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HOMOLOGY AND MODULAR CLASSES OF LIE ALGEBROIDS

by Janusz GRABOWSKI,
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ABSTRACT. — For a Lie algebroid, divergences chosen in a classical way lead to a uniquely defined homology theory. They define also, in a natural way, modular classes of certain Lie algebroid morphisms. This approach, applied for the anchor map, recovers the concept of modular class due to S. Evens, J.-H. Lu, and A. Weinstein.


1. Introduction

Homology of a Lie algebroid structure on a vector bundle $E$ over $M$ are usually considered as homology of the corresponding Batalin-Vilkovisky algebra associated with a chosen generating operator $\partial$ for the Schouten-Nijenhuis bracket on multisecions of $E$. The generating operators that are homology operators, i.e. $\partial^2 = 0$, can be identified with flat $E$-connections on $\Lambda^{\text{top}} E$ (see [18]) or divergence operators (flat right $E$-connections on $M \times \mathbb{R}$, see [8]). The problem is that the homology group depends on the choice of the generating operator (flat connection, divergence) and no one seems to be privileged. For instance, if a Lie algebroid on $T^*M$ associated

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with a Poisson tensor $P$ on $M$ is concerned, then the traditional Poisson homology is defined in terms of the Koszul-Brylinski homology operator $\partial_P = [d, i_P]$. However, the Poisson homology groups may differ from the homology groups obtained by means of 1-densities on $M$. The celebrated modular class of the Poisson structure [16] measures this difference. Analogous statement is valid for triangular Lie bialgebroids [10].

The concept of a Lie algebroid divergence, so a generating operator, associated with a ‘volume form’, i.e. nowhere-vanishing section of $\bigwedge^{\text{top}} E^*$, is completely classical (see [10], [18]). Less-known seems to be the fact that we can use ‘odd-forms’ instead of forms (cf. [2]) with same formulas for divergence and that such nowhere-vanishing volume odd-forms always exist. The point is that the homology groups obtained in this way are all isomorphic, independently on the choice of the volume odd-form. This makes the homology of a Lie algebroid a well-defined notion. From this point of view the Poisson homology is not the homology of the associated Lie algebroid $T^*M$ but a deformed version of the latter, exactly as the exterior differential $d^\phi \mu = d\mu + \phi \wedge \mu$ of Witten [17] is a deformation of the standard de Rham differential.

In this language, the modular class of a Lie algebroid morphism $\kappa : E_1 \to E_2$ covering the identity on $M$ is defined as the class of the difference between the pull-back of a divergence on $E_2$ and a divergence on $E_1$, both associated with volume odd-forms. In the case when $\kappa : E \to TM$ is the anchor map, we recognize the standard modular class of a Lie algebroid [3] but it is clear that other (canonical) morphisms will lead to other (canonical) modular classes.

2. Divergences and generating operators

2.1. Lie algebroids and their cohomology

Let $\tau : E \to M$ be a vector bundle. Let $A^i(E) = \text{Sec}(\bigwedge^i E)$ for $i = 0, 1, 2, \ldots$, let $A^i(E) = \{0\}$ for $i < 0$, and denote by $A(E) = \bigoplus_{i \in \mathbb{Z}} A^i(E)$ the Grassmann algebra of multisections of $E$. It is a graded commutative associative algebra with respect to the wedge product.

There are different ways to define a Lie algebroid structure on $E$. We prefer to see it as a linear graded Poisson structure on $A(E)$ (see [7]), i.e., a graded bilinear operation $[\cdot, \cdot]$ on $A(E)$ of degree $-1$ with the following properties:
(a) Graded anticommutativity:
\[ [a, b] = -(-1)^{(|a|-1)(|b|-1)} [b, a]. \]

(b) The graded Jacobi identity:
\[ [a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)} [b, [a, c]]. \]

(c) The graded Leibniz rule:
\[ [a, b \wedge c] = [a, b] \wedge c + (-1)^{|a|} b \wedge [a, c]. \]

This bracket is just the Schouten bracket associated with the standard Lie algebroid bracket on sections of \( E \). It is well known that such brackets are in bijective correspondence with de Rham differentials \( d \) on the Grassmann algebra \( \mathcal{A}(E^*) \) of multisections of the dual bundle \( E^* \) which are described by the formula

\[
(2.1) \quad d\mu(X_0, \ldots, X_n) = \sum_i (-1)^i [X_i, \mu(X_0, \ldots, \hat{i}, \ldots, X_n)] \\
+ \sum_{k<l} (-1)^{k+l} \mu([X_k, X_l], X_0, \ldots, \hat{k}, \ldots, \hat{l}, \ldots, X_n)
\]

where the \( X_i \) are sections of \( E \). We will refer to elements of \( \mathcal{A}(E^*) \) as forms. Since \( d \) is a derivation on \( \mathcal{A}(E^*) \) of degree 1 with \( d^2 = 0 \), it defines the corresponding de Rham cohomology \( H^*(E, d) \) of the Lie algebroid in the obvious way.

### 2.2. Generating operators and divergences

The definition of the homology of a Lie algebroid is more delicate than that of cohomology. The standard approach is via generating operators for the Schouten bracket \([\cdot, \cdot] \). By this we mean an operator \( \partial \) of degree \(-1\) on \( \mathcal{A}(E) \) which satisfies

\[
(2.2) \quad [a, b] = (-1)^{|a|} (\partial (a \wedge b) - \partial (a) \wedge b - (-1)^{|a|} a \wedge \partial (b)).
\]

The idea of a generating operator goes back to the work by Koszul [13]. A generating operator which is a homology operator, i.e. \( \partial^2 = 0 \), gives rise to the so called Batalin-Vilkovisky algebra. Remark that the leading sign \((-1)^{|a|}\) serves to produce graded antisymmetry with respects to the degrees shifted by \(-1\) out of graded symmetry. One could equally well use \((-1)^{|b|}\) instead of \((-1)^{|a|}\), or one could use the obstruction for \( \partial \) to be a graded right derivation in the parentheses instead of a graded left one as we did. We shall stick to the standard conventions.
It is clear from Eq. (2.2) and from the properties of the Schouten bracket that $\partial$ is then a second order differential operator on the graded commutative associative algebra $A(E)$, which is completely determined by its restriction to $\text{Sec}(E)$. In fact, it is easy to see (cf. [8]) that

\begin{equation}
\partial(X_1 \wedge \cdots \wedge X_n) = \sum_i (-1)^{i+1} \partial(X_i) X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_n
+ \sum_{k<l} (-1)^{k+l} [X_k, X_l] \wedge X_1 \wedge \cdots \hat{X}_k \cdots \wedge X_n
\end{equation}

for $X_1, \ldots, X_n \in \text{Sec}(E)$, which looks completely dual to Eq. (2.1). From Eq. (2.2) we get the following property of $\partial$:

\begin{equation}
- \partial(fX) = -f \partial(X) + [X, f] \quad \text{for } X \in \text{Sec}(E), f \in C^\infty(M).
\end{equation}

Since $[X, f] = \rho(X)(f)$, where $\rho : E \to TM$ is the anchor map of the Lie algebroid structure on $E$, the operator $-\partial$ has the algebraic property of a divergence. Conversely, Eq. (2.3) defines a generating operator for $[\cdot, \cdot]$ if only Eq. (2.4) is satisfied, i.e., generating operators can be identified with divergences. We may express this by $\text{div} \leftrightarrow \partial_{\text{div}}$. But a true divergence $\text{div} : \text{Sec}(E) \to C^\infty(M)$ satisfies besides Eq. (2.4) a cocycle condition

\begin{equation}
\text{div}([X, Y]) = [\text{div}(X), Y] + [X, \text{div}(Y)], \quad X, Y \in \text{Sec}(E),
\end{equation}

which is equivalent (see [8]) to the fact that the corresponding generating operator $\partial_{\text{div}}$ is a homology operator: $(\partial_{\text{div}})^2 = 0$. Note that divergences can be used in construction of generating operators also in the supersymmetric case (cf. [12]).

From now on we will fix the Lie algebroid structure on $E$, and we will denote by $\text{Gen}(E)$ the set of generating operators for $[\cdot, \cdot]$ which are homology operators, and by $\text{Div}(E)$ the canonically isomorphic (by Eq. (2.3)) set of divergences for the Lie algebroid satisfying Eq. (2.4) and Eq. (2.5). The problem is that there does not exist a canonical divergence, thus no canonical generating operator.

The set $\text{Div}(E)$ can be identified with the set of all flat $E$-connections on $\bigwedge^{\text{top}} E^*$, i.e., operators $\nabla : \text{Sec}(E) \times \text{Sec}(\bigwedge^{\text{top}} E^*) \to \text{Sec}(\bigwedge^{\text{top}} E^*)$ which satisfy

\begin{enumerate}
\item $\nabla_X f \mu = f \nabla_X \mu$,
\item $\nabla_X (f \mu) = f \nabla_X \mu + \rho(X)(f) \mu$,
\item $[\nabla_X, \nabla_Y] = \nabla_{[X, Y]}$.
\end{enumerate}

The identification is via

\begin{equation}
\mathcal{L}_X \mu - \nabla_X \mu = \text{div}(X) \mu
\end{equation}
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(cf. [10, (50)]), where $L_X = d i_X + i_X d$ is the Lie derivative. Note that Eq. (2.6) is independent of the choice of the section $\mu \in \text{Sec}(\Lambda^\text{top}(E^*))$. We can use $\Lambda^\text{top}(E)$ instead of $\Lambda^\text{top}(E^*)$ and get the identification of $\text{Div}(E)$ with the set of flat $E$-connection on $\Lambda^\text{top}(E)$ by (see [18])

$$L_X \Lambda - \nabla_X \Lambda = \text{div}(X)\Lambda. \quad (2.7)$$

Of course, additional structures on $E$ as, e.g., a Riemannian metric (smoothly arranged scalar products on fibers of $E$), may furnish a distinguished divergence on $E$. Fixing a metric we can distinguish a canonical torsionfree connection $\nabla$ on $E$—the Levi-Civita connection for the Lie algebroid—in the standard way. It satisfies the standard Bianchi and Ricci identities (see [15]) and induces a connection on $\Lambda^\text{top}(E)$ for which the generating operator $\partial_{\nabla}$ has the local form (see [18])

$$\partial_{\nabla}^2 = \sum_{k,j} i(\alpha^j)\nabla_{X_j} i(\alpha^k)\nabla_{X_k} = \sum_{k,j} i(\alpha^j)i(\alpha^k)(\nabla_{X_j}\nabla_{X_k} - \nabla_{\nabla_{X_j}X_k}),$$

$\partial_{\nabla}^2 = 0$ is equivalent to

$$\sum_{j,k} i(\alpha^j)i(\alpha^k)R(X_j, X_k) = 0, \quad (2.8)$$

where $R$ is the curvature tensor of $\nabla$. For a Levi-Civita connection $\nabla$ the generating operator $\partial_{\nabla}$ is really a homology operator due to the following lemma.

**Lemma 2.1.** — A torsionfree connection $\nabla$ on $E$ satisfies simultaneously the Bianchi and the Ricci identity if and only if Eq. (2.8) holds for dual local frames $X_k$ and $\alpha^k$ of $E$ and $E^*$, respectively.

**Proof.** — Eq. (2.8) is equivalent to $\sum_{j,k} R(X_j, X_k)^*(\alpha^k \wedge \alpha^j \wedge \omega) = 0$ for all forms $\omega$. It suffices to check this for $\omega$ a function or a 1-form due to the derivation property of contractions. For $\omega$ a function $f$ we have

$$\sum_{j,k} R(X_j, X_k)^*(f \alpha^k \wedge \alpha^j) =$$

$$= \sum_{j,k} f\left(R(X_j, X_k)^*(\alpha^k) \wedge \alpha^j + \alpha^k \wedge R(X_j, X_k)^*(\alpha^j)\right)$$

$$= 2f \sum_{s,j,k} R_{jks}^k \alpha^s \wedge \alpha^j$$

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and this vanishes for all $f$ if and only if $R_{jks}^k$ is symmetric in $(j, s)$, i.e., if the Ricci identity holds. For $\omega$ a 1-form, say $\alpha^i$, we have

$$\sum_{j,k} R(X_j, X_k)^*(\alpha^k \wedge \alpha^j \wedge \alpha^i) =$$

$$\sum_{j,k} \left( R(X_j, X_k)^*(\alpha^k \wedge \alpha^j) \wedge \alpha^i + \alpha^k \wedge \alpha^j \wedge R(X_j, X_k)^*(\alpha^i) \right)$$

$$= 0 + \sum_{j,k,s} R_{jks}^j \alpha^k \wedge \alpha^j \wedge \alpha^s$$

and this vanishes for all $i$ if and only if $\sum_{\text{cycl}(j,k,s)} R_{jks}^i = 0$, i.e., if the first Bianchi identity holds.

Corollary 2.2. — Any Levi-Civita connection for a Riemannian metric on a Lie algebroid $E$ induces a flat connection on $\bigwedge^{\text{top}} E$, thus also on $\bigwedge^{\text{top}} E^*$. 

3. Homology of the Lie algebroid

3.1. Getting divergences from odd forms

There is no distinguished divergence for the Lie algebroid structure on $E$, but there is a distinguished subset of divergences which we may obtain in a classical way. Firstly, suppose that the line bundle $\bigwedge^{\text{top}} E^*$ is trivializable. So we can choose a vector volume, i.e., a nowhere vanishing section $\mu \in \text{Sec}(\bigwedge^{\text{top}} E^*)$. Then the formula

$$\mathcal{L}_X \mu = \text{div}_\mu(X) \mu, \text{ where } X \in \text{Sec}(E),$$

defines a divergence $\text{div}_\mu$. We observe that $\text{div}_{-\mu} = \text{div}_\mu$. Thus for the non-orientable case we look for sections of a bundle over $M$ which locally consists of non-ordered pairs $\{\mu_\alpha, -\mu_\alpha\}$ for an open cover $M = \bigcup_\alpha U_\alpha$ such that the sets $\{\mu_\alpha, -\mu_\alpha\}$ and $\{\mu_\beta, -\mu_\beta\}$ coincide when restricted to $U_\alpha \cap U_\beta$. The fundamental observation is that such global sections always exist and define global divergences. This is because they can be viewed as sections of the bundle $|\text{Vol}|_E = (\bigwedge^{\text{top}} E^*)_0/\mathbb{Z}_2$, where $(\bigwedge^{\text{top}} E^*)_0/\mathbb{Z}_2$ is the bundle $\bigwedge^{\text{top}} E^*$ with the zero section removed and divided by the obvious $\mathbb{Z}_2$-action of passing to the opposite vector. The bundle $|\text{Vol}|_E$ is a 1-dimensional affine bundle modelled on the vector bundle $M \times \mathbb{R}$, and also a principal $\mathbb{R}$-bundle where $t \in \mathbb{R}$ acts by scalar multiplication with $e^t$. Since it has a contractible fiber, sections always exist. Note that sections
\[ |\mu| \text{ of } |\text{Vol}|_E \text{ are particular cases of odd forms, [2]}: \text{Let } p : \tilde{M} \to M \text{ be the two-fold covering of } M \text{ on which } p^*E \text{ is oriented, namely the set of vectors of length 1 in the line bundle over } M \text{ with cocycle of transition functions } \text{sign } \det(\phi_{\alpha\beta}), \text{ where } \phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V) \text{ is the cocycle of transition functions for the vector bundle } E. \text{ Then the odd forms are those forms on } p^*E \text{ which are in the } -1 \text{ eigenspace of the natural vector bundle isomorphism which covers the decktransformation of } \tilde{M}. \text{ So odd forms are certain sections of a line bundle over a two-fold covering of the base manifold } M. \text{ This is related but complementary to the construction of the line bundle (over } M) \text{ of densities which involve the cocycle of transition functions } |\det(\phi_{\alpha\beta})|. \text{ For example, any Riemannian metric } g \text{ on the vector bundle } E \text{ induces an odd volume form } |\mu|_g \in \text{Sec}(|\text{Vol}|_E) \simeq \text{Sec}(|\text{Vol}|_{E^*}) \text{ which locally is represented by the wedge product of any orthonormal basis of local sections of } E \text{ (thus } E^*). \text{ Note that such product is independent on the choice of the basis modulo sign, so our odd volume is well defined.}

For the definition of a divergence } \text{div}_{|\mu|} \text{ associated to } |\mu| \in \text{Sec}(|\text{Vol}|_E) \text{ we will write simply}

(3.2) \quad \mathcal{L}_X|\mu| = \text{div}_{|\mu|}(X)|\mu| \text{ for } X \in \text{Sec}(E).

\text{Note that the distinguished set } \text{Div}_0(E) \text{ of divergences obtained in this way from sections of } |\text{Vol}|_E \text{ corresponds (in the sense of Eq. (2.6)) to the set of those flat connections on } \Lambda^\text{top} E^* \text{ whose holonomy group equals } \mathbb{Z}_2: \text{ Associate the horizontal leaf } |\mu| \text{ to such a connection, and note that a positive multiple of } |\mu| \text{ gives rise to the same divergence.}

\text{In the case of a vector bundle Riemannian metric } g \text{ on } E \text{ a natural question arises about the relation between the divergence } \text{div}_{|\mu|_g} \text{ associated with the odd volume } |\mu|_g \text{ induced by the metric } g \text{ and the divergence } \text{div}_{\nabla_g} \text{ induced by the flat Levi-Civita connection } \nabla_g \text{ on } \Lambda^\text{top} E^* \simeq \Lambda^\text{top} E.

\text{THEOREM 3.1. — For any vector bundle Riemannian metric } g \text{ on } E

\text{div}_{|\mu|_g} = \text{div}_{\nabla_g}.

\text{Proof. — Let } X_1, \ldots, X_n \text{ be an orthonormal basis of local sections of } E \text{ and } \alpha^k = g(X_k, \cdot) \text{ be the dual basis of local sections of } E^*, \text{ so that } |\mu|_g \text{ is locally represented by } \alpha^1 \wedge \cdots \wedge \alpha^n.
For any local section $X$ of $E$

$$\operatorname{div}_{|\mu|}(X) = -\langle L_X(\alpha^1 \wedge \cdots \wedge \alpha^n), X_1 \wedge \cdots \wedge X_n)\rangle$$

$$= \sum_k \langle \alpha^k, [X, X_k] \rangle$$

$$= \sum_k \langle \alpha^k, \nabla_X X_k - \nabla_{X_k} X \rangle$$

$$= \sum_k g(X_k, \nabla_X X_k) - \sum_k i(\alpha^k) \nabla_X X_k.$$ 

But $-\sum_k i(\alpha^k) \nabla_X X_k = \operatorname{div}_{\nabla g}(X)$ and

$$2 \sum_k g(X_k, \nabla_X X_k) = \sum_k \rho(X) g(X_k, X_k) - \sum_k \nabla_X (g(X_k, X_k) = 0,$$

where $\rho : E \to TM$ is the anchor of the Lie algebroid on $E$, since $\nabla$ is Levi-Civita ($\nabla g = 0$).

### 3.2. The generating operator for an odd form

The corresponding generating operator $\partial_{|\mu|}$ for the divergence of a non-vanishing odd form $|\mu|$ can be defined explicitly by

$$\mathcal{L}_a |\mu| = -i(\partial_{|\mu|}(a))|\mu|,$$

where $\mathcal{L}_a = i_a d - (-1)^{|a|} d i_a$ is the Lie differential associated with $a \in \mathcal{A}(|a|(E)$ so that

$$i(\partial_{|\mu|}(a))|\mu| = (-1)^{|a|} d i_a |\mu|.$$ 

In other words, locally over $U$ we have

$$(3.4) \quad \partial_{|\mu|}(a) = (-1)^{|a|} *_{\mu}^{-1} d *_{\mu} (a),$$

where $*_{\mu}$ is the isomorphism of $\mathcal{A}(E)|_U$ and $\mathcal{A}(E^*)|_U$ given by $*_{\mu}(a) = i_a \mu$, for a representative $\mu$ of $|\mu|$. Note that the right hand side of Eq. (3.4) depends only on $|\mu|$ and not on the choice of the representative, since $*_{\mu} d *_{\mu} = -d *_{\mu}$. Formula Eq. (3.4) gives immediately $\partial_{|\mu|}^2 = 0$, which also follows from the remark on flat connections above. So $\partial_{|\mu|}$ is a homology operator.

Moreover, it is also a generating operator. Namely, using standard calculus of Lie derivatives we get

$$\mathcal{L}_{a \wedge b} = i_b \mathcal{L}_a - (-1)^{|a|} i_{[a,b]} + (-1)^{|a| |b|} i_a \mathcal{L}_b$$
which can be rewritten in the form

\[(3.5) \quad i_{[a,b]} = (-1)^{|a|} \left( -\mathcal{L}_a \wedge b + i_b \mathcal{L}_a + (-1)^{|a|(|b|+1)} i_a \mathcal{L}_b \right). \]

When we apply Eq. (3.5) to $|\mu|$ we get

\[i_{[a,b]} |\mu| = (-1)^{|a|} \left( i(\partial_{|\mu|}(a \wedge b)) - i(\partial_{|\mu|}(a) \wedge b) - (-1)^{|a|} i(a \wedge \partial_{|\mu|}(b)) \right) |\mu| \]

which proves Eq. (2.2). Thus we get:

**Theorem 3.2.** — For any $|\mu| \in \text{Sec}(\text{Vol}|E)$ the formula

\[(3.6) \quad \mathcal{L}_a |\mu| = -i(\partial_{|\mu|}(a)) |\mu| \]

defines uniquely a generating operator $\partial_{|\mu|} \in \text{Gen}(E)$.

We remark that formula Eq. (3.6) in the case of trivializable $\bigwedge^\text{top} E^*$ has been already found in [10]. In this sense the formula is well known. What is stated in Theorem 3.2 is that Eq. (3.6) serves in general, as if the bundle $\bigwedge^\text{top} E^*$ were trivial, if we replace ordinary forms with odd volume forms.

### 3.3. Homology of the Lie algebroid

The homology operator of the form $\partial_{|\mu|}$ will be called the homology operator for the Lie algebroid $E$. The crucial point is that they all define the same homology. This is due to the fact that $\partial_{|\mu_1|}$ and $\partial_{|\mu_2|}$ differ by contraction with an exact 1-form.

In general, two divergences differ by contraction with a closed 1-form. Indeed, $(\text{div}_1 - \text{div}_2)(fX) = f(\text{div}_1 - \text{div}_2)(X)$, so $(\text{div}_1 - \text{div}_2)(X) = i_\phi X$ for a unique 1-form $\phi$. Moreover, Eq. (2.5) implies that $i_\phi [X,Y] = [i_\phi X, Y] + [X, i_\phi Y]$, so $\phi$ is closed. Since both sides are derivations we have

\[(3.7) \quad \partial_{\text{div}_2} - \partial_{\text{div}_1} = i_\phi. \]

But for any $|\mu_1|, |\mu_2| \in \text{Sec}(\text{Vol}|E)$ there exists a positive function $F = e^f$ such that $|\mu_2| = F |\mu_1|$. Then

\[\mathcal{L}_X |\mu_2| = \mathcal{L}_X (F |\mu_1|) = \mathcal{L}_X (F) |\mu_1| + F \mathcal{L}_X (|\mu_1|) \]

so that

\[\text{div}_{|\mu_2|}(X) |\mu_2| = \mathcal{L}_X (f) |\mu_2| + \text{div}_{|\mu_1|}(X) |\mu_2|, \]

i.e.,

\[\text{div}_{|\mu_2|} - \text{div}_{|\mu_1|} = i(df). \]
To see that the homology of $\partial_{\mu_1}$ and $\partial_{\mu_2}$ are the same, note first that $\partial_{\mu_2} = \partial_{\mu_1}a + idf a$. And then let us gauge $\mathcal{A}(E)$ by multiplication with $F = e^f$. This is an isomorphism of graded vector spaces and we have

$$e^f \partial_{\mu_1} e^{-f} a = \partial_{\mu_1}a + idf a = \partial_{\mu_2}a,$$

so $\partial_{\mu_1}$ and $\partial_{\mu_2}$ are graded conjugate operators.

This is just the dual picture of the well-known gauging of the de Rham differential by Witten [17], see also [7] for consequences in the theory of Lie algebroids. Thus we have proved (cf. [10, p.120]):

**Theorem 3.3.** — All homology operators for a Lie algebroid generate the same homology:

$$H^*(E, \partial_{\mu_1}) = H^*(E, \partial_{\mu_2}).$$

In the case of trivializable $\bigwedge^{\text{top}} E^*$, Eq. (3.4) gives Poincaré duality

$$H^*(E, d) \simeq H^{\text{top}-*}(E, \partial_{\mu_1}).$$

### 3.4. Remark

We got a well-defined Lie algebroid homology, in contrast with the standard approach when all generating operators are admitted. It is clear that adding a term $i\phi$ with $\phi$ a closed 1-from which is not exact, as in Eq. (3.7), will probably change the homology. But this could be understood as an a priori deformation, like in the case of the deformed de Rham differential of Witten [17]:

\[ (3.8) \quad d^\phi \eta = d\eta + \phi \wedge \eta. \]

Indeed, $i(i\phi a)\mu = -(-1)^{|a|} \phi \wedge i_\mu \mu$ implies $\ast_\mu i_\phi (a) = -(-1)^{|a|} e_\phi \ast_\mu (a)$, where $e_\phi \eta = \phi \wedge \eta$. Thus we get $(-1)^{|a|} \ast_\mu^{-1} (d + e_\phi) \ast_\mu (a) = (\partial_{\mu_1} - i_\phi)(a)$, so, at least in the the trivializable case, there is the Poincaré duality

$$H^*(E, d + e_\phi) \simeq H_{\text{top}-*}(E, \partial_{\mu_1} - i_\phi).$$

Note that the differentials $d^\phi$ appear as part of the Cartan differential calculus for Jacobi algebroids, see [9], [6], [7], so that there is a relation between generating operators for a Lie algebroid and the Jacobi algebroid structures associated with it.

### 4. Modular classes

#### 4.1. The modular class of a morphism

As we have shown, every Lie algebroid $E$ has a distinguished class $\text{Div}_0(E)$ of divergences obtained from sections of $|\text{Vol}|_E$. Such divergences
differ by contraction with an exact 1-form. Let now $\kappa : E_1 \to E_2$ be a morphism of Lie algebroids.

There is the induced map $\kappa^* : \text{Div}(E_2) \to \text{Div}(E_1)$ defined by

$$\kappa^*(\text{div}_2)(X_1) = \text{div}_2(\kappa(X_1)).$$

The fact that $\kappa^*$ maps divergences into divergences follows from $\kappa(fX) = f\kappa(X)$ and the fact that the Lie algebroid morphism respects the anchors, $\rho_1 = \rho_2 \circ \kappa$. The space $\kappa^*(\text{Div}_0(E_2)) \subset \text{Div}(E_1)$ consists of divergences which differ by insertion of an exact 1-form. Therefore, the cohomology class of the 1-form $\phi$ which is defined by the equation

$$(4.1) \quad \kappa^*(\text{div}_{E_2}) - \text{div}_{E_1} = i_\phi, \quad \text{for } \text{div}_{E_i} \in \text{Div}_0(E_i), \quad i = 1, 2,$$

does not depend on the choice of $\text{div}_{E_1}$ and $\text{div}_{E_2}$. We will call it the **modular class** of $\kappa$ and denote it by $\text{Mod}(\kappa)$. Thus we have:

**Theorem 4.1.** — For every Lie algebroid morphism

$${\begin{array}{c} E_1 \xrightarrow{\kappa} E_2 \\ \downarrow \tau_1 \quad \downarrow \tau_2 \\ M \end{array}}$$

the cohomology class $\text{Mod}(\kappa) = [\phi] \in H^1(E_1, d_{E_1})$ defined by $\phi$ in Eq. (4.1) is well defined independently of the choice of $\text{div}_{E_1} \in \text{Div}_0(E_1)$ and $\text{div}_{E_2} \in \text{Div}_0(E_2)$.

4.2. The modular class of a Lie algebroid

In the case when the morphism $\kappa = \rho : E \to TM$ is the anchor map of a Lie algebroid $E$, the modular class $\text{Mod}(\rho)$ is called the **modular class of the Lie algebroid** $E$ and it is denoted by $\text{Mod}(E)$. The idea that the modular class is associated with the difference between the Lie derivative action on $\bigwedge^\text{top}(E^*)$ and on $\bigwedge^\text{top} T^*M$ via the anchor map is, in fact, already present in [3]. Also the interpretation of the modular class as certain secondary characteristic class of a Lie algebroid, present in [4], is a quite similar. In [4] the trace of the difference of some connections is used instead of the difference of two divergences. We have

**Theorem 4.2.** — $\text{Mod}(E)$ is the modular class $\Theta_E$ in the sense of [3].
Proof. — The modular class $\Theta_E$ in the sense of [3] is defined as the class $[\phi]$ where $\phi$ is given by

\begin{equation}
\mathcal{L}_X (a) \otimes \mu + a \otimes \mathcal{L}_\rho(X) \mu = \langle X, \phi \rangle a \otimes \mu
\end{equation}

for all sections $a$ of $\Lambda^{top}(E)$ and $\mu$ of $\Lambda^{top}(T^*M)$, respectively. Let us take $|a^*| \in \text{Sec}(|\text{Vol}||E|$) and $|\mu| \in \text{Sec}(|\text{Vol}||T_M|)$, locally represented by $a^* \in \text{Sec}(\Lambda^{top}(E^*|U|))$ and $\mu \in \text{Sec}(\Lambda^{top}(T^*M|U|))$. Let $a$ be a local section of $\Lambda^{top}E$ dual to $a^*$. Then $\mathcal{L}_X (a) = -\text{div}_{|a^*|}(X)a$ and $\mathcal{L}_\rho(X)(\mu) = \rho^*(\text{div}_{|\mu|})(X)\mu$ so that Eq. (4.2) yields $i_\phi = \rho^*(\text{div}_{|\mu|}) - \text{div}_{|a^*|}$.

Note that in our approach the modular class $\text{Mod}(TM)$ of the canonical Lie algebroid $TM$ is trivial by definition. It is easy to see that the modular class of a base preserving morphism can be expressed in terms of the modular classes of the corresponding Lie algebroids.

**Theorem 4.3.** — For a base preserving morphism $\kappa : E_1 \to E_2$ of Lie algebroids

$$\text{Mod}(\kappa) = \text{Mod}(E_1) - \kappa^*(\text{Mod}(E_2)).$$

Proof. — Let $\rho_l : E_l \to TM$ be the anchor of $E_l$, $l = 1, 2$. Take $\text{div}_{E_l} \in \text{Div}_{0}(E_l)$, $l = 1, 2$, and $\text{div}_{TM} \in \text{Div}_{0}(TM)$. Since Mod$(E_l)$ is represented by $\eta_l$, $\eta_l = \text{div}_{E_l} - \rho_l^*(\text{div}_{TM})$ and $\rho_1 = \rho_2 \circ \kappa$, we can write

$$i_{\eta_1} = \text{div}_{E_1} - \rho_1^*(\text{div}_{TM})$$

$$= \text{div}_{E_1} - \kappa^*(\text{div}_{E_2}) + \kappa^*(\text{div}_{E_2} - \rho_2^*(\text{div}_{TM}))$$

$$= i_{\eta_2} + i_{\kappa^*(\eta_2)},$$

where $\eta_\kappa$ represents Mod$(\kappa)$. Thus $\eta_1 = \eta_\kappa + \eta_2$.

**4.3. The universal Lie algebroid**

For any vector bundle $\tau : E \to M$ there exists a universal Lie algebroid $\text{QD}(E)$ whose sections are the quasi-derivations on $E$, i.e., mappings $D : \text{Sec}(E) \to \text{Sec}(E)$ such that $D(fX) = f D(X) + \hat{D}(f)X$ for $f \in C^\infty(M)$ and $X \in \text{Sec}(E)$, where $\hat{D}$ is a vector field on $M$; see the survey article [5]. Quasi-derivations are known in the literature under various names: covariant differential operators [14], module derivations [15], derivative endomorphisms [11], etc. The Lie algebroid $\text{QD}(E)$ can be described as the Atiyah algebroid associated with the principal $GL(n, \mathbb{R})$-bundle Fr$(E)$ of frames in $E$, and quasi-derivations can be identified with the $GL(n, \mathbb{R})$-invariant vector fields on Fr$(E)$. The corresponding short exact Atiyah
sequence in this case is

\[ 0 \rightarrow \text{End}(E) \rightarrow \text{QD}(E) \rightarrow TM \rightarrow 0. \]

This observation shows that there is a modular class associated to every vector bundle \( E \), namely the modular class \( \text{Mod}(\text{QD}(E)) \), which is a vector bundle invariant.

It is also obvious that, viewing a flat \( E_0 \)-connection (representation) in a vector bundle \( E \) over \( M \) for a Lie algebroid \( E_0 \) over \( M \) as a Lie algebroid morphism \( \nabla : E_0 \rightarrow \text{QD}(E) \), one can define the modular class \( \text{Mod}(\nabla) \).

**Question.** How is \( \text{Mod}(\text{QD}(E)) \) related to other invariants of \( E \) (e.g.
characteristic classes)?

### 4.4. Remark

One can interpret the modular class \( \text{Mod}(E) \) of the Lie algebroid \( E \) as a “trace” of the adjoint representation. Indeed, if we fix local coordinates \( u^a \) on \( U \subset M \) a local frame \( X_i \) of local sections of \( E \) over \( U \), and the dual frame \( \alpha^i \) of \( E^* \), then the Lie algebroid structure is encoded in the “structure functions”

\[ [X_i, X_j] = \sum_k c^k_{ij} X_k, \quad \rho(X_i) = \sum_a \rho^a_i \partial u^a. \]

**Proposition.** The modular class \( \text{Mod}(E) \) is locally represented by the closed 1-form

\[ (4.3) \quad \phi = \sum_i \left( \sum_k c^k_{ik} X_k + \sum_a \frac{\partial \rho^a_i}{\partial u^a} \right) \alpha^i. \]

**Proof.** We insert into Eq. (4.2) the elements \( a = X_1 \wedge \cdots \wedge X_n \) and \( \mu = du^1 \wedge \cdots \wedge du^m \). Since

\[ \mathcal{L}_{X_i} a = \sum_k c^k_{ik} a \quad \text{and} \quad \mathcal{L}_{X_i} \mu = \sum_a \frac{\partial \rho^a_i}{\partial u^a} \mu, \]

we get

\[ \langle X_i, \phi \rangle a \otimes \mu = \left( \sum_k c^k_{ik} + \sum_a \frac{\partial \rho^a_i}{\partial u^a} \right) a \otimes \mu. \]

One could say that representing cohomology locally does not make much sense, e.g.
the modular class \( \text{Mod}(TM) \) is trivial so locally trivial. However, remember that for a general Lie algebroid the Poincaré lemma does not hold: closed forms need not be locally exact. In particular, for a Lie algebra (with structure constants), Eq. (4.3) says that the modular class is just the
trace of the adjoint representation. In any case, Eq. (4.3) gives us a closed form, which is not obvious on first sight. If $E$ is a trivial bundle, Eq. (4.3) gives us a globally defined modular class in local coordinates.

### 4.5. Remark

As we have already mentioned, the modular class of a Lie algebroid is the first characteristic class of R. L. Fernandes [4]. There are also higher classes, shown in [1] to be characteristic classes of the anchor map, interpreted as a representation “up to homotopy”. It is interesting if our idea can be adapted to describe these higher characteristic classes as well.

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