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Urtzi BUIJS & Aniceto MURILLO

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# BASIC CONSTRUCTIONS IN RATIONAL HOMOTOPY THEORY OF FUNCTION SPACES

by Urtzi BUIJS & Aniceto MURILLO (\*)

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ABSTRACT. — Via the Bousfield-Gugenheim realization functor, and starting from the Brown-Szcarba model of a function space, we give a functorial framework to describe basic objects and maps concerning the rational homotopy type of function spaces and its path components.

RÉSUMÉ. — Moyennant le foncteur de réalisation de Bousfield-Gugenheim et à l'aide du modèle de Brown Szcarba d'un espace de fonctions comme point de départ, on décrit les objets basiques et les applications relatives au type d'homotopie rationnelle des espaces fonctionnels et de leurs composantes arc-connexes.

## 1. Introduction

There are two choices to describe the equivalence between the homotopy categories of rational nilpotent CW-complexes of finite type and that of commutative differential graded algebras over the rationals. These choices depends on how to realize a given algebra. On one hand, one may proceed as in [15], considering a Sullivan model and then build its realization as a Postnikov tower whose topological description mimics the algebraic properties of the model. This is highly geometrical but quite unnatural. On the other hand, one can take a general algebra and realizing it as a simplicial set via the Bousfield-Gugenheim realization functor [1]. This might be less transparent but totally functorial.

In particular both options are available to the study of the rational homotopy type of the function (or mapping) space  $\mathcal{F}(X, Y)$ . The first approach

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was first followed by Thom [16] and then by the fundamental work of Haefliger [7] where he supplied a model for the path component of sections, homotopic to a fixed one, of a nilpotent bundle. In particular, a model of a given component of a mapping space is produced. Later on, this approach was further developed by other authors [4, 13, 14, 17]. However, due to the non functorial description of the Haefliger model, it is not easy to use it to describe further geometrical behaviour of function spaces.

Under the second, and more functorial approach, it is essential the paper of Brown and Szczarba [3] in which it is proved that the Bousfield-Gugenheim realization of a particular commutative differential  $\mathbb{Z}$ -graded algebra (CDGA henceforth), which already appears in [7], is homotopy equivalent to the (non connected!) function space  $\mathcal{F}(X, Y)$ . It is also shown how to obtain from this algebra a quotient CDGA whose realization is a given path component of the function space, and therefore, the Haefliger model of this component is retrieved.

In this paper we continue with the study under this natural approach, and present models for some of the basic objects and maps concerning function spaces. By a model of a non connected object (or a map between them) we mean a  $\mathbb{Z}$ -graded CDGA (or a morphism) whose simplicial realization has the same homotopy type as the singular simplicial approximation of the chosen object. We also present an explicit way of restricting algebraically to components obtaining Sullivan models of nilpotent spaces and maps relating functions spaces, retrieving in particular the Haefliger model for its components.

We now state our main results: For any function space  $\mathcal{F}(X, Y)$  we shall always assume  $X$  and  $Y$  to be CW-complexes with  $X$  finite, so that  $\mathcal{F}(X, Y)$  is itself a CW-complex [12], and with both  $X$  and  $Y$  nilpotent of finite type over  $\mathbb{Q}$ , so that the components of  $\mathcal{F}(X, Y)$  are also nilpotent and of finite  $\mathbb{Q}$ -type [9, Thm.2.5]. Given  $f: X \rightarrow Y$ , denote by  $\mathcal{F}(X, Y; f)$  the component of  $\mathcal{F}(X, Y)$  containing  $f$ . We first give an explicit model of the evaluation map which includes in particular the original result of Haefliger [7, Thm.3.2] and [10, Thm.1.1].

**THEOREM 1.1.** — *Let  $(\Lambda(V \otimes B_*), \tilde{d})$  be the Brown-Szczarba model for  $\mathcal{F}(X, Y)$  (see §2). Then,*

$$\omega: (\Lambda V, d) \rightarrow (\Lambda(V \otimes B_*), \tilde{d}) \otimes B, \quad \omega(v) = \sum_i (-1)^{\alpha|b_i|} (v \otimes \beta_i) \otimes b_i,$$

where  $\{b_i\}$  is a basis for  $B$ ,  $\{\beta_i\}$  its dual and  $\alpha(n)$  is the integer part of  $(n+1)/2$ , is a model of the evaluation map

$$\omega: \mathcal{F}(X, Y) \times X \rightarrow Y, \quad \omega(f, x) = f(x).$$

In particular, the map

$$\omega_0: (\Lambda V, d) \rightarrow (\Lambda(V \otimes B_*), \tilde{d}), \quad \omega_0(v) = v \otimes 1^*,$$

is a model of the evaluation of the base point  $\omega_0: \mathcal{F}(X, Y) \rightarrow Y, \omega_0(f) = f(x_0)$ .

Moreover, given  $\zeta: (\Lambda V, d) \longrightarrow (\Lambda W, d)$  a Sullivan model for  $f: X \rightarrow Y$ , there is an explicit morphism  $\xi: (\Lambda(V \otimes C_*), \tilde{d}) \longrightarrow (\Lambda(W \otimes C_*), \tilde{d})$  (see §3) between the models of  $\mathcal{F}(Z, Y)$  and  $\mathcal{F}(Z, X)$  satisfying:

**THEOREM 1.2.** — *The commutative square*

$$\begin{array}{ccc} (\Lambda(W \otimes C_*), \tilde{d}) \otimes C & \xleftarrow{\xi \otimes 1_C} & (\Lambda(V \otimes C_*), \tilde{d}) \otimes C \\ \omega \uparrow & & \uparrow \omega \\ (\Lambda W, d) & \xleftarrow{\zeta} & (\Lambda V, d) \end{array}$$

is a model of

$$\begin{array}{ccc} \mathcal{F}(Z, X) \times Z & \xrightarrow{(f)_* \times 1_Z} & \mathcal{F}(Z, Y) \times Z \\ \omega \downarrow & & \downarrow \omega \\ X & \xrightarrow{f} & Y. \end{array}$$

Next, we show how to obtain from these results the corresponding Sullivan models when considering a fixed component of the function space. Indeed, given  $(\Lambda(V \otimes B_*), \tilde{d})$  the model of  $\mathcal{F}(X, Y)$  and a map  $f: X \rightarrow Y$ , there is a Sullivan model (in fact this is the Haefliger model) of the component  $\mathcal{F}(X, Y; f)$  of the form  $(\Lambda S_f, \bar{d})$  in which  $S_f^{\geq 2} = (V \otimes B_*)^{\geq 2}$  and  $S_f^1$  is a quotient of  $(V \otimes B_*)^1$  (see §4). Then we prove:

**THEOREM 1.3.** — *The CDGA morphism*

$$\omega_0: (\Lambda V, d) \longrightarrow (\Lambda S_f, \bar{d}),$$

given by  $\omega_0(v) = v \otimes 1^*$  if  $v \in V^{\geq 2}$ , or its projection over  $S_f^1$  if  $v \in V^1$ , is a Sullivan model of the evaluation at the base point  $\omega_0: \mathcal{F}(X, Y; f) \longrightarrow Y$ .

Moreover, with the notation of Theorem 1.2,  $\xi$  induces a morphism  $\bar{\xi}$  for which the square

$$\begin{array}{ccc}
 (\Lambda S_g, \bar{d}) & \xleftarrow{\bar{\xi}} & (\Lambda S_{fg}, \bar{d}) \\
 \omega_0 \uparrow & & \uparrow \omega_0 \\
 (\Lambda W, d) & \xleftarrow{\zeta} & (\Lambda V, d)
 \end{array}$$

is a Sullivan model of

$$\begin{array}{ccc}
 \mathcal{F}(Z, X; g) & \xrightarrow{(f)_*} & \mathcal{F}(Z, Y; f \circ g) \\
 \omega_0 \downarrow & & \downarrow \omega_0 \\
 X & \xrightarrow{f} & Y.
 \end{array}$$

As in this result,  $\omega_0$  is a fibration with fibre  $\mathcal{F}^*(X, Y; f)$ , the space of pointed maps, we easily obtain the following consequence, a form of which already appeared in [4], [10] or [17].

**COROLLARY 1.4.** — *The CDGA  $(\Lambda(S_f/V), \bar{d})$  is a Sullivan model of  $\mathcal{F}^*(X, Y; f)$ .*

Finally, we translate the constructions above to rational homotopy groups by taking the homology of the indecomposables of the given sullivan models and maps.

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## 2. Notation, basic facts and tools

All spaces considered henceforth shall be of the homotopy type of connected CW-complexes. We will heavily rely on known results from homotopy theory of simplicial sets and from rational homotopy theory. For that [6] or [11] and [5] are respectively standard and excellent references. Here, mainly to set the framework in which we work and to fix notation, we recall some basics:

Denote by

$$\mathbf{Top} \xrightleftharpoons[S_*]{S_*} \mathbf{SimpSet}$$

the singular and realization adjoint functors relating the categories of simplicial sets and topological spaces. These induce equivalences between the (pointed if we wish) homotopy category of CW-complexes and the homotopy category of Kan complexes [12]. For each space  $X$  and each simplicial set  $K$ , we shall denote by  $\psi(K): K \xrightarrow{\cong} S_*|K|$  and  $\phi(X): |S_*X| \xrightarrow{\cong} X$  the corresponding weak equivalences.

On the other hand, if **CDGA** is the category of cochain  $\mathbb{Z}$ -graded commutative differential graded algebras, there are adjoint functors [1]

$$\mathbf{SimpSet} \begin{array}{c} \xrightarrow{A_{PL}} \\ \xleftarrow{\langle \cdot \rangle} \end{array} \mathbf{CDGA},$$

where:  $A_{PL}(X)$  is the CDGA of *piecewise-linear polynomial forms* on  $X$  with rational coefficients and  $\langle A \rangle$  is the *simplicial realization* of the CDGA  $A$ : For any  $q \geq 0$ ,  $\langle A \rangle_q = \text{hom}(A, (A_{PL})_q)$ , i.e., the set of morphisms from  $A$  to  $\Lambda(x_1, \dots, x_q, dx_1, \dots, dx_q)$ . The face and degeneracies operators are induced by their counterparts in the simplicial algebra  $(A_{PL})_*$  [1, 5].

The composition of the above pairs of adjoint functors produce equivalences when restricted to the homotopy category of rational nilpotent space of finite type over  $\mathbb{Q}$  and the homotopy category of connected CDGA's over  $\mathbb{Q}$  of the same homotopy type of Sullivan algebras of finite type [1, 5]. Abusing of notation we shall write  $A_{PL}(X) = A_{PL}(S_*X)$  and  $|A| = |\langle A \rangle|$  for any space  $X$  and any CDGA  $A$ . Given a non necessarily connected space  $Z$ , a CDGA  $A$  is a *model* of  $Z$  if  $S_*(Z)$  and  $\langle A \rangle$  are simplicial sets of the same homotopy type. As we are mainly concerned in this paper on mapping spaces, let us describe these objects whenever we take simplicial sets or CDGA's. Given  $K, L$  simplicial sets,  $\mathcal{F}(K, L)$  is the simplicial set defined by  $\mathcal{F}(K, L)_q = \text{hom}(K \times \Delta[q], L)$ , where  $\Delta[q] \in \mathbf{SimpSet}$  is the *standard  $n$ -simplex*: For each  $n \geq 0$ ,  $[n] = \{0, \dots, n\}$  and  $(\Delta[n])_q$  is the set of non decreasing maps  $[q] \rightarrow [n]$ . Call  $\Delta_q$  the fundamental simplex  $1_{[q]} = (0, 1, \dots, q) \in \Delta[q]_q$ . Faces and degeneracies operators on  $\mathcal{F}(K, L)$  are defined by  $\partial_i f = f \circ (1_K \times \delta_i)$  and  $s_i f = f \circ (1_K \times \sigma_i)$ . Whenever  $L$  is Kan,  $\mathcal{F}(K, L)$  is also Kan and it has the same homotopy type as  $S_*\mathcal{F}(|K|, |L|)$  via the equivalence

$$\alpha: \mathcal{F}(K, L) \xrightarrow{\cong} S_*\mathcal{F}(|K|, |L|),$$

in which  $\alpha(f) \in \Delta^q \mathcal{F}(|K|, |L|)$  is given by the exponential law applied to the composition

$$|K| \times \Delta^q \xrightarrow{\cong} |K| \times |\Delta[q]| \xrightarrow{\cong} |K \times \Delta[q]| \xrightarrow{|f|} |L|.$$

The evaluation map  $\omega: \mathcal{F}(K, L) \times K \rightarrow L$  is defined by  $\omega(f, \sigma) = f(\sigma, \Delta_q)$ . This map can be identified with  $S_*\omega: S_*(\mathcal{F}(|K|, |L|) \times |K|) \cong S_*\mathcal{F}(|K|, |L|) \times S_*|K| \rightarrow S_*|L|$  via the following result obtained by direct calculation:

PROPOSITION 2.1. — *The following diagram commutes:*

$$\begin{array}{ccc}
 S_*\mathcal{F}(|K|, |L|) \times S_*|K| & \xleftarrow[\cong]{\alpha \times \psi(K)} & \mathcal{F}(K, L) \times K \\
 S_*\omega \downarrow & & \downarrow \omega \\
 S_*|L| & \xleftarrow[\cong]{\psi(L)} & L.
 \end{array}$$

On the other hand given CDGA's  $A$  and  $B$  define  $\mathcal{F}(A, B) \in \mathbf{SimpSet}$  by  $\mathcal{F}(A, B)_q = \text{hom}(A, (A_{PL})_q \otimes B)$  with faces and degeneracies given by  $\partial_i f = (\partial_i \otimes 1_B) \circ f$ ,  $s_j f = (s_j \otimes 1_B) \circ f$ . If  $\phi: B \rightarrow C$  is a CDGA morphism, we denote by

$$\phi_*: \mathcal{F}(A, B) \rightarrow \mathcal{F}(A, C)$$

the simplicial map given which assigns to each morphism  $\psi: A \rightarrow (A_{PL})_q \otimes B$  the composition  $\phi_*(\psi) = (1_{(A_{PL})_q} \otimes \phi) \circ \psi$ . If  $A, B, C$  are connected CDGA's and  $\phi$  is a quasi-isomorphism, then  $\phi_*$  is a weak equivalence.

Next we recall the Brown-Szczarba model of the function space  $\mathcal{F}(X, Y)$ , with  $X$  as always a finite complex and both  $X$  and  $Y$  nilpotent of finite type (over  $\mathbb{Q}$ ). Consider  $A = (\Lambda V, d) \xrightarrow{\cong} A_{PL}(Y)$  a Sullivan model (non necessarily minimal!) of  $Y$  and  $B \xrightarrow{\cong} A_{PL}(X)$  a quasi-isomorphism with  $B$  a connected CDGA of finite type. Let  $B_* = \text{hom}(B, \mathbb{Q})$  be the differential graded coalgebra dual of  $B$ , and consider the CDGA  $\Lambda(A \otimes B_*)$  with the natural differential induced by the one on  $A$  and by the dual of the differential of  $B$ . Now, consider the differential ideal  $I \subset \Lambda(A \otimes B_*)$  generated by  $1 \otimes 1^* - 1$  and by the elements of the form

$$a_1 a_2 \otimes \beta - \sum_j (-1)^{|a_2||\beta_j'|} (a_1 \otimes \beta_j') (a_2 \otimes \beta_j''),$$

$a_1, a_2 \in A$ ,  $\beta \in B$ , and  $\Delta\beta = \sum_j \beta_j' \otimes \beta_j''$ . Then, the composition

$$\rho: \Lambda(V \otimes B_*) \subset \Lambda(A \otimes B_*) \longrightarrow \Lambda(A \otimes B_*)/I$$

is an isomorphism of graded algebras [3, Thm.1.2], and therefore, considering on  $\Lambda(V \otimes B_*)$  the differential  $\tilde{d} = \rho^{-1}d\rho$ ,  $\rho$  is also an isomorphism of CDGA's. Then,  $(\Lambda(V \otimes B_*), \tilde{d})$  is a model of  $\mathcal{F}(X, Y_{\mathbb{Q}})$  [3, Thm.1.3]. In other words,  $S_*\mathcal{F}(X, Y)$  and  $\langle (\Lambda(V \otimes B_*), \tilde{d}) \rangle$  are homotopy equivalent.

In order to explicitly determine  $\tilde{d}$  on  $v \otimes \beta \in V \otimes B_*$ , calculate  $(dv) \otimes \beta + (-1)^{|v|} v \otimes d\beta$  and then use the relations which generate the ideal  $I$  to express  $(dv) \otimes \beta$  as an element of  $\Lambda(V \otimes B_*)$ .

### 3. Compatible models for the evaluation map

In this section we properly state and prove Theorems 1.1 and 1.2. For that, fix as before  $\varphi: (\Lambda V, d) \xrightarrow{\cong} A_{PL}(Y)$  a Sullivan model of  $Y$  (i.e., a KS-complex non necessarily minimal), and  $\beta: B \xrightarrow{\cong} A_{PL}(X)$ , a quasi-isomorphism in which  $B$  is finite dimensional and satisfies an additional condition: the inclusion of the base point  $\{x_0\} \hookrightarrow X$  determines a canonical augmentation  $\varepsilon: A_{PL}(X) \rightarrow A_{PL}(x_0) \cong \mathbb{Q}$ . Thus we may, and will, choose  $B$  connected and  $\beta$  preserving augmentations, i.e.,  $\varepsilon \circ \beta(1) = 1$ . Finally, fix  $\{b_i\}$  (resp.  $\{\beta_i\}$ ) a basis for  $B$  (resp. the dual basis of  $B_*$ ) and denote by  $\alpha(n)$  the integer part of  $(n + 1)/2$ . Then we have:

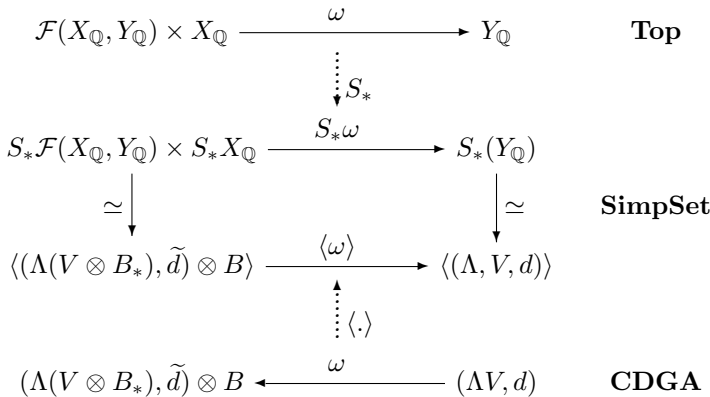
**THEOREM 3.1.** — *The CDGA morphism*

$$\omega: (\Lambda V, d) \rightarrow (\Lambda(V \otimes B_*), \tilde{d}) \otimes B, \quad \omega(v) = \sum_i (-1)^{\alpha|b_i|} (v \otimes \beta_i) \otimes b_i,$$

is a model of the evaluation map

$$\omega: \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}, \quad \omega(f, x) = f(x).$$

That is to say, there is a commutative diagram like the following in which the vertical arrows are equivalences:



As the inclusion  $i: \mathcal{F}(X, Y) \hookrightarrow \mathcal{F}(X, Y) \times X$  on the base point  $x_0$  is clearly modeled by  $(\Lambda(V \otimes B_*), \tilde{d}) \otimes B \xrightarrow{1 \otimes \varepsilon} (\Lambda(V \otimes B_*), \tilde{d})$ , and since  $\omega_0 = \omega \circ i$ , we immediately have:



COROLLARY 3.2. — *The CDGA morphism*

$$\omega_0: (\Lambda V, d) \longrightarrow (\Lambda(V \otimes B_*), \tilde{d}), \quad \omega_0(v) = v \otimes 1^*,$$

is a model of the evaluation map at the base point  $\omega_0: \mathcal{F}(X, Y_{\mathbb{Q}}) \longrightarrow Y_{\mathbb{Q}}$ ,  $\omega_0(g) = g(x_0)$ .

Note that in Theorem 3.1 we consider  $\mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$  while in Corollary 3.2 the evaluation map is defined over  $\mathcal{F}(X, Y_{\mathbb{Q}})$ : although there is a weak equivalence [14, Thm.2.3]  $\mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \simeq \mathcal{F}(X, Y_{\mathbb{Q}})$ , in the first result we evaluate at any point of  $X_{\mathbb{Q}}$  while in the second we simply do it at the base point of  $X$ .

*Proof.* — Start by choosing simplicial sets  $K = S_*(X_{\mathbb{Q}})$ ,  $L = S_*(Y_{\mathbb{Q}})$ . As  $Y$  is nilpotent, the natural map induced by the adjunction  $j: L \xrightarrow{\simeq} \langle A_{PL}(L) \rangle$ ,  $j(\sigma)(\Phi) = \Phi_{\sigma}$ , is an equivalence which may be composed with the map  $\langle \varphi \rangle: \langle A_{PL}(L) \rangle \xrightarrow{\simeq} \langle A \rangle$  to produce an equivalence  $h: L \xrightarrow{\simeq} \langle A \rangle$ . Apply just definitions to see that the following commutes:

$$(1) \quad \begin{array}{ccc} \mathcal{F}(K, L) \times K & \xrightarrow[h_* \times 1_K]{\simeq} & \mathcal{F}(K, \langle A \rangle) \times K \\ \omega \downarrow & & \downarrow \omega \\ L & \xrightarrow[h]{\simeq} & \langle A \rangle. \end{array}$$

Next, we use the homotopy equivalence  $\gamma: \mathcal{F}(A, A_{PL}(K)) \xrightarrow{\simeq} \mathcal{F}(K, \langle A \rangle)$  given in [2, Thm.20] which we now describe: Consider the morphism

$$A_{PL}(p_1) \otimes A_{PL}(p_2): A_{PL}(K) \otimes A_{PL}(\Delta[q]) \longrightarrow A_{PL}(K \times \Delta[q]),$$

induced by the projections  $K \xleftarrow{p_1} K \times \Delta[q] \xrightarrow{p_2} \Delta[q]$ , and the isomorphism of [5, Prop.10.4]  $\theta: (A_{PL})_q \longrightarrow A_{PL}(\Delta[q])$  which assigns to each  $\Omega \in (A_{PL})_q$  the only element  $\theta(\Omega) \in A_{PL}(\Delta[q])$  satisfying  $\theta(\Omega)(\Delta_q) = \Omega$ . Then, given  $\psi: A \longrightarrow (A_{PL})_q \otimes A_{PL}(K)$  an element of  $\mathcal{F}(A, A_{PL}(K))_q$ ,

$$\gamma(\psi) = \langle A_{PL}(p_1) \otimes A_{PL}(p_2) \circ \theta \otimes 1 \circ \psi \rangle \circ j: K \times \Delta[q] \longrightarrow \langle A \rangle.$$

As before,  $j: K \times \Delta[q] \xrightarrow{\simeq} \langle A_{PL}(K \times \Delta[q]) \rangle$  is the natural equivalence induced by adjunction. On the other hand, define a simplicial map

$$\bar{\omega}: \mathcal{F}(A, A_{PL}(K)) \times \langle A_{PL}(K) \rangle \longrightarrow \langle A \rangle$$

which sends each  $\psi: A \longrightarrow (A_{PL})_q \otimes A_{PL}(K)$  and  $\phi: A_{PL}(K) \rightarrow (A_{PL})_q$  to the composition

$$\bar{\omega}(\psi, \phi): A \xrightarrow{\psi} (A_{PL})_q \otimes A_{PL}(K) \xrightarrow{1 \cdot \phi} (A_{PL})_q.$$

It is then straightforward to check that the following commutes:

$$(2) \quad \begin{array}{ccc} \mathcal{F}(K, \langle A \rangle) \times K & \xleftarrow[\simeq]{\gamma \times j^{-1}} & \mathcal{F}(A, A_{PL}(K)) \times \langle A_{PL}(K) \rangle \\ \omega \downarrow & & \downarrow \bar{\omega} \\ \langle A \rangle & \xlongequal{\hspace{10em}} & \langle A \rangle. \end{array}$$

For the simplicial map  $\bar{\omega}: \mathcal{F}(A, B) \times \langle B \rangle \rightarrow \langle A \rangle$ , defined as before by

$$\bar{\omega}(\psi, \phi) : A \xrightarrow{\psi} (A_{PL})_q \otimes B \xrightarrow{1 \cdot \phi} (A_{PL})_q,$$

it is also easy to see the commutativity of:

$$(3) \quad \begin{array}{ccc} \mathcal{F}(A, A_{PL}(K)) \times \langle A_{PL}(K) \rangle & \xleftarrow[\simeq]{\beta_* \times \langle \beta \rangle} & \mathcal{F}(A, B) \times \langle B \rangle \\ \downarrow \bar{\omega} & & \downarrow \bar{\omega} \\ \langle A \rangle & \xlongequal{\hspace{10em}} & \langle A \rangle. \end{array}$$

Finally, consider the isomorphism of CDGA's

$$\rho: (\Lambda(V \otimes B_*), \tilde{d}) \xrightarrow{\cong} (\Lambda(A \otimes B_*)/I, d),$$

recalled in §2, and the simplicial isomorphism [3, Thm.3.1]

$$\Psi: \langle \Lambda(A \otimes B_*)/I \rangle \xrightarrow{\cong} \mathcal{F}(A, B),$$

defined as follows: for each  $\eta \in \langle \Lambda(A \otimes B_*)/I \rangle_q$  and each  $v \in V$ ,

$$\Psi(\eta)(v) = \sum_i (-1)^{\alpha|b_i|} \eta[v \otimes \beta_i] \otimes b_i.$$

On the other hand observe that

$$\langle \Lambda(A \otimes B_*)/I \otimes B \rangle \cong \langle \Lambda(A \otimes B_*)/I \rangle \times \langle B \rangle.$$

Then, under this identification, we may show that the following commutes:

$$(4) \quad \begin{array}{ccc} \langle \Lambda(A \otimes B_*)/I \rangle \times \langle B \rangle & \xrightarrow{\Psi \times 1_{\langle B \rangle}} & \mathcal{F}(A, B) \times \langle B \rangle \\ \downarrow \langle (\rho \otimes 1_B) \circ \omega \rangle & & \downarrow \bar{\omega} \\ \langle A \rangle & \xlongequal{\hspace{10em}} & \langle A \rangle. \end{array}$$

Indeed, given each  $\eta \in \langle \Lambda(A \otimes B_*)/I \rangle_q$ ,  $\phi \in \langle B \rangle$  and  $v \in V$ :

$$\begin{aligned} (\bar{\omega} \circ \Psi \times 1_{\langle B \rangle})(\eta, \phi)(v) &= \bar{\omega}(\Psi(\eta), \phi)(v) = \sum_i (-1)^{\alpha|b_i|} \eta[v \otimes \beta_i] \cdot \phi(b_i) \\ &= \langle (\rho \otimes 1_B) \circ \omega \rangle(\eta, \phi)(v). \end{aligned}$$

To finish, use diagrams (1), (2), (3) and (4), together with proposition 2.1, to obtain the following diagram which completes the proof:

$$\begin{array}{ccc} \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times X_{\mathbb{Q}} & \xrightarrow{\omega} & Y_{\mathbb{Q}} & \text{Top} \\ & & \downarrow S_* & \\ S_* \mathcal{F}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times S_* X_{\mathbb{Q}} & \xrightarrow{S_*(\omega)} & S_*(Y_{\mathbb{Q}}) & \\ \uparrow \simeq & & \uparrow \simeq & \\ \mathcal{F}(K, L) \times K & \xrightarrow{\omega} & L & \\ h_* \times 1_K \downarrow \simeq & & \simeq \downarrow h & \\ \mathcal{F}(K, \langle A \rangle) \times K & \xrightarrow{\omega} & \langle A \rangle & \\ \gamma \times j^{-1} \uparrow \simeq & & \parallel & \\ \mathcal{F}(A, A_{PL}(K)) \times \langle A_{PL}(K) \rangle & \xrightarrow{\bar{\omega}} & \langle A \rangle & \text{SimpSet} \\ \beta_* \times \langle \beta \rangle \uparrow \simeq & & \parallel & \\ \mathcal{F}(A, B) \times \langle B \rangle & \xrightarrow{\bar{\omega}} & \langle A \rangle & \\ \Psi \times 1_{\langle B \rangle} \uparrow \cong & & \parallel & \\ \langle \Lambda(A \otimes B_*)/I \rangle \times \langle B \rangle & \xrightarrow{\langle (\rho \otimes 1_B) \circ \omega \rangle} & \langle A \rangle & \\ \langle \rho \rangle \times 1_{\langle B \rangle} \downarrow \cong & & \parallel & \\ \langle \Lambda(V \otimes B_*) \rangle \times \langle B \rangle & \xrightarrow{\langle \omega \rangle} & \langle \Lambda V \rangle & \\ & & \uparrow \langle \cdot \rangle & \\ \Lambda(V \otimes B_*) \otimes B & \xleftarrow{\omega} & \Lambda V & \text{AGDC} \end{array}$$

□

Next, fix a map  $f: X \rightarrow Y$  between nilpotent complexes of finite type over  $\mathbb{Q}$  and let  $Z$  be a finite nilpotent complex. Let

$$A = (\Lambda W, d) \xrightarrow[\simeq]{\varphi} A_{PL}(X) \quad \text{and} \quad B = (\Lambda V, d) \xrightarrow[\simeq]{\psi} A_{PL}(Y)$$

be Sullivan models (again non necessarily minimal!) of  $X$  and  $Y$  respectively, let  $C = (C, \delta) \xrightarrow[\simeq]{\nu} A_{PL}(Z)$  be a quasi-isomorphism with  $C$  finite dimensional connected of finite type, and let  $\zeta: (\Lambda V, d) \longrightarrow (\Lambda W, d)$  be a Sullivan model for  $f$ . Define

$$\xi: (\Lambda(V \otimes C_*), \tilde{d}) \longrightarrow (\Lambda(W \otimes C_*), \tilde{d}), \quad \xi(v \otimes c) = \rho^{-1}[\zeta(v) \otimes c],$$

being  $(\Lambda(V \otimes C_*), \tilde{d})$  and  $(\Lambda(W \otimes C_*), \tilde{d})$  the models of  $\mathcal{F}(Z, Y_{\mathbb{Q}})$  and  $\mathcal{F}(Z, X_{\mathbb{Q}})$  respectively, and  $\rho: (\Lambda(W \otimes C_*), \tilde{d}) \xrightarrow{\cong} (\Lambda(A \otimes C_*), d)/I$  the CDGA isomorphism described in §2. In other words, to compute effectively  $\xi(v \otimes c)$  use the relations which define  $I$  to express  $\zeta(v) \otimes c$  as an element of  $\Lambda(V \otimes C_*)$ . For instance, if  $\zeta(v) = w_1 w_2$  and  $\Delta c = \sum_i c'_i \otimes c''_i$ ,  $\xi(v \otimes c) = \sum_i (-1)^{|w_2||c'_i|} (w_1 \otimes c'_i)(w_2 \otimes c''_i)$ .

Then, we prove this more explicit version of Theorem 1.2:

**THEOREM 3.3.** — *The commutative square*

$$\begin{array}{ccc} (\Lambda(W \otimes C_*) \otimes C, \tilde{d}) & \xleftarrow{\xi \otimes 1_C} & (\Lambda(V \otimes C_*) \otimes C, \tilde{d}) \\ \omega \uparrow & & \uparrow \omega \\ (\Lambda W, d) & \xleftarrow{\zeta} & (\Lambda V, d) \end{array}$$

is a model of

$$\begin{array}{ccc} \mathcal{F}(Z_{\mathbb{Q}}, X_{\mathbb{Q}}) \times Z_{\mathbb{Q}} & \xrightarrow{(f_{\mathbb{Q}})_* \times 1} & \mathcal{F}(Z_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times Z_{\mathbb{Q}} \\ \omega \downarrow & & \downarrow \omega \\ X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}}, \end{array}$$

i.e., there exists a commutative cube of simplicial sets as the following in which the vertical arrows are homotopy equivalences:

$$\begin{array}{ccccc}
 S_*(\mathcal{F}(Z_{\mathbb{Q}}, X_{\mathbb{Q}}) \times Z_{\mathbb{Q}}) & \xrightarrow{S_*((f_{\mathbb{Q}})_* \times 1)} & S_*(\mathcal{F}(Z_{\mathbb{Q}}, Y_{\mathbb{Q}}) \times Z_{\mathbb{Q}}) & & \\
 \downarrow \simeq & \searrow S_*(\omega) & \downarrow \simeq & \searrow S_*(\omega) & \\
 & S_*(X_{\mathbb{Q}}) & \xrightarrow{S_*(f_{\mathbb{Q}})} & S_*(Y_{\mathbb{Q}}) & \\
 \downarrow \simeq & & & & \downarrow \simeq \\
 \langle \Lambda(W \otimes C_*) \otimes C \rangle & \xrightarrow{\langle \xi \otimes 1_C \rangle} & \langle \Lambda(V \otimes C_*) \otimes C \rangle & & \\
 \downarrow \langle \omega \rangle & & \downarrow \langle \omega \rangle & & \\
 \langle \Lambda W, d \rangle & \xrightarrow{\langle \zeta \rangle} & \langle \Lambda V, d \rangle & & 
 \end{array}$$

As in Corollary 3.2, evaluating at the base point we immediately obtain:

COROLLARY 3.4. — *The commutative square*

$$\begin{array}{ccc}
 (\Lambda(W \otimes C_*), \tilde{d}) & \xleftarrow{\xi} & (\Lambda(V \otimes C_*), \tilde{d}) \\
 \omega_0 \uparrow & & \uparrow \omega_0 \\
 (\Lambda W, d) & \xleftarrow{\zeta} & (\Lambda V, d)
 \end{array}$$

is a model of

$$\begin{array}{ccc}
 \mathcal{F}(Z, X_{\mathbb{Q}}) & \xrightarrow{(f_{\mathbb{Q}})_*} & \mathcal{F}(Z, Y_{\mathbb{Q}}) \\
 \omega_0 \downarrow & & \downarrow \omega_0 \\
 X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}}.
 \end{array}$$

*Proof.* — Fix  $N = S_*(Z_{\mathbb{Q}})$ ,  $K = S_*(X_{\mathbb{Q}})$ ,  $L = S_*(Y_{\mathbb{Q}})$  and  $\lambda = S_*(f_{\mathbb{Q}}): K \rightarrow L$ . □

Then we have:

LEMMA 3.5. — *The following commutes:*

$$\begin{array}{ccc}
 S_*\mathcal{F}(|N|, |K|) \times S_*|N| & \xrightarrow{S_*|\lambda|_* \times 1} & S_*\mathcal{F}(|N|, |L|) \times S_*|N| \\
 \downarrow S_*(\omega) & \swarrow \alpha \times \psi(N) \cong & \downarrow S_*(\omega) \\
 S_*|K| & \xrightarrow{\lambda_* \times 1} & S_*|L| \\
 \downarrow \omega & \xrightarrow{S_*(|\lambda|)} & \downarrow \omega \\
 K & \xrightarrow{\lambda} & L
 \end{array}$$

*Proof.* — The lateral faces commute by Proposition 2.1, while the back and bottom faces are trivially commutative. For the front face observe that given  $g: N \times \Delta[q] \rightarrow K$  in  $\mathcal{F}(N, K)_q$ ,  $(\lambda \circ \omega)(g, \sigma) = \lambda(g(\sigma, \Delta_q)) = \omega \circ (\lambda_* \times 1)(g, \sigma)$ . Finally, to check the commutativity of the top face, choose as before  $g \in \mathcal{F}(N, K)_q$  and note that  $\alpha(g) \in S_q\mathcal{F}(|N|, |K|)$  is the map  $\Delta^q \rightarrow \mathcal{F}(|N|, |K|)$  associated by the exponential law to

$$\tau: |N| \times \Delta^q \xrightarrow{\cong} |N| \times |\Delta[q]| \xrightarrow{\cong} |N \times \Delta[q]| \xrightarrow{|g|} |K|.$$

Taking into account that  $|\lambda_*(g)| = |\lambda \circ g| = |\lambda| \circ |g|$ , we observe that  $\alpha(\lambda_*(g)) \in S_q\mathcal{F}(|N|, |L|)$  is again associated by the exponential law to

$$\tau \circ |\lambda|: |N| \times \Delta^q \xrightarrow{1 \times \iota} |N| \times |\Delta[q]| \xrightarrow{\bar{\eta}} |N \times \Delta[q]| \xrightarrow{|\lambda| \circ |g|} |L|.$$

Then,  $(S_*(|\lambda|_*) \circ \alpha)(g) = |\lambda|_* \circ \alpha(g) = \alpha(\lambda_*(g)) = (\alpha \circ \lambda_*)(g)$ . □

LEMMA 3.6. — *The following commutes:*

$$\begin{array}{ccc}
 \mathcal{F}(N, K) \times N & \xrightarrow{\lambda_* \times 1} & \mathcal{F}(N, L) \times N \\
 \downarrow \omega & \searrow (h_K)_* \times 1 \cong & \downarrow \omega \\
 K & \xrightarrow{\langle \zeta \rangle_* \times 1} & L \\
 \downarrow \omega & \xrightarrow{\lambda} & \downarrow \omega \\
 \langle A \rangle & \xrightarrow{\langle \zeta \rangle} & \langle B \rangle
 \end{array}$$

*Proof.* — Side faces are squares of type (1); commutativity of the back face has been checked in the above lemma while that of the front and bottom face are trivial. For the top face, given  $g \in \mathcal{F}(N, K)_q$ ,  $(h_L)_*(\lambda_*(g)) = h_L \circ \lambda \circ g$ , while  $\langle \zeta \rangle_*((h_K)_*(g)) = \langle \zeta \rangle \circ h_K \circ g$ . But these are equal in view of the following commutative diagram in which the composition of the vertical arrows are precisely  $h_K$  and  $h_L$ :

$$\begin{array}{ccc}
 K & \xrightarrow{\lambda} & L \\
 \cong \downarrow & & \downarrow \cong \\
 \langle A_{PL}(S_*(X)) \rangle & \xrightarrow{\langle A_{PL}(S_*(f)) \rangle} & \langle A_{PL}(S_*(Y)) \rangle \\
 \langle \varphi \rangle \downarrow & & \downarrow \langle \psi \rangle \\
 \langle A \rangle & \xrightarrow{\langle \zeta \rangle} & \langle B \rangle.
 \end{array}$$

□

LEMMA 3.7. — *The following commutes:*

$$\begin{array}{ccccc}
 \mathcal{F}(N, \langle A \rangle) \times N & \xrightarrow{\langle \zeta \rangle_* \times 1} & \mathcal{F}(N, \langle B \rangle) \times N & & \\
 \omega_A \downarrow & \swarrow \gamma_A \times j^{-1} \cong & \omega_B \downarrow & \nwarrow \gamma_B \times j^{-1} \cong & \\
 \mathcal{F}(A, A_{PL}(N)) \times \langle A_{PL}(N) \rangle & \xrightarrow{\zeta^* \times 1} & \mathcal{F}(B, A_{PL}(N)) \times \langle A_{PL}(N) \rangle & & \\
 \bar{\omega}_A \downarrow & & \downarrow & & \downarrow \bar{\omega}_B \\
 \langle A \rangle & \xrightarrow{\langle \zeta \rangle} & \langle B \rangle & & \\
 \searrow & & \searrow & & \\
 \langle A \rangle & \xrightarrow{\langle \zeta \rangle} & \langle B \rangle. & & 
 \end{array}$$

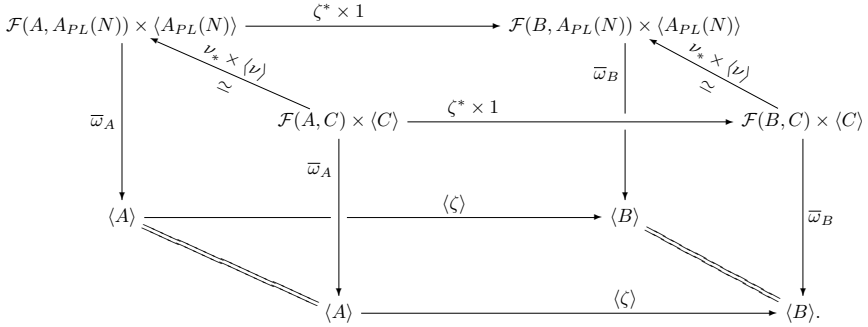
*Proof.* — The lateral squares are as in (2). For the front face, given  $g \in \mathcal{F}(A, A_{PL}(N))_q$ , both  $(\bar{\omega}_B \circ \zeta^* \times 1)(g, \phi)$  and  $(\langle \zeta \rangle \circ \bar{\omega}_A)(g, \phi)$  are the same composition

$$B \xrightarrow{\zeta} A \xrightarrow{g} (A_{PL})_q \otimes A_{PL}(N) \xrightarrow{1 \cdot \phi} (A_{PL})_q.$$

For the top face recall that  $\gamma_A: \mathcal{F}(A, A_{PL}(N)) \rightarrow \mathcal{F}(N, \langle A \rangle)$  assigns to each  $g: A \rightarrow A_{PL}(N) \otimes (A_{PL})_q$  the simplicial map  $\gamma(g) = \langle A_{PL}(p_1) \otimes A_{PL}(p_2) \circ 1 \otimes \theta \circ g \rangle \circ j$  in which  $j: N \times \Delta[q] \rightarrow \langle A_{PL}(N \times \Delta[q]) \rangle$  is induced by adjunction. Taking into account that  $\langle \zeta^*(g) \rangle = \langle g \circ \zeta \rangle = \langle \zeta \rangle \circ \langle g \rangle = \langle \zeta \rangle_*(g)$ ,

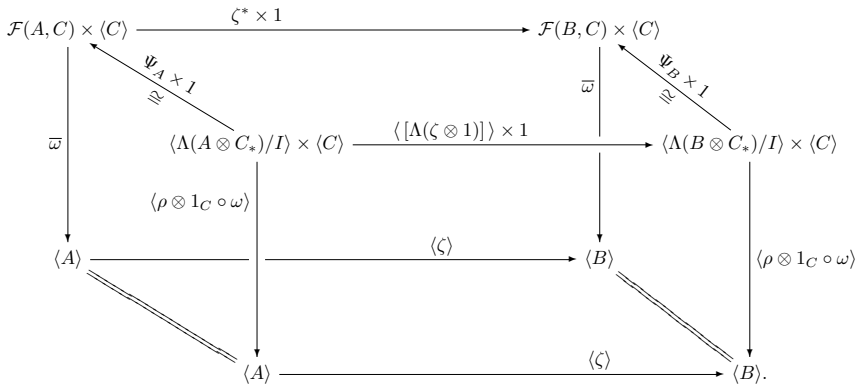
it is immediate to see that both  $(\langle\zeta\rangle_* \circ \gamma_A)(g)$  and  $(\gamma_B \circ \zeta^*)(g)$  correspond to the same map  $N \times \Delta[q] \xrightarrow{\gamma(g)} \langle A \rangle \xrightarrow{\langle\zeta\rangle} \langle B \rangle$ .  $\square$

LEMMA 3.8. — *The following commutes:*



*Proof.* — Lateral faces are squares like (3). The front face has been checked in the past lemma and the rest of the faces trivially commute.  $\square$

LEMMA 3.9. — *The following commutes:*



*Proof.* — First, observe that the morphism  $\Lambda(\zeta \otimes 1): \Lambda(B \otimes C_*) \rightarrow \Lambda(A \otimes C_*)$  sends  $I_B$  into  $I_A$  and therefore it induces  $[\Lambda(\zeta \otimes 1)]: \Lambda(B \otimes C_*)/I_B \rightarrow \Lambda(A \otimes C_*)/I_A$ . Now, the back face has been checked in the past lemma while lateral faces are squares like (4). For the top face observe that given  $g \in \langle \Lambda(A \otimes C_*) / I_A \rangle_q$  and  $b \in B$ ,  $\zeta^*(\Psi_A(g))(b) = (\Psi_A(g) \circ \zeta)(b) = \sum_i (-1)^{\alpha(c_i)} g([\zeta b \otimes c_i^*]) \otimes c_i$ .



On the other hand,

$$\begin{aligned}
 \Psi_B(\langle [\Lambda(\zeta \otimes 1)] \rangle(g))(b) &= \sum_i (-1)^{\alpha(|c_i|)} \langle [\Lambda(\zeta \otimes 1)] \rangle(g)([b \otimes c_i^*]) \otimes c_i \\
 &= \sum_i (-1)^{\alpha(|c_i|)} (g \circ [\Lambda(\zeta \otimes 1)])([b \otimes c_i^*]) \otimes c_i \\
 &= \sum_i (-1)^{\alpha(|c_i|)} g([\zeta b \otimes c_i^*]) \otimes c_i.
 \end{aligned}$$

Finally, the square

$$\begin{array}{ccc}
 \Lambda(A \otimes C_*)/I_A \otimes C & \xleftarrow{[\Lambda(\zeta \otimes 1)] \otimes 1_C} & \Lambda(B \otimes C_*)/I_B \otimes C \\
 \rho \otimes 1_C \circ \omega \uparrow & & \uparrow \rho \otimes 1_C \circ \omega \\
 A & \xleftarrow{\zeta} & B,
 \end{array}$$

is trivially commutative, so is the front face. □

We shall also need an additional cube whose commutativity is trivial:

LEMMA 3.10. — *The following commutes:*

$$\begin{array}{ccccc}
 \langle \Lambda(A \otimes C_*)/I \otimes C \rangle & \xrightarrow{\langle [\Lambda(\zeta \otimes 1)] \otimes 1_C \rangle} & \langle \Lambda(B \otimes C_*)/I \otimes C \rangle & & \\
 \downarrow \langle \rho \otimes 1_C \circ \omega \rangle & \searrow \langle \rho \otimes 1_C \rangle \cong & \downarrow \langle \rho \otimes 1_C \circ \omega \rangle & \searrow \langle \rho \otimes 1_C \rangle \cong & \\
 & \langle \Lambda(W \otimes C_*) \otimes C \rangle & \xrightarrow{\langle \xi \otimes 1_C \rangle} & \langle \Lambda(V \otimes C_*) \otimes C \rangle & \\
 & \downarrow \langle \omega \rangle & & \downarrow \langle \omega \rangle & \\
 \langle A \rangle & \xrightarrow{\langle \zeta \rangle} & \langle B \rangle & & \\
 \downarrow \langle \omega \rangle & & \downarrow \langle \omega \rangle & & \\
 \langle \Lambda W \rangle & \xrightarrow{\langle \zeta \rangle} & \langle \Lambda V \rangle & & 
 \end{array}$$

To finish the proof of Theorem 3.3 join the cubes of past lemmas by the back and front face to obtain the required commutative diagram.

### 4. The restriction to components

Here we restrict the results of past sections to the components of function spaces to get, in particular, Theorem 1.3 of the introduction. For this we

need some algebraic tools: let  $(\Lambda W, d)$  be a CDGA in which  $W$  is  $\mathbb{Z}$ -graded, and let  $u: \Lambda W \rightarrow \mathbb{Q}$  be an augmentation. In  $(\Lambda W, d)$  we consider as in [3] the differential ideal  $K_u$  generated by  $A_1 \cup A_2 \cup A_3$ , where  $A_1 = (\Lambda W)^{<0}$ ,  $A_2 = d(\Lambda W)^0$  and  $A_3 = \{\alpha - u(\alpha) : \alpha \in (\Lambda W)^0\}$ .

LEMMA 4.1. — *The ideal  $K_u$  coincides with  $K_u'$  generated by  $A_1' \cup A_2' \cup A_3'$ , where  $A_1' = W^{<0}$ ,  $A_2' = dW^0$  and  $A_3' = \{w - u(w) : w \in W^0\}$ .*

*Proof.* — The inclusion  $K_u' \subset K_u$  is trivial, as it is  $A_1 \subset K_u'$ .

Let  $\Phi = \alpha - u(\alpha) \in A_3$  and write  $\alpha = a + b$ ,  $a \in \Lambda^+(W^{<0}) \cdot \Lambda W$ ,  $b \in \Lambda W^0$ . Then,  $\alpha - u(\alpha) = a + b - u(a) - u(b) = a + b - u(b)$ . As  $a \in A_1 \subset K_u'$ , it remains to see that  $b - u(b) \in K_u'$ . Assume  $b \in \Lambda^n W^0$  and argue by induction on  $n$ . For  $n = 1$ , trivially  $b - u(b) \in A_3' \subset K_u'$ . Let  $b = b_1 \cdots b_n$ ,  $b_i \in W^0$ . Then,  $b - u(b) = b_1((b_2 \cdots b_n) - u(b_2 \cdots b_n)) + u(b_2 \cdots b_n)(b_1 - u(b_1)) \in K_u'$  by induction hypothesis.

Finally, let  $\Phi = d\alpha$ ,  $\alpha \in (\Lambda W)^0$ , be a generator of  $A_2$  and write  $\alpha = a + b + c$  where  $a \in (\Lambda^+ W^{<-1}) \cdot \Lambda W$ ,  $b \in \Lambda(W^0)$  and  $c \in \Lambda(W^0) \cdot W^{-1} \cdot W^1$ . Obviously  $da, db \in K_u'$ . Write  $c = \sum \Phi c_1 c_2$ ,  $\Phi \in \Lambda(W^0)$ ,  $|c_1| = -1$  and  $|c_2| = 1$ . Thus,  $dc = \sum d\Phi c_1 c_2 + \Phi dc_1 c_2 - \Phi c_1 dc_2$ . While the first and third set of summands are trivially in  $K_u'$ ,  $u(dc_1) = du(c_1) = 0$  and therefore  $\phi dc_1 c_2 = \phi(dc_1 - u(dc_1))c_2$  is also in  $K_u'$ . □

Next, we see that  $(\Lambda W, d)/K_u$  is itself a free commutative graded algebra. Let  $\widetilde{K}_u$  be the ideal of  $(\Lambda W, d)$  generated by  $A_1 \cup A_3$  and observe that the projection  $\rho: W^1 \xrightarrow{\cong} (\Lambda W/\widetilde{K}_u)^1$  is a vector space isomorphism. Consider the linear map

$$\partial: W^0 \xrightarrow{d} (\Lambda W)^1 \twoheadrightarrow (\Lambda W/\widetilde{K}_u)^1 \xrightarrow{\rho^{-1}} W^1$$

and call  $\overline{W}^1$  a complement of the image of this map,  $W^1 = \partial W^0 \oplus \overline{W}^1$ .

In which follows, given  $\Phi = \alpha \cdot \Psi$ ,  $\alpha \in (\Lambda^+ W^0)$  and  $\Psi \in \Lambda W^{>0}$ , we denote by  $\Phi/u$  the element  $u(\alpha)\Psi$ . Hence, if for  $w \in W^0$ ,  $dw = \Phi_0 + \Phi_1 + \Phi_2$ , with  $\Phi_0 \in (\Lambda^+ W^{<0}) \cdot (\Lambda W)$ ,  $\Phi_1 \in (\Lambda^+ W^0) \cdot W^1$ ,  $\Phi_2 \in W^1$ , then  $\partial(w) = \Phi_1/u + \Phi_2$ .

PROPOSITION 4.2. — *For a certain  $\bar{d}$ ,  $(\Lambda W/K_u, d) \cong \Lambda(\overline{W}^1 \oplus W^{\geq 2}, \bar{d})$ .*

*Proof.* — It is easy to see that the surjective morphism of graded algebras  $\varphi: \Lambda W \rightarrow \Lambda(\overline{W}^1 \oplus W^{\geq 2})$ , given by

$$\varphi(w) = \begin{cases} 0 & \text{if } |w| < 0 \text{ or } w \in \partial W^0, \\ u(w) & \text{if } |w| = 0, \\ w & \text{otherwise,} \end{cases}$$

has  $K_u$  as kernel so it induces the required isomorphism. Let us call it  $\bar{\varphi}$ . To finish, define  $\bar{d} = \bar{\varphi} \circ d \circ \bar{\varphi}^{-1}$ . □

*Remark 4.3.* — (1) To effectively compute  $\bar{d}$ , choose  $w \in \Lambda(\bar{W}^1 \oplus W^{\geq 2})$  and write  $dw = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3$ , in which  $\Phi_0 \in \Lambda^+W^{<0} \cdot \Lambda W$ ,  $\Phi_1 \in \Lambda^+(\partial W^0) \cdot \Lambda W^{\geq 0}$ ,  $\Phi_2 \in (\Lambda^+W^0) \cdot \Lambda(\bar{W}^1 \oplus W^{\geq 2})$  and  $\Phi_3 \in \Lambda(\bar{W}^1 \oplus W^{\geq 2})$ . Then,  $\bar{d}w = \bar{\varphi}d\bar{\varphi}^{-1}w = \varphi(\Phi_2 + \Phi_3) = \Phi_2/u + \Phi_3$ .

(2) Note that if we have in  $W$  a basis  $w_i$  for which  $dw_i \in \Lambda W_{<i}$ , then the image of this basis in  $\Lambda(\bar{W}^1 \oplus W^{\geq 2}, \bar{d})$  makes it a Sullivan model. However, even when  $d$  is decomposable in  $\Lambda W$ ,  $\bar{d}$  might not be, i.e.,  $\Lambda(\bar{W}^1 \oplus W^{\geq 2}, \bar{d})$  is not necessarily minimal. This depends on  $u$ . In fact, as we just remarked, for each  $w \in W$ ,  $\Phi_2/u$  could be the linear part of  $\bar{d}w$ .

Next, consider again the function space  $\mathcal{F}(X, Y)$  and recall that, if  $h: Y \rightarrow Y_{\mathbb{Q}}$  denotes rationalization, then  $\mathcal{F}(X, Y; f) \rightarrow \mathcal{F}(X, Y_{\mathbb{Q}}; hf)$  is also the rationalization map [9, Thm.3.11], i.e.,  $\mathcal{F}(X, Y; f)_{\mathbb{Q}} = \mathcal{F}(X, Y_{\mathbb{Q}}; hf)$ .

On the other hand, recall that for a simplicial set  $L$  and a 0-simplex  $x_0 \in L_0$ , the component of  $L$  containing  $x_0$  is the subsimplicial set  $L_{x_0}$  in which

$$(L_{x_0})_q = \{x \in L_q \mid \partial^{i_1} \dots \partial^{i_q} x = x_0, \text{ for some } i_1, \dots, i_q\}.$$

Then, given a non necessarily connected CW-complex  $X$ , if we denote by  $\mathcal{C}(X; x_0)$  the path component of  $X$  containing  $x_0$ , there exists a homotopy equivalence  $(S_*X)_{x_0} \xrightarrow{\simeq} S_*\mathcal{C}(X; x_0)$ .

Let  $(\Lambda(V \otimes B_*), \bar{d})$  be the model of  $\mathcal{F}(X, Y)$  and let  $h \circ f \in \mathcal{F}(X, Y_{\mathbb{Q}})$  considered also as a 0-simplex of  $S_0\mathcal{F}(X, Y_{\mathbb{Q}})$ . As before, denote by  $S_*\mathcal{F}(X, Y_{\mathbb{Q}})_{hf}$  the component of  $S_*\mathcal{F}(X, Y_{\mathbb{Q}})$  containing  $hf$ , and by  $u$  the 0-simplex of  $\langle \Lambda(V \otimes B_*) \rangle$ , i.e., the CDGA morphism  $u: (\Lambda(V \otimes B_*), \bar{d}) \rightarrow \mathbb{Q}$ , which corresponds to  $hf$  through the equivalence  $S_*\mathcal{F}(X, Y_{\mathbb{Q}}) \xrightarrow{\simeq} \langle \Lambda(V \otimes B_*) \rangle$ .

We first prove a generalization of [3, Thm.6.1]:

PROPOSITION 4.4. — *The projection  $(\Lambda(V \otimes B_*), \bar{d}) \rightarrow (\Lambda(V \otimes B_*), \bar{d})/K_u$  induces a homotopy equivalence*

$$\langle \Lambda(V \otimes B_*)/K_u \rangle \xrightarrow{\simeq} \langle \Lambda(V \otimes B_*) \rangle_u$$

which makes the following commutative:

$$\begin{array}{ccc} \langle \Lambda(V \otimes B_*) \rangle_u & \hookrightarrow & \langle \Lambda(V \otimes B_*) \rangle \\ \uparrow \simeq & & \parallel \\ \langle \Lambda(V \otimes B_*)/K_u \rangle & \rightarrow & \langle \Lambda(V \otimes B_*) \rangle. \end{array}$$

*Proof.* — We shall need the following version of [8, Thm. 2.2] for  $(\Lambda(V \otimes B_*), \tilde{d})$ :

Write  $B_* = A \oplus \delta A \oplus C$  in which  $C \cong H(B_*)$  and choose a basis  $\{a_j, b_j, c_k\}$  with  $\delta a_j = b_j$  and  $\delta c_k = 0$ . Also, as  $\Lambda V$  is a Sullivan model choose  $\{v_i\}$  a basis for  $V$  satisfying  $dv_i \in \Lambda(V_{<i})$ . Then [3, Prop.5.1], there is a CDGA isomorphism  $\eta: (\Lambda(V \otimes B_*), \tilde{d}) \xrightarrow{\cong} \Lambda(U \oplus dU) \otimes (\Lambda W, d)$  in which:

- (1)  $U$  is generated by  $u_{ij} = v_i \otimes a_j$ .
- (2)  $W$  is generated by  $w_{ik} = v_i \otimes c_k + x_{ik}$ , for some  $x_{ik} \in \Lambda(V_{<i} \otimes B_*)$ .
- (3) If the differential in  $\Lambda V$  is decomposable, the one in  $\Lambda W$  is also decomposable.

Consider now  $u: \Lambda(V \otimes B_*) \rightarrow \mathbb{Q}$  and  $u' = u \circ \eta^{-1}$ , and let  $K_u$  and  $K_{u'}$  be the associated differential ideals of  $(\Lambda(V \otimes B_*), \tilde{d})$  and  $(\Lambda W, d) \otimes \Lambda(U \oplus dU)$  respectively. With the aid of Lemma 4.1 is easy to see that  $\eta(K_u) = K_{u'}$ .

On the other hand, consider in  $(\Lambda W, d)$  and  $\Lambda(U \oplus dU)$  the ideals  $K_p$  and  $K_q$  associated to the augmentations  $p: (\Lambda W, d) \rightarrow (\Lambda W, d) \otimes \Lambda(U \oplus dU) \xrightarrow{u'} \mathbb{Q}$  and  $q: \Lambda(U \oplus dU) \rightarrow (\Lambda W, d) \otimes \Lambda(U \oplus dU) \xrightarrow{u'} \mathbb{Q}$  respectively. Again, it is immediate to check that  $K_{u'} = K_p \otimes K_q$ , and thus,

$$\begin{aligned} (\Lambda(V \otimes B_*), \tilde{d})/K_u &\cong (\Lambda W, d) \otimes \Lambda(U \oplus dU)/K_{u'} \\ &\cong (\Lambda W, d)/K_p \otimes \Lambda(U \oplus dU)/K_q. \end{aligned}$$

Next, observe that  $\Lambda(U \oplus dU)/K_q \cong \Lambda(U^{\geq 1} \oplus d(U^{\geq 1}))$ , and therefore, it is acyclic. Indeed, the surjective morphism  $\varphi: \Lambda(U \oplus dU) \rightarrow \Lambda(U^{\geq 1} \oplus d(U^{\geq 1}))$ ,

$$\varphi(z) = \begin{cases} 0 & \text{if } z \in U^{<0} \text{ or } z \in d(U^{\leq 0}), \\ u(z) & \text{if } z \in U^0, \\ z & \text{if } z \in U^{\geq 1}, \end{cases}$$

has  $K_q$  as kernel. Then:

$$\begin{aligned} \langle (\Lambda(V \otimes B_*), \tilde{d})/K_u \rangle &\cong \langle (\Lambda W, d)/K_p \otimes \Lambda(U \oplus dU)/K_q \rangle \\ &\cong \langle (\Lambda W, d)/K_p \rangle \times \langle \Lambda(U \oplus dU)/K_q \rangle \\ &\simeq \langle (\Lambda W, d)/K_p \rangle. \end{aligned}$$

Finally, in [3, Thm.6.1] is proved that the projection  $\Lambda W \rightarrow \Lambda W/K_p$  induces an equivalence  $\langle \Lambda W/K_p \rangle \xrightarrow{\cong} \langle \Lambda W \rangle_p$ . Therefore, as  $\langle \Lambda(V \otimes B_*) \rangle_u \simeq \langle \Lambda W \rangle_p$  we get an equivalence  $\langle \Lambda(V \otimes B_*)/K_u \rangle \xrightarrow{\cong} \langle \Lambda(V \otimes B_*) \rangle_u$  which satisfies the required property.  $\square$

Then, we have:

**THEOREM 4.5.** — *The projection  $(\Lambda(V \otimes B_*), \tilde{d}) \rightarrow (\Lambda(V \otimes B_*), \tilde{d})/K_u$  is a model of  $i: \mathcal{F}(X, Y; f) \hookrightarrow \mathcal{F}(X, Y)$ .*

*Proof.* — Indeed, we have the following diagram

$$\begin{array}{ccc}
 \mathcal{F}(X, Y_{\mathbb{Q}}; h \circ f) & \xhookrightarrow{i} & \mathcal{F}(X, Y_{\mathbb{Q}}) & \text{Top} \\
 & & \downarrow S_* & \\
 S_*\mathcal{F}(X, Y_{\mathbb{Q}}; h \circ f) & \xhookrightarrow{S_*i} & S_*\mathcal{F}(X, Y_{\mathbb{Q}}) & \\
 \uparrow \simeq & & \parallel & \\
 S_*\mathcal{F}(X, Y_{\mathbb{Q}})_{hf} & \hookrightarrow & S_*\mathcal{F}(X, Y_{\mathbb{Q}}) & \\
 \downarrow \simeq & & \downarrow \simeq & \text{SimpSet} \\
 \langle \Lambda(V \otimes B_*) \rangle_u & \hookrightarrow & \langle \Lambda(V \otimes B_*) \rangle & \\
 \uparrow \simeq & & \parallel & \\
 \langle \Lambda(V \otimes B_*)/K_u \rangle & \longrightarrow & \langle \Lambda(V \otimes B_*) \rangle & \\
 & \uparrow \langle \cdot \rangle & & \\
 \Lambda(V \otimes B_*)/K_u & \longleftarrow & \Lambda(V \otimes B_*) & \text{AGDC}
 \end{array}$$

in which the top and middle square are trivially commutative and the bottom is given in Proposition 4.4. □

The next results prove Theorem 1.3 and Corollary 1.4.

**THEOREM 4.6.** — *The CDGA morphism*

$$\omega_0: (\Lambda V, d) \longrightarrow (\Lambda(V \otimes B_*), \tilde{d})/K_u, \quad \omega(v) = [v \otimes 1^*],$$

is a model of  $\omega_0: \mathcal{F}(X, Y; f) \longrightarrow Y$ .

*Proof.* — Write  $\omega_0$  as the composition

$$\mathcal{F}(X, Y; f) \xhookrightarrow{i} \mathcal{F}(X, Y) \xrightarrow{\omega_0} Y$$

and apply Theorems 3.1 and 4.5. □

In particular, via Proposition 4.2, we have:

**COROLLARY 4.7.** — *The CDGA morphism*

$$\omega_0: (\Lambda V, d) \longrightarrow (\Lambda S_f, \bar{d}) = (\Lambda(\overline{V \otimes B_*^{-1}} \oplus (V \otimes B_*)^{\geq 2}), \bar{d}),$$

$\omega_0(v) = v \otimes 1^*$  if  $v \in V^{\geq 2}$ , or its projection over  $\overline{V \otimes B_*^{-1}}$  if  $v \in V^1$ , is a Sullivan model of the evaluation at the base point  $\omega_0: \mathcal{F}(X, Y; f) \rightarrow Y$ .

Observe that, while  $\omega_0(v)$  could vanish if  $|v| = 1$ , when  $(\Lambda V, d)$  is 1-connected,

$$\omega_0: (\Lambda V, d) \longrightarrow (\Lambda S_f, \bar{d}) = (\Lambda(\overline{V \otimes B_*^1} \oplus (V \otimes B_*)^{\geq 2}), \bar{d}),$$

is a KS-extension or a relative Sullivan algebra. The fibre is of the form

$$(\Lambda(\overline{V \otimes B_*^1} \oplus (V \otimes B_*)^{\geq 2}) / (V \otimes 1^*), \bar{d}) = (\Lambda(S_f/V), \bar{d}).$$

Hence, as the fibre of  $\omega_0: \mathcal{F}(X, Y; f) \rightarrow Y$  is  $\mathcal{F}^*(X, Y; f)$ , the path component of  $f$  of the space of pointed maps, we immediately obtain:

**COROLLARY 4.8.** — *For a 1-connected space  $Y$ ,  $(\Lambda(S_f/V), \bar{d})$  is a Sullivan model of  $\mathcal{F}^*(X, Y; f)$ .*

Finally, let  $X, Y, Z$  and  $f: X \rightarrow Y$  be as in Theorem 3.3. Again, we denote by  $\zeta: (\Lambda V, d) \rightarrow (\Lambda W, d)$  a model of  $f$  and  $\xi: \Lambda(V \otimes C_*) \rightarrow \Lambda(W \otimes C_*)$  as defined in Theorems 1.2 or 3.3. Let  $u_g: \Lambda(W \otimes C_*) \rightarrow \mathbb{Q}$  and  $u_{fg}: \Lambda(V \otimes C_*) \rightarrow \mathbb{Q}$  be the 0-simplices of  $\langle \Lambda(W \otimes C_*) \rangle$  and  $\langle \Lambda(V \otimes C_*) \rangle$  corresponding to  $hg: Z \rightarrow X_{\mathbb{Q}}$  and  $hfg = f_{\mathbb{Q}}hg: Z \rightarrow Y_{\mathbb{Q}}$  respectively. Finally

**THEOREM 4.9.** — *The square*

$$\begin{array}{ccc} (\Lambda S_g, \bar{d}) & \xleftarrow{\bar{\xi}} & (\Lambda S_{fg}, \bar{d}) \\ \omega_0 \uparrow & & \uparrow \omega_0 \\ (\Lambda W, d) & \xleftarrow{\zeta} & (\Lambda V, d) \end{array}$$

is a Sullivan model of

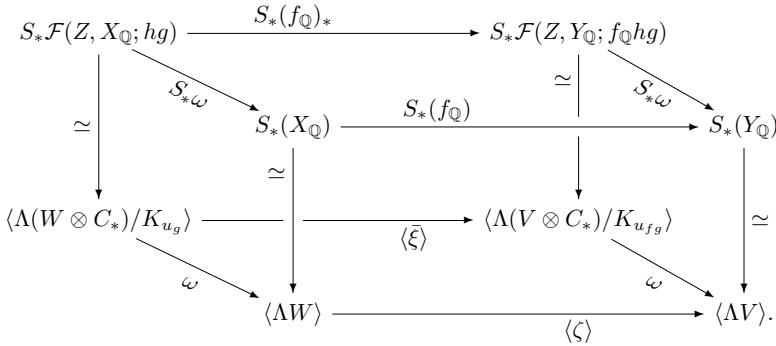
$$\begin{array}{ccc} \mathcal{F}(Z, X; g) & \xrightarrow{(f)_*} & \mathcal{F}(Z, Y; fg) \\ \omega_0 \downarrow & & \downarrow \omega_0 \\ X & \xrightarrow{f} & Y, \end{array}$$

*Proof.* — First, observe that the  $\Lambda(W \otimes C_*)/K_{u_g} \xleftarrow{\bar{\xi}} \Lambda(V \otimes C_*)/K_{u_{fg}}$  is well defined, i.e.,  $\xi(K_{u_{fg}}) \subset K_{u_g}$ . For that note that  $u_{fg}$  is the composition  $u_g \xi$ . If  $A_1', A_2'$  and  $A_3'$  is the set of generators of  $K_{u_{fg}}$  given in Lemma 4.1, then  $\xi(A_1') \subset K_{u_g}$  trivially. Let  $\tilde{d}(w \otimes c)$ ,  $w \otimes c \in (W \otimes C_*)^0$ , an element of  $A_2'$ . Then,  $\xi(\tilde{d}(w \otimes c)) = \tilde{d}\xi(w \otimes c) = \tilde{d}\rho^{-1}[\zeta w \otimes c]$ . Write  $\rho^{-1}[\zeta w \otimes c] = a + b$ ,  $a \in \Lambda^+(V \otimes C_*)^{<0} \cdot \Lambda(V \otimes C_*)$ ,  $b \in \Lambda(V \otimes C_*)^0$ . Hence,

$\tilde{d}(a), \tilde{d}(b) \in K_{u_g}$ . Finally consider  $w \otimes c - u_{fg}(w \otimes c)$ ,  $w \otimes c \in (W \otimes C_*)^0$ , a generator of  $A_3'$ . Then,

$$\xi(w \otimes c - u_{fg}(w \otimes c)) = \xi(w \otimes c) - u_{fg}(w \otimes c) = \xi(w \otimes c) - u_g(\xi(w \otimes c)),$$

which again, by Lemma 4.1, lives in  $K_{u_g}$ . To finish the proof we have to show the existence of a commutative cube in **SimpSet** as the following, in which the vertical arrows are homotopy equivalences:



But this is immediate: Lateral faces are obtained by Theorem 4.6. The commutativity of the back face is an exercise and the rest trivially commutes. □

Finally, we describe the behavior at the homotopy group level of the distinguished maps studied in past sections. Let  $(\Lambda S_f, \bar{d}) \cong (\Lambda(V \otimes B_*)/K_u, d)$  be the Sullivan model of the component  $\mathcal{F}(X, Y; f)$  given in Corollary 4.7. Recall that  $u: \Lambda(V \otimes B_*) \rightarrow \mathbb{Q}$  is the CDGA morphism corresponding to  $f$  and that  $(\Lambda S_f, \bar{d}) = (\Lambda(\overline{V \otimes B_*^{-1}} \oplus (V \otimes B_*)^{\geq 2}), \bar{d})$ . Now, by classical facts on rational homotopy theory of nilpotent spaces, we have:

**THEOREM 4.10.** — (1)  $\pi_*\mathcal{F}(X, Y; f)_{\mathbb{Q}}$  is naturally isomorphic to the dual of  $H^*(S_f, Q(\bar{d})) \cong H^*(\overline{V \otimes B_*^{-1}} \oplus (V \otimes B_*)^{\geq 2}, Q(\bar{d}))$ , being  $S \cong Q(\Lambda S_f) = \Lambda S_f / (\Lambda^+ S_f \cdot \Lambda^+ S_f)$  the space of indecomposables.

(2) Moreover, the morphism  $\pi_*(\omega_0): \pi_*\mathcal{F}(X, Y; f)_{\mathbb{Q}} \rightarrow \pi_*(Y_{\mathbb{Q}})$  is dual of

$$H^*(Q(\omega_0)): H^*(V, Q(d)) \rightarrow H^*(\overline{V \otimes B_*^{-1}} \oplus (V \otimes B_*)^{\geq 2}, Q(\bar{d})),$$

$$H^*(Q(\omega_0))(v) = [v \otimes 1^*].$$

(3) Under the conditions and with the notation of Theorems 3.3 or 4.9, the morphism induced in homotopy groups by

$$\begin{array}{ccc}
 \mathcal{F}(Z, X; g)_{\mathbb{Q}} & \xrightarrow{(f_*)_{\mathbb{Q}}} & \mathcal{F}(Z, Y; fg)_{\mathbb{Q}} \\
 \omega_0 \downarrow & & \downarrow \omega_0 \\
 X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}},
 \end{array}$$

is dual of

$$\begin{array}{ccc}
 H^*(S_g, Q(\bar{d})) & \xleftarrow{H^*(Q(\xi))} & H^*(S_{fg}, Q(\bar{d})) \\
 H^*(Q(\omega_0)) \uparrow & & \uparrow H^*(Q(\omega_0)) \\
 H^*(W, Q(d)) & \xleftarrow{H^*(Q(\zeta))} & H^*(V, Q(d)).
 \end{array}$$

Also, taking homology of indecomposables in Corollary 4.8 we obtain:

COROLLARY 4.11. —  $\pi_*\mathcal{F}^*(X, Y; f)_{\mathbb{Q}}$  is naturally isomorphic to the dual of  $H^*(S_f/V, Q(\bar{d}))$ .

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Urtzi BUIJS & Aniceto MURILLO  
Universidad de Málaga  
Departamento de Algebra  
Geometría y Topología  
Ap. 59, 29080 Málaga (Spain)  
urtzi@agt.cie.uma.es  
aniceto@agt.cie.uma.es