcal{E}_{\mathbb{L}}^{\pm} \subset \{(\alpha, \nu) \in \mathbb{L}^2 \mid |\alpha|_o = |\nu|_o > 1\}$ . Now since under the involution  $(\alpha, \nu) \mapsto (\alpha', \nu') = (-\alpha, 4\alpha^3 + \nu)$ , a parameter  $(\alpha, \nu)$  such that  $|\alpha|_o = |\nu|_o > 1$  maps to one such that  $|\alpha'|_o^3 = |\nu'|_o$ , the last assertion of the lemma follows. □

*Proof of Theorem 1.3.* — Note that  $\partial\mathcal{S}_{\mathbb{L}} \subset \mathcal{E}_{\mathbb{L}}$  since both  $\mathcal{S}_{\mathbb{L}}$  and  $\mathcal{C}_{\mathbb{L}}$  are open. Let  $\mathcal{A}_{\mathbb{L}}^{\pm}$  be the set of polynomials  $\varphi_{\alpha, \nu}$  in  $\mathcal{E}_{\mathbb{L}}^{\pm}$  such that the infraconnected component of  $K(\varphi_{\alpha, \nu})$  which contains  $\mp\alpha$  is not periodic. In view of the parameter space involution to establish that  $\partial\mathcal{S}_{\mathbb{L}} = \mathcal{A}_{\mathbb{L}}$  it suffices to prove that  $\partial\mathcal{S}_{\mathbb{L}} \cap \mathcal{E}_{\mathbb{L}}^- = \mathcal{A}_{\mathbb{L}}^-$ .

Fix  $\alpha > 1$  and observe that

$$\frac{1}{\alpha} \varphi_{\alpha, \nu}(\alpha\zeta) = \alpha^2(\zeta - 1)^2(\zeta + 2) + \nu = \psi_{\nu}(\zeta).$$

We studied such one-parameter families  $\psi_{\nu}$  in Section 5. In particular, Corollary 5.15 shows that if the infraconnected component of  $\omega^+ = +1$  is not periodic under  $\psi_{\nu}$ , then there exists a sequence  $\{\nu_n\}$  converging to  $\nu$  such that  $\psi_{\nu_n}$  has both critical points  $\omega^{\pm} = \pm 1$  escaping. It follows that  $\mathcal{A}_{\mathbb{L}}^- \subset \partial\mathcal{S}_{\mathbb{L}} \cap \mathcal{E}_{\mathbb{L}}^-$ .

To finish the proof it suffices to show that  $\mathcal{E}_{\mathbb{L}}^- \setminus \mathcal{A}_{\mathbb{L}}^-$  is open in  $\mathbb{L}^2$ . In fact, let  $\varphi_{\alpha, \nu} \in \mathcal{E}_{\mathbb{L}}^- \setminus \mathcal{A}_{\mathbb{L}}^-$  and denote by  $U$  the infraconnected component of  $K(\varphi_{\alpha, \nu})$  which contains  $\alpha$ . From Theorem 1.1 we have that  $U$  is a periodic closed ball, say of period  $p$  and radius  $r > 0$ . Let  $\mathcal{V}$  be an open neighborhood of  $\varphi_{\alpha, \nu}$  such that

$$|\varphi_{\alpha', \nu'}^p(\zeta) - \varphi_{\alpha, \nu}^p(\zeta)|_o < r$$

for all  $\zeta \in U$  and all  $(\alpha', \nu') \in \mathcal{V}$ . It follows that  $\varphi_{\alpha', \nu'}^p(U) = U$  for all  $(\alpha', \nu') \in \mathcal{V}$ . Hence,  $\mathcal{V} \subset \mathcal{E}_{\mathbb{L}}^- \setminus \mathcal{A}_{\mathbb{L}}^-$ . □

*Proof of Corollary 1.4.* — (i)  $\implies$  (ii)  $\implies$  (iii) follows from Corollary 2.12.

(iii)  $\implies$  (iv). If all cycles of  $\varphi$  are repelling, then  $\varphi$  lies in the closure of  $\mathcal{S}_{\mathbb{L}}$ . Otherwise, there would exist an infraconnected component  $U$  of  $K(\varphi)$  containing a critical point which is a periodic closed ball of some period  $p \geq 1$ . According to Lemma 2.4, we would have that  $\varphi^p$  has a fixed point in  $U$  which, by Schwarz Lemma, would be non-repelling. Thus, we obtain a contradiction with (iii).

(iv)  $\implies$  (i). This follows from combining theorems 1.1 and 1.3.  $\square$

### 7. Complex cubic polynomials

The aim of this section is to prove theorems 1.5 and 1.6. Recall that in complex cubic polynomial dynamics we work in the parameter space  $\mathcal{P}_{\mathbb{C}}$  of polynomials of the form

$$f_{a,v}(z) = (z - a)^2(z + 2a) + v, \quad (a, v) \in \mathbb{C}^2.$$

Parameter space is subdivided into three sets according to how many critical points of  $f_{a,v}$  escape to infinity:

- The connectedness locus  $\mathcal{C}_{\mathbb{C}}$  is the subset of  $\mathcal{P}_{\mathbb{C}}$  formed by all parameters  $(a, v)$  for which none of the critical points of  $f_{a,v}$  escape to infinity. The connectedness locus is a compact and connected subset of parameter space (see [10]).
- The set  $\mathcal{E}_{\mathbb{C}}$  formed by all  $(a, v)$  such that  $f_{a,v}$  has one critical point in the basin of infinity and one critical point in the filled Julia set. Thus,  $\mathcal{E}_{\mathbb{C}}$  is the disjoint union of  $\mathcal{E}_{\mathbb{C}}^-$  and  $\mathcal{E}_{\mathbb{C}}^+$  where  $\mathcal{E}_{\mathbb{C}}^-$  (resp.  $\mathcal{E}_{\mathbb{C}}^+$ ) is formed by all  $(a, v) \in \mathcal{E}_{\mathbb{C}}$  such that  $-a$  (resp.  $+a$ ) escapes to  $\infty$ .
- The shift locus  $\mathcal{S}_{\mathbb{C}}$  consisting of the parameters  $(a, v)$  corresponding to polynomials  $f_{a,v}$  such that all their critical points are in the basin of infinity.

As mentioned in the introduction, periodic curves will play a central rôle in establishing a relation between dynamics over  $\mathbb{L}$  and dynamics over  $\mathbb{C}$ . Recall that  $\text{Per}(n)$  is the algebraic subset of  $\mathcal{P}_{\mathbb{C}} \equiv \mathbb{C}^2$  formed by all parameters such that the critical point  $+a$  has exact period  $n$ . Periodic curves of cubic polynomials have been studied by Milnor in [23] and their analogue in the parameter space of quadratic rational functions have been intensively studied by Rees [26]. It is known that  $\text{Per}(n)$  is a smooth submanifold of  $\mathbb{C}^2$ , for all  $n$ , and it is an open problem to determine whether  $\text{Per}(n)$  is connected (i.e., irreducible). We will be particularly interested in the portion of  $\text{Per}(n)$  outside the connectedness locus  $\mathcal{C}_{\mathbb{C}}$ . An *end* of  $\text{Per}(n)$  is, by definition, a connected component of  $\text{Per}(n) \setminus \mathcal{C}_{\mathbb{C}}$ .

Since our aim is to study  $\mathcal{E}_\mathbb{C}^-$  near infinity, it is convenient to compactify  $\mathcal{P}_\mathbb{C}$  by adding a line  $\mathcal{L}_\infty$  at infinity and to identify the resulting space with  $\mathbb{CP}^2$ . That is,  $\mathbb{CP}^2 \equiv \mathcal{P}_\mathbb{C} \cup \mathcal{L}_\infty$  where  $\mathcal{P}_\mathbb{C} \equiv \{[a : v : 1] \in \mathbb{CP}^2\}$  and  $\mathcal{L}_\infty \equiv \{[a : v : 0] \in \mathbb{CP}^2\}$ .

Let us now outline the contents of this section.

In Subsection 7.1 we introduce some definitions from complex polynomial dynamics and give a brief summary about the combinatorial structure of the dynamical plane of cubic polynomials in a neighborhood  $\mathcal{V}$  of  $\mathcal{E}_\mathbb{C}^-$ . In particular, following Branner and Hubbard, we introduce dynamical disks and annuli of level  $n$  for polynomials in  $\mathcal{V}$ .

In Subsection 7.2, to simplify the exposition, we isolate the proof of three lemmas which consist of easy calculations that we will need afterward. In particular, these lemmas show that the closure of  $\mathcal{E}_\mathbb{C}^-$  in  $\mathbb{CP}^2$  intersects the line  $\mathcal{L}_\infty$  at  $\{[1 : 1 : 0], [1 : -2 : 0]\}$ . Note that the line  $\{[a : a : 1]\}$  (resp.  $\{[a : -2a : 1]\}$ ) is  $\text{Per}(1)$  (resp. all  $f_{a,v}$  such that  $+a$  is a prefixed critical point).

In Subsection 7.3, roughly speaking, we show that Puiseux series of ends of periodic curves have a “radius of convergence” uniformly bounded away from zero (after passing to the universal cover). More precisely, we show that there exists  $\varepsilon > 0$  such that if  $\nu = \sum_{j \geq 0} a_j t^{j/m} \in \mathbb{Q}^a \langle\langle t \rangle\rangle$  is the Puiseux series of an end  $\mathcal{F}$  of a periodic curve, then

$$h \mapsto [1 : \nu(e^{2\pi i h}) : e^{2\pi i h}]$$

is a well defined map from

$$\mathbb{H}_\varepsilon = \left\{ h \in \mathbb{C} \mid \text{Im } h > -\frac{\log \varepsilon}{2\pi} \right\}$$

onto  $\mathcal{F} \cap V_\varepsilon$  which is a conformal universal cover, where

$$V_\varepsilon = \{[1 : \bar{v} : \bar{a}] \mid 0 < |\bar{a}| < \varepsilon\}.$$

In Subsection 7.7 we let  $\text{Per}_\mathbb{L}$  be the set formed by all Puiseux series of ends of periodic curves and consider its closure  $\overline{\text{Per}_\mathbb{L}}$  in  $\mathbb{L}$ . We show the map from  $\mathbb{H}_\varepsilon \times \text{Per}_\mathbb{L}$  onto  $\cup \text{Per}(n) \cap V_\varepsilon$ , given by the previous subsection, extends to a map  $\Phi : \mathbb{H}_\varepsilon \times \overline{\text{Per}_\mathbb{L}} \rightarrow V_\varepsilon$  which is holomorphic in the first coordinate and continuous in the second.

In Subsection 7.5 we study the lack of injectivity of  $\Phi$ .

In Subsection 7.6 we establish that  $\text{Per}_\mathbb{L}$  has a clear non-Archimedean dynamical interpretation. Namely, we consider the one-parameter family  $\psi_\nu(\zeta) = t^{-2}(\zeta - 1)^2(\zeta + 2) + \nu \in \mathbb{L}[\zeta]$  and show that  $\text{Per}_\mathbb{L}$  consists of all parameters  $\nu \in \mathbb{L}$  such that the critical point  $+1$  is periodic under  $\psi_\nu$ .

In Subsection 7.7, we use the compactness given by Theorem 5.3 to show that the map  $\nu \rightarrow \Phi(\cdot, \nu)$  is continuous from the subspace topology of  $\overline{\text{Per}}_{\mathbb{L}} \subset \mathbb{L}$  to the sup-norm topology on functions. After recalling some facts about almost periodic functions, we conclude that each  $\nu \in \overline{\text{Per}}_{\mathbb{L}}$  corresponds to the Fourier series of an almost periodic function.

In order to proceed it is necessary to go into a more detailed analysis of the relation between the dynamics of  $\psi_\nu$  over  $\mathbb{L}$  and that of the complex polynomial  $f_{\Phi(h,\nu)}$ , for  $\nu \in \overline{\text{Per}}_{\mathbb{L}}$ . In Subsection 7.8 it is shown that the marked grids of  $\psi_\nu$  and  $f_{\Phi(h,\nu)}$  agree for all  $\nu \in \overline{\text{Per}}_{\mathbb{L}}$ .

In Subsection 7.9, using that  $\mathcal{NR}_{\mathbb{L}} = \overline{\text{Per}}_{\mathbb{L}} \setminus \text{Per}_{\mathbb{L}}$  and some results from complex dynamics, we prove theorems 1.5 and 1.6.

### 7.1. Dynamical disks and annuli for complex cubic polynomials

Important tools to study the dynamics of complex polynomials are the Green function and the Böttcher map. Given a degree  $d \geq 2$  monic polynomial  $f : \mathbb{C} \rightarrow \mathbb{C}$ , the Green function

$$G_f : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$$

$$z \mapsto \lim_{n \rightarrow \infty} \frac{\log_+ |f^n(z)|}{d^n}$$

is a well defined continuous function which vanishes in  $K(f)$  and is harmonic in  $\mathbb{C} \setminus K(f)$ . The Böttcher map  $\phi_f : \mathcal{B}_\nabla(f) \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$  is a conformal isomorphism from the basin of infinity  $\mathcal{B}_\nabla(f)$  under the gradient flow  $\nabla G_f$  into  $\mathbb{C} \setminus \overline{\mathbb{D}}$  which conjugates  $f$  with  $z \mapsto z^d$  (i.e.  $\phi_f(f(z)) = \phi_f(z)^d$  for all  $z \in \mathcal{B}_\nabla(f)$ ) and is asymptotic to the identity at infinity (i.e.  $\phi_f(z) = z + o(z)$  as  $|z| \rightarrow \infty$ ).

For general background on polynomial dynamics see Section 18 in [24].

Recall that we work in the parameter space  $\mathcal{P}_{\mathbb{C}}$  of complex cubic polynomials of the form

$$f_{a,v}(z) = (z - a)^2(z + 2a) + v, \quad (a, v) \in \mathbb{C}^2.$$

We now specialize on the subset  $\mathcal{V} \subset \mathcal{P}_{\mathbb{C}}$  where the fastest escaping critical point is  $-a$ . That is,

$$\mathcal{V} = \{(a, v) \mid -a \notin K(f_{a,v}), G_{f_{a,v}}(+a) < G_{f_{a,v}}(-a)\}.$$

Note that  $\mathcal{V}$  is a neighborhood of  $\mathcal{E}_{\mathbb{C}}^-$ .

Following Branner and Hubbard [10, 11] we summarize the basic combinatorial structure of the dynamical plane of polynomials in  $\mathcal{V}$ . Let  $(a, v) \in \mathcal{V}$  and  $f = f_{a,v}$ . It follows that

$$D_0^f = \{z \mid G_f(z) < 3G_f(-a)\}$$



is a topological open disk which we call the *dynamical disk of level 0* of  $f$ . The set  $\{z \mid G_f(z) < 3^{-n+1}G_f(-a)\}$  is a finite disjoint union of open topological disks that we call *dynamical disks of level  $n$* . Equivalently, a dynamical disk of level  $n$  is a connected component of  $f^{-n}(D_0^f)$ . A point  $z$  is a *level  $n$  point* if  $G_f(z) < 3^{-n+1}G_f(-a)$  and the disk of level  $n$  containing  $z$  is denoted  $D_n^f(z)$ . The *level 0 annulus* is

$$A_0^f = \{z \mid G_f(-a) < G_f(z) < 3G_f(-a)\}$$

and has modulus  $\text{mod } A = \pi^{-1}G_f(-a)$  since it is conformally isomorphic to  $\{z \mid e^{G_f(-a)} < |z| < e^{3G_f(-a)}\}$ . The *level  $n$  annulus* around a level  $n$  point  $z$  is

$$A_n^f(z) = D_n^f(z) \setminus \{w \mid G_f(w) \leq 3^{-n}G_f(-a)\}.$$

A level  $n$  annulus  $A_n^f$  is said to be *critical* if  $A_n^f = A_n^f(+a)$ .

If  $z$  is a level  $n+1$  point, then  $f(D_{n+1}^f(z)) = D_n^f(f(z))$  and  $f : D_{n+1}^f(z) \rightarrow D_n^f(f(z))$  is a branched covering of degree 2 if  $+a \in D_{n+1}^f(z)$  and of degree 1 otherwise. Now if  $+a$  is a level  $n+1$  point, then  $f : A_{n+1}^f(z) \rightarrow A_n^f(f(z))$  is a covering map of degree 2 when  $A_{n+1}^f(z)$  is critical and of degree 1 otherwise.

The *marked grids for  $f$*  are defined similarly than in Definition 4.4 and satisfy the rules stated in Proposition 4.5.

According to [10] the set  $\{(a, v) \mid G_{f_{a,v}}(\pm a) \leq \rho\}$  is compact, for all  $\rho \geq 0$ . Moreover,  $G_{f_{a,v}}$  depends continuously on  $(a, v)$ .

LEMMA 7.1. — *Suppose that  $U \subset \mathcal{V}$  is an open and connected set so that for all  $(a, v) \in U$  the critical point  $+a$  is a level  $n$  point of  $f_{a,v}$ . Then the level  $n$  critical marked grids of all  $f_{a,v} \in U$  coincide.*

*Proof.* — Given  $k = 1, \dots, n$ ; the set consisting of all  $(a, v) \in U$  such that  $f_{a,v}^k(+a)$  and  $+a$  are in a common (resp. different) level  $\ell \leq n - k$  disk is open in  $U$ . Since  $U$  is connected, the lemma follows. □

### 7.2. Preliminaries

In this subsection we establish three preliminary lemmas which are rather easy calculations related to parameters in a neighborhood of  $\{[1 : 1 : 0], [1 : -2 : 0]\} \subset \mathbb{CP}^2$ .

LEMMA 7.2. — *Suppose that  $(a, v) \in \mathbb{C}^2$  is such that  $|a| > 2$  and either  $|v + 2a| < |a|$  or  $|v - a| < |a|$ . Then:*

- (i)  $|f^n(-a)| \geq |a|^{3^n} (\sqrt{2})^{-3^{n-1}-1}$ .
- (ii)  $\log |a| - \frac{1}{6} \log 2 \leq G_{f_{a,v}}(-a) \leq \log |a| + \frac{1}{2} \log \frac{3}{2}$ .
- (iii)  $G_{f_{a,v}}(+a) \leq \frac{1}{3} \log |a| + \frac{1}{18} \log 25 \cdot 32$ .

*Proof.* — Let  $f = f_{a,v}$ . First note that

$$(7.1) \quad |z| > 3|a| \implies |f(z)| > 3|a|.$$

In fact, using that  $|v| < 3|a|$  we obtain  $|f(z)| \geq |z|^3(1 - |\frac{a}{z}|^2 - 2|\frac{a}{z}| - |\frac{v}{z^3}|) \geq 12|z| \geq 3|a|$ . Similarly we have:

$$(7.2) \quad |z| > 3|a| \implies \frac{1}{2}|z|^3 \leq |f(z)| \leq \frac{3}{2}|z|^3.$$

Also,

$$(7.3) \quad |z| < 3|a| \implies |f(z)| < 40|a|^3,$$

since  $|f(z)| \leq |a|^3(2 + |\frac{v}{|a|^3}| + 3|\frac{z}{a}| + |\frac{z}{a}|^3) \leq 40|a|^3$ . Inductively applying equation (7.2) we have:

$$(7.4) \quad 3|a| < |z| < 40|a|^3 \implies |f^n(z)| < 40^{3^n} |a|^{3^{n+1}} \left(\frac{3}{2}\right)^{\frac{3^n-1-1}{2}}$$

Now we prove (i). Since  $|f(-a)| = |4a^3 + v| \geq |a|^3 > 3|-a|$ , by induction, from equation (7.2) we conclude that for all  $n$ :

$$|f^n(-a)| \geq |a|^{3^n} \left(\frac{1}{2}\right)^{\frac{3^n-1-1}{2}}.$$

From here, taking logarithms, dividing by  $3^n$  and passing to the limit we obtain the lower bound of (ii). For the upper bound of (ii) note that  $3|-a| < |f(-a)| \leq 7|a|^3 < 40|a|^3$ . Hence, applying equation (7.4) the desired upper bound follows.

For (iii), either  $G_f(a) = 0$  or there exists  $n_0 \geq 1$  such that  $|f^n(a)| \leq 3|a|$  for  $n \leq n_0$  and  $3|a| < |f^{n_0+1}(a)| < 40|a|^3$ . In the latter case, let  $w = f^{n_0}(a)$  and from equation (7.4) conclude that  $G_f(w) \leq 3 \log |a| + \frac{1}{2} \log 25 \cdot 32$ . Since  $G_f(w) = 3^{n_0+1}G_f(a) \geq 9G_f(a)$  the upper bound of (iii) follows.  $\square$

LEMMA 7.3. — *Let  $f = f_{a,v}$  and suppose that  $G_f(-a) \geq 3G_f(v)$ .*

- (i) *If  $D_1^f(v) = D_1^f(a)$ , then*

$$\left| \frac{v}{a} - 1 \right| \leq \frac{16}{e^{G_f(-a)} - 16}.$$

- (ii) *If  $D_1^f(v) = D_1^f(-2a)$ , then*

$$\left| \frac{v}{a} + 2 \right| \leq \frac{16}{e^{2G_f(-a)} - 16}.$$

*Proof.* — Since both assertions follow along the same lines, we just prove (i). So suppose that  $D_1^f(v) = D_1^f(a)$ . Hence,  $2\pi \bmod A_1^f(v) = G_f(-a)$  and,  $A_1^f(v)$  separates  $a$  and  $v$  from  $-a$  and  $\infty$ . Consider the Möebius transformation  $\Gamma(z) = -a(z + 2a)^{-1}$  and note that  $\Gamma(A_1^f(v))$  separates  $-1$  and  $0$  from  $\infty$  and  $\Gamma(v) = (v/a - 1)^{-1}$ . According to Chapter III in [1], for some function  $\Psi$ :

$$\frac{1}{2\pi}G_f(-a) \leq \frac{1}{2\pi}\Psi(|\Gamma(v)|) \leq \frac{1}{2\pi} \log 16(|\Gamma(v)| + 1).$$

From where the desired inequality immediately follows. □

LEMMA 7.4. — For  $\rho > 0$  let  $E_\rho$  be the set formed by all  $(a, v) \in \mathcal{P}_\mathbb{C}$  such that

- (i)  $G_{f_{a,v}}(-a) \geq 3G_{f_{a,v}}(v)$ .
- (ii)  $D_1^{f_{a,v}}(v) = D_1^{f_{a,v}}(a)$  (resp.  $D_1^{f_{a,v}}(v) = D_1^{f_{a,v}}(-2a)$ ).
- (iii)  $G_{f_{a,v}}(-a) > \rho$ .

For any neighborhood  $W$  of  $[1 : 1 : 0]$  (resp.  $[1 : -2 : 0]$ ) in  $\mathbb{C}P^2 \equiv \mathcal{P}_\mathbb{C} \cup \mathcal{L}_\infty$ , there exists  $\hat{\rho}$  such that  $E_{\hat{\rho}} \subset W$ .

*Proof.* — Without loss of generality we prove the lemma for a neighborhood  $W$  of  $[1 : 1 : 0]$ . Take  $\rho_0 > 0$  such that (i), (ii) and  $G_{f_{a,v}}(-a) > \rho_0$  imply that  $|v - a| < |a|$  (see Lemma 7.3). Let  $\rho_1 \geq \rho_0$  be such that  $G_{f_{a,v}}(-a) \leq \rho_1$  for all  $(a, v)$  such that  $|a| \leq 2$  and  $|v| \leq 6$ . It follows that  $G_{f_{a,v}}(-a) > \rho_1$  implies that  $|a| > 2$  and  $|v - a| < |a|$ . Now let  $\delta > 0$  be such that  $\{[1 : \bar{v} : \bar{a}] \mid |\bar{a}| < \delta, |\bar{v} - 1| < \delta\} \subset W$ . By Lemma 7.2 (ii) and Lemma 7.3, there exists  $\hat{\rho}$  such that  $G_{f_{a,v}}(-a) > \hat{\rho}$  implies that  $|a|^{-1} = |\bar{a}| < \delta$  and  $|v/a - 1| = |\bar{v} - 1| < \delta$ . □

As mentioned above, given  $\rho \geq 0$ , according to Branner and Hubbard [10], the set formed by polynomials  $(a, v)$  such that  $G_{f_{a,v}}(\pm a) \leq \rho$  is compact. Since for all  $(a, v) \in \mathcal{E}_\mathbb{C}^-$  we have that the critical value  $v = f_{a,v}(+a)$  has escape rate  $G_{f_{a,v}}(v) = 0$  and  $v$  lies in one of the two level one disks  $D_1^{f_{a,v}}(a), D_1^{f_{a,v}}(-2a)$ , the previous lemma shows that the portion of  $\mathcal{E}_\mathbb{C}^-$  outside a large enough compact subset of  $\mathcal{P}_\mathbb{C}$  is contained in an arbitrarily small neighborhood of  $\{[1 : 1 : 0], [1 : -2 : 0]\}$ . Hence, we have the following corollary.

COROLLARY 7.5. — The intersection of the closure of  $\mathcal{E}_\mathbb{C}^-$  in  $\mathbb{C}P^2$  with the line at infinity  $\mathcal{L}_\infty$  is  $\{[1 : 1 : 0], [1 : -2 : 0]\}$ .

### 7.3. Geometry of periodic curves

We are particularly interested in the geometry of periodic curves near the line at infinity. Recall that an end  $\mathcal{F}$  of a periodic curve  $\text{Per}(n)$  is a connected component of  $\text{Per}(n) \setminus \mathcal{C}_C$ . As already proven in [23], each end is conformally isomorphic to the punctured unit disk (also see Lemma 7.8 below). Moreover, the closure of an end  $\mathcal{F}$  contains a unique point  $x$  in  $\mathcal{L}_\infty$  which must be either  $[1 : 1 : 0]$  or  $[1 : -2 : 0]$ . Furthermore,  $\mathcal{F} \cup \{x\}$  is an irreducible analytic set which determines, and it is determined, by a unique branch of  $\text{Per}(n)$  at  $x$  (see Corollary 7.9). We will simply say that  $\mathcal{F}$  is an end at  $x$ .

The aim of this subsection is to establish the following proposition and corollary.

**PROPOSITION 7.6.** — *There exist coordinates  $(u, w) \in \mathbb{D} \times \mathbb{D}$  in a neighborhood  $U$  of  $[1 : 1 : 0]$  such that  $\mathcal{L}_\infty \supset \{u = 0\}$ , and for any end  $\mathcal{F}$  of  $\text{Per}(n)$  at  $[1 : 1 : 0]$ , the projection  $\mathcal{F} \rightarrow \mathbb{D}$  to the first coordinate is a finite unramified cover over  $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ .*

Although the proposition is stated in a neighborhood of  $[1 : 1 : 0]$  a similar result, proven with similar arguments, holds in a neighborhood of  $[1 : -2 : 0]$ .

Before proving the proposition we state an important consequence regarding Puiseux series of branches of  $\text{Per}(n)$  at  $\mathcal{L}_\infty$ . Here, a Puiseux series of the branch of  $\text{Per}(n)$  corresponding to an end  $\mathcal{F}$  is a series  $\nu = \sum_{j \geq 0} a_j t^{j/m} \in \mathbb{Q}^a \langle\langle t \rangle\rangle$  such that  $T \rightarrow [1 : \sum_{j \geq 0} a_j T^j : T^m]$  is a well defined conformal isomorphism from  $\{0 < |T| < \delta\}$  onto  $\mathcal{F} \cap V$  for some  $\delta > 0$  and some neighborhood  $V$  of  $\mathcal{L}_\infty$ . For more about Puiseux series of singularities of algebraic curves see [12]. Our next result shows that Puiseux series defining ends of periodic curves have a common domain of convergence (modulo passing to the universal cover).

Recall that

$$\mathbb{H}_\varepsilon = \left\{ h \in \mathbb{C} \mid \text{Im } h > -\frac{\log \varepsilon}{2\pi} \right\}.$$

**COROLLARY 7.7.** — *There exists  $\varepsilon > 0$  such that for any Puiseux series  $\nu = \sum_{j \geq 0} a_j t^{j/m}$  defining an end  $\mathcal{F}$  of  $\text{Per}(n)$  for some  $n$ , the map*

$$h \rightarrow \nu(e^{2\pi i h}) = \sum_{j \geq 0} a_j e^{2\pi i h j/m}$$

*is well defined on  $\mathbb{H}_\varepsilon$ . Moreover, the map*

$$h \rightarrow [1 : \nu(e^{2\pi i h}) : e^{2\pi i h}]$$

induces a conformal isomorphism of  $\mathbb{H}_\varepsilon$  modulo  $z \rightarrow z + m$  onto  $\mathcal{F} \cap V_\varepsilon$  where  $V_\varepsilon = \{[1 : \bar{v} : \bar{a}] \mid 0 < |\bar{a}| < \varepsilon\}$ .

We first show how to deduce the corollary from the proposition and then proceed to prove the latter.

*Proof.* — Without loss of generality we prove the corollary for all ends at  $[1 : 1 : 0]$ . To simplify notation write  $x = [1 : 1 : 0]$ . Let  $V$  be a neighborhood of  $x$  and  $\Psi : V \rightarrow \mathbb{D} \times \mathbb{D}$  be a coordinate as in the proposition. Given any end  $\mathcal{F}$  at  $x$  of any periodic curve, the projection to the first coordinate  $\Pi_1 : \Psi(\mathcal{F} \cap V) \rightarrow \mathbb{D}^*$  is a finite degree covering. Thus, there exists a holomorphic map  $\tau_{\mathcal{F}} : \mathbb{H} \rightarrow \mathbb{D}$  such that

$$\begin{aligned} \mathbb{H} &\rightarrow \Psi(\mathcal{F} \cap V) \\ h &\mapsto (e^{2\pi ih}, \tau_{\mathcal{F}}(h)) \end{aligned}$$

is an universal cover.

First we show that the tangent lines to  $\Psi(\mathcal{F} \cap V)$  are uniformly bounded away from the vertical line.

*Claim.* — Given  $\rho_0 > 0$ , there exists a constant  $C > 0$  such that

$$\left| \frac{d}{dh} \tau_{\mathcal{F}}(h) / \frac{d}{dh} e^{2\pi ih} \right| \leq C$$

for all ends  $\mathcal{F}$  at  $x$  and all  $h \in \mathbb{H}$  such that  $\text{Im } h \geq \rho_0$ .

In fact, consider the family  $\mathcal{T}$  of all holomorphic functions  $\tau : \mathbb{H} \rightarrow \mathbb{D}$  such that  $h \mapsto (e^{2\pi ih}, \tau_{\mathcal{F}}(h))$  is an universal covering of  $\Psi(V \cap \mathcal{F})$  for some end  $\mathcal{F}$  at  $x$ . Observe that  $\mathcal{T}$  is a normal family and that if  $\tau \in \mathcal{T}$ , then  $\tau(\cdot + k) \in \mathcal{T}$  for all  $k \in \mathbb{Z}$ . Hence, there exists  $C > 0$  such that

$$\left| \frac{d}{dh} \tau(x + i\rho_0) \right| \leq 2\pi C e^{-2\pi\rho_0},$$

for all  $x \in \mathbb{R}$  and all  $\tau \in \mathcal{T}$ . In order to apply the Maximum Principle we fix an end  $\mathcal{F}$  and a map  $\tau_{\mathcal{F}}$  as above. Denote the degree of  $\Pi_1 : \Psi(\mathcal{F} \cap V) \rightarrow \mathbb{D}^*$  by  $m$ . It follows that there exists  $\tilde{\tau}_{\mathcal{F}} : \mathbb{D}^* \rightarrow \mathbb{D}$  such that

$$\mathbb{D}^* \ni T \mapsto (T^m, \tilde{\tau}_{\mathcal{F}}(T)) \in \Psi(\mathcal{F} \cap V)$$

is a conformal isomorphism and that  $\tau_{\mathcal{F}}(h) = \tilde{\tau}_{\mathcal{F}}(T)$  where  $T = e^{2\pi ih/m}$ . By Lemma 7.3,  $\tilde{\tau}_{\mathcal{F}}(T) = cT^m + \text{higher order terms}$ . Therefore,

$$\frac{d}{dT} \tilde{\tau}_{\mathcal{F}}(T) / (mT^{m-1})$$

extends to a holomorphic function in  $\mathbb{D}$  which coincides with

$$\frac{d}{dh} \tau_{\mathcal{F}}(h) / \frac{d}{dh} e^{2\pi ih}$$

when  $T = e^{2\pi ih/m}$ . The desired inequality follows after applying the Maximum Principle. Thus, we have finished the proof of the claim.

Since  $V \subset \{[1 : \bar{v} : \bar{a}] \in \mathbb{C}\mathbb{P}^2\}$  we may use  $(\bar{a}, \bar{v})$  coordinates in  $V$  and write  $\Psi(\bar{a}, \bar{v}) = (u(\bar{a}, \bar{v}), w(\bar{a}, \bar{v}))$ . By the proposition,  $\partial_{\bar{v}}w(x) \neq 0$  and  $\partial_{\bar{a}}u(x) = 0$ . Therefore, shrinking  $V$  if necessary, we have that

$$|\partial_{\bar{v}}w(y)/\partial_{\bar{a}}w(y)| > C$$

for all  $y \in V$  (we allow the left hand side to take the value  $\infty$ ). In view of Lemma 7.4 there exists  $\varepsilon > 0$  small enough so that for any end  $\mathcal{F}$  at  $x$  we have that  $\mathcal{F} \cap V_\varepsilon \subset V$  where

$$V_\varepsilon = \{[1 : \bar{v} : \bar{a}] \mid 0 < |\bar{a}| < \varepsilon\}.$$

Now given an end  $\mathcal{F}$  at  $x$  the projection

$$\Pi_{\bar{a}} : \mathcal{F} \cap V_\varepsilon \rightarrow \mathbb{D}_\varepsilon^* = \{\bar{a} \in \mathbb{C} \mid 0 < |\bar{a}| < \varepsilon\}$$

is an unramified covering of finite degree, say  $m$ . In fact, this map is a local homeomorphism since the derivative of  $\Pi_{\bar{a}}$  restricted to the tangent space of  $\mathcal{F} \cap V_\varepsilon$  has rank 1 at every point. Taking into account that every degree  $m$  covering of  $\{0 < |T| < \varepsilon\}$  is equivalent to  $T \mapsto T^m$ , there exists a series  $\sum_{j \geq 0} a_j T^j$  convergent in  $|T|^m < \varepsilon$  such that

$$T \rightarrow [1 : \sum_{j \geq 0} a_j T^j : T^m]$$

is a conformal isomorphism from  $0 < |T|^m < \varepsilon$  onto  $\mathcal{F} \cap V_\varepsilon$ . Therefore,

$$\sum_{j \geq 0} a_j t^{j/m}$$

is a Puiseux series of  $\mathcal{F}$ . All Puiseux series of  $\mathcal{F}$  converge in the same domain, since any other Puiseux series is of the form

$$\sum_{j \geq 0} a_j e^{2\pi ijk/m} t^{j/m}$$

for some  $k = 0, \dots, m - 1$ . For  $h \in \mathbb{H}_\varepsilon$ , the corollary easily follows making the substitution  $T = e^{2\pi ih/m}$  in the above parametrization. □

The proof of the proposition relies on changing coordinates near  $[1 : 1 : 0]$  with the aid of the Böttcher map  $\phi_{f_{a,v}}$  evaluated at a cocritical value. Whenever  $a \neq 0$ , the cocritical value of  $-a$  is  $2a$  (i.e.,  $f_{a,v}(-a) = f_{a,v}(2a)$  but  $-a \neq 2a$ ). A nice observation due Branner and Hubbard is that the

cocritical value  $2a$  is always in the domain of definition  $\mathcal{B}_\nabla(f_{a,v})$  of the Böttcher map  $\phi_{f_{a,v}}$  provided that  $G_f(-a) > G_f(a)$ . Moreover, the map

$$\begin{aligned} \{f_{a,v} \mid G_{f_{a,v}}(-a) > G_{f_{a,v}}(+a)\} &\rightarrow \mathbb{C} \setminus \overline{\mathbb{D}} \\ (a, v) &\mapsto \phi_{f_{a,v}}(2a) \end{aligned}$$

is holomorphic.

LEMMA 7.8. — *If  $\mathcal{F}$  is an end of  $\text{Per}(n)$ , then the map*

$$\begin{aligned} \phi_{\mathcal{F}} : \mathcal{F} &\rightarrow \mathbb{C} \setminus \overline{\mathbb{D}} \\ (a, v) &\mapsto \phi_{f_{a,v}}(2a) \end{aligned}$$

is an unramified covering of finite degree.

*Proof.* — From Branner and Hubbard’s wringing construction [10] it follows that  $\phi_{\mathcal{F}}$  is a local homeomorphism (compare with [23]). Note that  $\phi_{\mathcal{F}}^{-1}(w_0)$  is closed and bounded. To prove that this set is finite it suffices to show that every point is isolated. In fact, if  $(a_0, v_0) \in \phi_{\mathcal{F}}^{-1}(w_0)$ , then  $\phi_{\mathcal{F}}$  extends holomorphically to a map  $\phi$  on a neighborhood  $U_0$  of  $(a_0, v_0)$  in  $\mathbb{C}^2$  and therefore  $\phi^{-1}(w_0) \cap \text{Per}(n)$  is discrete in  $U_0$ .  $\square$

COROLLARY 7.9. — *If  $\mathcal{F}$  is an end of  $\text{Per}(n)$ , then the closure of  $\mathcal{F}$  in  $\mathbb{C}\mathbb{P}^2$  intersects the line at infinity at exactly one point  $x$ . The germ of the analytic set  $\mathcal{F} \cup \{x\}$  at  $x$  is irreducible.*

*Proof.* — Since  $\phi_{\mathcal{F}}$  is a finite degree covering of a punctured disk, it follows that  $\mathcal{F}$  is conformally isomorphic to a punctured disk and  $\overline{\mathcal{F}} \cap \mathcal{L}_\infty$  consists of exactly one point  $x$ . Moreover, the germ of  $\mathcal{F} \cup \{x\}$  at  $x$  is irreducible since a fundamental system of (punctured) neighborhoods of  $x$  in  $\mathcal{F}$  is given by the connected sets  $\phi_{\mathcal{F}}^{-1}(\{|w| > k\})$  where  $k \in \mathbb{N}$ .  $\square$

LEMMA 7.10. — *There exists a neighborhood  $U$  of  $[1 : 1 : 0]$  such that*

$$\begin{aligned} \Psi : U &\rightarrow \mathbb{C}^2 \\ y &\mapsto \begin{cases} \left( \frac{1}{\phi_{f_{a,v}}(2a)}, \frac{v}{a} - 1 \right) & \text{if } y = [a : v : 1] \in \mathcal{P}_{\mathbb{C}} \\ \left( 0, \frac{v}{a} - 1 \right) & \text{if } y = [a : v : 0] \in \mathcal{L}_\infty \end{cases} \end{aligned}$$

is a well defined injective function and  $\Psi : U \rightarrow \Psi(U)$  is biholomorphic.

*Proof.* — Let  $U_0 = \{[1 : \bar{v} : \bar{a}] \mid |\bar{a}| < 1/3, |\bar{v} - 1| < 1\}$ . By Lemma 7.2 (ii) and (iii),  $\Psi : U_0 \rightarrow \mathbb{C}^2$  is well defined. Moreover,  $\Psi$  is holomorphic in  $U_0 \cap \mathcal{P}_{\mathbb{C}} = U_0 \setminus \mathcal{L}_\infty$  and continuous in  $U_0$ . It follows that  $\Psi$  is holomorphic in  $U_0$ .

To show that  $\Psi$  is biholomorphic in a smaller open set it suffices to show that the derivative  $D\Psi$  has full rank at  $[1 : 1 : 0]$ . In coordinates  $(\bar{a}, \bar{v})$  in the domain,  $D\Psi_{(1,1)}$  has the form:

$$\begin{bmatrix} \partial_{\bar{a}}\Psi(1, 1) & \star \\ 0 & 1 \end{bmatrix}$$

where

$$\partial_{\bar{a}}\Psi(1, 1) = \lim_{|a| \rightarrow \infty} \frac{a}{\phi_{f_{(a,a)}}(2a)}.$$

Thus it is enough to show that  $\partial_{\bar{a}}\Psi(1, 1) \neq 0$ . In fact, let  $f_a = f_{(a,a)}$  and  $\phi_a = \phi_{f_a}$ . Then

$$\phi_a(2a) = \lim_{n \rightarrow \infty} 2a \left( \frac{f_a(2a)}{(2a)^3} \right)^{1/3} \cdot \left( \frac{f_a^2(2a)}{(f_a(2a))^3} \right)^{1/3^2} \cdots \left( \frac{f_a^n(2a)}{(f_a^{n-1}(2a))^{3^{n-1}}} \right)^{1/3^{n-1}}.$$

For  $|a|$  sufficiently large, the above limit is uniform. In fact, since  $f_a(-a) = f_a(2a)$  we have that

$$\begin{aligned} \log \left| \frac{f_a^i(2a)}{f_a^{i-1}(2a)^{3^{i-1}}} \right|^{1/3^i} &= \frac{1}{3^i} \log \left| 1 - 3 \left( \frac{a}{f_a^i(2a)} \right)^2 + 2 \left( \frac{a}{f_a^i(2a)} \right)^3 + \frac{a}{(f_a^i(2a))^3} \right| \\ &\leq \frac{K}{3^i} \end{aligned}$$

for some  $K \geq 0$ . Now observe that:

$$\lim_{|a| \rightarrow \infty} \frac{f_a(2a)}{(2a)^3} = \lim_{|a| \rightarrow \infty} \frac{4a^3 + a}{8a^3} = \frac{1}{2}$$

and, for all  $n \geq 2$ ,

$$\lim_{|a| \rightarrow \infty} \frac{f_a^n(2a)}{(f_a^{n-1}(2a))^{3^{n-1}}} = 1.$$

Therefore, switching the order of the limits  $|a| \rightarrow \infty$  and  $n \rightarrow \infty$  we obtain that  $\partial_{\bar{a}}\Psi(1, 1) = 2^{-2/3}$  and the lemma follows. □

*Proof of Proposition 7.6.* — Consider a neighborhood  $U$  as in the previous lemma. We may assume that  $\Psi(U)$  is a small polydisk  $\{(u, w) \mid |u| < \delta_1, |w| < \delta_2\}$ . By Lemma 7.4, we may choose a sufficiently small  $\delta > 0$  such that  $0 < \delta < \delta_1$  and  $\mathcal{F} \cap \{f_{a,v} \mid G_{f_{a,v}}(-a) > 1/\delta\} \subset U$  for all ends  $\mathcal{F}$  at  $[1 : 1 : 0]$ . Now shrink  $U$  by declaring  $U = \Psi^{-1}\{(u, w) \mid |u| < \delta, |w| < \delta_2\}$  and rescale the image of  $\Psi$  to obtain the desired change of coordinates. □



**7.4. Continuous extension**

Let  $\text{Per}_{\mathbb{L}}(n) \subset \mathbb{L}$  be the set of Puiseux series of ends of the period  $n$  curve  $\text{Per}(n)$ . Consider

$$\text{Per}_{\mathbb{L}} = \cup_{n \geq 1} \text{Per}_{\mathbb{L}}(n)$$

and its closure  $\overline{\text{Per}}_{\mathbb{L}}$  in  $\mathbb{L}$ . Throughout fix  $\varepsilon > 0$  small so that the conclusion of Corollary 7.7 holds. The aim of this subsection is to extend the map

$$\begin{aligned} \mathbb{H}_{\varepsilon} \times \text{Per}_{\mathbb{L}} &\rightarrow V_{\varepsilon} \\ (h, \nu) &\mapsto [1 : \nu(e^{2\pi ih}) : e^{2\pi ih}] \end{aligned}$$

given by the above mentioned corollary to a map defined on  $\mathbb{H}_{\varepsilon} \times \overline{\text{Per}}_{\mathbb{L}}$ .

PROPOSITION 7.11. — *There exists a unique continuous map*

$$\Phi : \mathbb{H}_{\varepsilon} \times \overline{\text{Per}}_{\mathbb{L}} \rightarrow V_{\varepsilon}$$

which extends the map of  $\mathbb{H}_{\varepsilon} \times \text{Per}_{\mathbb{L}}$  naturally defined by Corollary 7.7. Moreover,  $\Phi(\cdot, \nu)$  is holomorphic. Furthermore, if  $\overline{\text{Per}}_{\mathbb{L}} \supset \{\nu_n\}$  converges to  $\nu$ , then  $\Phi(\cdot, \nu_n)$  converges locally uniformly to  $\Phi(\cdot, \nu)$ .

In Subsection 7.7 we will upgrade the local uniform convergence of the proposition to uniform convergence.

The proof relies on the following Fourier analysis lemma.

LEMMA 7.12. — *Let  $\{f_n : \mathbb{R} \rightarrow \mathbb{C}\}$  be a uniformly bounded sequence of functions that converges locally uniformly to a function  $f$ . For all  $n \geq 1$ , suppose*

$$f_n(x) = \sum_{\Lambda_n \ni \lambda \geq M_n} a_{\lambda}^{(n)} e^{2\pi i \lambda x}$$

where  $\Lambda_n$  is a discrete subset of  $\mathbb{R}$ , the sum converges uniformly and

$$M_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

*Proof.* — Denote by  $S(\mathbb{R})$  the Schwarz space (see VI.4.1 in [21]). To show that  $f \equiv 0$  it is sufficient to prove that  $\int fg = 0$  for all  $g \in Z$  where

$$Z = \{g \in S(\mathbb{R}) \mid \hat{g} \text{ has compact support}\}$$

and  $\hat{g}$  denotes the Fourier transform of  $g$ . In fact,  $Z$  is dense in  $S(\mathbb{R})$  (see [19] II.1.6) and  $g \mapsto \int fg$  is a continuous functional.

Observe that if

$$h(x) = \sum_{\lambda \in \Lambda} a_{\lambda} e^{2\pi i \lambda x}$$

where  $\Lambda$  is a discrete and bounded below subset of  $\mathbb{R}$  and the sum converges uniformly, then

$$\int_{\mathbb{R}} h(x)g(x)dx = \sum_{\lambda \in \Lambda} a_{\lambda} \hat{g}(\lambda)$$

for all  $g \in S(\mathbb{R})$ .

Given  $g \in Z$  the support of  $\hat{g}$  is contained in  $[-R, R]$  for some  $R > 0$ . Hence, there exists  $N$  such that  $M_n > R$  for all  $n > N$ . It follows that

$$\begin{aligned} \int_{\mathbb{R}} f(x) \cdot g(x)dx &= \lim_{n \rightarrow \infty} \int f_n(x) \cdot g(x)dx \\ &= \lim_{n \rightarrow \infty} \sum_{\lambda \in \Lambda_n} a_{\lambda}^{(n)} \hat{g}(\lambda) = 0 \end{aligned}$$

since  $\sum_{\lambda \in \Lambda_n} a_{\lambda}^{(n)} \hat{g}(\lambda) = 0$  for all  $n > N$ . □

*Proof of Proposition 7.11.* — Given  $\nu \in \overline{\text{Per}}_{\mathbb{L}}$ , consider a sequence  $\{\nu_n\} \subset \text{Per}_{\mathbb{L}}$  which converges to  $\nu$ . Let  $\bar{v}_n(h) = \nu_n(e^{2\pi i h})$  and observe that the family of maps  $\{\bar{v}_n : \mathbb{H}_{\varepsilon} \rightarrow \mathbb{C}\}$  is a normal family. Thus, we may suppose that a subsequence  $\{\bar{v}_{n_k}\}$  converges locally uniformly to some function  $\bar{v}$ . We claim that  $\{\bar{v}_n\}$  converges locally uniformly to  $\bar{v}$ . In fact, by the previous lemma, if  $\{\bar{v}_{m_k}\}$  is any convergent subsequence, then

$$(\bar{v}_{n_k} - \bar{v}_{m_k}) \left( x - \frac{i}{2\pi} \log \frac{\epsilon}{2} \right)$$

converges locally uniformly to zero, as  $k \rightarrow \infty$ . Therefore,  $\{\bar{v}_n\}$  converges locally uniformly to  $\bar{v}$  and the proposition follows. □

### 7.5. Injectivity

To study the lack of injectivity of  $\Phi$  we consider the unique automorphism  $\sigma : \mathbb{L} \rightarrow \mathbb{L}$  such that  $\sigma(t^{1/m}) = e^{2\pi i/m} t^{1/m}$  for all  $m \geq 1$ . Note that  $\sigma$  is a Galois automorphism of  $\mathbb{L}$  over  $\mathbb{Q}^a((t))$ . In particular,  $\sigma$  is an isometry of  $\mathbb{L}$ .

If  $\nu \in \text{Per}_{\mathbb{L}}$  is a Puiseux series of an end  $\mathcal{F}$ , then it easily follows that  $\sigma(\nu)$  is also a Puiseux series of  $\mathcal{F}$  and  $\Phi(h - 1, \sigma(\nu)) = \Phi(h, \nu)$ . Therefore,  $\sigma(\text{Per}_{\mathbb{L}}) = \text{Per}_{\mathbb{L}}$ . Since  $\sigma$  is an isometry, we have that  $\sigma(\overline{\text{Per}}_{\mathbb{L}}) = \overline{\text{Per}}_{\mathbb{L}}$  and

$$\Phi(h - 1, \sigma(\nu)) = \Phi(h, \nu)$$

for all  $\nu \in \overline{\text{Per}}_{\mathbb{L}}$ . Our next results shows that injectivity of  $\Phi$  only fails in the above manner.

PROPOSITION 7.13. — *Let  $\overline{\Sigma}$  be  $\mathbb{H}_\varepsilon \times \overline{\text{Per}}_\mathbb{L}$  modulo  $(h, \nu) \sim (h-1, \sigma(\nu))$ . Let  $\varpi : \mathbb{H}_\varepsilon \times \overline{\text{Per}}_\mathbb{L} \rightarrow \overline{\Sigma}$  be the natural projection. Then there exists a unique map  $\Phi_{\overline{\Sigma}} : \overline{\Sigma} \rightarrow V_\varepsilon$  such that  $\Phi = \Phi_{\overline{\Sigma}} \circ \varpi$ . Moreover, the map  $\Phi_{\overline{\Sigma}}$  is injective.*

*Proof.* — Suppose that  $(h, \nu)$  and  $(h', \nu')$  are elements of  $\mathbb{H}_\varepsilon \times \overline{\text{Per}}_\mathbb{L}$  such that  $\Phi(h, \nu) = \Phi(h', \nu')$ . From the definition of  $\Phi$ , it follows that  $h' = h - k$  for some  $k \in \mathbb{Z}$ . To prove the proposition is sufficient to show that  $\nu' = \sigma^k(\nu)$ . For this consider sequences  $\{\nu_n\}, \{\nu'_n\} \subset \text{Per}_\mathbb{L}$  converging to  $\nu$  and  $\nu'$  respectively. In  $\{[1 : \bar{v} : \bar{a}] \in \mathbb{C}\mathbb{P}^2\}$ , denote by  $\Pi_{\bar{v}}$  the projection onto the  $\bar{v}$ -coordinate (i.e.,  $\Pi_{\bar{v}}([1 : \bar{v} : \bar{a}]) = \bar{v}$ ). By Proposition 7.11,

$$\Pi_{\bar{v}}(\Phi(\cdot - k, \sigma^k(\nu'_n)) - \Phi(\cdot, \nu_n))$$

converges locally uniformly to

$$\Pi_{\bar{v}}(\Phi(\cdot - k, \sigma^k(\nu')) - \Phi(\cdot, \nu)),$$

which has value 0 at  $h$ . By the Argument Principle,  $\Pi_{\bar{v}}(\Phi(\cdot - k, \sigma^k(\nu'_n)) - \Phi(\cdot, \nu_n))$  has a value 0 at some  $h_n$  close to  $h$  for all  $n$  sufficiently large. Since ends of periodic curves are either equal or disjoint,  $\nu_n$  and  $\sigma^k(\nu'_n)$  are Puiseux series of a common end  $\mathcal{F}_n$  for all  $n$  sufficiently large. Moreover, these Puiseux series “evaluated” at the point  $h_n$  coincide. Therefore,  $\sigma^k(\nu'_n) = \nu_n$  for all  $n$  large enough. Passing to the limit,  $\sigma^k(\nu') = \nu$ .  $\square$

### 7.6. $\text{Per}_\mathbb{L}$ from a Puiseux series dynamics viewpoint

To continue we need to give a Puiseux series dynamics interpretation of  $\text{Per}_\mathbb{L}$ . In this subsection we establish a basic relation between iteration of complex cubic polynomials near  $\mathcal{L}_\infty$  and Puiseux series dynamics. For this we consider the *one-parameter family*  $\{\psi_\nu \mid \nu \in \mathbb{L}\}$  of cubic polynomials with coefficients in  $\mathbb{L}$  given by:

$$\begin{aligned} \psi_\nu \quad \mathbb{L} &\rightarrow \mathbb{L} \\ \zeta &\mapsto t^{-2}(\zeta - 1)^2(\zeta + 2) + \nu. \end{aligned}$$

This is a particular case of the one-parameter families already considered in Section 5.

Before stating a rigorous relation of  $\psi_\nu$  with iteration of complex cubic polynomials, let us give a more intuitive and informal discussion. The reader that dislikes this sort of discussions may proceed to the next paragraph. Near  $\mathcal{L}_\infty$  it is convenient to use coordinates  $(\bar{a}, \bar{v}) = (1/a, v/a)$  in parameter space  $\mathcal{P}_\mathbb{C}$  and to make the change of coordinates  $\bar{z} = z/a$  in the dynamical plane. In this new coordinates  $f_{a,v}(z)$  becomes

$$\bar{z} \mapsto \bar{a}^{-2}(\bar{z} - 1)^2(\bar{z} + 2) + \bar{v}.$$

Then we think of  $\bar{v}$  and  $\bar{z}$  as “functions” of  $\bar{a}$ . So after declaring  $\bar{a} = t$  transcendental over  $\mathbb{C}$  we may replace  $\bar{z}$  by  $\zeta \in \mathbb{L}$  and  $\bar{v}$  by  $\nu \in \mathbb{L}$  to obtain  $\psi_\nu$ . In certain sense,  $\psi_\nu$  can be thought as the action of irreducible germs of analytic sets (at points in  $\mathcal{L}_\infty$ ) on irreducible germs of analytic sets at points in the dynamical plane  $\mathbb{C}$  (here we view  $\mathbb{C}$  as contained in the two dimensional space with coordinates  $(\bar{a}, \bar{z})$ ).

To give a rigorous discussion we restrict to the elements of  $\mathbb{L}$  which are “convergent along certain suitably chosen sums” in  $\mathbb{H}_\varepsilon$ . More precisely, we say that  $\zeta = \sum a_\lambda t^\lambda$  converges in  $\mathbb{H}_\varepsilon$  if

$$s_n = \sum_{n! \lambda \in \mathbb{Z}} a_\lambda e^{2\pi i \lambda h}$$

converges uniformly in  $\mathbb{H}_\varepsilon$  and  $\{s_n(h)\}$  converges for all  $h \in \mathbb{H}_\varepsilon$ . In this case we denote the limit of  $s_n(h)$  by  $\zeta(e^{2\pi i h})$  (compare with [6] Chapter I.8). Our next result contains the key formula to move between dynamics over  $\mathbb{L}$  and dynamics over  $\mathbb{C}$ .

LEMMA 7.14. — *Suppose that  $\nu$  and  $\zeta$  converge in  $\mathbb{H}_\varepsilon$ . Let  $h \in \mathbb{H}_\varepsilon$  and write  $T = e^{2\pi i h}$ . Consider the cubic polynomial*

$$f_T = f_{1/T, \nu(T)/T}.$$

Then

$$[\psi_\nu(\zeta)](T) = T f_T \left( \frac{\zeta(T)}{T} \right).$$

*Proof.* —  $[\psi_\nu(\zeta)](T) = T^{-2}(\zeta(T) - 1)^2(\zeta(T) + 2) + \nu(T) = T((\zeta(T)/T - 1/T)^2(\zeta(T)/T + 2/T) + \nu(T)/T)$ . □

Algebraically (e.g., see [12]), Puiseux series of ends of  $\text{Per}(n)$  are characterized as solutions in  $\nu \in \mathbb{L}$  of the equations

$$f_{1/t, \nu/t}^n(1/t) - 1/t = 0,$$

$$f_{1/t, \nu/t}^m(1/t) - 1/t \neq 0 \quad \text{for all } m \in \mathbb{N} \text{ such that } m \text{ divides } n.$$

Note that any solution  $\nu$  of the above equations belongs to  $\mathbb{Q}^a \langle\langle t \rangle\rangle$  since  $\mathbb{Q}^a \langle\langle t \rangle\rangle$  is algebraically closed. After changing coordinates to  $\bar{\zeta} = \zeta/t$ , the map  $\bar{\zeta} \mapsto f_{1/t, \nu/t}(\bar{\zeta})$  becomes  $\psi_\nu(\zeta)$ . Therefore,  $\nu$  is a solution of the above equations if and only if  $\omega^+ = +1$  is periodic and has period exactly  $n$  under  $\psi_\nu$ . Hence we have the following Puiseux series dynamics characterization of the elements of  $\text{Per}_\mathbb{L}$ .

COROLLARY 7.15. — *An element  $\nu$  of  $\mathbb{L}$  is the Puiseux series of an end of  $\text{Per}(n)$  if and only if the critical point  $\omega^+$  is periodic of period exactly  $n$  under  $\psi_\nu$ .*

Now from Corollary 5.6 we obtain the following.

COROLLARY 7.16. — *The closure of  $\text{Per}_{\mathbb{L}}$  in  $\mathbb{L}$  is  $\text{Per}_{\mathbb{L}} \cup \mathcal{NR}_{\mathbb{L}}$ .*

### 7.7. Almost periodic functions

In this subsection we upgrade the local uniform convergence of Proposition 7.11 to uniform convergence. Then we extract some conclusions regarding almost periodic functions. More precisely we will prove the following result.

PROPOSITION 7.17. — *If  $\nu \in \overline{\text{Per}_{\mathbb{L}}}$  and  $\{\nu_n\} \subset \overline{\text{Per}_{\mathbb{L}}}$  is a sequence converging to  $\nu$ , then  $\mathbb{H}_{\varepsilon} \ni h \mapsto \Phi(h, \nu_n)$  converges uniformly to  $h \mapsto \Phi(h, \nu)$ .*

The key to prove the proposition is the compactness result given by Theorem 5.3.

LEMMA 7.18. — *If  $\nu \in \overline{\text{Per}_{\mathbb{L}}}$ , then  $\{\sigma^n(\nu) \mid n \in \mathbb{N}\}$  has compact closure.*

*Proof.* — If  $\nu \in \text{Per}_{\mathbb{L}}$ , then  $\{\sigma^n(\nu) \mid n \in \mathbb{N}\}$  is finite. Since  $\sigma \circ \psi_{\nu} = \psi_{\sigma(\nu)} \circ \sigma$ , if  $\nu \in \mathcal{NR}_{\mathbb{L}}$  has critical marked grid  $\mathbf{M}$ , then  $\sigma(\nu)$  also has critical marked grid  $\mathbf{M}$ . Hence,  $\{\sigma^n(\nu) \mid n \in \mathbb{N}\} \subset C_{\mathbf{M}}$  where  $C_{\mathbf{M}}$  is the compact set (see Theorem 5.3) consisting of all  $\eta$  such that the corresponding critical marked is  $\mathbf{M}$ . □

*Proof of Proposition 7.17.* — Recall that  $\Pi_{\bar{v}} : [1 : \bar{v} : \bar{a}] \mapsto \bar{v}$  denotes the projection to the  $\bar{v}$ -coordinate. To simplify notation let  $\bar{v}_n : \mathbb{H}_{\varepsilon} \rightarrow \mathbb{C}$  be the holomorphic function defined by  $\bar{v}_n(h) = \Pi_{\bar{v}} \circ \Phi(h, \nu_n)$  and similarly let  $\bar{v}$  be the function defined by  $\bar{v}(h) = \Pi_{\bar{v}} \circ \Phi(h, \nu)$ . By the Maximum principle it is enough to show that  $\bar{v}_n$  converges uniformly to  $\bar{v}$  in  $\{\text{Im } h = -(\log \varepsilon)/2\pi\}$ . We proceed by contradiction and suppose that there exists  $\delta > 0$ ,  $n_k \rightarrow \infty$ ,  $x_k \in \mathbb{R}$  such that

$$|\bar{v}_{n_k}(x_k - i \log \varepsilon/2\pi) - \bar{v}(x_k - i \log \varepsilon/2\pi)| > \delta.$$

By passing to a subsequence we may also assume that there exists  $x_{\infty} \in \mathbb{R}$ ,  $M_k \in \mathbb{Z}$  and  $|\Delta_k| < 1/2$  such that

$$x_k = x_{\infty} + M_k + \Delta_k$$

and  $\Delta_k \rightarrow 0$ . From the compactness given by the previous lemma, we may pass to a subsequence and assume that  $\sigma^{M_k}(\nu) \rightarrow \hat{\nu}$ . Thus,  $\sigma^{M_k}(\nu_{n_k}) \rightarrow \hat{\nu}$ . Let  $\bar{v}_{\hat{\nu}}(h) = \Pi_{\bar{v}} \circ \Phi(h, \hat{\nu})$ . It follows that

$$\bar{v}_{n_k}(x_k - i \log \varepsilon/2\pi) = \bar{v}_{\sigma^{M_k} \nu_{n_k}}(x_{\infty} + \Delta_k - i \log \varepsilon/2\pi) \rightarrow \bar{v}_{\hat{\nu}}(x_{\infty} - i \log \varepsilon/2\pi)$$

and similarly

$$\bar{v}_\nu(x_k - i \log \epsilon / 2\pi) = \bar{v}_{\sigma^{M_k} \nu}(x_\infty + \Delta_k - i \log \epsilon / 2\pi) \rightarrow \bar{v}_{\bar{\nu}}(x_\infty - i \log \epsilon / 2\pi)$$

which is a contradiction. □

The uniform limit of a sequence of periodic functions is an almost periodic function in the sense of Bohr. To interpret  $\nu \in \mathcal{NR}_{\mathbb{L}}$  as a Fourier series of the almost periodic function  $\bar{v} : h \mapsto \nu(e^{2\pi i h})$  we briefly summarize some facts about almost periodic functions and refer the reader for a detailed discussion to [6]. A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is *almost periodic* if the family  $\{f(\cdot + \delta) \mid \delta \in \mathbb{R}\}$  is precompact in the sup-norm. That is, every sequence in this family has a subsequence which converges uniformly in  $\mathbb{R}$  (i.e., in the sup-norm). An analytic function  $f : \mathbb{H}_\epsilon \rightarrow \mathbb{C}$  is *almost periodic* if  $f(\cdot + iy) : \mathbb{R} \rightarrow \mathbb{C}$  is almost periodic for all  $y > -(2\pi)^{-1} \log \epsilon$ . If  $f : \mathbb{H}_\epsilon \rightarrow \mathbb{C}$  is an analytic almost periodic function, then

$$a_\lambda = \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R e^{-2\pi i \lambda x} f(x + iy) dx$$

exists for all  $\lambda \in \mathbb{R}$  and  $a_\lambda$  is independent of  $y > -(2\pi)^{-1} \log \epsilon$ . Moreover,  $a_\lambda$  is non-zero for at most countably many  $\lambda \in \mathbb{R}$ . Furthermore, the *Fourier series of  $f$* :

$$\sum_{\lambda \in \mathbb{R}} a_\lambda e^{-2\pi i \lambda T}$$

converges to  $f$  in the norm  $\|\cdot\|_M$  defined in the space of analytic almost periodic functions with domain  $\mathbb{H}_\epsilon$  by:

$$\|g\|_M^2 = \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R |g(x + iy)|^2 dx$$

for any  $y > -(2\pi)^{-1} \log \epsilon$ . Also, if  $f$  is the limit of periodic functions, then the periodic functions

$$s_n = \sum_{n! \lambda \in \mathbb{Z}} a_\lambda e^{-2\pi i \lambda T}$$

converge uniformly to  $f$  as  $n \rightarrow \infty$ . (See [6] Chapter I.8).

Since, as mentioned above, the uniform limit of a sequence of periodic functions is an almost periodic function and the corresponding Fourier coefficients also converge we obtain the following result.

**COROLLARY 7.19.** — *If  $\nu = \sum a_\lambda t^\lambda \in \overline{\text{Per}}_{\mathbb{L}}$ , then  $\mathbb{H}_\epsilon \ni h \mapsto \Phi(h, \nu)$  is an analytic almost periodic function in the sense of Bohr with Fourier series:*

$$\sum a_\lambda e^{2\pi i \lambda h}.$$

Moreover,

$$s_n = \sum_{n! \lambda \in \mathbb{Z}} a_\lambda e^{-2\pi i \lambda h}$$

is an analytic periodic function, for all  $n \geq 0$ ; and  $\{s_n(\cdot)\}$  converges uniformly to  $\Pi_{\bar{\nu}} \circ \Phi(\cdot, \nu)$  in  $\mathbb{H}_\varepsilon$ .

### 7.8. Marked grid correspondence

To show that  $\Phi(\mathbb{H}_\varepsilon \times \mathcal{NR}_\mathbb{L})$  consists of all non-renormalizable cubic polynomials in a neighborhood of  $\mathcal{L}_\infty$ , we need to establish a correspondence between the dynamics of  $\psi_\nu$  and the one of the complex cubic polynomial corresponding to the parameter  $\Phi(h, \nu)$ .

PROPOSITION 7.20. — *For any  $\nu \in \overline{\text{Per}}_\mathbb{L}$ , the critical marked grid of  $\psi_\nu$  is the same as the critical marked grid of  $f_{\Phi(h, \nu)}$ , for all  $h \in \mathbb{H}_\varepsilon$ .*

The main step to prove the proposition is to establish the assertion for  $\nu \in \text{Per}_\mathbb{L}$ .

LEMMA 7.21. — *For any  $\nu \in \text{Per}_\mathbb{L}$ , the critical marked grid of  $\psi_\nu$  is the same as the critical marked grid of  $f_{\Phi(h, \nu)}$  for all  $h \in \mathbb{H}_\varepsilon$ .*

Let us first show how to deduce the proposition from the lemma.

*Proof of Proposition 7.20* (assuming Lemma 7.21). — For  $\nu \in \overline{\text{Per}}_\mathbb{L}$ , denote by  $\mathbf{M}^\nu$  the critical marked grid of  $\psi_\nu$ . Fix  $h \in \mathbb{H}_\varepsilon$  and let  $\mathbf{M}^{f_\nu}$  be the critical marked grid of  $f_\nu$ , where  $f_\nu$  denotes the complex cubic polynomial corresponding to  $\Phi(h, \nu)$ .

Given  $\hat{\nu} \in \mathcal{NR}_\mathbb{L}$ , let  $\{\nu_k\} \subset \text{Per}_\mathbb{L}$  be a sequence converging to  $\hat{\nu}$ . Fix  $n \geq 0$  and denote with a subscript  $n$  the corresponding level  $n$  marked grids. Since  $f_{\nu_k}$  converges to  $f_{\hat{\nu}}$ , for  $k$  sufficiently large,  $\mathbf{M}_n^{f_{\hat{\nu}}} = \mathbf{M}_n^{f_{\nu_k}}$  (Lemma 7.1). Assuming the lemma, we have that  $\mathbf{M}_n^{f_{\nu_k}} = \mathbf{M}_n^{\nu_k}$ . By Proposition 5.5, for  $k$  large,  $\mathbf{M}_n^{\nu_k} = \mathbf{M}_n^{\hat{\nu}}$ . Hence,  $\mathbf{M}_n^{f_{\hat{\nu}}} = \mathbf{M}_n^{\hat{\nu}}$  for all  $n$ , and the proposition follows. □

The rest of this subsection is devoted to the proof of Lemma 7.21. We will need the following fact.

LEMMA 7.22. — *Let  $\mathcal{F}$  be an end of a periodic curve. Then the marked grid  $\mathbf{M}^\mathcal{F}$  of  $+a$  under  $f_{a, \nu}$  is independent of  $f_{a, \nu} \in \mathcal{F}$ .*

*Proof.* — Since  $+a \in K(f_{a,v})$  for all  $f_{a,v} \in \mathcal{F}$ , by Lemma 7.1, the subset of  $\mathcal{F}$  where the  $(\ell, k)$  position of the critical marked grid is marked (resp. unmarked) is open, and therefore closed in  $\mathcal{F}$ . □

To prove Lemma 7.21, we fix  $\nu \in \text{Per}_{\mathbb{L}}$  and denote by  $\mathcal{F}$  the corresponding end of a periodic curve. For  $h \in \mathbb{H}_{\varepsilon}$ , let

$$a(h) = e^{-2\pi ih}, \quad v(h) = \frac{\nu(e^{2\pi ih})}{e^{2\pi ih}}.$$

To simplify notation, let

$$f_h = f_{a(h),v(h)}$$

be the polynomial corresponding to  $\Phi(h, \nu)$ . The Green function of  $f_h$  will be denoted by  $G_h$ , a level  $n$  dynamical disk by  $D_n^h(z)$  and a level  $n$  annulus by  $A_n^h(z)$ . The level 0 annulus will be denoted by  $A_0^h$ . The critical marked grid of  $f_h$  only depends on  $\mathcal{F}$  so we denote it by  $\mathbf{M}^{\mathcal{F}}$ .

For  $k \geq 0$ , we let  $\nu_k = \psi_{\nu}^k(\omega^+)$  and  $f_h^k(a(h)) = v_k(h)$ . From Lemma 7.14 we have that  $v_k(h) = e^{-2\pi ih} \cdot \nu_k(e^{2\pi ih})$ .

LEMMA 7.23. — *The following hold:*

(i)

$$\frac{\pi \bmod A_0^h}{\log |a(h)|} = \frac{G_h(-a(h))}{\log |a(h)|} \rightarrow 1 \text{ as } \text{Im } h \rightarrow +\infty.$$

(ii) *Let  $q = 1$  or  $-2$ . Assume that  $z_1, z_2 : \mathbb{H}_{\varepsilon} \rightarrow \mathbb{C}$  are functions such that, for all  $h \in \mathbb{H}_{\varepsilon}$ ,  $z_1(h)$  is a level 2 point under  $f_h$ ,  $z_1(h) \in D_1^h(qa(h))$  and there exists an annulus  $A^h$  which separates  $\{z_1(h), z_2(h)\}$  from  $\{-a(h), \infty\}$ . If there exists  $S > 0$  such that, for all  $h \in \mathbb{H}_{\varepsilon}$ ,*

$$\frac{\bmod A^h}{\bmod A_0^h} \geq S,$$

then

$$\left| \frac{z_1(h)}{a(h)} - \frac{z_2(h)}{a(h)} \right| = O(|a(h)|^{-2S})$$

as  $\text{Im } h \rightarrow +\infty$ .

*Proof.* — Since (i) is a direct consequence of Lemma 7.2 we proceed to prove (ii). For this consider the Möebius transformation  $\Gamma_h(z) = (a(h) + z_1(h))(z - z_1(h))^{-1}$ . The annulus  $\Gamma_h(A^h)$  separates  $\{-1, 0\}$  from  $\infty$  and

$$\left( 1 + \frac{z_1(h)}{a(h)} \right) \left( \frac{z_1(h)}{a(h)} - \frac{z_2(h)}{a(h)} \right)^{-1}.$$



From Chapter III in [1] we have that

$$S \bmod A_0^h \leq \bmod A^h \leq \frac{1}{2\pi} \log 16(\Gamma_h(z_1(h)) + 1).$$

From (i) we conclude that for  $\text{Im } h$  sufficiently large:

$$(7.5) \quad \left| \frac{z_1(h)}{a(h)} - \frac{z_2(h)}{a(h)} \right| \leq \frac{16}{a(h)^{2S} - 16} \left( 1 + \frac{z_1(h)}{a(h)} \right)$$

Now replace in the previous equation  $z_1(h)$  by  $qa(h)$ ,  $z_2(h)$  by  $z_1(h)$ ,  $A^h$  by the annulus  $A_1^h(qa(h))$  of modulus  $|q + 1|^{-1} \bmod A_0^h$  and conclude that, as  $\text{Im } h \rightarrow \infty$ ,

$$1 + \frac{z_1(h)}{a(h)} \rightarrow q + 1.$$

Combining this with equation (7.5), part (ii) of the lemma follows. □

For any  $j, k \geq 0$ , the level  $n$  disks  $D_n^h(v_j(h))$  and  $D_n^h(v_k(h))$  are either equal for all  $h \in \mathbb{H}_\varepsilon$  or distinct for all  $h \in \mathbb{H}_\varepsilon$ . If we denote by  $M_{\ell,k}^{\mathcal{F}}$  the entries of  $\mathbf{M}^{\mathcal{F}}$ , then

$$\bmod A_\ell^h(v_k(h)) = 2^{-S_\ell} \bmod A^h$$

where  $S_\ell = \sum_{k=0}^{\ell-1} M_{\ell-k,j+k}^{\mathcal{F}}$ . By the Grötzsch inequality (see [1]) it follows that

$$\bmod D_1^h(v_k(h)) \setminus \overline{D_n^h(v_k(h))} \geq \sum_{\ell=1}^n \bmod A_\ell^h(v_k(h)).$$

Taking  $A^h = D_1^h(v_k(h)) \setminus \overline{D_n^h(v_k(h))}$  in the previous lemma we immediately obtain the following.

**COROLLARY 7.24.** — *If for some  $j, k \geq 0$  and  $n \geq 1$  we have that  $D_n^h(v_j(h)) = D_n^h(v_k(h))$ , then*

$$\text{ord}(\nu_j - \nu_k) \geq \frac{2}{\bmod A_0^h} \sum_{\ell=1}^n \bmod A_\ell^h(v_j(h)).$$

Lemma 7.21 is implied by the following.

**LEMMA 7.25.** — *For all  $n \geq 1$  and  $j, k \geq 0$  we have that  $D_n^\nu(\nu_j) = D_n^\nu(\nu_k)$  if and only if  $D_n^h(v_j(h)) = D_n^h(v_k(h))$ .*

*Proof.* — We proceed by induction on  $n$ .

For  $n = 1$ , if  $D_1^h(v_j(h)) = D_1^h(a(h))$ , then  $\text{ord}(\nu_j - \omega^+) \geq 1$ . Hence  $\nu_j \in D_1^\nu(\omega^+)$ . If  $D_1^h(v_j(h)) = D_1^h(-2a(h))$ , then  $\text{ord}(\nu_j - (-2)) \geq 2$ . Therefore,  $\nu_j \in D_1^\nu(-2)$ . It follows that  $D_1^h(v_j(h)) = D_1^h(v_k(h))$  if and only if  $D_1^\nu(\nu_j) = D_1^\nu(\nu_k)$ .

Now suppose that the lemma is true for  $n$ . Note that this implies that

$$\text{mod } A_\ell^\nu(\nu_j) = \frac{2\pi}{G_h(-a(h))} \text{mod } A_\ell^h(v_j(h))$$

for all  $\ell \leq n$ . Therefore, if  $D_{n+1}^h(v_j(h)) = D_{n+1}^h(v_k(h))$  then

$$\begin{aligned} \text{ord}(\nu_j - \nu_k) &\geq \sum_{\ell=1}^{n+1} \text{mod } A_\ell^h(v_j(h)) > \sum_{\ell=1}^n \text{mod } A_\ell^h(v_j(h)) \\ &= \sum_{\ell=1}^n \text{mod } A_\ell^\nu(\nu_j) = -\log r_n^\nu(\nu_j). \end{aligned}$$

Hence,  $|\nu_j - \nu_k|_o < r_n^\nu(\nu_j)$  and  $\nu_j, \nu_k$  belong to the same element of the affine partition associated to  $D_n^\nu(\nu_j)$ . By Lemma 4.2 (i),  $\nu_k \in D_{n+1}^\nu(\nu_j)$ .

To finish the proof it is sufficient to show that if  $D_n^h(v_j(h)) = D_n^h(v_k(h))$  and  $D_{n+1}^h(v_j(h)) \neq D_{n+1}^h(v_k(h))$ , then  $D_{n+1}^\nu(\nu_j) \neq D_{n+1}^\nu(\nu_k)$ . There are two cases.

*Case 1.*  $D_n^h(v_{j+1}(h)) \neq D_n^h(v_{k+1}(h))$ : By the inductive hypothesis,  $D_n^\nu(\nu_{j+1}) \neq D_n^\nu(\nu_{k+1})$ . Thus,  $D_{n+1}^\nu(\nu_j) \neq D_{n+1}^\nu(\nu_k)$ .

*Case 2.*  $D_n^h(v_{j+1}(h)) = D_n^h(v_{k+1}(h))$ : In this case  $j, k > 0$  and  $f_h : D_n^h(v_j(h)) \rightarrow D_{n-1}^h(v_{j+1}(h))$  has degree 2. Thus, there exist  $v'_j(h)$  and  $v'_k(h)$  in  $D_n^h(v_j(h))$  distinct from  $v_j(h)$  and  $v_k(h)$ , respectively, such that  $f_h(v'_j(h)) = v_{j+1}(h)$  and  $f_h(v'_k(h)) = v_{k+1}(h)$ . Similarly,  $\psi_\nu : D_n^\nu(\nu_j) \rightarrow D_{n-1}^\nu(\nu_{j+1})$  has degree 2 and there exist  $\nu'_j$  and  $\nu'_k$  in  $D_n^\nu(\nu_j)$  distinct from  $\nu_j$  and  $\nu_k$ , respectively, such that  $\psi_\nu(\nu'_j) = \nu_{j+1}$  and  $\psi_\nu(\nu'_k) = \nu_{k+1}$ . It follows that  $\nu'_j(e^{2\pi i h}) = e^{2\pi i h} v'_j(h)$  and  $\nu'_k(e^{2\pi i h}) = e^{2\pi i h} v'_k(h)$ . We claim that

$$\text{ord}(\nu'_j - \nu_k) > \text{ord}(\nu_j - \nu_k).$$

In fact, let

$$\Gamma_h(z) = \frac{z - v_k(h)}{v_j(h) - v_k(h)}.$$

Since  $v'_j(h) \in D_{n+1}^h(v_k(h))$  the annulus  $A_{n+1}^h(v_k(h))$  separates  $v'_j$  and  $v_k$  from  $v_j$  and  $\infty$ . Therefore,  $\Gamma(v'_j) \rightarrow 0$  as  $\text{Im } h \rightarrow \infty$  because  $\text{mod } A_{n+1}^h(v_k(h)) \rightarrow \infty$ . Thus  $\text{ord}(\nu'_j - \nu_k) > \text{ord}(\nu_j - \nu_k)$ . From here we conclude that  $D_{n+1}^\nu(\nu_j) \neq D_{n+1}^\nu(\nu_k)$ . For otherwise  $D_{n+1}^\nu(\nu'_j) \neq D_{n+1}^\nu(\nu_k)$  and

$$\text{ord}(\nu_j - \nu_k) > \sum_{\ell=1}^n A_\ell^\nu(\nu_j) = \text{ord}(\nu'_j - \nu_k).$$

□

**7.9. Proofs of theorems 1.5 and 1.6**

Since  $\overline{\text{Per}}_{\mathbb{L}} \supset \mathcal{NR}_{\mathbb{L}}$ , parts (i), (ii) and (iv) of Theorem 1.5 follow from Propositions 7.11, 7.17 and 7.20. Hence, to finish the proof of Theorem 1.5 we just need to show that  $\Phi(\mathbb{H}_{\varepsilon} \times \mathcal{NR}_{\mathbb{L}}) = \mathcal{NR}_{\mathbb{C}} \cap V_{\varepsilon}$ . For this purpose we will need the following two lemmas.

LEMMA 7.26. — *Let  $\text{Per} = \cup \text{Per}(n)$ . Then  $\overline{\text{Per}}$  contains  $\mathcal{NR}_{\mathbb{C}} \setminus \text{int}\mathcal{NR}_{\mathbb{C}}$ .*

*Proof.* — Given  $(a_0, v_0) \in \mathcal{NR}_{\mathbb{C}} \setminus \text{int}\mathcal{NR}_{\mathbb{C}}$  and a small connected open neighborhood  $U$  of  $(a_0, v_0)$  in  $\mathcal{P}_{\mathbb{C}}$  we must show that  $U \cap \text{Per} \neq \emptyset$ . First we claim that the family  $\{U \ni (a, v) \mapsto f_{a,v}^n(a) \mid n \geq 1\}$  is not a normal family. Otherwise,  $f_{a,v}^n(a)$  would be uniformly bounded, for all  $n$ , since  $a_0 \in K(f_{a_0,v_0})$ . Hence, for all  $(a, v) \in U$  we would have that  $a \in K(f_{a,v})$ . Therefore we would have that the critical marked grid  $\mathbf{M}$  of  $f_{a_0,v_0}$  coincides with that of  $f_{a,v}$  for all  $(a, v) \in U$  (see Lemma 7.1). This would imply that  $U \subset \mathcal{NR}_{\mathbb{C}}$  which is a contradiction with  $f \in \mathcal{NR}_{\mathbb{C}} \setminus \text{int}\mathcal{NR}_{\mathbb{C}}$ .

Given  $(a, v) \in U$ , let  $a'$  be the preimage of  $a$  in  $D_1^{a,v}(-2a)$  that is not critical. It follows that for some  $n \geq 1$  the map  $U \ni (a, v) \mapsto (f_{a,v}^n(a) - a')/(a - a')$  must contain 0 or 1 in its image. Hence, there exists  $(a, v) \in U$  such that  $f_{a,v}^n(a) = a'$  or  $f_{a,v}^n(a) = a$  for some  $n$ . Thus,  $U$  contains points in a periodic curve. □

LEMMA 7.27. — *The image of  $\Phi : \mathbb{H}_{\varepsilon} \times \mathcal{NR}_{\mathbb{L}} \rightarrow V_{\varepsilon}$  is the set of all  $(a, v) \in \mathcal{NR}_{\mathbb{C}} \setminus \text{int}\mathcal{NR}_{\mathbb{C}}$  such that  $|a| > 1/\varepsilon$ .*

*Proof.* — Suppose that  $(a, v) \in \mathcal{NR}_{\mathbb{C}} \setminus \text{int}\mathcal{NR}_{\mathbb{C}}$ . By the previous lemma there exists a sequence  $\{(a_k, v_k)\}$  converging to  $(a, v)$  such that  $a_k$  is periodic under  $f_{a_k,v_k}$ . It follows that  $(a_k, v_k) = \Phi(h_k, \nu_k)$  for some  $h_k \in \mathbb{H}_{\varepsilon}$  and  $\nu_k \in \text{Per}_{\mathbb{L}}$ . We may assume that  $h_k$  converges to  $h \in \mathbb{H}_{\varepsilon}$ . Denote by  $\mathbf{M}$  the critical marked grid of  $f_{a,v}$  and by  $\mathbf{M}_{n+1}$  the corresponding level  $n + 1$  grid. By Lemma 7.1 and Proposition 7.20, there exists  $k_0$  such that the level  $n + 1$  critical marked grid  $M_{n+1}^{\nu_k}$  of  $\psi_{\nu_k}$  coincides with  $\mathbf{M}_{n+1}$  for all  $k \geq k_0$ . According to Proposition 5.5, there are finitely many level  $n$  parameter balls such that the level  $n + 1$  critical marked grids of parameters in these balls coincides with  $\mathbf{M}_{n+1}$ . Consider a nested sequence  $\{\mathcal{D}_n\}$  of parameter balls so that each contains infinitely many elements of the sequence  $\{\nu_k\}$ . In view of Proposition 5.5, the radius of  $\mathcal{D}_n$  converges to 0 as  $n$  goes to  $\infty$ . Therefore,  $\cap \mathcal{D}_n$  is a singleton, say  $\{\nu\} \subset \mathcal{NR}_{\mathbb{L}}$ , and there is a subsequence of  $\{\nu_k\}$  converging to  $\nu$ . It follows that  $\Phi(h, \nu) = (a, v)$  □

Although Branner and Hubbard already established that  $\text{int } \mathcal{NR}_{\mathbb{C}}$  is empty, we provide a proof which relies on Puiseux series dynamics. Nevertheless, this proof is not independent from Branner and Hubbard's work. With the following result we finish the proof of Theorem 1.5.

**COROLLARY 7.28.** —  $\text{int } \mathcal{NR}_{\mathbb{C}} \cap \{|a| > 1/\epsilon\} = \emptyset$ . Thus,  $\Phi(\mathbb{H}_{\epsilon} \times \mathcal{NR}_{\mathbb{L}}) = \mathcal{NR}_{\mathbb{C}} \cap V_{\epsilon}$ .

*Proof.* — If  $\text{int } \mathcal{NR}_{\mathbb{C}} \neq \emptyset$ , then there exists  $h \in \mathbb{H}_{\epsilon}$  and an aperiodic critical marked grid  $\mathbf{M}$  such that the closed and bounded set

$$S_{\mathbf{M}} = \{v \in \mathbb{C} \mid a = e^{-2\pi ih} \text{ and } \mathbf{M}^{a,v}(a) = \mathbf{M}\}$$

has non-empty interior. So it is sufficient to show that  $S_{\mathbf{M}}$  is totally disconnected.

According to Theorem 5.3, the set  $C_{\mathbf{M}}$  of all parameters  $\nu$  such that  $\psi_{\nu}$  has critical marked grid  $\mathbf{M}$  is compact, non-empty and totally disconnected. By the previous results,  $\Phi(h, \cdot) : C_{\mathbf{M}} \rightarrow S_{\mathbf{M}}$  is continuous, one-to-one and contains  $\partial S_{\mathbf{M}}$ . But since  $C_{\mathbf{M}}$  is compact,  $\Phi(h, \cdot)$  is a homeomorphism between the totally disconnected set  $C_{\mathbf{M}}$  and its image. Therefore,  $\partial S_{\mathbf{M}}$  is totally disconnected and hence  $S_{\mathbf{M}}$  is totally disconnected.  $\square$

*Proof of Theorem 1.6.* — It follows from Proposition 7.13 that  $\Phi_{\Sigma}$  is a well defined injective map. In view of Corollary 7.28, we have that  $\Phi_{\Sigma}$  is onto and, by Proposition 7.11,  $\Phi_{\Sigma}$  is continuous. It remains to show that the inverse  $\Phi_{\Sigma}^{-1}$  is also continuous. For this let  $X$  be the preimage of a closed set under  $\varpi$ . We must show that  $\Phi(X)$  is closed. In fact, consider a sequence  $\{(h_k, \nu_k)\} \subset X$  such that  $\Phi(h_k, \nu_k) \rightarrow (a, v) \in \mathcal{NR}_{\mathbb{C}}$ . By Subsection 7.5, we may assume that  $h_k \rightarrow h$ . Then, as in the proof of Lemma 7.27, by passing to a subsequence we may assume that  $\{\nu_k\}$  converges to some  $\nu \in \mathcal{NR}_{\mathbb{L}}$ . It follows that  $(h, \nu) \in X$  and  $(a, v) \in \Phi(X)$ .  $\square$

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Manuscrit reçu le 4 octobre 2004,  
révisé le 11 juillet 2005,  
accepté le 19 septembre 2005.

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