



ANNALES

DE

L'INSTITUT FOURIER

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Tome 56, n° 4 (2006), p. 1165-1205.

http://aif.cedram.org/item?id=AIF_2006__56_4_1165_0

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A LIMIT LINEAR SERIES MODULI SCHEME

by Brian OSSERMAN (*)

ABSTRACT. — We develop a new, more functorial construction for the basic theory of limit linear series, which provides a compactification of the Eisenbud-Harris theory. In an appendix, in order to obtain the necessary dimensional lower bounds on our limit linear series scheme we develop a theory of “linked Grassmannians”; these are schemes parametrizing sub-bundles of a sequence of vector bundles, which map into one another under fixed maps of the ambient bundles.

RÉSUMÉ. — Nous produisons une construction fonctorielle nouvelle pour la théorie fondamentale des séries linéaires limites, qui nous donne une compactification de la théorie de Eisenbud-Harris. Dans un appendice, pour obtenir les bornes inférieures dimensionnelles nécessaires pour notre schéma de séries linéaires limites nous produisons une théorie de “grassmaniennes liées” ; ce sont des schémas paramétrisant sous-fibrés d’une suite de fibrés vectoriels, qui se transforment sous les transformations données des fibrés ambiants.

1. Introduction

The Eisenbud-Harris theory of limit linear series of [9] is a powerful tool for degeneration arguments on curves, with applications to the Kodaira dimension of moduli spaces of curves, and analysis of Weierstrass points on curves, as well as new arguments for results such as the Gieseker-Petri theorem. In this paper, we give a new construction for limit linear series, very much in the spirit of Eisenbud and Harris’ theory, but more functorial in nature, and involving a substantially new approach which appears better suited to generalization to higher-dimensional varieties and higher-rank vector bundles. The application of the theory of limit linear series in positive characteristic is fundamental to [18]; we should remark that we do not

Keywords: Limit linear series, compactification, linked Grassmannian.

Math. classification: 14H51.

(*) This paper was partially supported by a fellowship from the Japan Society for the Promotion of Science.

see any obstructions to the original construction of Eisenbud and Harris working in characteristic p , but the independence of characteristic is more transparent in the functorial setting. We begin with an overview of the basic ideas of limit linear series; for those unfamiliar with linear series, the actual definitions and notation are all recalled below.

While our main theorem (see Theorem 5.3) is too technical to state in an introduction, we can outline the main concepts involved. The basic idea of limit linear series is to analyze how linear series behave as a family of smooth curves X/B degenerates to a nodal curve X_0 ; a key distinguishing feature of the theory is that rather than standard deformation-theoretic techniques to obtain results from the degeneration, a simple dimension count on the special fiber produces results immediately.

More specifically, recall that a proper, geometrically reduced and connected nodal curve with smooth components is said to be of *compact type* if the dual graph is a tree, or equivalently if the (connected component of the) Picard scheme is proper. Now, if X_0 is not of compact type, line bundles on the smooth curves may not limit to a line bundle on the nodal fiber, as the Picard scheme of the family (and specifically of the nodal fiber) will not be proper. On the other hand, if the nodal fiber is reducible, limiting line bundles will exist, but will not be unique, as one can always twist by one of the components of the reducible fiber to get a new line bundle, isomorphic away from the nodal fiber to the original one. However, this turns out to be the only ambiguity. To explain the approach to this issue, we consider for simplicity the case of \mathfrak{g}_d^r 's where the family X/B has smooth general fiber, and X_0 consists of two smooth components Y and Z , meeting at a single node P . Given a line bundle of degree d on X , we will say it has degree $(i, d - i)$ on X_0 if it restricts to a line bundle of degree i on Y and degree $d - i$ on Z . Eisenbud and Harris approached the problem by considering the linear series obtained by looking at the two possible limit line bundles obtained by requiring degrees $(d, 0)$ and $(0, d)$ on X_0 . Since the degree 0 components cannot contribute anything to the space of global sections chosen for the \mathfrak{g}_d^r , this is equivalent to specifying a \mathfrak{g}_d^r on each of Y and Z ; they showed that if such a pair arises as a limit of \mathfrak{g}_d^r 's from the smooth fibers, it will satisfy the ramification condition

$$(1.1) \quad a_i^Y(P) + a_{r-i}^Z(P) \geq d,$$

for all i , where $\{a_i^Y(P)\}_i$ and $\{a_i^Z(P)\}_i$ are the vanishing sequences on Y and Z at the node. Eisenbud and Harris refer to such pairs on a nodal fiber as *crude limit series*, and when the inequality is replaced by an equality, as *refined limit series*.

Eisenbud and Harris' moduli scheme construction requires restriction to refined limit series, and as such is not generally proper, and is also necessarily disconnected, being constructed as a disjoint union over the different possible ramification indices at the nodes. Moreover, the necessity to specify ramification indices makes it unsuitable for generalizing from curves to higher-dimensional varieties. The basic idea of our construction is to remember not just the line bundles of degree $(d, 0)$ and $(0, d)$ on X_0 , but also the $d - 1$ line bundles of degree $(i, d - i)$ that lie in between. One can then replace the ramification condition with a simpler compatibility condition on the corresponding spaces of global sections, yielding a very functorial approach to constructing the moduli scheme. Further, one can show a high degree of compatibility with Eisenbud and Harris' construction: in particular, for a curve (of compact type) over a field, our construction contains the Eisenbud-Harris version as an open subscheme.

We begin in Section 2 with a review of the basics of linear series, but in arbitrary characteristic. In Section 3 we give the precise conditions on the families of curves we will consider, and show that such families may be constructed as necessary. In Section 4 we define the limit linear series functors we will consider, and our main theorem, the representability of these functors, is proved in Section 5; we conclude with corollaries as in Eisenbud and Harris on smoothing linear series from the special fiber when the dimension is as expected, including in the cases of positive and mixed characteristic. We compare our theory to that of Eisenbud and Harris in Section 6, and conclude with some further questions in Section 7. Finally, in Appendix A we develop a theory of linked Grassmannian schemes, which parametrize collections of sub-bundles of a sequence of vector bundles linked together by maps between the bundles; this is used in the construction of the limit linear series scheme in the main theorem, and in particular to obtain the necessary lower bound on its dimension.

The work here is of course entirely inspired by Eisenbud and Harris' original construction in [9]. Attempts to generalize this theory have thus far been sparse, but include, for instance, work of Esteves [10] and of Teixidor i Bigas [1] to generalize to certain curves not of compact type and higher-rank vector bundles respectively.

The contents of this paper form a portion of the author's 2004 PhD thesis at MIT, under the direction of Johan de Jong.

Acknowledgements. — I would like to thank Johan de Jong for his tireless and invaluable guidance. I would also like to thank Jason Starr, Joe Harris, Max Lieblich, and Steve Kleiman for their many helpful discussions.

2. Linear series in arbitrary characteristic

Before getting into the technical definitions related to the central construction, we begin with a few preliminary definitions and lemmas in the case of a smooth, proper, geometrically integral curve C of genus g , over a field k of any characteristic.

First, recall:

DEFINITION 2.1. — *If \mathcal{L} is a line bundle of degree d on C , and V an $(r + 1)$ -dimensional subspace of $H^0(C, \mathcal{L})$, we call the pair (\mathcal{L}, V) a linear series of degree d and dimension r on C , or a \mathfrak{g}_d^r . Given (\mathcal{L}, V) a \mathfrak{g}_d^r on C , and a point P of C , there is a unique sequence of $r + 1$ increasing integers $a_i^{(\mathcal{L}, V)}(P)$ called the vanishing sequence of (\mathcal{L}, V) at P , given by the orders of vanishing at P of sections in V . We also define $\alpha_i^{(\mathcal{L}, V)}(P) := a_i^{(\mathcal{L}, V)}(P) - i$, the ramification sequence of (\mathcal{L}, V) at P . (\mathcal{L}, V) is said to be unramified at P if all $\alpha_i^{(\mathcal{L}, V)}(P)$ are zero; otherwise, it is ramified at P .*

Warning 2.2. — Since the ramification and vanishing sequences are equivalent data, we tend to refer to conditions stated in terms of either one simply as “ramification conditions.” We will also drop the (\mathcal{L}, V) superscript or replace it as appropriate, particularly when we have a linear series on each component of a reducible curve, when we will tend to simply use the component to indicate which series we are referring to.

The following definitions, being tailored to characteristic p , may be less standard:

DEFINITION 2.3. — *We say a linear series (\mathcal{L}, V) on C is separable if it is not everywhere ramified. Otherwise, it is inseparable. At a point P , we say that (\mathcal{L}, V) is tamely ramified if the characteristic is 0 or if the vanishing orders $a_i(P)$ are maximally distributed mod p (in particular, this holds at any unramified point). Otherwise, we say that (\mathcal{L}, V) is wildly ramified at P .*

The following result is a characteristic- p version of a standard Plücker formula, whose proof simply adapts standard techniques:

PROPOSITION 2.4. — *Let C be a smooth, proper, geometrically integral curve of genus g over a field k , and (\mathcal{L}, V) a \mathfrak{g}_d^r on C . Then either (\mathcal{L}, V) is inseparable, or we have the inequality*

$$\sum_{P \in C} \sum_i \alpha_i(P) \leq (r + 1)d + \binom{r + 1}{2}(2g - 2).$$

Furthermore, this will be an equality if and only if (\mathcal{L}, V) is everywhere tamely ramified; in particular, in this case inseparability is impossible.

Proof. — We simply use the argument of [8, Prop. 1.1]. Even though it is intended for characteristic 0, the proof follows through equally well in characteristic p for our modified statement, noting that their “Taylor expansion” map is defined independent of characteristic, and their formulas then hold on a formal level. Indeed, their argument shows that if (\mathcal{L}, V) induces a non-zero section $s(\mathcal{L}, V)$ of $\mathcal{L}^{\otimes r+1} \otimes (\Omega_C^1)^{\otimes \binom{r+1}{2}}$, we get the desired inequality, with equality if and only if the determinant of their Lemma 1.2 is non-zero at all P (where, as in the proof of the proposition, $X_j := \alpha_j^{(\mathcal{L}, V)}(P)$). In fact, if this determinant is non-zero anywhere, we see also that $s(\mathcal{L}, V)$ has finite order of vanishing at that point, and cannot be the zero section. Next, their same lemma shows that their determinant will be non-zero at a point P if and only if (\mathcal{L}, V) is tamely ramified at P . This means that if we show that inseparability corresponds precisely to having $s(\mathcal{L}, V) = 0$, we are done. But this also follows trivially, since on the one hand any unramified point is in particular tamely ramified, and will in fact give a non-vanishing point of $s(\mathcal{L}, V)$, and on the other hand, if $s(\mathcal{L}, V)$ is non-zero, we have seen that we can get only finitely many ramification points. □

Note that because vanishing sequences are bounded by d , if $d < p$, then wild ramification is not possible, so the previous proposition immediately implies:

COROLLARY 2.5. — *Wildly ramified or inseparable linear series of degree d are only possible when $d \geq p$.*

Finally, we have the notation:

DEFINITION 2.6. — *Given n points P_i and n ramification sequences $\alpha^i = \{\alpha_j^i\}_j$, we write $\rho := \rho(g, r, d; \alpha^i) := (r + 1)(d - r) - rg - \sum_{i,j} \alpha_j^i$. This is the expected dimension of linear series of degree d and dimension r on a curve of genus g , with at least the specified ramification at the P_i .*

3. Smoothing families

In this section we describe the families of curves whose limit linear series we will study, called “smoothing families”, and then give some basic existence results. While the definition of a smoothing family is rather technical,

we expect that most applications will involve smoothing a given reducible curve over a one-dimensional base, so we conclude with a theorem giving the existence of such families satisfying all our technical conditions, given the desired reducible fiber. However, we work over a fairly arbitrary base, because this allows the use of arguments in the universal setting, in negative expected dimension, and in certain pathological cases where expected dimension is satisfied, but only over a positive-dimensional “special fiber”.

Our central technical definition is:

DEFINITION 3.1. — *A morphism of schemes $\pi : X \rightarrow B$, together with sections $P_1, \dots, P_n : B \rightarrow X$ constitutes a smoothing family if:*

- (I) *B is regular and connected;*
- (II) *π is flat and proper;*
- (III) *The fibers of π are genus- g curves of compact type;*
- (IV) *The images of the P_i are disjoint and contained in the smooth locus of π ;*
- (V) *Each connected component Δ' of the singular locus of π maps isomorphically onto its scheme-theoretic image Δ in B , and furthermore $X|_{\pi^{-1}\Delta}$ breaks into two (not necessarily irreducible) components intersecting along Δ' ;*
- (VI) *Any point in the singular locus of π which is smoothed in the generic fiber is regular in the total space of X ;*
- (VII) *There exist sections D_i contained in the smooth locus of π such that every irreducible component of any geometric fiber of π meets at least one of the D_i .*

We begin with a lemma on two methods of obtaining new smoothing families from a given one:

LEMMA 3.2. — *Let $X/B, P_i$ be a smoothing family. Then*

- (i) *If $B' \rightarrow B$ is either a k -valued point of B for any field k , a localization of B , or a smooth morphism with B' connected, then base change to B' gives a new smoothing family.*
- (ii) *If Δ' is a node of X/B which is not smoothed in the generic fiber, let Y, Z be the components of X with $Y \cup Z = X, Y \cap Z = \Delta'$. Then restriction to Y or Z gives a new smoothing family.*

Proof. — For (i), the only properties of a smoothing family not preserved under arbitrary base change are (I) and (VI), which are easily checked in our specific cases.

For (ii), the only condition that isn't immediately clear is that flatness is preserved. However, this follows from the exact sequence of sheaves on X

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \oplus \mathcal{O}_Z \rightarrow \mathcal{O}_{Y \cap Z} \rightarrow 0$$

together with the hypothesized flatness of \mathcal{O}_X and $\mathcal{O}_{Y \cap Z}$ over \mathcal{O}_B . □

We now proceed to develop some results on construction of smoothing families.

LEMMA 3.3. — *Let $\pi : X \rightarrow B$ be a family satisfying conditions (I)-(III) and (VI) of a smoothing family, \bar{X}_0 a chosen geometric fiber of π mapping to a point $P \in B$, and \bar{P}_i smooth closed points on \bar{X}_0 with images in X_P having residue fields separable extensions of $\kappa(P)$. Suppose further that each component Δ' of the singular locus of X/B is flat over its image Δ in B . Then there is an étale base change of π and sections P_i specializing to the \bar{P}_i which yield a smoothing family still containing \bar{X}_0 as a geometric fiber, and with the same geometric generic fiber as π .*

Proof. — We can Zariski localize B to avoid any components of the singular locus not occurring in X_0 , and to insure that the sections we will construct are disjoint and in the smooth locus. First, in addition to the \bar{P}_i , choose one smooth closed point \bar{D}_i on each component of X_0 , each having field of definition a separable extension of $\kappa(P)$ (this is possible by [2, Prop. 2.2.16 and Cor. 2.2.13]). Next, by [2, Prop. 2.2.14], after possible étale base change we can find the desired sections P_i and D_i of π , each going through the corresponding \bar{P}_i or \bar{D}_i . All that remains is to show that we can obtain condition (V) as well. Since the singular locus of a family of nodal curves is finite and unramified over the base, our flatness hypothesis implies that each connected component Δ' is étale over its image Δ , and using [2, Prop. 2.3.8 b)] together with [19, Cor. V.1, p. 52] in the case that $\Delta \neq B$, after an étale base change Δ' will map isomorphically to Δ , giving the first half of (V).

Finally, we need to make sure that X breaks into components around each node. For each connected component Δ' of the singular locus of π , it suffices to produce an étale base change which causes the generic fiber X_1^Δ of $X|_\Delta$ to break. By hypothesis, X_1^Δ breaks geometrically, and it will break into components over an intermediate field K if and only if the geometric components are $\text{Gal}(\bar{K}/K)$ -invariant. But this may be accomplished after étale base change by [17, Lem. 4.2], again using [19, Cor. V.1, p. 52] in the case $\Delta \neq B$. □

For typical applications of limit linear series, we expect that the following theorem, which follows fairly easily from a theorem of Winters, will render irrelevant the technical hypotheses of our smoothing families:

THEOREM 3.4. — *Let X_0 be any curve of compact type over an algebraically closed field k , and $\overline{P}_1, \dots, \overline{P}_n$ distinct smooth closed points. Then X_0 may be placed into a smoothing family X/B with sections P_i specializing to the \overline{P}_i , where B is a curve over k , and where the generic fiber of X over B is smooth.*

Proof. — Setting all $m_i = 1$, we can apply [20, Prop. 4.2] to obtain a proper map over $\text{Spec } k$ from some regular surface \tilde{X} to some regular curve \tilde{B} , having X_0 as a fiber. This must automatically be flat, and if we localize \tilde{B} we can assume all fibers are at most nodal. We then claim that the generic fiber X_1 must be smooth: indeed, all the local rings are regular by hypothesis, so by [7, Cor. 16.21] the residue fields of any non-smooth points would have to be inseparable over $K(B)$, which cannot happen in the case of nodes, since they are always unramified. Since the base field is algebraically closed, we need not worry about separability of residue field extensions on the closed fiber. Finally, our nodes are all isolated points, so the map to their image is a finite, unramified map of local schemes with algebraically closed residue field, and hence an isomorphism. Therefore, we can apply the preceding lemma to obtain our desired smoothing family. \square

Remark 3.5. — There are a number of differences between our definition of a smoothing family, and the one used in Eisenbud and Harris' original construction in [9]. None of these are due to the different construction. Extra conditions such as the reducedness of B and the regularity of X at smoothed nodes are in fact necessary to ensure that certain closed subschemes are actually Cartier divisors, and the condition that X break into distinct components above the nodes is likewise tacitly assumed, but not automatic. The regularity of B is necessary to make the sort of dimension-count arguments employed in the construction. Conversely, the hypotheses on the characteristic (or even existence) of a base field appears to be unnecessary in their construction, as does the hypothesis that the relatively ample divisor be disjoint from the ramification sections. The only hypothesis we include here that may be truly gratuitous is that the relatively ample divisor be composed of global sections, but it is convenient and, as we have shown, not difficult to achieve.

Remark 3.6. — We do not claim that the moduli scheme could not be constructed under weaker hypotheses, but merely that our hypotheses are

those which are necessary for our particular argument. It seems quite likely that one could drop many of the hypotheses on both X and B if one carried out the construction in a universal setting and then pulled back the result to arbitrary families.

Remark 3.7. — It is not true that condition (VI) of a smoothing family is preserved under base change by arbitrary closed immersions $B' \rightarrow B$, even when B' is regular and connected. For example, consider any smoothing family with $B = \mathbb{A}_k^2$, and having a node Δ' with Δ given by the x -axis. Then if B' is the parabola $y = x^2$, base change to B' will create a singularity in X above the origin.

Remark 3.8. — In fact, the hypotheses for a smoothing family π imply that every connected component of the singular locus of π is regular, and in particular irreducible and reduced. However, we will not need this, so we do not pursue it.

4. The relative \mathcal{G}_d^r functor

Given, in addition to a smoothing family, integers r, d , and ramification sequences $\alpha^i := \{\alpha_j^i\}_j$ for each of our P_i , we will associate a \mathcal{G}_d^r functor to our smoothing family; this functor will initially appear to include a lot of extraneous data, but we will show that it actually gives the “right” functor, at least in the sense that it associates a reasonable set to any geometric point of B .

However, before defining the functor, we give some preliminary lemmas and definitions. In order to ensure that our functor is globally well-defined, we will need the following easily-verified lemma.

LEMMA 4.1. — *Let $\pi : X \rightarrow B$ be a proper morphism with geometrically reduced and connected fibers, \mathcal{L} and \mathcal{L}' two isomorphic line bundles on X , and V and V' sub-modules of $\pi_*\mathcal{L}$ and $\pi_*\mathcal{L}'$ respectively. Then the property that “ V maps into V' ” is independent of the choice of isomorphism between \mathcal{L} and \mathcal{L}' .*

We also describe a generalized notion of sub-bundle:

DEFINITION 4.2. — *Let $\pi : X \rightarrow B$ be a morphism of schemes, and \mathcal{L} a line bundle on X . A sub-sheaf V is defined to be a sub-bundle of $\pi_*\mathcal{L}$ if in addition to V being a locally free sheaf, for any $S \rightarrow B$, the map $V_S \rightarrow \pi_{S*}\mathcal{L}_S$ remains injective.*

Note that in this definition, we are pushing forward the pullback of \mathcal{L} , and not the other way around. The required sheaf map is gotten by composing the induced map $V_S \rightarrow (\pi_*\mathcal{L})_S$ with the natural map $(\pi_*\mathcal{L})_S \rightarrow \pi_{S*}\mathcal{L}_S$.

Finally, we define ramification conditions in this context.

DEFINITION 4.3. — *Let X/B be a proper relative curve with line bundle \mathcal{L} of degree d . Let V be a sub-bundle of $\pi_*\mathcal{L}$ on X/B of rank r , and P a smooth section of π . Consider the sequence of maps*

$$V \rightarrow \pi_*\mathcal{L}|_{(d+1)P} \rightarrow \pi_*\mathcal{L}|_{dP} \rightarrow \dots, \pi_*\mathcal{L}|_P \rightarrow 0$$

We denote by β_m the composition map $V \rightarrow \pi_*\mathcal{L}|_{mP}$. Given a sequence of r increasing integers a_j between 0 and d , we say that V has vanishing sequence at least $\{a_j\}_j$, or ramification sequence at least $\{a_j - j\}_j$, if $\text{rk } \beta_m \leq j$ for all $m \leq a_j^i$.

Finally, to simplify notation, and because it will be enough for inductive degenerations, we will restrict our families to reducible curves with only two components.

SITUATION 4.4. — *We assume that X/B is a smoothing family with at most one node (in the sense that the singular locus of π is irreducible). If there is a node, we introduce some notation: denote by Δ' the singular locus of π , and Δ its image in B ; by hypothesis, π maps Δ' isomorphically to Δ . We now distinguish three cases: case (1) is that there is no node; case (2) is that Δ is all of B ; and case (3) is that Δ is a Cartier divisor on B . In cases (2) and (3), we denote by Y and Z the components of $X|_{\pi^{-1}\Delta}$, necessarily smooth and intersecting along Δ' .*

We observe that with the specified hypotheses, these three cases are all the possibilities: indeed, since B is regular, completing and examining the universal deformation of a nodal curve described in [6, p. 82] easily shows that if Δ is non-empty, it is locally generated principally. Note that with no hypotheses on the base, this would not be true; in fact, one can construct a families of nodal curves over a quadric cone having a node whose image is the union of three lines through the cone point.

In case (2), we will make use of the natural morphism (actually an isomorphism onto a connected component) $\text{Pic}^{d-i}(Y_T/T) \times \text{Pic}^i(Z_T/T) \rightarrow \text{Pic}^d(X_T/T)$ for any i and any T over B , in order to think of a pair of line bundles $\mathcal{L}_Y, \mathcal{L}_Z$ on Y_T and Z_T as a line bundle on X_T , which we will denote $(\mathcal{L}_Y, \mathcal{L}_Z)$. Note that this is only defined up to agreement locally on

the base, but this will not be a problem as we will consider the sheafified Picard functor.

In case (3), by the nonsingularity hypothesis, Y and Z are Cartier divisors in X , so we have associated line bundles on X , $\mathcal{O}_X(Y)$ and $\mathcal{O}_X(Z)$. Moreover, because Δ is a Cartier divisor on B , and $\mathcal{O}_X(Y + Z) \cong \mathcal{O}_X(\pi^*\Delta) \cong \pi^*\mathcal{O}_B(\Delta)$, we have that locally on B , $\mathcal{O}_X(Y + Z) \cong \mathcal{O}_X$.

Given a morphism $f : T \rightarrow B$, we denote by a subscript T the various pullbacks under f . We now describe our functor.

DEFINITION 4.5. — *The functor $\mathcal{G}_d^r(X/B, \{(P_i, \alpha^i)\}_i)$ associates to T/B the set of objects described as follows, modulo the equivalence induced by tensoring by pullbacks of line bundles on T .*

- Case (1):** a line bundle \mathcal{L} of degree d on X_T , together with a rank r sub-bundle V of $\pi_{T*}\mathcal{L}$, having ramification sequence at least α^i along the $P_{i,T}$.
- Case (2):** a line bundle \mathcal{L} of degree d on X_T , which has degree d when restricted to Y_T , and degree 0 on Z_T , together with rank r sub-bundles V_0, \dots, V_d of $\pi_{T*}\mathcal{L}^i$, where $\mathcal{L}^i := (\mathcal{L}|_{Y_T}(-i\Delta'_T), \mathcal{L}|_{Z_T}(i\Delta'_T))$. Each V_i must map to V_{i+1} under the natural map given by inclusion on Z_T and 0 on Y_T , and each V_i must map to V_{i-1} under inclusion on Y_T and 0 on Z_T . Finally, we impose ramification along the $P_{i,T}$ as in case (1), with the caveat that we impose it only on V_0 if P_i is on Y , and only on V_d if P_i is on Z .
- Case (3):** a line bundle \mathcal{L} of degree d on X_T , which has degree d when restricted to Y_T , and degree 0 on Z_T , together with rank r sub-bundles V_0, \dots, V_d of $\pi_{T*}(\mathcal{L}^i)$, where $\mathcal{L}^i := (\mathcal{L} \otimes \mathcal{O}_X(Y)^{\otimes i})$. Each V_i must map to V_{i+1} under the natural map $\pi_{T*}(\mathcal{L}^i) \rightarrow \pi_{T*}(\mathcal{L}^{i+1})$ obtained by pushing forward the inclusion induced by tensoring with the effective divisor Y . Further, locally on T , we have $\mathcal{O}_X(Y + Z)_T \cong \mathcal{O}_{X,T}$, and we require that V_i map to V_{i-1} under the map induced by this isomorphism and tensoring with Z . Finally, we impose the desired ramification along the $P_{i,T}$ as in the first two cases, imposing it only on V_0 if P_i specializes to Y , and only on V_d if P_i specializes to Z .

Remark 4.6. — By [2, Prop. 8.1.4], under the hypotheses of our smoothing families the standard sheafified Picard functor simply reduces to $T \mapsto \mathcal{P}ic(X_T)/\mathcal{P}ic(T)$ used above, and in particular when the sheafified functor is representable, there is a universal line bundle.

Remark 4.7. — By Lemma 4.1, our functor is well-defined despite the fact that we cannot distinguish line bundles on X which are isomorphic locally on T . By the same token, the compatibility condition on V_i in case (3) is independent of choice of local isomorphisms $\mathcal{O}_X(Y + Z) \cong \mathcal{O}_X$, and it isn't hard to see that the definition of \mathcal{G}_d^r in cases (2) and (3) is independent of the choice of Y and Z .

We also have:

DEFINITION 4.8. — $\mathcal{G}_d^{r,\text{sep}}$ is the subfunctor of \mathcal{G}_d^r consisting of those linear series which are separable in every fiber. This is self-explanatory for smooth curves, while for reducible curves we require both $V_0|_Y$ and $V_d|_Z$ to be separable.

One can verify quite directly that the \mathcal{G}_d^r we have defined is in fact a functor. However, since we defined it differently in three separate cases, we also need to check:

LEMMA 4.9. — \mathcal{G}_d^r and $\mathcal{G}_d^{r,\text{sep}}$ are compatible with base change.

Proof. — Suppose we pull back from X/B to a new smoothing family X'/B' . The only case with anything to check is if case (3) pulls back to case (1) or (2). For the former, we note that $\mathcal{O}_X(Y)$ and $\mathcal{O}_X(Z)$ pull back to the trivial bundle, so all the maps between the \mathcal{L}^i are isomorphisms. For the latter, the point is that $\mathcal{O}_X(Y)$ clearly pulls back to $\mathcal{O}_{Z'}(\Delta')$ on Z' , from which one checks that locally on the base, $\mathcal{O}_X(Z)$ pulls back to $\mathcal{O}_{Z'}(-\Delta')$, and similarly with Y and Z switched.

Lastly, because $\mathcal{G}_d^{r,\text{sep}}$ was defined as a sub-functor of \mathcal{G}_d^r in terms of behavior on fibers, it immediately follows that it too is compatible with base change. \square

5. Representability

The main theorem is the representability of our \mathcal{G}_d^r functors, together with a lower bound on its dimension.

However, to ease the pain of the proof, we begin with some technical lemmas before proceeding to the statement and proof of the main theorem.

We begin with some compatibility checks on our notion of sub-bundle:

LEMMA 5.1. — *Our notion of sub-bundle has the following desirable properties:*

- (i) Suppose we have \mathcal{L} such that $\pi_*\mathcal{L}$ is locally free, and the higher derived pushforward functors vanish. Then our definition of sub-bundle of $\pi_*\mathcal{L}$ is equivalent to the usual one (that is, a locally free sub-sheaf with locally free quotient).
- (ii) If D is an effective Cartier divisor on X , flat over B , and \mathcal{L} any line bundle on X , then a sub-sheaf V of $\pi_*\mathcal{L}$ is a sub-bundle of $\pi_*\mathcal{L}$ if and only if it is a sub-bundle of $\pi_*\mathcal{L}(D)$ under the natural inclusion.
- (iii) Let V_1, V_2 be sub-bundles of rank r of $\pi_*\mathcal{L}$ in our sense, and suppose $V_1 \subset V_2$. Then $V_1 = V_2$.

Proof. — For (i), we first note that by [13, Cor. 6.9.9] (see also [13, 6.2.1]), the natural map $(\pi_*\mathcal{L})_S \rightarrow \pi_*\mathcal{L}_S$ is an isomorphism. Now, if the quotient $Q := \pi_*\mathcal{L}/V$ is locally free, we have $V_S \hookrightarrow (\pi_*\mathcal{L})_S$ for any S over B , and therefore $V_S \hookrightarrow \pi_*\mathcal{L}_S$, as desired.

Conversely, $V_S \hookrightarrow \pi_*\mathcal{L}_S$ for all S means that the injectivity of $V \rightarrow \pi_*\mathcal{L}$ is preserved under base change; this in turn implies that $\text{Tor}_S^1(\mathcal{O}_S, Q_S) = 0$ for all S , since $\pi_*\mathcal{L}$ has vanishing Tor. By [7, Prop. 6.1], we conclude that Q is flat, hence locally free, completing the proof of (i).

Assertion (ii) will follow immediately if we show that $\pi_*\mathcal{L}_S \rightarrow \pi_*\mathcal{L}(D)_S$ is injective for all S . However, by the flatness of D over B , the cokernel of $\mathcal{L} \hookrightarrow \mathcal{L}(D)$ is flat over B , so injectivity of this map is preserved under base change, and applying π_* gives the desired result.

Finally, (iii) is straightforward: let $Q = V_2/V_1$, and let $b \in B$ be any point of B . If we base change to $\text{Spec } \kappa(b)$, we get find from the definition of sub-bundle that $V_{1b} \hookrightarrow V_{2b}$, so since their dimension is the same, we get $Q_b = 0$, and by Nakayama’s lemma we conclude $Q = 0$ and $V_1 = V_2$, as asserted. □

We also have a lemma illustrating how we will use our sections D_i :

LEMMA 5.2. — *Let X/B be a smoothing family, and \mathcal{L}^i any finite collection of line bundles on X , of degree d . Then there exists an effective divisor D on X satisfying:*

- (i) D is flat over B , and supported in the smooth locus of π .
- (ii) $\pi_*\mathcal{L}^i(D)$ is locally free and $R^1\pi_*\mathcal{L}^i(D) = 0$ for all i .
- (iii) $\pi_*\mathcal{L}^i(D) \rightarrow \pi_*\mathcal{L}^i(D)|_{mP_j}$ is surjective for all i, j and all $m \leq d+1$.
- (iv) In case (3) of Situation 4.4, D may be written as $D^Y + D^Z$, with $D^Y|_{\pi^{-1}\Delta}$ contained in Y , and similarly for D^Z .

Furthermore, étale locally on B we may require that D is disjoint from the P_i as well.

Proof. — With D_i any collection of sections as in the definition of a smoothing family, let $D' = \sum_i D_i$; then D' is π -ample, so locally on B , for ℓ sufficiently large, $\ell D'$ will have the desired properties. Since B is Noetherian, we can choose such an ℓ globally.

The étale-local disjointness assertion is obtained by constructing new sections étale locally as in the proof of Lemma 3.3, choosing the \overline{D}_i to be distinct from the \overline{P}_i . □

We can now prove our central result:

THEOREM 5.3. — *If $\pi : X \rightarrow B$, $P_1, \dots, P_n : B \rightarrow X$ is a smoothing family satisfying the two-component hypothesis of Situation 4.4, and $\alpha^i := \{\alpha_j^i\}_j$ ramification sequences, then $\mathcal{G}_d^r = \mathcal{G}_d^r(X/B; \{(P_i, \alpha^i)\}_i)$ is represented by a scheme G_d^r , compatible with base change to any other smoothing family. This scheme is projective, and if it is non-empty, the local ring at any point $x \in G_d^r$ closed in its fiber over $b \in B$ has dimension at least $\dim \mathcal{O}_{B,b} + \rho$, where $\rho = \rho(g, r, d; \alpha^i)$ as in Definition 2.6. Furthermore, $\mathcal{G}_d^{r, \text{sep}}$ is also representable, and is naturally an open subscheme of G_d^r .*

Proof. — Once the \mathcal{G}_d^r functor has been defined, the proof of its representability is long but for the most part extremely straightforward, using nothing more than the well-known representability of the various functors in terms of which we have described \mathcal{G}_d^r . The one trick, borrowed from Eisenbud and Harris, is to twist a universal line bundle \mathcal{L} by a high power of an ample divisor so sub-bundles of its pushforward are parametrized by a standard Grassmannian scheme. The dimension count is an altogether different story; it is harder than in Eisenbud and Harris' construction, and is essentially the subject of Appendix A.

We note that our functor is visibly a Zariski sheaf, so we can check representability Zariski locally on B . Furthermore, it is clear that imposing ramification conditions, consisting of imposing rank conditions on sequences of maps of locally free sheaves, give closed subfunctors, so it suffices to check representability without imposing ramification.

As in defining the functor, we have three cases to consider. The first is the simplest. We start in this case with the relative Picard scheme $P = \text{Pic}^d(X/B)$, obtained for instance from $\text{Pic}^0(X/B)$ (see [2, Thm. 9.4.1]) by twisting by sections to obtain the desired degree. We denote by $\widetilde{\mathcal{L}}$ the universal line bundle on $X \times_B P$. Let D be the divisor provided by Lemma 5.2 for $\widetilde{\mathcal{L}}$, viewing $X \times_B P$ as a smoothing family over P . Now, we let G be the relative Grassmannian scheme of $\pi_{P*}(\widetilde{\mathcal{L}}(D))$. We define our G_d^r scheme to be the closed subscheme of G cut out by the condition that any sub-bundle V of $\pi_{P*}(\widetilde{\mathcal{L}}(D))$ vanishes on D , which naturally gives a closed

subscheme cut out locally by minors. This completes the construction in the first case.

Now, in the second case, we use the Picard schemes $P^i := \text{Pic}^{d-i,i}(X/B)$, the schemes parametrizing line bundles on X with degrees $d - i$ and i when restricted to Y and Z respectively. These may be constructed from $\text{Pic}^0(X/B)$ as before, except twisting by sections on Y and Z separately to obtain the desired bi-degrees. The P^i are all naturally isomorphic to one another by twisting the line bundles on each component by Δ' and $-\Delta'$; in particular, we can identify all of them with a fixed P over B .

On each P^i , we have a universal line bundle \mathcal{L}^i , and just as in the first case, we take a very ample divisor D obtained from Lemma 5.2 for the $\widetilde{\mathcal{L}}^i$, twist \mathcal{L}^i by D , and then construct Grassmannian bundles G^i , this time one for each \mathcal{L}^i . Denoting by G the product of all these Grassmannians over P , we take the closed subscheme inside G cut out by, as in the first case, vanishing on D . Here, we actually write $D = D^Y + D^Z$, where D^Y and D^Z are supported on Y and Z respectively, and impose vanishing along D^Y only in G^0 , and along D^Z only in G^d . Finally, we make use of the construction Lemma A.3 to add the requirement that the V_i each map into V_{i+1} on Z and V_{i-1} on Y under the natural maps, also as in the definition of the functor. This completes the construction in the second case.

In the final case, the first step is to work sufficiently locally on B that Δ is principal, so that $\mathcal{O}_B(\Delta) \cong \mathcal{O}_B$ and $\mathcal{O}_X(Y + Z) \cong \mathcal{O}_X$, and fix a choice of this isomorphism. The rest proceeds very similarly to the second case: our Picard schemes P^i are described identically, and constructed from $\text{Pic}^0(X/B)$ by twisting separately by sections specializing to Y and Z . To describe isomorphisms between the P^i , we now tensor as necessary by $\mathcal{O}_X(Y)$. Replacing the maps between the $\widetilde{\mathcal{L}}^i$ with the appropriate maps for this case, the rest of the construction then proceeds identically to the previous case.

Because in each case the construction used only Picard schemes, Grassmannians, fiber products, and closed subschemes obtained by bounding the rank of maps between vector bundles, it nearly follows from the standard representability theorems for these functors that the G_d^r scheme we have constructed represents the \mathcal{G}_d^r functor. We do need to note that in the second and third cases, our conditions for vanishing along D actually imply that all V_i vanish along D : in the second case, this follows simply because D^Y and D^Z are disjoint from Δ' ; in the third case, we have similarly that D^Y is disjoint from Z and D^Z disjoint from Y . By Lemma 5.1, our definition of sub-bundle is compatible with the usual definition for the

Grassmannian functor. We have thus proven representability. It easily follows from the projectivity of Grassmannians and Picard schemes that our G_d^r scheme is projective over B . Lastly, compatibility with base change has already been proven in Lemma 4.9.

We now verify that the moduli scheme we have constructed has the desired lower bound on its dimension. Our ambient scheme G is a product of Grassmannians over a Picard scheme, so since B was assumed to be regular, we conclude that G is regular. Hence, in order to bound the codimension of G_d^r in G it suffices to consider the codimensions of each condition cutting it out. We denote by d' the rank of $\pi_{P_*}\widetilde{\mathcal{L}}^i(D)$; we have $d' = d + \deg D + 1 - g$.

Now, in the first case, vanishing along D imposes $(\operatorname{rk} V)(\operatorname{rk} \pi_{P_*}\widetilde{\mathcal{L}}|_D) = (r+1)(\deg D)$ conditions. Next, we consider the codimension of the ramification conditions. For this calculation, it suffices to work étale locally, so by Lemma 5.2 we may assume that D is disjoint from the P_i , in which case it follows that ramification conditions imposed on sub-bundles of $\mathcal{L}(D)$ are equivalent to the desired ramification for sub-bundles of \mathcal{L} .

Since the evaluation maps $\pi_{P_*}\widetilde{\mathcal{L}}(D) \rightarrow \pi_{P_*}(\widetilde{\mathcal{L}}(D)|_{jP_i})$ are surjective, each condition defines a Schubert cycle. In particular, by [4, Thm. 6.3, Cor. 5.12 (b)] the imposition of ramification at P_i gives an integral subscheme of codimension $\sum_j (a_j^i - j) = \sum_j (\alpha_j^i)$ inside G . Thus the total codimension of any component of G_d^r inside G is at most $(r+1)(\deg D) + \sum_{i,j} (\alpha_j^i)$.

In the second and third cases, the only real difference is that we replace the Grassmannian with the linked Grassmannian of Appendix A; it is easily verified that because the maps on $\pi_{P_*}\mathcal{L}^i$ are induced from maps on the \mathcal{L}^i , they satisfy the conditions of a linked Grassmannian (Definition A.4). Note that in the case of a reducible fiber, everything in the kernel of f_i really is in the image of g_i and vice versa, because the \mathcal{L}^i have all been constructed to be sufficiently ample. Then it follows from Theorem A.15 that every component of the linked Grassmannian has codimension $d(r+1)(d' - r - 1)$, and the rest of the calculation proceeds the same way, with the minor exception that we have to compute vanishing on D^Y and D^Z separately and use $\deg D^Y + \deg D^Z = \deg D$.

Finally, we conclude the desired dimension statement, noting that G is catenary, and smooth over B of relative dimension $(r+1)(d' - r - 1) + g$ in the first case and $(d+1)(r+1)(d' - r - 1) + g$ in the second and third cases.

Lastly, we need to show that the sub-functor of separable limit series is representable by an open subscheme. Denote by $\widetilde{\mathcal{F}}^i$ the universal sub-bundles of $\pi_{P_*}\widetilde{\mathcal{L}}$ on our G_d^r scheme. As in the proof of Proposition 2.4,

we can construct a map $\widetilde{\mathcal{F}}^i \otimes \mathcal{O}_{X \times_B G_d^r} \rightarrow \mathcal{P}^r(\widetilde{\mathcal{L}})$ where \mathcal{P}^r denotes the bundle of principal parts of order r ; taking $(r + 1)$ st exterior powers gives a map $s^{\text{univ}} : \det(\widetilde{\mathcal{F}}^i) \rightarrow \widetilde{\mathcal{L}}^{\otimes r+1} \otimes (\Omega_{X/B}^1)^{\otimes \binom{r+1}{2}}$; we already noted that in the smooth case, our separable subscheme is the image under π_P in G_d^r of the complement of the closed subscheme cut out as the kernel of s^{univ} . For the second and third cases, we restrict to the smooth locus to avoid the problem that $\Omega_{X/B}^1$ is no longer locally free. It is then easy to check that the same construction applied to \mathcal{F}^0 and \mathcal{F}^d will give the subscheme of linear series which are separable on Y and Z respectively, and their intersection gives the desired $G_d^{r,\text{sep}}$ subscheme. \square

Our first application is the same regeneration/smoothing theorem due to Eisenbud-Harris, except that now it *a priori* gives results on smoothings of crude limit series as well, and we are also able to include upper bounds of dimensions of general fibers in certain cases.

We have:

COROLLARY 5.4. — *In the situation of Theorem 5.3, suppose that $\rho \geq 0$, that U is any open subscheme of our G_d^r scheme, and that for some point $b \in B$, the fiber of U over b has the expected dimension ρ . Then every point of the fiber may be smoothed to nearby points. Specifically:*

- (i) *The map from U to B is open at any point in the fiber over b , and for any component Z of U whose image contains b , the generic fiber of Z over B has dimension ρ .*
- (ii) *If further U is closed in G_d^r , then there is a neighborhood V of b such that the preimage of V in U is open over V , and for each component Z of U , every component of every fiber of Z over V has dimension precisely ρ .*

In particular, if X_0 is a curve of compact type (with two components) over an algebraically closed field, with $\overline{P}_1, \dots, \overline{P}_n$ distinct smooth closed points of X_0 , α^i any collection of ramification sequences, and U_0 any open subset of $G_d^r(X_0/k; \{(P_i, \alpha^i)\}_i)$ having expected dimension ρ , then there exists a smooth curve X_1 over a one-dimensional function field k' over k , specializing to X_0 , with points P_i specializing to the \overline{P}_i , and such that every point of U_0 smooths to X_1 ; if further $U_0 = G_d^r(X_0/k; \{(\overline{P}_i, \alpha^i)\}_i)$, then $G_d^r(X_1/k'; \{(P_i, \alpha^i)\}_i)$ also has dimension ρ .

Proof. — For (i), let $x \in Z$ be any closed point in the fiber of Z over b , and η the generic point of Z . Say η maps to ξ ; then the dimension of the fiber of Z over ξ is at most ρ , by [14, Thm. 13.1.3], and at least ρ by Theorem 5.3 after base change to ξ . But because the fiber over b has

dimension ρ , and by Theorem 5.3 we have $\dim \mathcal{O}_{Z,x} \geq \dim \mathcal{O}_{B,b} + \rho$, the image of Z cannot have dimension less than $\dim \mathcal{O}_{B,b}$, so ξ is the generic point of B .

For the openness assertion, it suffices to prove that the image of U contains a neighborhood of b , since if we replace U by any neighborhood of a point of the fiber of U over b , the hypotheses of our corollary are still satisfied. Let b_1 be a point of B , specializing to b ; let B_1 be the closure of b_1 in B , and consider the base change $U_1 \rightarrow B_1$. If B_1 has codimension c in B , then every component of U_1 would have codimension at most c in U , so if we restrict to a component Z_1 of U_1 passing through x , we have $\dim \mathcal{O}_{Z_1,x} \geq \dim \mathcal{O}_{B_1,b} + \rho$, so arguing as before we see that b_1 must be in the image of U . Now, by constructibility of the image, $f(U)$ must contain some neighborhood of b , as desired.

For (ii), if U is closed in G_d^r we have that it is proper over B , and every component Z of U either contains b in its image, or is supported on a closed subset of B away from b . If Z maps to b , we can apply (i) to conclude that Z maps surjectively to B , and by [14, Cor. 13.1.5] the locus on B of fibers of Z having a component of dimension greater than ρ is closed, so taking its complement and intersecting over the finitely many components of U gives a V of the desired form. Openness then follows from (i) and the fact that all fibers over V have dimension ρ .

Finally, given an X_0 as described, we can apply Theorem 3.4 to place X_0 into a smoothing family X/B with generic fiber X_1 ; the desired assertions then follow immediately from the main assertions of the corollary. \square

The finite case is particularly nice, but we put off any discussion of it until after we have introduced the language of Eisenbud-Harris limit series in the next section.

Even without knowing anything about the separable locus being closed, which in general seems to be a subtle issue, we can still obtain results on lifting from characteristic p to characteristic 0. However, note that the expected dimension hypothesis in the following corollary is not only key to the argument, but at least in some cases both non-vacuous and necessary for the validity of the conclusion. See in particular [18, Prop. 5.4, Rem. 8.3].

In any case, our machinery now easily yields:

COROLLARY 5.5. — *In the situation of Theorem 5.3, suppose that $\rho \geq 0$, that B is a mixed-characteristic DVR, and that the special fiber of some U open inside G_d^r has the expected dimension ρ . Then every point x_0 of U in the special fiber may be lifted to characteristic 0, in the sense that*

there will be a point x_1 of the generic fiber of U (and in particular of G_d^r) specializing to x_0 .

In particular, suppose that X_0 is a smooth, proper curve over a perfect field k of characteristic p , with $\bar{P}_1, \dots, \bar{P}_n$ distinct closed points of X_0 , α^i any collection of ramification sequences, and U_0 any open subset of $G_d^r(X_0/k; \{(\bar{P}_i, \alpha^i)\}_i)$ having expected dimension ρ ; then there exists a smooth curve X_1 over the fraction field K of the Witt vectors of k , specializing to X_0 , with points P_i specializing to the \bar{P}_i , and such that every point of U_0 may be lifted to a point of $G_d^r(X_1/K; \{(P_i, \alpha^i)\}_i)$.

Proof. — The first assertion follows immediately from the openness proven in Corollary 5.4.

For the second assertion, let A be the Witt vectors of k ; then by [3, 11, Thm 1.1] we can find an X over $\text{Spec } A$ whose special fiber is X_0 , and since A is complete and the \bar{P}_i are smooth points we can lift them to sections P_i of X . Applying the first assertion then gives the desired result. \square

Remark 5.6. — This last corollary, dealing only with smooth curves, has nothing to do with limit linear series, and only uses the elementary lower bound on the dimension of a standard G_d^r space with imposed ramification.

6. Comparison to Eisenbud-Harris theory

This is all well and good, but our description of the limit series associated to a reducible curve is rather cumbersome, so we now establish the relationship to Eisenbud and Harris' limit series in this situation.

SITUATION 6.1. — X/B is a smoothing family with X reducible; specifically, falling into case (2) of Situation 4.4.

The first step is to consider the “forgetful” map from our $\mathcal{G}_d^r(X/B)$ functor into the product of $\mathcal{G}_d^r(Y/B)$ and $\mathcal{G}_d^r(Z/B)$.

We have:

LEMMA 6.2. — In Situation 6.1, given any T -valued point $\{(\mathcal{L}^i, V_i)\}_i$ of $G_d^r(X/B)$, the pair $((\mathcal{L}^0|_{Y_T}, V_0|_{Y_T}), (\mathcal{L}^d|_{Z_T}, V_d|_{Z_T}))$ gives a T -valued point of $G_d^r(Y/B) \times G_d^r(Z/B)$. In particular, this defines a morphism $FR : G_d^r(X/B) \rightarrow G_d^r(Y/B) \times_B G_d^r(Z/B)$. A limit series in $G_d^r(X/B)$ is separable if and only if its image under FR is separable in both $G_d^r(Y/B)$ and $G_d^r(Z/B)$.

Proof. — It is clearly enough to show that given T/B , and $\{(\mathcal{L}^i, V_i)\}_i$ a T -valued point of $G_d^r(X/B)$, then $V_0|_{Y_T}$ is a sub-bundle of $\mathcal{L}^0|_{Y_T}$ and correspondingly for Z_T and V_d . This follows immediately from our definition of sub-bundle, since if a section of V_{0S} vanishes on Y_S for any S over T , it defines a section of the negative line bundle $\mathcal{L}^0|_Z(-\Delta')$ and hence vanishes on Z_S as well, and similarly for V_{dS} with Y and Z switched.

The statement on separability is immediate from the definition of separability of a limit series on a reducible curve. □

Notation 6.3. — In the same situation as the previous lemma, we denote the image under FR of $\{(\mathcal{L}^i, V_i)\}_i$ by $((\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z))$. Further, we denote by V_i^Y the image of V_i inside $\pi_*(\mathcal{L}^i|_Y) \cong \pi_*(\mathcal{L}^Y(-i\Delta'))$, and similarly for Z .

LEMMA 6.4. — *In the same situation as the previous lemma, we have the following additional observations (and consequent notation):*

- (i) V_i^Y injects naturally into V^Y , and similarly for Z ;
- (ii) V_i^Y will be contained in $\ker \beta_i^Y \subset V^Y$, where $\beta_i^Y : V^Y \rightarrow \pi_{T*} \mathcal{L}^Y|_{i\Delta'}$ is the natural i th order evaluation map at Δ' , and V_i^Z will similarly be contained in $\ker \beta_{d-i}^Z \subset V^Z$;
- (iii) The induced map $V_i \rightarrow V^Y \oplus V^Z$ in fact exhibits V_i as a sub-bundle (in the usual sense) of $V^Y \oplus V^Z$.

Proof. — Assertions (i) and (ii) are clear. For (iii), it suffices to show that $V_i \rightarrow V^Y \oplus V^Z$ is injective after any base change $S \rightarrow T$, and this is easily verified from the definitions. □

DEFINITION 6.5. — *In case (2) of Situation 4.4, we define an Eisenbud-Harris (crude) limit series on X to be a pair $((\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z))$ in $G_d^r(Y) \times_B G_d^r(Z)$ satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') \geq d$ for all i (see below). The closed subscheme of $G_d^r(Y) \times_B G_d^r(Z)$ obtained by these ramification conditions will be denoted $G_{d,EH}^r(X/B)$. We also define $G_{d,EH}^{r,sep}(X/B) \subset G_{d,EH}^r(X/B)$ to be the open subscheme of limit series which are separable on each component, and $G_{d,EH}^{r,ref}(X)$ to be the open subscheme of refined Eisenbud-Harris limit series satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') = d$ for all i , or more precisely, the complement of the closed subscheme satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') > d$ for some i .*

We remark that these ramification conditions do in fact give a canonical closed subscheme structure: for each sequence of $r + 1$ non-decreasing integers $0 \leq a_i \leq d$, we get a closed subscheme defined by the conditions

$a_i^Y(\Delta') \geq a_i$, $a_{r-i}^Z(\Delta') \geq d - a_i$; there are only finitely many such sequences, so the union of the closed subschemes obtained over each of them is again a closed subscheme. However, this definition gives us trouble when we attempt to show that our $G_d^r(X/B)$ maps into $G_{d,EH}^r(X/B)$, as it is difficult to describe the T -valued points of a union of schemes in terms of the T -valued points of the individual schemes. As a result, we settle for the following slightly weaker statement.

PROPOSITION 6.6. — *We have the following facts about the image of $FR : G_d^r(X/B) \rightarrow G_d^r(Y/B) \times_B G_d^r(Z/B)$:*

- (i) *FR has set-theoretic image precisely $G_{d,EH}^r(X/B)$;*
- (ii) *Scheme-theoretically, $G_d^r(X/B)$ maps into the closed subscheme satisfying for all j*

$$a_j^Y(\Delta') + a_{r-j}^Z(\Delta') \geq d - 1;$$

- (iii) *The open subscheme of $G_d^r(X/B)$ mapping set-theoretically into $G_{d,EH}^{r, \text{ref}}(X)$ actually maps scheme-theoretically into $G_{d,EH}^{r, \text{ref}}(X) \subset G_{d,EH}^r(X/B)$.*

Proof. — In general, for a T -valued pair $((\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z))$, define a_j^Y to be the largest integer i with $\text{rk } \beta_i^Y \leq j$ everywhere on T , and similarly for Z . The set-theoretic statement may be checked point by point, and is equivalent to saying that when $T = \text{Spec } k$ for some k , $((\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z))$ is in the image of FR if and only if $a_j^Y + a_{r-j}^Z \geq d$ for all j . For (ii), it is enough to check that for arbitrary local T , $a_j^Y + a_{r-j}^Z \geq d - 1$ for all j , and for (iii), we want to show in this case that if the point obtained by restriction to the closed point of T satisfies $a_j^Y + a_{r-j}^Z = d$ for all j , then the entire T -valued point does. In all cases, we make use of the fact from Lemma 6.4 that V_i^Y may be considered as lying inside $\ker \beta_i^Y$, and V_i^Z in $\ker \beta_{d-i}^Z$. Conceptually, the basic idea is that for V_i to maintain rank $r + 1$ at each i , the ranks of $\ker \beta_i^Y$ and $\ker \beta_{d-i}^Z$ must add up to at least $r + 1$, and looking at $i = a_j^Y$ for different j should yield the desired inequalities. As we will see, this works over a field, but is not quite so nice for a more general T .

Now, for the set-theoretic statement (i), suppose we have a k -valued point $\{(\mathcal{L}^i, V_i)\}_i$ of $G_d^r(X/B)$; we first show that it maps into $G_{d,EH}^r(X/B)$. Since V_i is glued from subspaces of $\ker \beta_i^Y$ and $\ker \beta_{d-i}^Z$ and has dimension $r + 1$, we conclude that $\dim \ker \beta_i^Y + \dim \ker \beta_{d-i}^Z \geq r + 1$, so $\text{rk } \beta_i^Y + \text{rk } \beta_{d-i}^Z \leq r + 1$, and it follows that $a_j^Y + a_{r+1-j}^Z \geq d$ for all j . On the other hand, for a given j , set $i = a_j^Y$; we know that $\text{rk } \beta_{i+1}^Y > j$, so one of the sections in V_i^Y is non-vanishing at Δ' when considered as a section of $\mathcal{L}^i(-i\Delta')$, and to

use it in V_i , it must be glued to a section of V_i^Z similarly non-vanishing at Δ' . Thus, $\dim V_i < \dim \ker \beta_i^Y + \dim \ker \beta_{d-i}^Z$, so our earlier argument gives $\text{rk } \beta_i^Y + \text{rk } \beta_{d-i}^Z < r + 1$, hence $a_j^Y + a_{r-j}^Z \geq d$. For later use, note that when $a_j^Y + a_{r-j}^Z = d$ for all j , this argument shows that we have $\ker \beta_i^Y = V_i^Y$ and $\ker \beta_{d-i}^Z = V_i^Z$ for all i , and in particular when $i = a_j^Y = d - a_{r-j}^Z$, we get $\dim V_i^Y + \dim V_i^Z = r + 2$.

Conversely, given a $((\mathcal{L}^Y, V^Y), (\mathcal{L}^Z, V^Z))$ satisfying the Eisenbud-Harris inequalities, we construct the \mathcal{L}^i by gluing $\mathcal{L}^Y(-i\Delta')$ and $\mathcal{L}^Z((i-d)\Delta')$, and set $V_0 = V^Y$, $V_d = V^Z$. Note that sections in V^Y which vanish at Δ' are extended by 0 along Z . If there is a non-vanishing section, then $a_0^Y = 0$, so $a_r^Z \geq d$, and \mathcal{L}^Z is necessarily $\mathcal{O}_{Z_T}(d\Delta')$, so $\mathcal{L}^i|_Z \cong \mathcal{O}_{Z_T}$, and we can (uniquely) extend sections not vanishing at Δ' , also. We also observe that this implies that we have V_0 mapping into V^Z under iterations of f_i . By symmetry, we can make the same arguments for V^Z to get our V_d . Now we inductively construct each V_i for $i = 1, 2, \dots, d - 1$ in terms of V_{i-1} and V_d . Our induction hypothesis will be that V_{i-1} is linked to the previous V_j and V_d under iterates of f_j and g_j , and furthermore that each V_j has a basis of sections each of which is either non-vanishing at Δ' , or vanishes uniformly on either Y or Z , with at most one basis element in the first category. We denote the number of each of these by r_j^1 , r_j^2 , and r_j^3 respectively, where we have r_j^1 always 0 or 1, and $r_j^1 + r_j^2 + r_j^3 = r + 1$ for all j . Finally, we also impose in our induction hypothesis that r_j^3 is always the maximal possible value, which is $\dim \ker \beta_{j+1}^Y$. Note that since this is non-increasing, if we construct a V_i with $r_i^3 = r_{i-1}^3$, maximality is automatically satisfied.

Now, for general i , suppose we have constructed the V_j up to V_{i-1} satisfying our induction hypothesis. To construct V_i , the basis elements vanishing on Y must contain $f_{i-1}(V_{i-1})$, which is an $(r_{i-1}^1 + r_{i-1}^2)$ -dimensional space, and of course they must map into V^Z . Since $f_{i-1}(V_{i-1})$ maps into V^Z , we can choose $r_{i-1}^1 + r_{i-1}^2$ such sections, by taking any basis of $f_{i-1}(V_{i-1})$. Next, the basis elements vanishing on Z must be contained in $g_{i-1}^{-1}(V_{i-1})$, and we choose them to be a basis of the subspace of $g_{i-1}^{-1}(V_{i-1})$ vanishing on Z . This is at most an r_{i-1}^3 -dimensional space, with equality if all of the r_{i-1}^3 basis elements of V_{i-1} vanish to order greater than one at Δ' . If there was a section vanishing to order exactly one at Δ' , the subspace of $g_{i-1}^{-1}(V_{i-1})$ vanishing on Z will instead be $(r_{i-1}^3 - 1)$ -dimensional. Now, by our induction hypothesis, the iterated image of V_d under the g_j is contained in V_{i-1} , necessarily in the span of the basis elements vanishing on Z . Moreover, since $i < d$, this image lies in the subspace of V_{i-1} vanishing

to order at least 2 at Δ' , so it is automatically contained in the span of the basis elements we have chosen for V_i which vanish on Z . Now, if we had r_{i-1}^3 such basis elements, we are done. If not, we had a section of V_{i-1} vanishing on Z and vanishing to first order at Δ' on Y , so it follows that $\dim \ker \beta_i^Y = \dim \ker \beta_{i+1}^Y + 1$, and therefore that $a_{r+1-r_{i-1}^3}^Y = i$; in particular, the required maximality of $r_i^3 = r_{i-1}^3 - 1$ is satisfied. It also follows that $a_{r_{i-1}^3}^Z \geq d - i$; if it is equal, we can find a section of V^Z vanishing to order precisely $d - i$ at Δ' , which we could glue to our final section of $g_{i-1}^{-1}(V_{i-1})$ to obtain our $(r + 1)$ st generator for V_i , which will be non-vanishing at Δ' . Otherwise, we have $a_{r_{i-1}^3}^Z > d - i$, so following through the definitions, $\dim \ker \beta_{d-i+1}^Z \geq r + 2 - r_{i-1}^3 = 1 + r_{i-1}^1 + r_{i-1}^2$, and we can choose an $(r_{i-1}^1 + r_{i-1}^2 + 1)$ st generator vanishing on Y to be our $(r + 1)$ st generator for V_i . This completes the proof of the set-theoretic surjectivity of FR onto $G_{d,\text{EH}}^r(X/B)$.

For the scheme-theoretic statements, let $T = \text{Spec } A$ where A is any local ring with maximal ideal \mathfrak{m} . By Lemma 6.4, for each i we have $0 \rightarrow V_i \rightarrow V^Y \oplus V^Z \rightarrow Q \rightarrow 0$ for some free Q . Working modulo \mathfrak{m} and then using Nakayama's lemma, we find that for some j depending on i , we can construct sub-bundles of V_i whose images in V_i^Y and V_i^Z are sub-bundles of V^Y and V^Z of rank $r + 1 - j$ and j . These are contained in $\ker \beta_i^Y$ and $\ker \beta_{d-i}^Z$, and we conclude that $a_j^Y + a_{r+1-j}^Z \geq d$. As in the fields case, if we set $i = a_j^Y$, then by hypothesis $\text{rk } \beta_{i+1}^Y$ is not less than or equal to j on all of T , so our constructed sub-bundle of V^Y for $i + 1$ could have rank at most $r - j$, and the sub-bundle of V^Z would have to have rank at least $j + 1$, giving $\text{rk } \beta_{d-i-1}^Z \leq r - j$ on T , and yielding the inequality $a_j^Y + a_{r-j}^Z \geq d - 1$ of statement (ii).

Finally, for statement (iii), we need only combine this argument with our earlier observation that at the closed point, where by hypothesis we had $a_j^Y + a_{r-j}^Z = d$ for all j , when $i = a_j^Y$ we necessarily have $\dim V_i^Y + \dim V_i^Z = r + 2$; thus we can in fact choose a basis of V_i (still modulo \mathfrak{m}) which has rank $r + 1 - j$ in V_i^Y and rank $j + 1$ (rather than j) in V_i^Z , and we then get the desired inequality $a_j^Y + a_{r+1-j}^Z \geq d$ for the entire T -valued point, as desired. Note that the subscheme of G_d^r in question is open simply because G_d^r is known to map set-theoretically into $G_{d,\text{EH}}^r$, and $G_{d,\text{EH}}^{r,\text{ref}}(X)$ is open inside $G_{d,\text{EH}}^r$. □

We observe that for a crude Eisenbud-Harris limit series, there may be many ways of filling in the intermediate V_i from V_0 and V_0 , so the fiber of FR may be positive-dimensional. However, the situation is easier to get a

handle on for the open subset of refined series which Eisenbud and Harris used in their construction. Indeed, we show that the space of refined limit series is actually isomorphic to an open subscheme of our G_d^r scheme.

PROPOSITION 6.7. — *Suppose that (\mathcal{L}^Y, V^Y) and (\mathcal{L}^Z, V^Z) form a T -valued point of $G_{d,\text{EH}}^{r,\text{ref}}(X/B)$. Then we have that (\mathcal{L}^Y, V^Y) and (\mathcal{L}^Z, V^Z) are the image of a unique T -valued point under FR .*

Proof. — It clearly suffices to handle the case that T is connected, so we make this hypothesis. In this case, we see that we get unique vanishing sequences at Δ' for V^Y and V^Z , in the sense that some sequence is satisfied everywhere on T , with no stronger ramification index satisfied anywhere on T . Indeed, the subscheme of T satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') > d$ is empty by hypothesis, and because ramification conditions are closed, the ramification sequences obtained at any point of T persist in an open and closed neighborhood. Now, if β_i^Y is the evaluation map $V^Y \rightarrow \pi_{T*}\mathcal{L}^Y|_{i\Delta'}$ and similarly for β_i^Z , this immediately implies that each β_i^Y and β_i^Z has rank determined exactly by the vanishing sequences, in the strong sense that for some j , the closed subscheme where the rank is less than or equal to j is all of T , but the closed subscheme where the rank is strictly less than j is empty. It follows (see, e.g., [7, Prop. 20.8]) that the images of the β_i^Y and β_i^Z are locally free, with locally free quotients. If we denote by $\{a_j\}_j$ the vanishing sequence at Δ' for V^Y , we also note that $\ker \beta_i^Y$ will have rank $r + 1 - j$ if $a_{j-1} < i \leq a_j$, and following through the definitions we see that $\ker \beta_{d-i}^Z$ will have rank j if $a_{j-1} \leq i < a_j$, so we find that $\ker \beta_i^Y = \ker \beta_{i+1}^Y$ if and only if $\ker \beta_{d-i}^Z = \ker \beta_{d-i+1}^Z$, and $\text{rk } \ker \beta_i^Y + \text{rk } \ker \beta_{d-i+1}^Z = r + 1$ for all i .

The main idea is to construct the V_i as the subspace of $\ker \beta_i^Y \oplus \ker \beta_{d-i}^Z$ which agree on the two maps given by evaluation at Δ' . This would then be unique by Lemmas 6.4 and 5.1, so we need only show existence. We work locally on the base, so that $\mathcal{L}^Y(-i\Delta')|_{\Delta'} \cong \mathcal{L}^Z(-i'\Delta')|_{\Delta'} \cong \mathcal{O}_{\Delta'} \cong \mathcal{O}_T$ for all i, i' , and fix a choice of these isomorphisms. As prescribed for gluing together line bundles defined on components, we define \mathcal{L}^i by the short exact sequence

$$0 \rightarrow \mathcal{L}^i \rightarrow \mathcal{L}^Y(-i\Delta') \oplus \mathcal{L}^Z((i-d)\Delta') \rightarrow \mathcal{O}_{\Delta'} \rightarrow 0,$$

and pushforward gives us

$$0 \rightarrow \pi_{T*}\mathcal{L}^i \rightarrow \pi_{T*}\mathcal{L}^Y(-i\Delta') \oplus \pi_{T*}\mathcal{L}^Z((i-d)\Delta') \rightarrow \mathcal{O}_T.$$

We then define V_i to be the kernel of the induced map, so that:

$$0 \rightarrow V_i \rightarrow \ker \beta_i^Y \oplus \ker \beta_{d-i}^Z \rightarrow \mathcal{O}_T.$$

We have to show that V_i is a sub-bundle of $\pi_{T*}\mathcal{L}^i$ of the correct rank. We first observe that the image $\beta_{i+1}^Y(\ker \beta_i^Y)$ has locally free quotient in $\pi_{T*}\mathcal{L}^Y|_{(i+1)\Delta'}$, and similarly for Z : this image is inside $\text{im } \beta_{i+1}^Y$ by definition, and the quotient is easily seen to be isomorphic to $\text{im } \beta_i^Y$, via the map $\pi_{T*}\mathcal{L}^Y|_{(i+1)\Delta'} \rightarrow \pi_{T*}\mathcal{L}^Y|_{i\Delta'}$. Thus $\beta_{i+1}^Y(\ker \beta_i^Y)$ is a sub-bundle of a sub-bundle, and must itself be a sub-bundle of $\pi_{T*}\mathcal{L}^Y|_{(i+1)\Delta'}$. Now, we can factor β_{i+1}^Y restricted to $\ker \beta_i^Y$ as

$$\ker \beta_i^Y \rightarrow \pi_{T*}\mathcal{L}^Y(-i\Delta')|_{\Delta'} \hookrightarrow \pi_{T*}\mathcal{L}^Y|_{(i+1)\Delta'}$$

and we just showed that the cokernel of the composition is locally free; since $\mathcal{L}^Y(-i\Delta')|_{\Delta'}$ is a line bundle, this means the first map must be either zero or surjective, with surjectivity precisely when $\text{rk } \ker \beta_i^Y = \text{rk } \ker \beta_{i+1}^Y + 1$, and the ranks equal otherwise. We obtain the corresponding result for Z , and immediately conclude that V_i is a sub-bundle of $\ker \beta_i^Y \oplus \ker \beta_{d-i}^Z$, with equality if and only if both $\ker \beta_i^Y = \ker \beta_{i+1}^Y$ and $\ker \beta_{d-i}^Z = \ker \beta_{d-i+1}^Z$, and corank one otherwise. Thus, our hypotheses imply that V_i has rank $r + 1$. The last observation is that V_i being a sub-bundle of $V^Y \oplus V^Z$ implies that it is a sub-bundle (in our generalized sense) of $\pi_{T*}\mathcal{L}^i$, but this follows easily from the fact that V^Y and V^Z are sub-bundles of $\pi_{T*}\mathcal{L}^Y$ and $\pi_{T*}\mathcal{L}^Z$. □

We immediately conclude:

COROLLARY 6.8. — *The map $FR : G_d^r(X/B) \rightarrow G_d^r(Y/B) \times G_d^r(Z/B)$ induces an isomorphism from an open subscheme $G_d^r(X/B)$ onto $G_{d,\text{EH}}^{r,\text{ref}}(X/B)$, and on the corresponding separable subschemes of these.*

We may therefore think of the scheme of refined Eisenbud-Harris limit series as forming an open subscheme of our G_d^r scheme itself:

DEFINITION 6.9. — *We say that a point of G_d^r is a refined limit series if it maps under FR to $G_{d,\text{EH}}^{r,\text{ref}}$, and we denote the open subscheme of refined limit series by $G_d^{r,\text{ref}} \subset G_d^r$.*

We also have the following trivial observation.

COROLLARY 6.10. — *Lemma 6.2, Propositions 6.6 and 6.7, and Corollary 6.8 all hold when ramification conditions are imposed.*

Proof. — Indeed, we specified ramification solely on V_0 or V_d depending on whether the relevant section was on Y or Z , so the ramification conditions are visibly compatible with FR . □

Since in practice it is less cumbersome to work with Eisenbud-Harris series on a given reducible curve, we state our main corollary for the finite

case of Theorem 5.3 in a situation where one can (nearly) restrict attention entirely to the Eisenbud-Harris series. We now drop the hypothesis that we are in case (2), and for notational convenience define:

DEFINITION 6.11. — *In case (1), i.e., with X/B smooth, we simply define $G_{d,\text{EH}}^r(X/B)$ to be equal to $G_d^r(X/B)$ and similarly for $G_{d,\text{EH}}^{r,\text{sep}}(X/B)$ and $G_{d,\text{EH}}^{r,\text{ref}}(X/B)$.*

COROLLARY 6.12. — *In the situation of Theorem 5.3, suppose that $B = \text{Spec } A$ with A a DVR having algebraically closed residue field, and $\rho = 0$, and denote by X_0 and X_1 the special and generic fibers of X/B . We omit the ramification conditions from our notation for the sake of brevity. Then consider the following conditions:*

- (I) $G_{d,\text{EH}}^{r,\text{sep}}(X_0) \subset G_{d,\text{EH}}^{r,\text{ref}}(X_0)$
- (II) $G_{d,\text{EH}}^{r,\text{sep}}(X_0)$ consists of m reduced points for some $m > 0$.
- (III) For any DVR A' , and any A' -valued point of $G_d^r(X)$ such that the induced map $\text{Spec } A' \rightarrow \text{Spec } A$ is flat and the generic point of $\text{Spec } A'$ maps into $G_d^{r,\text{sep}}(X)$, then the closed point of $\text{Spec } A'$ maps into $G_d^{r,\text{sep}}(X)$ as well.

If (I) and (II) hold, we have that the $G_d^{r,\text{sep}}(X_1)$ geometrically contains at least m points; if further (III) holds, then $G_d^{r,\text{sep}}(X)$ is finite étale over B , and the geometric generic fiber also consists of exactly m reduced points.

Proof. — First, we have by virtue of (I) and Corollary 6.8 that $G_{d,\text{EH}}^{r,\text{sep}}(X_0) \cong G_d^{r,\text{sep}}(X_0)$. Setting $U = G_d^{r,\text{sep}}(X/B)$, if we choose any point x in the special fiber, applying Corollary 5.4 we find that any component Z of U passing through x maps dominantly to B with 0-dimensional generic fiber. To count the number of points, we can take Z to be reduced, in which case it is flat over B , and we obtain the first assertion.

In the case that (III) holds, we claim that U is in fact proper over B . We begin by noting that the generic fiber must be 0-dimensional: by the preceding arguments, this will follow if we show that every component Z of U meets the special fiber of U , and this follows from the properness of $G_d^r(X/B)$ and condition (III). Now, it suffices to show that U is closed in $G_d^r(X)$, so choose $y \in U$, $y' \in G_d^r(X)$ distinct points with y specializing to y' ; by [12, Prop. 7.1.9] we can find a DVR A' and a map $\text{Spec } A' \rightarrow G_d^r(X)$ with the generic point mapping to y and the special point mapping to y' . The image of $\text{Spec } A'$ cannot be contained in either the special or generic fiber by 0-dimensionality. Therefore, it gives a flat map $\text{Spec } A' \rightarrow \text{Spec } A$, and hypothesis (III) implies $y' \in U$ as well, yielding properness of U . Given this, since U is unramified in the special fiber,

it must be unramified over B ; thus, the fibers are reduced, and the lemma which follows gives flatness, so we conclude the desired finite étaleness. \square

The first statement of the following lemma was provided by Max Lieblich.

LEMMA 6.13. — *Let $f : X \rightarrow Y$ be a morphism, with all fibers of f reduced. Then if $f' : X_{\text{red}} \rightarrow Y$ is flat, we have that X is reduced. In particular, if X is irreducible, $Y = \text{Spec } A$ for some DVR A , and f is dominant (still having reduced fibers), then f is flat.*

Proof. — The first assertion follows from Nakayama’s lemma together with consideration of the exact sequence

$$0 \rightarrow \mathcal{N}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}} \rightarrow 0,$$

where \mathcal{N}_X is the sheaf of nilpotents inside \mathcal{O}_X . For the second assertion, we apply the standard criterion for flatness over a DVR twice: first to X_{red} so we can use the lemma to conclude that X is reduced, and then again to X itself. \square

Remark 6.14. — Note that even if G_d^r maps scheme-theoretically into $G_{d,\text{EH}}^r$, the statement of Proposition 6.6 would be false if we replaced $d-1$ in the inequality by d : we would expect it to fail precisely at the intersection of the closed subschemes defined by different choices of a_j^Y and a_j^Z with $a_j^Y + a_{r-j}^Z = d$. Indeed, it is easy enough to write down examples where one does not have the desired inequality: the simplest case is $Y \cong Z \cong \mathbb{P}^1$, with affine coordinate functions y and z vanishing at the node, $d = 2$, $r = 0$, and a $\text{Spec } k[\epsilon]/(\epsilon^2)$ -valued limit series given by

$$(y^2 + \epsilon y, 0), (y + \epsilon, \epsilon), (\epsilon, z + \epsilon).$$

Here we are identifying sections of $\mathcal{O}(d')$ with polynomials of degree d' , and for each i specifying pairs of sections of degree $2 - i$ and i on the components, required to agree at $y = z = 0$; the inclusion maps are then given by multiplication by y or z on the appropriate component, and 0 on the other component. This has $a_0^Y = 1$, $a_0^Z = 0$; note that modulo ϵ , it has $a_0^Y = 2$, $a_0^Z = 1$, so it does not correspond to a refined series, as is required by the proof of Proposition 6.7 (iii).

Remark 6.15. — Continuing along the same line of reasoning, we see that the scheme-theoretic statement of Proposition 6.6 is actually surprisingly strong; indeed, it implies that if there are two components of the locus of refined series which meet at a point of G_d^r , then if the components have vanishing sequences at the node given by a_j^Y, a_j^Z , and $a_j'^Y, a_j'^Z$, we must have $|a_j^Y - a_j'^Y| \leq 1, |a_j^Z - a_j'^Z| \leq 1$ for all j .

Remark 6.16. — Note that it is not hard to deduce from the proof of Proposition 6.6 that any refined point is the image of an exact point of the linked Grassmannian used in the construction (see Definition A.10). However, the converse is false. Indeed, there exist non-refined points for which there is an exact point above them, and there exist others for which there isn't. We see both already in the simplest case of $Y \cong Z \cong \mathbb{P}^1$, and $d = 2, r = 0$. In the notation of Remark 6.14, we could consider $(y^2, 0)$, (y, z) , $(0, z^2)$. One checks that this is not refined, but is an exact point. On the other hand, if we start with the two pairs $(y^2, 0)$, $(0, z^2 + z)$, it is easy to see that up to scalar the only way to fill in the middle pair is with $(y, 0)$, and this point is not exact.

7. Further questions

This construction brings up a number of natural further questions, and we briefly set out a few of them. First, as mentioned earlier, the Eisenbud-Harris limit series scheme on a reducible curve was never connected. However, in our case it seems as though the crude limit series ought to serve as bridges between components of refined limit series with differing ramification sequences at the node. In fact, at first blush it may appear based on dimension-counting that crude limit series should simply be the closure of the refined series in many cases, and this may well be true in the Eisenbud-Harris scenario of only looking at a \mathfrak{g}_d^r on each component, but because our crude series will often map with positive-dimensional fibers to the Eisenbud-Harris crude series, the geometry is not entirely clear. For similar reasons, even though our construction *a priori* gives results on smoothing of crude series, the expected dimension hypothesis for all limit series will not follow immediately from having the expected dimension of refined series. We can therefore reasonably ask:

QUESTION 7.1. — *When is the space of limit series on a reducible curve connected?*

QUESTION 7.2. — *When is the space of refined limit series dense in the space of all limit series?*

QUESTION 7.3. — *In characteristic 0, what can we say about the dimension of spaces of crude limit series, and by extension their smoothability? In particular, can we smooth a “general” crude series, as we can in the case of refined series (the latter follows from [9, Thm. 4.5])?*

We remark that bounding the dimension of crude series on a reducible curve, given an understanding of dimensions of linear series on each component, should be a combinatorial problem, and if the bound is restrictive enough to imply that on a general curve the crude series have dimension at most as large as the dimension of refined series, it will follow that for a general reducible curve, we can always apply the strong form (that is, part (ii)) of Corollary 5.4 to our entire G_d^r space. In particular, we can actually make use of the properness of the constructed G_d^r scheme to obtain direct arguments for theorems such as Brill-Noether, without requiring arguments involving blowing up the family, as used in [16, p. 261].

Given our inability to adequately describe the T -valued points of $G_{d,\text{EH}}^r(X/B)$, we can also ask:

QUESTION 7.4. — *Does $G_d^r(X/B)$ actually map scheme-theoretically into $G_{d,\text{EH}}^r(X/B)$? Is it scheme-theoretically surjective?*

In applications, an important direction of generalization is specified ramification along one or more unspecified smooth sections; this may now be accomplished just as with the case of the Eisenbud-Harris theory, by looking at positive-dimensional “special fibers” and allowing ρ to become negative; see [16, p. 270] for an example.

Finally, the transparency of the construction presented here offers various possibilities for generalization beyond the setting of linear series on curves of compact type. One direction of generalization is to replace curves by higher-dimensional varieties. To carry out our main theorem in this setting seems at this point to be just a formality, but its application presents considerable challenges, the most formidable of which is that the “expected dimension” hypothesis of our main theorem is suddenly more of a burden in dimension higher than one. This is amply demonstrated by the interpolation problem (see [5] and [11]), where one sees first that expected dimension for general ramification points need not hold, even for zero-dimensional linear series on \mathbb{P}^2 , and second, that standard degeneration arguments have thus far failed to succeed in describing when exactly the expected dimension is in fact correct.

One could also hope to generalize from line bundles to vector bundles. This should not be too difficult, but in order to set up inductive degeneration arguments, one would then need some description of the limit objects on the reducible curve which would play the role of the Eisenbud-Harris description of limit series. Lastly, it might be possible to adapt the construction to work on curves not of compact type, but in this setting one

may find that the functor for a given family, after restriction to the reducible special fiber, will still depend on the geometry of the entire family. This would potentially complicate the situation considerably.

Appendix A. The linked Grassmannian scheme

In this appendix, we develop a theory of a moduli scheme parametrizing collections of sub-bundles of vector bundles on a base scheme, linked together via maps between the vector bundles. Representability by a proper scheme is easy and true quite generally; however, to obtain dimension formulas will require more hypotheses and more work. These hypotheses, while reasonably natural and easy to state, are motivated by the idea that the vector bundle maps are induced as pushforwards of certain maps of sufficiently ample line bundles on a scheme proper over the base scheme, as in the situation of the limit linear series theory of the present paper, and its natural generalization to higher-dimensional varieties.

We first specify the objects we will study in more detail; for the remainder of this appendix, we will be in:

SITUATION A.1. — *Let S be any base scheme, and $\mathcal{E}_1, \dots, \mathcal{E}_n$ vector bundles on S , each of rank d . We have maps $f_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$ and $g_i : \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$, and a positive integer $r < d$.*

The functor we wish to study may now be easily described:

DEFINITION A.2. — *In this situation, we have the functor $\mathcal{L}\mathcal{G}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$, associating to each S -scheme T the set of sub-bundles V_1, \dots, V_n of $\mathcal{E}_{1,T}, \dots, \mathcal{E}_{n,T}$ of rank r and satisfying $f_{i,T}(V_i) \subset V_{i+1}$, $g_{i,T}(V_{i+1}) \subset V_i$ for all i .*

Then without any further hypotheses, we have:

LEMMA A.3. — *$\mathcal{L}\mathcal{G}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ is representable by a projective scheme LG over S , which is naturally a closed subscheme of a product G of Grassmannian schemes over S ; G is smooth and projective over S of relative dimension $nr(d - r)$.*

Proof. — Let G_i be the schemes of Grassmannians of rank r sub-bundles of the \mathcal{E}_i , and G the product of the G_i over S . Denote our projection maps from G to each G_i by π_i , and the maps from each G_i to S by ϕ_i . Let \mathcal{F}_i be the universal sub-bundles on each G_i .

Then each f_i induces a map

$$\pi_i^* \mathcal{F}_i \rightarrow \pi_i^* \phi_i^* \mathcal{E}_i = \pi_{i+1}^* \phi_{i+1}^* \mathcal{E}_i \xrightarrow{f_i} \pi_{i+1}^* \phi_{i+1}^* \mathcal{E}_{i+1} \rightarrow \pi_{i+1}^* \phi_{i+1}^* \mathcal{E}_{i+1} / \pi_{i+1}^* \mathcal{F}_{i+1}$$

on G , and the kernel of this map is a closed subscheme which imposes precisely the condition that $f_i(V_i) \subset V_{i+1}$. Similarly, g_i induces a map $\pi_{i+1}^* \mathcal{F}_{i+1} \rightarrow \pi_i^* \phi_i^* \mathcal{E}_i / \pi_i^* \mathcal{F}_i$ on G whose kernel imposes the condition $g_i(V_{i+1}) \subset V_i$. Taking the intersection of these closed subschemes for all f_i and g_i thus gives a scheme representing $\mathcal{LG}(r, \{E_i\}_i, \{f_i, g_i\}_i)$, which as a closed subscheme of G is projective over S . \square

However, in order to say anything of substance about the LG scheme itself, and in particular to get the necessary lower bound on dimension, we need to make a number of additional hypotheses. We define:

DEFINITION A.4. — *In Situation A.1, we say that $\mathcal{LG}(r, \{E_i\}_i, \{f_i, g_i\}_i)$ is a linked Grassmannian of length n if S is integral and Cohen-Macaulay, and the following additional conditions on the f_i and g_i are satisfied:*

- (I) *There exists some $s \in \mathcal{O}_S$ such that $f_i g_i = g_i f_i$ is scalar multiplication by s for all i .*
- (II) *Wherever s vanishes, the kernel of f_i is precisely equal to the image of g_i , and vice versa. More precisely, for any i and given any two integers r_1 and r_2 such that $r_1 + r_2 < d$, then the closed subscheme of S obtained as the locus where f_i has rank less than or equal to r_1 and g_i has rank less than or equal to r_2 is empty.*
- (III) *At any point of S , $\text{im } f_i \cap \ker f_{i+1} = 0$, and $\text{im } g_{i+1} \cap \ker g_i = 0$. More precisely, for any integer r_1 , and any i , we have locally closed subschemes of S corresponding to the locus where f_i has rank exactly r_1 , and $f_{i+1} f_i$ has rank less than or equal to $r_1 - 1$, and similarly for the g_i . Then we require simply that all of these subschemes be empty.*

Remark A.5. — The hypothesis that S is integral and Cohen-Macaulay is unnecessary for most of our analysis. We use it only in the dimension theory portion of the argument, to ensure that LG is catenary.

From this point on, we strengthen Situation A.1.

SITUATION A.6. — *We suppose that LG is a linked Grassmannian, and we denote its structure map to S by π .*

The following lemma will be convenient for constructing and manipulating points of LG :

LEMMA A.7. — *Let $\{V_i\}_i$ be a k -valued point of LG , and suppose $s = 0$ in k . Then for any i , we can decompose V_i as $f_{i-1}(V_{i-1}) \oplus \ker f_i|_{V_i} \oplus C$ for some complementary subspace $C \subset V_i$. Indeed, if we specify any $C' \subset \ker g_{i-1}|_{V_i}$ which intersects $f_{i-1}(V_{i-1})$ trivially, we may choose $C = C' \oplus C''$ for some C'' .*

Proof. — Clearly, it suffices to show that for any C' as in the statement, we have that $f_{i-1}(V_{i-1}) \oplus \ker f_i|_{V_i} \oplus C'$ injects into V_i . But suppose we have $v_1 \in f_{i-1}(V_{i-1})$, $v_2 \in \ker f_i|_{V_i}$, and $v_3 \in C'$, such that $v_1 + v_2 + v_3 = 0$. If we apply g_{i-1} , by hypothesis $g_{i-1}(v_3) = 0$, and $g_{i-1}(v_1) = 0$ because v_1 is in the image of f_{i-1} and we assumed $s = 0$. So we find that $g_{i-1}(v_2) = 0$, which we claim implies $v_2 = 0$: indeed, $v_2 \in \ker f_i$ by hypothesis, so by condition (II) of a linked Grassmannian it is in the image of g_i , and by condition (III), it cannot map to 0 under g_{i-1} unless it is 0. Hence $v_2 = 0$, so $v_1 + v_3 = 0$, and since we assumed that C' was disjoint from $f_{i-1}(V_{i-1})$, we get $v_1 = v_3 = 0$ as well. \square

In order to make inductive arguments convenient, we define:

DEFINITION A.8. — *If LG is a linked Grassmannian of length n , and n' any positive integer less than n , we have the truncation map from LG to the linked Grassmannian of length n' obtained by forgetting all \mathcal{E}_i , f_i , and g_i for all $i > n'$.*

We will want to know that the truncation map is always surjective, even on certain classes of families:

LEMMA A.9. — *The truncation map is surjective for all n' . Further, in the case that the base is a point, let $x = \{V_i\}_i$ be any point of LG , and suppose we have a family $\tilde{x}_{n'} = \{\tilde{V}_i\}_{i \leq n'}$ (that is, a scheme-valued point of the restricted LG scheme) specializing to the truncation of x to length n' , and such that $\tilde{V}_{n'}$ may be written as $\tilde{C}_{n'} \oplus \ker f_{n'}|_{V_{n'}}$ for some family $\tilde{C}_{n'}$. Then $\tilde{x}_{n'}$ may be lifted to a family \tilde{x} of length n , specializing to x , possibly after a Zariski localization of the base of the family.*

Proof. — Surjectivity may be checked on points, and given the description of the \mathcal{LG} functor, it suffices to handle the case $n = 2$, $n' = 1$. Over a point, we may consider $\mathcal{E}_1 = \mathcal{E}_2 = E$ to be a single d -dimensional vector space, and f_1 and g_1 to be self-maps of E . Let V_1 be a vector space of dimension r inside E ; we just need to show that there exists a V_2 of dimension r inside E such that $f_1(V_1) \subset V_2$, and $g_1(V_2) \subset V_1$, or equivalently, such that $f_1(V_1) \subset V_2 \subset g_1^{-1}(V_1)$. Now, $\dim f_1(V_1) \leq r$ and $f_1(V_1) \subset g_1^{-1}(V_1)$ by hypothesis, so it suffices to observe that $\dim g_1^{-1}(V_1) =$

$\dim \ker g_1 + \dim(V_1 \cap \text{im } g_1)$, and the codimension of $\text{im } g_1$ in E and hence V_1 is bounded by $\dim \ker g_1$, so we conclude that $\dim g_1^{-1}(V_1) \geq r$.

For the second assertion, it suffices to show that we can lift to a $\tilde{V}_{n'+1}$ of the form $\tilde{C}_{n'+1} \oplus \ker f_{n'+1}|_{V_{n'+1}}$ and specializing to the truncation of x to length $n' + 1$, since then we can iterate until we have lifted all the way to length n . Thanks to Lemma A.7, we can write $V_{n'} = C_{n'} \oplus \ker f_{n'}|_{V_{n'}}$, and $V_{n'+1} = f_{n'}(V_{n'}) \oplus \ker f_{n'+1}|_{V_{n'+1}} \oplus C_{n'+1}$ for some $C_{n'}$ and $C_{n'+1}$, with $\tilde{C}_{n'}$ specializing to $C_{n'}$, and in particular, having full rank under $f_{n'}$ except possibly on a closed subset of the base supported away from x , where the rank could drop. Away from this locus on the base, if we replace $V_{n'+1}$ by $f_{n'}\tilde{V}_{n'} \oplus \ker f_{n'+1}|_{V_{n'+1}} \oplus C_{n'+1}$ (that is, if we set $\tilde{C}_{n'+1} = f_{n'}\tilde{V}_{n'} \oplus C_{n'+1}$), noting that $f_{n'}\tilde{V}_{n'} = f_{n'}\tilde{C}_{n'}$, we obtain a lifting with the desired properties. □

The key notion for getting a handle on the LG scheme is the following:

DEFINITION A.10. — *We say that a point of a linked Grassmannian scheme is exact if the corresponding collection of vector spaces V_i satisfy the conditions that $\ker g_i|_{V_{i+1}} \subset f_i(V_i)$ and $\ker f_i|_{V_i} \subset g_i(V_{i+1})$ for all i .*

The last part of assertion (ii) of the following lemma is gratuitous, but it follows immediately from the argument for the rest, and may perhaps shed some little light on the overall situation.

LEMMA A.11. — *We have the following description of exact points:*

- (i) *The exact points form an open subscheme of LG, and are naturally described as the complement of the closed subscheme on which $\text{rk } f_i|_{V_i} + \text{rk } g_i|_{V_{i+1}} < r$ for some i .*
- (ii) *In the case $s = 0$, we find that we can describe exact points as those with $\text{rk } f_i(V_i) + \text{rk } g_i(V_{i+1}) = r$ for all i , even for arbitrary scheme-valued points, and we also find that an exact point has $f_i(V_i)$ a sub-bundle of V_{i+1} , and $g_i(V_{i+1})$ a sub-bundle of V_i for all i .*

Proof. — We certainly get a closed subscheme as described, simply by taking the union over all i and r_1, r_2 with $r_1 + r_2 < r$ of the loci described by $\text{rk } f_i|_{V_i} \leq r_1$ and $\text{rk } g_i|_{V_{i+1}} \leq r_2$. We immediately see that the points of this set are precisely the complement of the exact points, since outside the locus where s vanishes, both f_i and g_i are invertible, and correspondingly all points are exact; on the other hand, if s vanishes at our point, we have $\text{im } f_i \subset \ker g_i$ and $\text{im } g_i \subset \ker f_i$ for all i , so we already have that $\dim f_i(V_i) + \dim g_i(V_{i+1}) \leq r$ and we get strict inequality if and only if these containments are strict.

For the second part, if a T -valued point satisfies $\text{rk } f_i|_{V_i} + \text{rk } g_i|_{V_{i+1}} = r$ for all i on all of T , by definition it is in the complement of the closed subscheme of non-exact points defined above, so it is certainly exact. Conversely, for the other direction it suffices to work over local rings, so suppose we have a T -valued point (V_i) of LG where T is local, and the point (\bar{V}_i) of LG at the closed point of T is exact. Since s is zero on T , then for any given i we have $\ker \bar{f}_i|_{\bar{V}_i} = \bar{g}_i(\bar{V}_{i+1})$ and vice versa; in particular, if we choose $\bar{v}_1, \dots, \bar{v}_{r_1} \in \bar{V}_i$ such that the $\bar{f}_i(\bar{v}_j)$ form a basis of $\bar{f}_i(\bar{V}_i)$, and $\bar{v}'_1, \dots, \bar{v}'_{r_2} \in \bar{V}_{i+1}$ such that the $\bar{g}_i(\bar{v}'_j)$ form a basis of $\bar{g}_i(\bar{V}_{i+1})$, we find that $r_1 + r_2 = r$, and we obtain a basis \bar{e}_i (resp., \bar{e}'_i) for \bar{V}_i (resp., \bar{V}_{i+1}) given by the \bar{v}_j and $g_i(\bar{v}'_j)$ (resp., \bar{v}'_j and $f_i(\bar{v}_j)$). By Nakayama's lemma, we can lift this situation to the local ring, and easily check that the desired assertions follow. \square

Our main technical lemma for this appendix is:

LEMMA A.12. — *We have the following statements on exact points:*

- (i) *The exact points are dense in LG, and indeed dense in every fiber.*
- (ii) *Given any exact point $x \in \text{LG}$, let y be its image in S , suppose A is a local ring, and A' a quotient of A . Let $T = \text{Spec } A$, and $T' = \text{Spec } A'$. Then given any commutative diagram containing the solid arrows of*

$$\begin{array}{ccc}
 T' & \xrightarrow{f} & \text{LG} \\
 \downarrow & \nearrow & \downarrow \\
 T & \longrightarrow & S
 \end{array}$$

with the closed point of T' mapping to x , the dashed arrow may also be filled in. In particular, x is a smooth point of LG over S .

Proof. — For (i), To see that the exact points are dense in every fiber, suppose we have a non-exact point; we just observed that this corresponds to a set of V_i such that for at least one i , we have $\dim f_i(V_i) + \dim g_i(V_{i+1}) < r$. In particular, we are in the situation where $f_i g_i = g_i f_i = 0$. Now, choose the smallest i such that $\dim f_i(V_i) + \dim g_i(V_{i+1}) < r$, and truncate our linked Grassmannian to $i + 1$; here, we show that there are nearby points in the fiber such that the condition $\dim f_i(V_i) + \dim g_i(V_{i+1}) = r$ is satisfied. We leave V_1 through V_i unmodified. By hypothesis, there are vectors in V_{i+1} in the kernel of g_i which are not in $f_i(V_i)$, and vice versa; indeed, we see that $r' := \dim \ker g_i|_{V_{i+1}} - \dim f_i(V_i) = \dim \ker f_i|_{V_i} - \dim g_i(V_{i+1}) = r - \dim f_i(V_i) - \dim g_i(V_{i+1})$. Choose C_i and C_{i+1} in $\ker f_i|_{V_i}$ and $\ker g_i|_{V_{i+1}}$ of dimension r' , intersecting $g_i(V_{i+1})$ and $f_i(V_i)$ trivially; we have that together with these spaces, they must complete the span of $\ker f_i|_{V_i}$ and

ker $g_i|_{V_{i+1}}$ respectively. Since $C_i \subset \ker f_i|_{V_i}$, it is in $\text{im } g_i$, and we can find $e_1, \dots, e_{r'} \in \mathcal{E}_{i+1}$, whose span is necessarily disjoint from V_{i+1} , and which map to a basis of C_i under g_i . By Lemma A.7, we can write $V_{i+1} = f_i(V_i) \oplus \ker f_{i+1}|_{V_{i+1}} \oplus C_{i+1} \oplus C''$ for some C'' . If we take any basis $e'_1, \dots, e'_{r'}$ for C_{i+1} , we can make a family \tilde{V}_{i+1} over \mathbb{A}^1 by replacing C_{i+1} with the span of $e'_i + te_i$ for all i , as t varies.

Now, \tilde{V}_{i+1} specializes to V_{i+1} at $t = 0$, and we see that it always remains linked to V_1, \dots, V_i , left unmodified: it certainly maps into V_i under g_i , since we are modifying basis elements by the e_i , which were chosen to map into V_i ; on the other hand, our construction leaves the summand $f_i(V_i)$ unmodified, so f_i certainly maps V_i into any member of \tilde{V}_{i+1} . We also observe that we now have $\dim f_i(V_i) + \dim g_i(\tilde{V}_{i+1}) = r$ whenever $t \neq 0$: indeed, C_{i+1} was in the kernel of g_i for $t = 0$, so we still have $g_i(\tilde{V}_{i+1}) \supset g_i(V_{i+1})$; for any $t \neq 0$, C_{i+1} maps isomorphically to C_i under g_i ; finally, since we chose C_i to, together with $g_i(V_{i+1})$, span $\ker f_i|_{V_i}$, we find that for any $t \neq 0$, $g_i(\tilde{V}_{i+1}) = \ker f_i|_{V_i}$, giving the desired exactness at i . Now, by Lemma A.9, we can lift this family to a family \tilde{V}_j for all j , specializing to our given point, but now satisfying $\dim f_i(V_i) + \dim g_i(V_{i+1}) = r$ for a general point in the family; we conclude that the points which are non-exact at the i th step (but exact for $j < i$) are in the closure of those which are exact through the i th step, and by induction are actually in the closure of the points which are exact at all steps.

For assertion (ii), $f(T')$ corresponds to a collection $\{V_i\}_i$ over A' ; Our \mathcal{E}_i are now all free modules of rank d over A , and we simply want to produce free A -submodules \tilde{V}_i (with free quotients) linked by the f_i and g_i and restricting to the given V_i in the quotient ring A' . To do this, denote by \bar{V}_i the collection of subspaces over the residue field of A' corresponding to x , and let r_i, r'_i be the dimensions of $f_i(\bar{V}_i), g_i(\bar{V}_{i+1})$ respectively for each i .

We begin by choosing bases \bar{e}_j^i of \bar{V}_i , and lifting appropriately. If our f_i and g_i are invertible at the closed point, which is to say, if the s from condition (I) of a linked Grassmannian is non-zero in $\kappa(y)$, we simply choose an arbitrary basis \bar{e}_j^1 of \bar{V}_1 , and take its images under the f_i . We then lift the \bar{e}_j^1 to V_1 , and take images under the f_i , to obtain bases of the V_i , and lift by the same process to \mathcal{E}_i , defining sub-modules \tilde{V}_i .

Otherwise, if we had $s = 0$ in $\kappa(y)$, for each i we choose \bar{e}_j^i in three categories: first, $r - r_{i-1} - r'_i$ elements which are linearly independent from the span of $f_{i-1}(\bar{V}_{i-1}) \cup g_i(\bar{V}_{i+1})$; second, r_{i-1} elements generating $f_{i-1}(\bar{V}_{i-1})$; and third, r'_i elements generating $g_i(\bar{V}_{i+1})$. Noting that even without exactness, since $s = 0$, we have $r = \dim f_{i-1}(\bar{V}_{i-1}) + \dim \ker f_{i-1}|_{\bar{V}_{i-1}} \geq$

$r_{i-1} + r'_{i-1} \geq r_{i-1} + r'_i$, and $f_{i-1}(\bar{V}_{i-1}) \subset \ker g_{i-1}|_{\bar{V}_i}$ which is disjoint from $g_i(\bar{V}_{i-1})$, so we see that this is possible. Moreover, by choosing the first category for all i first, we can inductively construct the basis elements in the second and third categories to be images under f_{i-1} and g_i of basis elements already chosen, moving from $i = 1$ to $i = n$ for the second category, and the opposite direction for the third. Next, choose lifts e_j^i to the V_i , using the same process of lifting all \bar{e}_j^i in the first category first, and defining the rest as iterated images under f_{i-1} and g_i . Finally, lift the e_j^i to $\tilde{e}_j^i \in \mathcal{E}_i$, once again via the same process, and define \tilde{V}_i to be the span of the \tilde{e}_j^i .

By Nakayama’s lemma, the e_j^i constructed in either case give free generators for the V_i . Applying Nakayama’s lemma again, we find that the \tilde{V}_i are sub-bundles of \mathcal{E}_i of rank r , and clearly they specialize to the V_i , so we need only check that they are linked. In the case that s was non-zero in $\kappa(y)$, the \tilde{V}_i are linked under the f_i by construction, and must likewise be linked under the g_i , since g_i is a unit times the inverse of f_i . In the case where s was zero in $\kappa(y)$, take any \tilde{e}_j^i for $i < n$; we show that its image under f_i is a scalar multiple of $\tilde{e}_{j'}^{i+1}$ for some j' . We now apply exactness, to note that either $f_i(\tilde{e}_j^i) = 0$ and $\tilde{e}_j^i = g_i(\tilde{e}_{j'}^{i+1})$, or we defined $\tilde{e}_{j'}^{i+1} = f_i(\tilde{e}_j^i)$, for some j' . In the latter case, we are done, while in the former case we simply observe that $f_i(\tilde{e}_j^i) = s(\tilde{e}_{j'}^{i+1})$. The same argument works for the g_i , so we have constructed a map from T to LG lifting f , which by [15, Prop. 17.14.2] completes the proof of part (ii). \square

The following proposition provides a strong converse to part (ii) of the above lemma:

PROPOSITION A.13. — *The non-exact points of a fiber are precisely the intersections of the components of that fiber.*

Proof. — Since the exact points are smooth, they are certainly not in any intersection of components. For the other direction, we first make the following observation: because ranks can only drop under specialization, given two exact points $\{V_i\}_i$ and $\{V'_i\}_i$, with $r_i := \dim f_i(V_i)$ and $r'_i := \dim f_i(V'_i)$, if some $r_i \neq r'_i$, then the two points must lie on distinct components of LG . Thus, to show that any non-exact point is in the intersection of components, it suffices to exhibit it as the specialization of two different exact points with distinct r_i .

Looking at the proof of Lemma A.12 part (i), we see that any point which is non-exact at i_0 , with i_0 minimal, can be expressed as the specialization of an exact point with r_i unchanged for all $i \leq i_0$; however, upon closer

examination, we see that in fact the process leaves all the r_i unchanged, simply increasing the dimensions of the $g_i(V_{i+1})$ as necessary to make the points exact. On the other hand, we note that the linked Grassmannian situation is completely symmetric in the f_i and g_i , so now that we have shown that any point can be written as the specialization of an exact point with the $\dim f_i(V_i)$ unchanged, it follows by symmetry that there is another exact point specializing to our given point, leaving the dimensions of the $g_i(V_{i+1})$ intact, and therefore necessarily increasing at least some of the r_i . This then expresses our non-exact point as lying in the intersection of two components, as desired. \square

We can also use the smoothness at exact points to compute the dimension of fibers of LG:

LEMMA A.14. — *The fibers of LG over S have every component of dimension precisely $r(d - r)$.*

Proof. — In view of Lemma A.12, we can compute the dimension of any component of the fiber by showing that its tangent space at any exact point has the desired dimension. Since we are only looking at a fiber, we set $S = \text{Spec } k$. If $s \neq 0$ in k , $\text{LG} \cong \mathbb{G}(r, d)$, and is smooth of dimension $r(d - r)$, so there is nothing to show. Otherwise, suppose we have a collection of V_i corresponding to an exact point. Then $\ker f_i|_{V_i} = g_i(V_{i+1})$ for all i , so we use Lemma A.7 to write each V_i as $f_{i-1}(V_{i-1}) \oplus g_i(V_{i+1}) \oplus C_i$ for some complementary space C_i . Our first assertion is that the dimensions d_i of the C_i add up to r . Indeed, if we let $r_i = \dim f_i(V_i)$, and $r'_i = \dim g_i(V_{i+1})$, we have $r_i = r - r'_i$ from exactness, and for $1 < i < n$, $d_i = r - r_{i-1} - r'_i = r_i - r_{i-1}$, with $d_1 = r - r'_1 = r_1$ and $d_n = r - r_{n-1}$, so we see we indeed have $\sum_i d_i = r$.

The next claim is that first-order deformations of the V_i inside of LG correspond precisely to first-order deformations of each C_i individually inside \mathcal{E}_i , taken modulo deformations of the C_i which remain inside V_i . Any deformation of the C_i together will yield a deformation of the V_i : we use our direct sum decomposition to inductively define the induced deformation, obtaining deformations of $f_i(V_i)$ as the image of the deformation of C_{i-1} together with the (inductively obtained) deformation of $f_{i-1}(V_{i-1})$, and similarly for the $g_i(V_{i+1})$. Moreover, since each $f_i(V_i)$ is spanned by $f_{i-1}(V_{i-1})$ together with C_{i-1} , this is the only possible way to obtain a deformation of the V_i given deformations of the C_i . Clearly, two deformations of the C_i will yield equivalent deformations of V_i if and only if their difference is a deformation of the C_i inside of its V_i . Finally, any deformation

of the V_i may be expressed (non-uniquely) as a deformation of its summands, and in particular gives a deformation of the C_i , at least up to the same equivalence relation. Since the deformation of the V_i induced by the deformations of the C_i was unique, this must invert our first construction, completing the proof of the claim.

Now we are done: first-order deformations of any given C_i are given by the tangent space to $\mathbb{G}(d_i, d)$, which is a variety smooth of dimension $d_i(d - d_i)$, so has $d_i(d - d_i)$ -dimensional tangent space at any point. Similarly, the space of deformations of C_i inside of V_i has dimension $d_i(r - d_i)$; the difference is $d_i(d - r)$. Thus, the total dimension of our tangent space is $\sum_i d_i(d - r) = r(d - r)$, as asserted. \square

We now have all the tools to prove our main result:

THEOREM A.15. — *A linked Grassmannian scheme is a closed subscheme of the obvious product of Grassmannian schemes over S ; it is projective over S , and each component has codimension $(n - 1)r(d - r)$ inside the product, and maps surjectively to S . If s is non-zero, then LG is also irreducible.*

Proof. — We already have that the linked Grassmannian is projective over S , and lies inside the obvious product of Grassmannians, which we denote by G . It is easy to see each component maps dominantly onto S , since the exact points are both smooth and dense by Lemma A.12.

For the dimension statement, given any component of LG, let x be an exact point of LG on the specified component, and not on any other component, and s the image of x in S . Since S is Cohen-Macaulay, everything is catenary, so codimensions can be computed naively for irreducible spaces. By Lemma A.14, we have that $\mathcal{O}_{\text{LG}, x}$ is smooth over $\mathcal{O}_{S, s}$ of relative dimension $r(d - r)$, and in particular integral. Similarly, $\mathcal{O}_{G, x}$ is locally affine over $\mathcal{O}_{S, s}$, hence integral and smooth of relative dimension $nr(d - r)$. The desired codimension statement then follows from [15, Prop. 17.5.8 (i)].

Finally, when s is non-zero, over the open subset of S where s is invertible, the fibers are all simply Grassmannians of dimension $r(d - r)$; since the map is proper, we conclude that LG is irreducible over this locus, of dimension $r(d - r)$. On the other hand, since every component maps dominantly to S , there cannot be any component of LG contained in the locus where s vanishes, yielding the desired irreducibility. \square

Warning A.16. — Lemma A.11 sounds quite innocuous, but there are some pitfalls to be aware of. Consider the simple example of $n = d = 2$,

$r = 1$, $S = \text{Spec } k$, $\mathcal{E}_1 = \mathcal{E}_2 = k^2$, $f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $f_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. In

this case, if V_1 is generated by $v_1 = \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$ and V_2 by $v_2 = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$, we find the condition for them to be linked is simply that $X_0Y_1 = 0$, and it is easy enough to check that we actually get that LG is scheme-theoretically cut out by this equation inside $\mathbb{P}^1 \times \mathbb{P}^1$, giving a pair of \mathbb{P}^1 's attached at $X_0 = Y_1 = 0$, which is the only non-exact point. Our lemma has shown that deformations have to behave well at the exact points, but if we consider the T -valued point for $T = \text{Spec } k[\epsilon]/(\epsilon^2)$ with V_1 generated by $v_1 = \begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$ and V_2 generated by $v_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$, we note two pathologies:

First, this point actually satisfies our initial set-theoretic description of an exact point, that $\ker g_1|_{V_2} \subset f_1(V_1)$ and vice versa, as both images and kernels will be given precisely by ϵv_i . So this description, while dealing with both $s = 0$ and s invertible simultaneously, is only valid from a set-theoretic point of view.

Second, while we have shown that at any (scheme-valued) exact point, there will be an r_1 and r_2 with $r_1 + r_2 = r$ and $\text{rk } f_1|_{V_1} \leq r_1$, $\text{rk } g_1|_{V_2} \leq r_2$, we see that by allowing the ranks to drop at the closed point, we actually allow them to increase on the local ring level. Specifically, in our case $r = 1$, so either r_1 or r_2 would have to be 0, but neither f_1 nor g_1 is the zero map. Of course, this makes perfect sense geometrically, as the node will necessarily have tangent vectors which don't point along either branch, but it underscores the fact that the T -valued points of a union of schemes is not simply the union of the T -valued points of the individual schemes.

We conclude with an example and some further questions which we have not pursued here because they are not necessary for our applications.

Example A.17. — We consider the situation of $S = \text{Spec } k$, $n = 2$. In this case, it is easy to describe the components explicitly, as well as to see their dimensions without invoking any deformation theory. We already know that if $s \neq 0$, we just get a Grassmannian, so we assume that $s = 0$. If we write $d_1 = \text{rk } f_1$, $d_2 = \text{rk } g_1$ (on the entire vector space), we have $d_1 + d_2 = d$ by condition (II) of a linked Grassmannian. We will see that there are $\min\{r + 1, d - r + 1, d_1 + 1, d_2 + 1\}$ components, each of dimension $r(d - r)$, and indexed by the dimension of $f_1(V_1)$ on general points.

Indeed, we saw in the proof of Lemma A.9 that the fiber of any point V_1 of G_1 under truncation is simply the Grassmannian of vector spaces V_2 con-

taining $f_1(V_1)$ and contained in $g_1^{-1}(V_1)$, which had dimension $\dim \ker g_1 + \dim(V_1 \cap \operatorname{im} g_1)$.

We need to see that this dimension depends only on the dimension of $f_1(V_1)$, which we will denote by r_1 . By condition (II) of a linked Grassmannian, $\ker g_1 = \operatorname{im} f_1$, and $\operatorname{im} g_1 = \ker f_1$, so we may write this as $d_1 + \dim(V_1 \cap \ker f_1)$. Furthermore, $\dim(V_1 \cap \ker f_1) = r - r_1$, so we can write everything in terms of r_1 , as desired. Specifically, we have a Grassmannian of r -dimensional subspaces of a $(d_1 + r - r_1)$ -dimensional space, containing an r_1 -dimensional space, and this has dimension $(r - r_1)(d_1 - r_1)$.

We now obtain our assertions without trouble: fix an $r_1 \leq \min\{r, d_1\}$ also satisfying $r_1 \geq \max\{0, r - d_2\}$, and consider the locally closed subset $G_1^{r_1}$ in G_1 with $\dim f_1(V_1) = r_1$. Note that the specified range is precisely the range for which this will be non-empty. Now, $G_1^{r_1}$ is an open subset of the locus in G_1 with $\dim f_1(V_1) \leq r_1$, which corresponds simply to a Schubert cycle, which is irreducible of codimension $(r - r_1)(d_1 - r_1)$. If we base change LG over G_1 to $G_1^{r_1}$, we get a proper map with irreducible equidimensional fibers, mapping surjectively to an irreducible base, so in fact LG becomes irreducible, and has dimension precisely $r(d - r)$. Since this dimension remains constant as r_1 decreases, and the codimension of $G_1^{r_1}$ increases as r_1 decreases, we find we must have exactly one irreducible component of LG for each choice of r_1 .

QUESTION A.18. — *Can we show that LG is flat over S ? That it is reduced?*

QUESTION A.19. — *Can we describe the components of LG for $n > 2$?*

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Manuscrit reçu le 8 novembre 2004,
révisé le 26 septembre 2005,
accepté le 7 novembre 2005.

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