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A SPECTRAL PALEY-WIENER THEOREM FOR THE HEISENBERG GROUP AND A SUPPORT THEOREM FOR THE TWISTED SPHERICAL MEANS ON \mathbb{C}^n

by E. K. NARAYANAN & S. THANGAVELU

Dedicated to Prof. U. B. Tewari on his sixtieth birthday

ABSTRACT. — We prove a spectral Paley-Wiener theorem for the Heisenberg group by means of a support theorem for the twisted spherical means on \mathbb{C}^n . If $f(z)e^{\frac{1}{4}|z|^2}$ is a Schwartz class function we show that f is supported in a ball of radius B in \mathbb{C}^n if and only if $f \times \mu_r(z) = 0$ for $r > B + |z|$ for all $z \in \mathbb{C}^n$. This is an analogue of Helgason's support theorem on Euclidean and hyperbolic spaces. When $n = 1$ we show that the two conditions $f \times \mu_r(z) = \mu_r \times f(z) = 0$ for $r > B + |z|$ imply a support theorem for a large class of functions with exponential growth. Surprisingly enough, this latter result does not generalize to higher dimensions.

RÉSUMÉ. — Nous prouvons un théorème de Paley-Wiener spectral pour le groupe d'Heisenberg en utilisant un théorème du support pour les moyennes sphériques tordues sur \mathbb{C}^n . Si $f(z)e^{\frac{1}{4}|z|^2}$ est une fonction dans la classe de Schwartz nous montrons que f a un support dans une boule de \mathbb{C}^n de rayon B si et seulement si $f \times \mu_r(z) = 0$ pour $r > B + |z|$ et pour tout $z \in \mathbb{C}^n$. C'est un analogue du théorème du support prouvé dans les contextes euclidiens et hyperboliques par Helgason. Lorsque $n = 1$ nous montrons que les deux conditions $f \times \mu_r(z) = \mu_r \times f(z) = 0$ pour $r > B + |z|$ impliquent un théorème du support pour une grande classe de fonctions à croissance exponentielle. Il est surprenant de constater que ce dernier résultat ne se généralise pas aux dimensions supérieures.

1. Introduction

The main result of this paper has its origin in a long paper of Strichartz [10] where he has given a new interpretation of harmonic analysis as spectral

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theory of Laplacians. The starting point is the spectral decomposition of a Laplacian Δ written in the form

$$f = \int_0^\infty P_\lambda f \, d\lambda$$

where $P_\lambda f$ are projections of f into the eigenspaces of Δ corresponding to the eigenvalues λ . In most cases of interest which include Euclidean spaces and Riemannian symmetric spaces associated to Lie groups, the operators $P_\lambda f$ are given by convolution with spherical functions. In [10] Strichartz has investigated how properties of f are translated into properties of $P_\lambda f$. One such result is the so called spectral Paley-Wiener theorem which characterizes compactly supported functions in terms of properties of $P_\lambda f$. Many works have been dedicated to this topic, see for example the papers by Bray [2], [3].

Our main concern in this paper is a spectral Paley-Wiener theorem for the Heisenberg group. To state our results let us quickly set up the notation:- details will be given in later sections. Let \mathbb{H}^n be the Heisenberg group which as a manifold is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \operatorname{Im} z \cdot \bar{w}).$$

On \mathbb{H}^n we consider the spectral resolution of the subLaplacian \mathcal{L} given by

$$f = (2\pi)^{-n-1} \sum_{k=0}^\infty \left(\int_{-\infty}^\infty f * e_k^\lambda |\lambda|^n \, d\lambda \right).$$

Here

$$e_k^\lambda(z, t) = e^{i\lambda t} \varphi_k^\lambda(z) = e^{i\lambda t} \varphi_k(\sqrt{|\lambda|}z)$$

where

$$\varphi_k(z) = L_k^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2}$$

are the Laguerre functions of type $(n - 1)$. The functions $f * e_k^\lambda(z, t)$ are generalized eigenfunctions of \mathcal{L} associated to the point $((2k + n)|\lambda|, \lambda)$ on the Heisenberg fan which is the spectrum of \mathcal{L} , see [13]. We are interested in characterizing functions $f(z, t)$ which are supported in $\{(z, t) : |z| \leq B, t \in \mathbb{R}\}$ in terms of properties of $f * e_k^\lambda$.

When $n = 1$, the following result can be found in [13] (see Theorem 2.4.1). To state the result we need some notation. Let $\Delta_+ \psi(k) = \psi(k + 1) - \psi(k)$, $\Delta_- \psi(k) = \psi(k) - \psi(k - 1)$ be the forward and backward finite difference operators. We let

$$\Delta \psi(k) = k \Delta_+ \Delta_- \psi(k) + n \Delta_+ \psi(k)$$

to stand for a second order finite difference operator (not to be confused with Laplacian). Using powers of Δ we define certain sequence spaces. For each $j \geq 0$ we let l_j^2 to stand for all sequences $\psi = (\psi(k))$ for which

$$\|\psi\|_{2,j} = \left(\sum_{k=0}^{\infty} |\Delta^j \psi(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

Given a continuous integrable function f on \mathbb{H}^n , define $f_m(z, t) = z^m f(z, t)$ or $\bar{z}^m f(z, t)$ depending on $m \in \mathbb{Z}$ is nonnegative or negative. We let

$$\psi_m(k) = \psi_m(k, z, t, \lambda) = f_m * e_k^\lambda(z, t).$$

Finally, let $L^{p,q}(\mathbb{H}^n)$, $1 \leq p < \infty$, $1 \leq q \leq 2$, consists of functions for which

$$\|f\|_{p,q}^p = \int_{\mathbb{C}^n} \left(\int_{\mathbb{R}} |f(z, t)|^q dt \right)^{\frac{p}{q}} dz < \infty.$$

With these notations the following theorem was proved in [13].

THEOREM 1.1. — *A function $f \in L^{p,q}(\mathbb{H}^n)$, $1 \leq p < \infty$, $1 \leq q \leq 2$ is supported in $\{(z, t) : |z| \leq B, t \in \mathbb{R}\}$ if and only if for every m and $j \geq 0$ the sequence $(\psi_m(k)) = \psi_m \in l_j^2$ and*

$$\|\psi_m\|_{2,j} \leq C_m |\lambda|^{j-1} 2^{-j} (B + |z|)^{2j}.$$

Here C_m is a constant depending only on m .

An examination of the proof of this result shows that the condition on the functions f_m have been introduced for technical reasons and one is naturally led to the conjecture that the theorem is true with conditions only on $f * e_k^\lambda$. In the Euclidean and symmetric space cases the condition is only on $P_\lambda f$. We show in this paper that this conjecture is true once we assume certain decay conditions on f . Here is our main result:

THEOREM 1.2. — *Let $f \in L^{p,q}(\mathbb{H}^n)$, $1 \leq p < \infty$, $1 \leq q \leq 2$ is such that $f^\lambda(z) e^{\frac{1}{4}|\lambda||z|^2}$ is a Schwartz class function on \mathbb{C}^n for every $\lambda \in \mathbb{R}$. Then f is supported in $\{(z, t) : |z| \leq B, t \in \mathbb{R}\}$ if and only if for every $j \geq 0$ the sequence $\psi_\lambda = (\psi_\lambda(k))$ where $\psi_\lambda(k) = f * e_k^\lambda(z, t)$ belongs to l_j^2 and*

$$\|\psi_\lambda\|_{2,j} \leq C \left(\frac{1}{2}|\lambda|\right)^j (B + |z|)^{2j}.$$

In the above theorem $f^\lambda(z)$ stands for the partial inverse Fourier transform of $f(z, t)$ in the t variable. The above theorem is essentially a theorem for the z variable and so we can assume $\lambda = 1$. Then we are led to characterizing compactly supported functions on \mathbb{C}^n in terms of properties of the

spectral projections $f \rightarrow f \times \varphi_k$ where

$$f \times \varphi_k(z) = \int_{\mathbb{C}^n} f(z-w) e^{\frac{i}{2} \operatorname{Im}(z \cdot \bar{w})} \varphi_k(w) dw$$

is the twisted convolution of f with φ_k . As in [13] everything is then reduced to the following support theorem for twisted spherical means. Let μ_r stand for the normalized surface measure on the sphere $S_r = \{z \in \mathbb{C}^n : |z| = r\}$. Define

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w) e^{\frac{i}{2} \operatorname{Im}(z \cdot \bar{w})} d\mu_r(w).$$

Without the exponential factor we have the ordinary spherical means $f * \mu_r$ which has been studied by several authors, see [2], [4] and [5]. For a support theorem for the spherical means we refer to [2] and [5]. Here we have the following support theorem for the twisted spherical means.

THEOREM 1.3. — *Let f be a function on \mathbb{C}^n such that $f(z)e^{\frac{1}{4}|z|^2}$ is in the Schwartz class. Then f is supported in $|z| \leq B$ if and only if $f \times \mu_r(z) = 0$ for $r > B + |z|$ for every $z \in \mathbb{C}^n$.*

A proof of this theorem is given in section 3. Note that the function f is assumed to have enough decay. Even in the Euclidean case the support theorem for the spherical means does not hold true without some decay assumption on f as can be seen from [5]. However when $n = 1$ we can greatly relax the condition on f .

THEOREM 1.4. — *Let f be a locally integrable function on \mathbb{C} such that*

$$|f(z)| \leq C e^{\frac{1}{4}(1-\epsilon)|z|^2}$$

for some $\epsilon > 0$. Then f is supported in $|z| \leq B$ if and only if $f \times \mu_r(z) = \mu_r \times f(z) = 0$ for $r > B + |z|$ for every $z \in \mathbb{C}$.

Note that we have allowed f to have growth as opposed to the decay in the previous theorem. It turns out that this result is not true when $n \geq 2$. A counter example for $n = 2$ is given in section 4.

2. Preliminaries

On the Heisenberg group \mathbb{H}^n we consider the left invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \quad j = 1, 2, \dots, n,$$

and $T = \frac{\partial}{\partial t}$. They form a basis for the Heisenberg Lie algebra. The sub-Laplacian \mathcal{L} which plays the role of the Laplacian for the subelliptic realm is defined by

$$(2.1) \quad \mathcal{L} = - \sum_{j=1}^n (X_j^2 + Y_j^2).$$

The functions e_k^λ defined in the introduction are eigenfunctions of \mathcal{L} with eigenvalues $(2k + n)|\lambda|$ and the spectral resolution of \mathcal{L} is written as

$$(2.2) \quad f(z, t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e_k^\lambda(z, t) |\lambda|^n d\lambda.$$

We let μ_r stand for the normalized surface measure on $\{z \in \mathbb{C}^n : |z| = r\}$ which is also considered as a measure on $\{(z, 0) : |z| = r\} \subset \mathbb{H}^n$. The spherical means of a function f on \mathbb{H}^n are defined by

$$(2.3) \quad f * \mu_r(z, t) = \int_{|w|=r} f((z, t)(-w, 0)) d\mu_r(w).$$

The spherical means $f * \mu_r$ is given by the expansion

$$f * \mu_r(z, t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \int_{-\infty}^{\infty} e_k^\lambda(r, 0) f * e_k^\lambda(z, t) |\lambda|^n d\lambda$$

where $e_k^\lambda(r, 0) = e_k^\lambda(w, 0)$ with $|w| = r$. This follows from the fact that $e_k^\lambda(z, t)$ are spherical functions on \mathbb{H}^n , see [13]. The above gives the relation between spherical means $f * \mu_r$ and the spectral projections $f * e_k^\lambda$. This has been used in [13] in the proof of Theorem 1.1. The same argument shows that in order to prove Theorem 1.2 we only need to prove Theorem 1.3.

A simple calculation shows that

$$\int_{-\infty}^{\infty} f * \mu_r * (z, t) e^{i\lambda t} dt = \int_{|w|=r} f^\lambda(z - w) e^{\frac{i\lambda}{2} \text{Im}(z \cdot \bar{w})} d\mu_r(w)$$

where f^λ is the inverse Fourier transform of f in the t variable. Defining the λ -twisted convolution of F and G by

$$F *_\lambda G(z) = \int_{\mathbb{C}^n} F(z - w) G(w) e^{\frac{i\lambda}{2} \text{Im}(z \cdot \bar{w})} dw$$

the above means

$$(f * \mu_r)^\lambda(z) = f^\lambda *_\lambda \mu_r(z).$$

This reduction allows us to concentrate on twisted spherical means $f^\lambda *_\lambda \mu_r(z)$ rather than $f * \mu_r$ and a scaling argument reduces everything to the case $\lambda = 1$. In this case we simply write $F \times G$ instead of $F *_1 G$ and call it the twisted convolution.

The vector fields X_j and Y_j on \mathbb{H}^n give rise to the complex vector fields $X_j \pm iY_j$ and applying them to functions of the form $f(z, t) = e^{-it}F(z)$ we are led to the vector fields

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{2}z_j, \quad j = 1, 2, \dots, n.$$

These $2n$ vector fields together with the identity generate an algebra which is isomorphic to the $(2n + 1)$ dimensional Heisenberg algebra. This algebra plays for the twisted convolution on \mathbb{C}^n a role analogous to that of the Lie algebra of left invariant vector fields on a Lie group. In fact it is easy to verify that

$$(2.4) \quad Z_j(f \times g) = f \times Z_jg, \quad \bar{Z}_j(f \times g) = f \times \bar{Z}_jg.$$

The special Hermite operator A also called the twisted Laplacian is defined by

$$A = -\frac{1}{2} \sum_{j=1}^n (Z_j\bar{Z}_j + \bar{Z}_jZ_j).$$

This is related to the subLaplacian by

$$\mathcal{L}(e^{-it}f(z)) = e^{-it}Af(z).$$

An easy calculation shows that L can be written in the form

$$(2.5) \quad A = -\Delta_z + \frac{1}{4}|z|^2 - iN$$

where $N = \sum_{j=1}^n (y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j})$ is the infinitesimal rotation operator. The spectral decomposition of A is given by the special Hermite expansion which can be put in the compact form

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k(z).$$

The operators $f \rightarrow f \times \varphi_k$ are the spectral projections and we have

$$A(f \times \varphi_k) = (2k + n)f \times \varphi_k.$$

For more details we refer to [12].

LEMMA 2.1. — *For $f \in L^2(\mathbb{C}^n)$ we have the expansion*

$$f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) f \times \varphi_k(z)$$

where $\varphi_k(r)$ stands for $L_k^{n-1}(\frac{1}{2}r^2) e^{-\frac{1}{4}r^2}$.

Proof. — See Theorem 2.4.4, page 84 of [13]. □

LEMMA 2.2. — *Let $f(z) = \sum_{k=0}^{\infty} c_k \varphi_k(z)$ be a smooth compactly supported radial function on \mathbb{C}^n . Then for every $N > 0$ there exists a positive constant C_N such that*

$$|c_k| \leq C_N (1 + k)^{-N}.$$

Proof. — Using the fact that $A\varphi_k = (2k + n)\varphi_k$ and applying the operator A repeatedly we have,

$$A^N f(z) = \sum (2k + n)^N c_k \varphi_k(z)$$

which yields, by the orthogonality of φ_k 's

$$\|A^N f\|_2^2 = C \sum (2k + n)^{2N} |c_k|^2 \frac{k! (n - 1)!}{(k + n - 1)!} < \infty$$

which finishes the proof. □

We require several results from the theory of bigraded spherical harmonics. For each pair of non-negative integers p and q let P_{pq} be the space of all polynomials P in z and \bar{z} of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Let $H_{pq} = \{P \in P_{pq} : \Delta P = 0\}$ where Δ is the standard Laplacian on \mathbb{C}^n . Elements of H_{pq} are called bigraded solid harmonics on \mathbb{C}^n . As polynomials in H_{pq} are homogeneous they are determined by their values on the unit sphere S^{2n-1} . We shall identify H_{pq} with its restriction to the unit sphere. Then $L^2(S^{2n-1})$ is the orthogonal direct sum of H_{pq} as p and q range over all non-negative integers. Given a continuous function f on \mathbb{C}^n we can expand the function $f(r\omega)$ where $r > 0$ and $\omega \in S^{2n-1}$ in terms of spherical harmonics

$$f(r\omega) = \sum_p \sum_q \sum_{j=1}^{d(p,q)} f_{pq}^j(r) S_{pq}^j(\omega)$$

where $\{S_{pq}^j\}$ form an orthonormal basis for the space H_{pq} and $d(p, q)$ is the dimension of H_{pq} . It is well known that the natural action of the unitary group $U(n)$ on the spaces H_{pq} is irreducible. If $\tau \in U(n)$, the invariance of H_{pq} implies

$$S_{pq}^i(\tau^{-1}z) = \sum_{j=0}^{d(p,q)} t_{ij}^{pq}(\tau) S_{pq}^j(z)$$

where $\tau \rightarrow t_{ij}^{pq}(\tau)$ are the matrix entries of the above representation.

We associate with each function $f \in L_{loc}(B_R)$ the Fourier series

$$(2.6) \quad f(z) = \sum_{p,q} \sum_{j=1}^{d(p,q)} f_{pq}^j(|z|) S_{pq}^j(\omega),$$

where $\omega = \frac{z}{|z|}$. Using the orthogonality of the matrix entries, we have

$$(2.7) \quad f_{pq}^j(|z|) S_{pq}^i(\omega) = d(p, q) \int_{U(n)} f(\tau^{-1}z) t_{ij}^{p,q}(\tau) d\tau$$

for $1 \leq i, j \leq d(p, q)$.

For $B > 0$, let us denote by V_B the class of continuous functions f satisfying the condition $f \times \mu_r(z) = 0$ whenever $r > |z| + B$.

LEMMA 2.3. — Suppose $f \in V_B$. Then

$$f_{pq}^j(|z|) Y_{pq}(\omega) \in V_B \quad \text{for any } Y_{pq} \in H_{pq}.$$

Proof. — From (2.7) we have

$$f_{pq}^j(|z|) S_{pq}^i(\omega) \in V_B.$$

Since i is arbitrary, the lemma follows. □

We shall also need the following Hecke-Bochner type identity.

LEMMA 2.4. — Let $f \in L^1(\mathbb{C}^n)$ be of the form $f = Pg$ where g is radial and $P \in H_{pq}$ is a solid harmonic. Then

$$f \times \varphi_k(z) = (2\pi)^{-n} P(z)g \times \varphi_{k-p}^{n+p+q-1}(\xi)$$

where

$$\varphi_k^{n+p+q-1}(\xi) = L_k^{n+p+q-1} \left(\frac{1}{2} |\xi|^2 \right) e^{-\frac{1}{4} |\xi|^2}, \quad \xi \in \mathbb{C}^{n+p+q}, \quad |\xi| = |z|$$

and the convolution on the right is on \mathbb{C}^{n+p+q} .

For a proof see [14], page 70.

3. A support theorem for twisted spherical means

In this section we prove Theorem 1.3. First we show that each $(p, 0)$ coefficient of the function f satisfies an appropriate confluent hypergeometric equation in the exterior of a ball. Then we use the decay condition to show that these coefficients must vanish.

LEMMA 3.1. — Let A_m be the special Hermite operator on \mathbb{C}^m and p be a fixed non negative integer. Let g be a radial function on \mathbb{C}^m which satisfies the equation

$$A_m g(z) = \lambda g(z) \quad \text{if } |z| \geq B$$

for some $\lambda \in \{2j + m - 2p : j = 0, 1, \dots, (p - 1)\}$. If $g(z)e^{\frac{1}{4}|z|^2}$ is a Schwartz class function then g has to vanish for $|z| \geq B$.

Proof. — As g is radial we have $Ng = 0$ where N is the rotation operator (see(2.2)). Now it is well known that the equation (see[1] for eg., p. 365–366)

$$(-\Delta + \frac{1}{4}|z|^2)g = \lambda g$$

can be reduced to a confluent hypergeometric equation using appropriate change of variables. Hence g can be written as a linear combination of two confluent hyper geometric functions whose asymptotics (see [7], p. 254–259) are not compatible with the decay condition on g . This forces g to vanish for $|z| \geq B$. □

Now assume that f satisfies the hypothesis in Theorem 1.3 and consider the expansion

$$(3.1) \quad f \times \mu_r(z) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) f \times \varphi_k(z)$$

which is zero for all $r > B + |z|$. Here $\varphi_k(r) = L_k^{n-1}(\frac{1}{2}r^2)e^{-\frac{1}{4}r^2}$.

Viewing the above as a radial function in r , we have by Lemma 2.2

$$(3.2) \quad |f \times \varphi_k(z)| \leq C_N(z) (1+k)^{-N} \quad \text{for every } N > 0.$$

Rewriting (3.1) as

$$(3.3) \quad f \times \mu_{\sqrt{2s}}(z) e^{\frac{1}{2}s} = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} f \times \varphi_k(z) L_k^{n-1}(s),$$

differentiating p times with respect to the variable s and using the identity $\frac{d}{dt} L_k^\alpha(t) = -L_{k-1}^{\alpha+1}(t)$ (see [11]) we obtain

$$\sum_{k=p}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} f \times \varphi_k(z) L_{k-p}^{n+p-1}(s) = 0$$

for $\sqrt{2s} > B + |z|$ (differentiating inside the summation is justified by(3.2)).

By Lemma 2.3 we may replace f with any of the terms in the spherical harmonic expansion. Let gP be such a term where g is radial and $P \in H_{p0}$

is a solid bigraded spherical harmonic. Then an application of Lemma 2.4 gives

$$(3.4) \quad \sum_{k=0}^{\infty} \left(\Pi_{j=0}^{(p-1)}(2k + 2p - 2j) \right) \frac{k!(n + p - 1)!}{(k + n + p - 1)!} \varphi_k^{n+p-1}(r)g \times \varphi_k^{n+p-1}(\xi) = 0$$

for $r > B + |\xi|$ on \mathbb{C}^{n+p} . We rewrite this as

$$(3.5) \quad \left(\Pi_{j=0}^{(p-1)} (A_{n+p} - (2j + n - p))g \right) \times \mu_r^{n+p}(\xi) = 0$$

for $r > B + |\xi|$ where μ_r^{n+p} is the normalized surface measure on the sphere of radius r in \mathbb{C}^{n+p} and A_{n+p} is the special Hermite operator on \mathbb{C}^{n+p} . Notice that $\left(\Pi_{j=0}^{(p-1)} (L_{n+p} - (2j + n - p))g \right)$ is a radial function. Hence evaluating the above at origin we obtain

$$(3.6) \quad \left(\Pi_{j=0}^{(p-1)} (L_{n+p} - (2j + n - p))g \right)(\xi) = 0$$

for $|\xi| > B$. Now using induction and the decay assumption in the hypothesis of Theorem 1.3, along with Lemma 3.1 we finish the proof.

Next we show that the (p, q) coefficients of f too must vanish for $|z| > B$. This is done by reducing it to the already done $(p, 0)$ case by means of differential operators. Let $W_j = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j = \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) + \frac{1}{2}(x_j + iy_j)$. Then a simple computation shows that $W_j(f \times \mu_r) = W_j f \times \mu_r$. So if $f \in V_B$ is smooth then $W_j f$ also belongs to V_B .

Now let $h(|z|)$ be one of the (p, q) coefficients of f in the spherical harmonic expansion. Then by Lemma 2.3 we have

$$g(z) = h(|z|) \frac{z_1^p \bar{z}_2^q}{|z|^{p+q}} \in V_B.$$

By the above remark we have $W_2 g \in V_B$. A simple calculation shows that $W_2 g$ is given by

$$\frac{z_1^p \bar{z}_2^{q-1} |z_2|^2}{|z|^{p+q+1}} \left(h'(s) - \frac{(p+q)}{s} h(s) + \frac{s}{2} h(s) \right) + \frac{2q}{s} h(s) \frac{z_1^p \bar{z}_2^{q-1}}{|z|^{p+q-1}}$$

where $s = |z|$. Projecting the function $\frac{z_1^p \bar{z}_2^{q-1} |z_2|^2}{|z|^{p+q+1}}$ to the span of $\frac{z_1^p \bar{z}_2^{q-1}}{|z|^{p+q-1}}$ and using Lemma 2.3 again we obtain that the function

$$(3.7) \quad \left[\frac{1}{(n+p+q-1)} \left(h'(s) - \frac{(p+q)}{s} h(s) + \frac{s}{2} h(s) \right) + \frac{2q}{s} h(s) \right] \frac{z_1^p \bar{z}_2^{q-1}}{|z|^{p+q-1}}$$

belongs to the class V_B .

We claim that each (p, q) coefficient of f must be of the form $e^{-\frac{s^2}{4}} P(\frac{1}{s})$, where $P(\frac{1}{s})$ is a polynomial in s and $\frac{1}{s^r}$ for r rational if $s > B$. To prove

this claim, first assume $q = 1$. Then the above gives

$$(3.8) \quad \left[\frac{1}{(n+p)} \left(h'(s) - \frac{(p+1)}{s} h(s) + \frac{s}{2} h(s) \right) + \frac{2}{s} h(s) \right] \frac{z_1^p}{|z|^p} \in V_B.$$

Notice that the assumptions on the original function f hold true for the above function as well. Hence from what we have seen for $(p, 0)$ coefficients it follows that (3.8) should vanish for $s > B$. Solving the linear differential equation we obtain that h is of the claimed form. A simple induction now proves the claim fully.

Finally the decay assumption on the function f implies that all spherical harmonic coefficients decay in a similar manner which forces all these terms to be zero outside a ball of radius B . This completes the proof of Theorem 1.3.

Remark. — Notice that the condition $f(z)e^{\frac{1}{4}|z|^2}$ is not translation invariant and so the smoothness assumption on the function f was necessary. However, if we assume a slightly stronger condition, namely that $|f(z)| \leq C e^{-(\frac{1}{4}+\epsilon)|z|^2}$, then we may convolve f on the right with a radial approximate identity and obtain the same result. These type of rapid decay conditions are natural for the study of twisted spherical means as can be seen from the earlier works (see [1] [6] and [9]).

4. Twisted spherical means on \mathbb{C}

This section is devoted to a proof Theorem 1.4. Convolution f on the right with a compactly supported radial approximate identity and reducing ϵ we may assume that f is smooth. As $\bar{f} \times \mu_r = \overline{\mu_r \times f}$ it follows that $\bar{f} \in V_B$ as well. Consequently we may assume that f is real valued. Our aim is to show that $\bar{z}^n f(z) \in V_B$ for all $n > 0$, following the method of Helgason, using Green’s formula (see [5] p. 107–108).

We have

$$f \times \mu_r(z) = \int_{|w|=r} f(z+w) e^{-\frac{i}{2} \text{Im} z \bar{w}} d\mu_r(w) = 0 \quad \text{for } r > |z| + B.$$

Therefore,

$$\int_{r_1 < |w| < r_2} f(z+w) e^{-\frac{i}{2} \text{Im} z \bar{w}} dw = 0 \quad \text{if } r_2 > r_1 > B + |z|.$$

Applying the operator $\partial = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ with respect to $z = x + iy$ we have

$$\int_{r_1 < |w| < r_2} \partial f(z+w) e^{-\frac{i}{2} \text{Im} z \bar{w}} + \int_{r_1 < |w| < r_2} f(z+w) e^{-\frac{i}{2} \text{Im} z \bar{w}} \left(-\frac{1}{2} i \bar{w}\right) dw = 0$$

for $r_2 > r_1 > B + |z|$. Hence

$$(4.1) \quad \int_{r_1 < |w| < r_2} \partial_w \left(f(z+w) e^{-\frac{i}{2} \text{Im } z \bar{w}} \right) dw \\ = \frac{1}{2} \int_{r_1 < |w| < r_2} \bar{w} f(z+w) e^{-\frac{i}{2} \text{Im } z \bar{w}} dw.$$

Using Green's formula on left hand side of(4.1) we have

$$\int_{|w|=r_2} \bar{w} f(z+w) e^{-\frac{i}{2} \text{Im } z \bar{w}} d\mu_{r_2}(w) - \int_{|w|=r_1} \bar{w} f(z+w) e^{-\frac{i}{2} \text{Im } z \bar{w}} d\mu_{r_2}(w) \\ = \frac{1}{2} \int_{r_1 < |w| < r_2} \bar{w} f(z+w) e^{-\frac{i}{2} \text{Im } z \bar{w}} dw.$$

Let $g(z) = \bar{z}f(z)$. Then the above implies

$$g \times \mu_t(z) - g \times \mu_s(z) = \frac{1}{2} \int_s^t g \times \mu_r(z) r dr$$

for $t > s > B + |z|$. This, in turn, implies that

$$(4.2) \quad g \times \mu_t(z) = C(z) e^{\frac{t^2}{4}} \quad \text{for } t > B + |z|.$$

But notice that g also satisfies the growth condition assumed on f which gives us the estimate, for a fixed z ,

$$|g \times \mu_t(z)| \leq C e^{(\frac{1}{4}-\delta)t^2}$$

for some $0 < \delta < \epsilon$. Comparing this estimate with(4.2) we conclude that $g \times \mu_t(z) = 0$ for $t > B + |z|$. Thus we have proved that if f is in the class V_B so is $g(z) = \bar{z}f(z)$. Repeating, we have $\bar{z}^n f(z) \in V_B$ for all $n \in \mathbb{N}$.

Evaluating the means of $\bar{z}^n f(z)$ at origin we have

$$(4.3) \quad \int_{|w|=r} \bar{w}^n f(w) d\mu_r(w) = 0 \quad \text{for } r > B.$$

Since f is real valued, taking complex conjugate in(4.3) we also have

$$(4.4) \quad \int_{|w|=r} w^n f(w) d\mu_r(w) = 0 \quad \text{for } r > B.$$

Now(4.3) and (4.4) clearly implies that $f(z)$ vanishes for $|z| > B$. This completes the proof of Theorem 1.4.

Next we show by an example that this result does not generalize to higher dimensions. We shall construct a function $h(z)$ on \mathbb{C}^2 which has exponential decay at infinity and satisfies

$$h \times \mu_r(z) = \mu_r \times h(z) = 0 \quad \text{for } r > |z|.$$

We make use of a result of Epstein and Kliener on the spherical means on \mathbb{R}^d in the proof. Let us briefly recall their result.

For a function f on \mathbb{R}^d we have the spherical harmonic expansion

$$f(x) = f(\rho\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl}(\rho) Y_k^l(\omega)$$

where $\rho = |x|$ and $\{Y_k^l(\omega) : l = 1, 2, \dots, d_k\}$ is an orthonormal basis for the space V_k of homogeneous harmonic polynomials of degree k restricted to the unit sphere. For each k the space V_k is invariant under the action of $SO(d)$. When $d = 2m$ for some m , it is invariant under the the action of the unitary group $U(m)$ as well, and under this action of $U(m)$ the space V_k breaks up into an orthogonal direct sum of H_{pq} 's where $p + q = k$.

Let σ_r stand for the normalized surface measure on the sphere of radius r centered at the origin contained in \mathbb{R}^d . The main result in[4] is the following:

THEOREM 4.1. — *A continuous function f on \mathbb{R}^d satisfies*

$$f * \sigma_r(x) = 0 \quad \text{for } r > |x| + B \quad \text{for all } x \in \mathbb{R}^d$$

if and only if

$$a_{kl}(\rho) = \sum_{i=0}^{k-1} \alpha_{kl}^i \rho^{k-d-2i}, \quad \alpha_{kl}^i \in \mathbb{C},$$

for all $k > 0, 1 \leq l \leq d_k$, and $a_0(\rho) = 0$ whenever $\rho > B$.

We shall also need another result on the spaces H_{pq} whose proof may be found in [8]. The space $H_{pq} \cdot H_{rs}$ is defined to be the vector space of finite sums $\sum f_i g_i$ with $f_i \in H_{pq}$ and $g_i \in H_{rs}$. For non-negative integers p, q, r, s define

$$\nu = \nu(p, q, r, s) = \min(p, s) + \min(r, q).$$

LEMMA 4.2. — $H_{pq} \cdot H_{rs} \subset \sum_{j=0}^{\nu} H_{(p+r-j)(q+s-j)}$ where $\nu = \nu(p, q, r, s)$.

Our example has the following simple form. Define h on \mathbb{C}^2 as

$$h(z) = e^{-\frac{1}{4}|z|^2} \frac{z_1 \bar{z}_2}{|z|^6}.$$

We first prove that $h \times \mu_r(z) = 0$ for $r > |z|$. Since $\mu_r \times h = \overline{\bar{h} \times \mu_r}$ and \bar{h} is of the same form it follows that h satisfies both the conditions.

Now,

$$h \times \mu_r(z) = \int_{|w|=r} \frac{(z_1 + w_1)(\bar{z}_2 + \bar{w}_2)}{|z + w|^6} e^{-\frac{1}{4}|z+w|^2} e^{-\frac{i}{2} \text{Im } z \cdot \bar{w}} d\mu_r(w).$$

Expanding the term $|z + w|^2$ and simplifying we see that it is enough to consider the integral

$$\int_{|w|=r} e^{-\frac{z \cdot \bar{w}}{2}} \frac{(z_1 + w_1)(\bar{z}_2 + \bar{w}_2)}{|z + w|^6} d\mu_r(w).$$

Expanding the exponential factor, we are led to terms of the form

$$\int_{|w|=r} \frac{(z_1 + w_1)(\bar{z}_2 + \bar{w}_2)}{|z + w|^6} \bar{w}_1^{\alpha_1} \bar{w}_2^{\alpha_2} d\mu_r(w) \quad \text{where } \alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}.$$

Writing $\bar{w}_1 = (\bar{z}_1 + \bar{w}_1 - \bar{z}_1)$ etc. and expanding again we see that it is enough to consider terms of the form

$$\int_{|w|=r} \frac{(z_1 + w_1)(\bar{z}_1 + \bar{w}_1)^{\beta_1} (\bar{z}_2 + \bar{w}_2)^{1+\beta_2}}{|z + w|^6} d\mu_r(w) \quad \text{where } \beta_1, \beta_2 \in \mathbb{N} \cup \{0\}.$$

Let $g(z) = (z_1 \bar{z}_1^{\beta_1} \bar{z}_2^{1+\beta_2}) / (|z|^6)$, then we need to show that $g * \mu_r(z) = 0$ for $r > |z|$ where $*$ stands for the Euclidean convolution. Writing

$$P(z) = z_1 \bar{z}_1^{\beta_1} \bar{z}_2^{1+\beta_2} = (z_1 \bar{z}_2^{1+\beta_2}) \bar{z}_1^{\beta_1}$$

and using Lemma 4.2 we have $P(z) = P_0(z) + |z|^2 P_1(z)$ if $\beta_1 \geq 1$ where P_0 is a solid harmonic of degree $2 + \beta_1 + \beta_2$ and P_1 is of degree $\beta_1 + \beta_2$. If $\beta_1 = 0$ then $P(z)$ belongs to $H_{1(1+\beta_2)}$. Now it is a matter of easy verification using Theorem 4.1 that each such term satisfies the convolution equation we want. This finishes the proof.

Remark. — Though the different behaviour of twisted spherical means in dimension one and higher dimensions is in sharp contrast with the known cases, namely, the Euclidean and symmetric spaces, it has a simple explanation. A close examination of Theorem 1.4 shows that the convolution conditions imply that the function f is orthogonal to holomorphic and anti-holomorphic monomials on any sphere of radius greater than B centered at the origin. When the dimension is one this implies that the function vanishes outside the ball of radius B while for the higher dimensional case this is clearly not sufficient.

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