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THE MEMBERSHIP PROBLEM FOR POLYNOMIAL IDEALS IN TERMS OF RESIDUE CURRENTS

by Mats ANDERSSON (*)

Abstract. — We find a relation between the vanishing of a globally defined residue current on $\mathbb{P}^n$ and solution of the membership problem with control of the polynomial degrees. Several classical results appear as special cases, such as Max Nöther's theorem, for which we also obtain a generalization. Furthermore there are some connections to effective versions of the Nullstellensatz. We also provide explicit integral representations of the solutions.

Résumé. — On obtient un lien entre l'annulation d'un courant résidu défini globalement sur $\mathbb{P}^n$ et la solvabilité du problème d'appartenance pour les idéaux de polynômes où l'on contrôle les degrés des polynômes. Plusieurs théorèmes classiques se déduisent comme cas particuliers, comme par exemple le théorème de Max Noether, dont on obtient de plus une généralisation. On trouve également des liens avec des versions effectives du Nullstellensatz. On donne aussi des représentations intégrales explicites des solutions.

1. Introduction

Let $F_1, \dots, F_m$ be polynomials in $\mathbb{C}^n$ and let Φ be a polynomial that vanishes on the common zero set of the F_j. By Hilbert's Nullstellensatz, for some power Φ^ν of Φ, one can find polynomials Q_j such that

$$\sum_j F_j Q_j = \Phi^\nu. \tag{1.1}$$

A lot of attention has been paid to find effective versions, i.e., control of ν and the degrees of Q_j in terms of the degrees of F_j. The breakthrough

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was in [12] where Brownawell obtained bounds on ν and $\deg Q_j$ not too far from the best possible, using a combination of algebraic and analytic methods, cf., Remark 1.10 below. Soon after that Kollár [21] obtained by purely algebraic methods the following optimal result.

Theorem (Kollár). — *Let $F_1, \ldots, F_n$ and Φ be polynomials in $\mathbb{C}^n$ of degrees d_j, and r, respectively, and assume that Φ vanishes on the common zero set of F_j. Then (if $d_j \neq 2$), one can find polynomials Q_j and a natural number s such that $\sum F_j Q_j = \Phi^\nu$, and such that $\nu \leqslant N(d_1 \cdots d_m)$ and $\deg F_j Q_j \leqslant (1+r)N(d_1 \cdots d_m)$; here $N(d_1 \cdots d_m) = d_1 \cdots d_m$ if $m \leqslant n$; for the case when $m > n$, see* [21].

In particular, if F_j have no common zeros in $\mathbb{C}^n$, then there are polynomials Q_j such that

$$\sum_j F_j Q_j = 1, \tag{1.2}$$

with

$$\deg F_j Q_j \leqslant N(d_1 \cdots d_m).$$

The restriction $d_j \neq 2$ has recently been removed by Jalonek, [20], in the case when $m = n$.

In [13] Brownawell gave a prime power version of Kollár's theorem which shed more geometric light on these questions, and there is a generalization to smooth algebraic manifolds in [16].

Kollár's result is optimal as long as one only makes assumptions of the degrees of F_j. However, if one imposes geometric conditions on the zero set one can get sharper results. For instance, assuming that $m = n+1$ and F_j have no common zero set even at infinity, then a classical theorem of Macaulay, [22], states that (1.2) has a solution such that $\deg F_j Q_j \leqslant \sum d_j - n$.

There is a related result due to Max Nöther, [23]; see also [18].

Theorem (Max Nöther, 1873). — *Assume that the zero set of $F_1, \ldots, F_n$ is discrete and contained in $\mathbb{C}^n$ and that Φ belongs to the ideal (F). Then there are polynomials Q_j such that*

$$\Phi = \sum_1^n F_j Q_j$$

and $\deg F_j Q_j \leqslant \deg \Phi$.

In this paper we present a more general result about solutions to the equation

$$\Phi = \sum_1^m F_j Q_j, \tag{1.3}$$

where $F_1, \dots, F_m$ are given polynomials in $\mathbb{C}^n$, with control of the degrees of $F_j Q_j$. It is formulated in terms of a residue current associated with F_j with support on their common zero set on $\mathbb{P}^n$, and the theorems of Macaulay and Max Nöther are simple consequences. We also provide explicit representation formulas of solutions.

If f_j denote homogenizations of F_j, i.e., $f_j(z) = z_0^{d_j} F_j(z'/z_0)$, where $d_j \geqslant \deg F_j$, (here $z = (z_0, z_1, \dots, z_n)$ and $z' = (z_1, \dots, z_n)$), then each f_j defines a global holomorphic section of the line bundle $L^{d_j} \to \mathbb{P}^n$, and hence $f = (f_1, \cdots, f_m)$ is a section of the rank m bundle $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$ (here L^s denotes the line bundle $\mathcal{O}(s)$). If $z \in \mathbb{C}^{n+1} \setminus \{0\}$ we let $[z]$ denote the corresponding point in $\mathbb{P}^n$ under the natural projection; however, we write $f(z)$ rather than $f([z])$. If E^* is equipped with the natural Hermitian structure, then

$$\|f(z)\|^2 = \sum_1^m \frac{|f_j(z)|^2}{|z|^{2d_j}}. \tag{1.4}$$

Following [3] we can define the residue current R^f which is an element in $\oplus_\ell \mathcal{D}'_{0,\ell}(\mathbb{P}^n, \Lambda^\ell E)$ and with support on the zero set

$$Z^f = \{[z] \in \mathbb{P}^n;\ f(z) = 0\}.$$

If we assume that the polynomials F_j have no common zeros in $\mathbb{C}^n$, then of course Z^f is a subset of the hyperplane at infinity. If $\operatorname{codim} Z^f = m$, i.e., f is locally a complete intersection, then R^f is a $(0, m)$-current with values in $\det E = L^{-\sum d_j}$; more precisely it can be identified with the Coleff-Herrera current

$$\Big[\bar\partial \frac{1}{f_1} \wedge \dots \wedge \bar\partial \frac{1}{f_m}\Big],$$

in $\mathbb{C}^{n+1} \setminus \{0\}$, see Section 2. We can now formulate our main result in this paper.

Theorem 1.1. — *Let $F_1, \dots, F_m$ be polynomials in $\mathbb{C}^n$, $\deg F_j \leqslant d_j$, let $f = (f_1, \cdots, f_m)$ be the corresponding section of $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$, and let R^f be the associated residue current. Moreover, assume* *that*

$$m \leqslant n \quad \text{or} \quad r \geqslant \sum_{j=1}^{n+1} d_j - n, \tag{1.5}$$

where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_m$. *Let* Φ *be a polynomial,* $\deg \Phi \leqslant r$, *and let* $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$ *denote its* r*-homogenization. If*

$$\phi R^f = 0, \tag{1.6}$$

then there are polynomials Q_j *such that* (1.3) *holds and* $\deg F_j Q_j \leqslant r$. *If* f *is a complete intersection (then the condition* (1.5) *is fulfilled) and there exist such polynomials* Q_j, *then* (1.6) *holds.*

It is clear that the conclusion about $\deg F_j Q_j$ cannot be improved. If $\Phi = 1$ the condition (1.6) means that F_j have no common zeros in $\mathbb{C}^n$ and that z_0^r annihilates the residue R^f at infinity. If Z^f is empty and $m = n+1$ (actually any $m \geqslant n+1$ works) we can choose $r = \sum d_j - n$ and hence we get a solution to the Bézout equation (1.2) such that $\deg F_j Q_j \leqslant \sum d_j - n$; thus we have obtained the theorem of Macaulay mentioned above.

We have the following generalization of Nöther's theorem.

THEOREM 1.2. — *Assume that the projective zero set of* $F_1, \ldots, F_m$ *has codimension* m *and that there is no irreducible component contained in the hyperplane at infinity. If* Φ *belongs to ideal* (F), *then there are polynomials* Q_j *such that* (1.3) *holds and* $\deg F_j Q_j \leqslant \deg \Phi$.

Proof. — Since $m \leqslant n$ the condition (1.5) is fulfilled so we can take $r = \deg \Phi$. Since $\Phi \in (F)$, ϕ is in the ideal (f) locally in $\mathbb{C}^n$ and since f is a complete intersection, $\phi R^f = 0$ in $\mathbb{C}^n$ by the duality theorem (see Section 2). If $m = n$, i.e., as in Nöther's theorem, Z^f is contained in $\mathbb{C}^n \subset \mathbb{P}^n$, so R^f has its support in $\mathbb{C}^n$ as well, and hence (1.6) holds in $\mathbb{P}^n$. Thus Theorem 1.1 provides the desired solution. In the general case the assumption means that the intersection of Z^f and the hyperplane at infinity has codimension $m+1$, and then Proposition 2.1 in Section 2 implies that $\phi R^f = 0$ in $\mathbb{P}^n$. $\square$

Remark 1.3. — Although this theorem is probably known before, we have not found it in the literature. A proof of Nöther's theorem by multivariable residue calculus has previously been obtained by Tsikh, [29]. In [30] is given an argument starting with a representation of Φ with the Cauchy-Weil formula. Making series expansion of the kernel and using Jacobi formulas (vanishing of certain residues as in [31]) and the duality theorem, one obtains Nöther's theorem. It is possible that one can prove the general form of Theorem 1.2 in a similar way, following the idea of [5] to add $n-k$ linear forms L such that (F, L) has no zeros at infinity, but we have not checked the details.

However some results related to Theorem 1.2 have appeared before. In [26], Proposition 2, it is assumed that f_j is a regular sequence in $\mathbb{P}^n$ but with no extra condition on the hyperplane at infinity. If Φ belongs to the ideal (F) as above, then there are Q_j solving (1.3) such that $\deg F_jQ_j \leqslant N + \deg\Phi$, where $N = \Pi_1^m \deg F_j$. To see this in our setting, recall that (see, e.g., [26] Lemma 2) if F_j is a regular sequence in $\mathcal{O}_x$, then

$$(\sqrt{(F)_x})^N \subset (F)_x. \tag{1.7}$$

Thus $z_0^{N+\deg\Phi}\Phi(z'/z_0)$ annihilates R^f in $\mathbb{P}^n$, and therefore the statement follows from Theorem 1.1.

Now let F_j be as in Theorem 1.2 and assume that Φ vanishes on their common zero set in $\mathbb{C}^n$. Then by (1.7), Φ^N belongs to (F). Therefore we get the following corollary of Theorem 1.2, which recently appeared in [17] under the slightly stronger assumption that F_j is a strictly regular sequence in $\mathbb{C}^n$.

Corollary 1.4. — *Assume that the projective zero set of $F_1, \dots, F_m$ has codimension m and that there is no component contained in the hyperplane at infinity. If Φ vanishes on the zero set of F in $\mathbb{C}^n$, then there are polynomials Q_j such that $\deg F_jQ_j \leqslant N\deg\Phi$ and $\sum F_jQ_j = \Phi^N$, where $N = (\deg F_1)\cdots(\deg F_m)$.*

If

$$\|\phi\| \leqslant C\,\|f\|, \tag{1.8}$$

then, see Section 2, $\phi^{\min(m,n)}R^f = 0$, and hence Theorem 1.1 implies

Corollary 1.5. — *Let F_j and Φ be as in Theorem 1.1, $r \geqslant \deg\Phi$, and assume that*

$$m \leqslant n \quad \text{or} \quad r\min(m,n) \geqslant \sum_1^{n+1} d_j - n.$$

If (1.8) holds, then there are polynomials Q_j such that

$$\sum F_jQ_j = \Phi^{\min(m,n)} \tag{1.9}$$

and $\deg F_jQ_j \leqslant r\min(m,n)$.

Since there are examples where f is a complete intersection and the full power $\min(m,n)$ of ϕ is needed to kill R^f, this result is then sharp.

Example 1.6. — Let M be a given positive integer. Take $F_j(z') = z_j^{Mm}$ in $\mathbb{C}^n$, $1 \leqslant j \leqslant m \leqslant n$, $\Phi(z') = (z_1 + \cdots z_m)^{Mm}$, and let f_j and ϕ be the homogenizations as before ($d_j = Mm$). Then (1.8) holds and hence the corollary states that (1.9) has a solution such that $\deg F_j Q_j \leqslant r \min(m,n) = Mm^2$. This is obvious also by a direct inspection, and one also immediately sees that Φ^{m-1} is not in the ideal (F). Thus the corollary is sharp.

It follows that $\phi^{m-1} R^f \neq 0$, and since f is a complete intersection in $\mathbb{P}^n$, in fact Z^f is the $n-m$-plane $\{[z] \in \mathbb{P}^n;\ z_1 = \cdots z_m = 0\}$, it also follows that $\phi^m R^f = 0$. One can also verify these residue conditions directly. In fact, in the standard affine coordinates z',

$$R^f = \Big[\bar\partial \frac{1}{z_1^{Mm}} \wedge \ldots \wedge \bar\partial \frac{1}{z_m^{Mm}}\Big] \wedge \epsilon,$$

where ϵ is a non-vanishing section of the line bundle $\det E$, see Section 2. Since this residue current is a tensor product of one-variable currents, the residue conditions follow from the one-variable equality $z\bar\partial[1/z^{p+1}] = \bar\partial[1/z^p]$.

Let F_j be polynomials with no common zeros in $\mathbb{C}^n$. Since the zero set of the section f (take $d_j = \deg F_j$) is then contained in the hyperplane at infinity it follows from Lojasiewicz' inequality that

$$\|z_0\|^M \leqslant C\|f\| \tag{1.10}$$

for some M, or equivalently,

$$\sum_1^m \frac{|F_j(z')|^2}{(1+|z'|^2)^{d_j}} \geqslant c\frac{1}{(1+|z'|^2)^M}. \tag{1.11}$$

Under this condition, $z_0^{M\min(m,n)} R^f = 0$, so we have

Corollary 1.7. — *Let $F_1, \ldots, F_m$ be polynomials in $\mathbb{C}^n$ of degrees d_j such that* (1.11) *(or equivalently* (1.10)*) holds for some number M, and assume that*

$$m \leqslant n \quad \text{or} \quad M\min(m,n) \geqslant \sum_1^{n+1} d_j - n.$$

Then there is a solution to $\sum F_j Q_j = 1$ with $\deg F_j Q_j \leqslant \min(m,n)M$.

Example 1.8. — Also Corollary 1.7 is essentially sharp. Let M be a given non-negative integer and take $F_j(z') = z_j^M$, $1 \leqslant j < m \leqslant n$, and $F_m(z') = (1+z_1+\cdots+z_{m-1})^M$. Then $f_j = z_j^M$ and $f_m = (z_0+\cdots+z_{m-1})^M$, so (1.11) holds. The corollary thus gives a solution to (1.2) with $\deg F_j Q_j \leqslant mM$.

Writing $1 = (1+z_1+\cdots+z_{m-1}) - z_1 - \cdots - z_{m-1}$ and taking the power $Mm - m + 1$ we get a solution to (1.2) with $\deg F_j Q_j = Mm - m + 1$, and it is easily seen to be the best possible, cf., Example 1.6. However, for large M, $Mm - m + 1$ is close to Mm.

Remark 1.9. — Given the estimate (1.11), one can obtain a solution to the Bézout equation (1.2) by a direct application of Skoda's L^2-estimate from [27], as is done in [12]. If we for simplicity assume that all $d_j = d$, then one gets a solution with $\deg Q_j \leqslant \min(m, n+1)M - d$, i.e., $\deg F_jQ_j \leqslant \min(m, n+1)M$. For $m \leqslant n$ this is the same as in Corollary 1.7 but for $m > n$ it is strictly weaker. This phenomenon is the same as in the original proof of Briançon-Skoda's theorem, [11]. Under the assumption $\|\phi\| \leqslant C\|f\|$, the (local) L^2-estimate immediately implies that $\phi^{\min(m,n+1)}$ belongs the the ideal (f) locally; to obtain the correct result when $m > n$, that the power n is enough, an additional argument is required. See also Section 2.

Remark 1.10. — The main step in Brownawell's paper [12] is to obtain good control of the power M in (1.11) in terms of the degrees of F_j, assuming that they have no common zeros in $\mathbb{C}^n$, and this is done by means of Chow forms, see also [28].

Kollár's theorem implies that the estimate (1.11) holds with

$$M = N(d_1, \ldots, d_m),$$

see [21], and this is in fact best possible. From this estimate one gets, via Corollary 1.7, a solution to (1.2) with $\deg F_jQ_j \leqslant \min(m, n)M$. In view of Kollár's theorem one has then "lost" the factor $\min(m, n)$.

Remark 1.11. — Kollár's theorem holds for any field. Berenstein and Yger, [6], have obtained explicit solutions to the Bézout equation (1.2) in subfields of $\mathbb{C}$, by means of integral formulas; see also [4] and the more recent survey article [30] for a thorough discussion.

Remark 1.12. — The condition (1.8) means that ϕ locally on $\mathbb{P}^n$ belongs to the integral closure of the ideal (f). In [19], Hickel proves that if Φ is in the integral closure of (F) in $\mathbb{C}^n$, then one can solve (assuming $m \leqslant n$ for simplicity) $\Phi^m = \sum F_jQ_j$ with $\deg(F_jQ_j) \leqslant m\deg\Phi + md_1\cdots d_m$. This result would follow from Theorem 1.1 if one could prove that the current $z_0^{md_1\cdots d_m}\phi^m R^f$ vanishes (ϕ is the $\deg\Phi$ homogenization of Φ). In $\mathbb{C}^n$ it vanishes since $|\Phi| \leqslant C|F|$ locally. If the zero set is contained in $\{z_0 = 0\}$, the current vanishes there by Kollár's theorem. We do not know whether it vanishes in the general case.

Theorem 1.1 is a special case of the following more general result, for which we formulate only the homogeneous version. Let δ_f denote the mapping $\mathcal{E}(\mathbb{P}^n, \Lambda^{\nu+1}E \otimes L^r) \to \mathcal{E}(\mathbb{P}^n, \Lambda^{\nu}E \otimes L^r)$ defined as interior multiplication with the section f of E^*. Thus for instance, if $q = (q_1, \dots, q_m)$ is a section of $E \otimes L^r$, then $\delta_f q$ is equal to the section $\sum_j f_j q_j$ of L^r. Moreover, let $\nabla_f = \delta_f - \bar\partial$.

Theorem 1.13. — *Let f be a holomorphic section of $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ and assume that $\ell \geqslant 0$ is given and that*

$$m - \ell \leqslant n \quad \text{or} \quad r \geqslant \sum_{j=1}^{n+\ell+1} d_j - n,$$

where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_m$. If $\phi \in \mathcal{O}(\mathbb{P}^n, \Lambda^\ell E \otimes L^r)$, then $\phi = \delta_f \psi$ for some $\psi \in \mathcal{O}(\mathbb{P}^n, \Lambda^{\ell+1} E \otimes L^r)$ if and only if

$$\nabla_f (w \wedge R^f) = \phi \wedge R^f \tag{1.12}$$

for some smooth w defined in a neighborhood of Z^f.

If $\ell > m - p$ ($p = \operatorname{codim} Z^f$) then the condition on ϕ is void; if $\ell = m - p$, it means that $\phi \wedge R^f = 0$, see the remarks after Theorem 2.3 below. If f is a complete intersection, then $m \leqslant n$ and therefore we have

Corollary 1.14. — *Let f be a holomorphic section of $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ that is a complete intersection, and assume that $r \geqslant 0$. If $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$, then $\phi = f \cdot q$ is solvable with $q \in \mathcal{O}(\mathbb{P}^n, E \otimes L^r)$ if and only if $\phi R^f = 0$.*

Proof of Theorem 1.1. — If the hypotheses in Theorem 1.1 are fulfilled, then Theorem 1.13 provides a section $q = (q_1, \dots, q_m)$ of $E \otimes L^r$ such that $\sum f_j q_j = \delta_f q = \phi$; here q_j are sections of $L^{-d_j + r}$. After dehomogenization this means that Q_j are polynomials such that $\deg F_j Q_j \leqslant r$. □

In Section 2 we recall the necessary background from [3] about the residue currents, and present a general result about the image of a holomorphic morphism f. Combined with well-known vanishing results for the line bundles $L^r \to \mathbb{P}^n$ it leads to a proof of Theorem 1.13.

In the last section we construct explicit integral representations of the solutions in Theorem 1.1. They give essentially the same results except for a small loss of precision. The construction is based on ideas in [1] and [3].

2. The residue current of a holomorphic section

Let $E \to X$ be a holomorphic Hermitian vector bundle of rank m over the n-dimensional complex manifold X, and let f be a holomorphic section of the dual bundle E^*, or in other words, a holomorphic morphism $f\colon E \to X \times \mathbb{C}$. Let

$$\mathcal{L}^r = \bigoplus_{\ell} \mathcal{D}'_{0,l+r}(X, \Lambda^\ell E);$$

we consider $\mathcal{L}^r$ as a subbundle to $\Lambda(T^*_{0,1} \oplus E)$, so that δ_f (i.e., interior multiplication with f) and $\bar\partial$ anticommutes. Then $\nabla_f = \delta_f - \bar\partial$ induces the complex $\to \mathcal{L}^{r-1} \to \mathcal{L}^r \to$. It is readily checked that ∇_f satisfies the Leibniz rule $\nabla_f(\alpha \wedge \beta) = \nabla_f \alpha \wedge \beta + (-1)^\nu \alpha \wedge \nabla_f \beta$, where ν is the total degree of β. Let s be the dual section of E of f so that in particular $\delta_f s = \|f\|^2$. In [3] we defined the current

$$R^f = \bar\partial \|f\|^{2\lambda} \wedge \frac{s}{\nabla_f s}\Big|_{\lambda=0};$$

for large $\operatorname{Re}\lambda$ the right hand side is integrable and therefore a well defined current, and by a nontrivial argument based on Hironaka's theorem one can make an analytic continuation to $\lambda = 0$. The resulting current is an element in $\mathcal{L}^0$ with support on $Z^f = \{z;\ f(z) = 0\}$ and it satisfies the basic equality

$$\nabla_f U^f = 1 - R^f, \tag{2.1}$$

where $U^f \in \mathcal{L}^{-1}$ is defined as

$$U^f = \|f\|^{2\lambda} \frac{s}{\nabla_f s}\Big|_{\lambda=0}.$$

Moreover,

$$R^f = R^f_{p,p} + \ldots + R^f_{m,m}, \tag{2.2}$$

where $p = \operatorname{codim} Z^f$; here lower index ℓ, q means that the current has bidegree $(0,q)$-form and takes values in $\Lambda^\ell E$.

Proposition 2.1. — *Assume that f defines a complete intersection and that h is a holomorphic section of some line bundle such that $\{h = 0\} \cap Z^f$ has codimension $m+1$. If ϕ is a holomorphic section such that $\phi R^f = 0$ in $X \setminus \{h = 0\}$, then $\phi R^f = 0$.*

Notice that since f is a complete intersection, $R^f = R^f_m$. The following lemma, which is the core of the proof, states that then R^f is robust in a certain sense.

Lemma 2.2. — *The current $|h|^{2\lambda} R^f$ has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ and*

$$|h|^{2\lambda} R^f|_{\lambda=0} = R^f.$$

Proof. — Clearly the statement is local. By Hironaka's theorem and a toric resolution we may assume that $f = f_0 f'$, where f_0 is a holomorphic function and f' is a non-vanishing section. In this way we can write the action of R^f on a test form ξ as a finite sum of terms like

$$\int \bar\partial\Big[\frac{1}{f_0^\ell}\Big] \wedge \alpha \wedge \tilde\xi\rho,$$

where $[1/f_0^\ell]$ is the principal value current, α is a $(0, m-1)$-form, $\tilde\xi$ is the pull-back of ξ in the given resolution, and ρ is a cut-off function. We may also assume that

$$f_0 = \tau_{k_1}^{\alpha_1} \cdots \tau_{k_\nu}^{\alpha_\nu},$$

in appropriate local coordinates τ_j, and therefore the integral is a sum of terms like

$$\int \Big[\Pi_{r\neq j} \frac{1}{\tau_{k_r}^{\alpha_r \ell}} \bar\partial \frac{1}{\tau_{k_j}^{\alpha_j \ell}}\Big] \wedge \alpha \wedge \tilde\xi\rho. \tag{2.3}$$

We may also assume that $h = \tau_{m_1}^{\beta_1} \cdots \tau_{m_\mu}^{\beta_\mu} u$, where $u \neq 0$. Thus $|h|^{2\lambda} R.\xi$ is a finite sum of terms like

$$\int |\tau_{m_1}|^{2\lambda\beta_1} \cdots |\tau_{m_\mu}|^{2\lambda\beta_\mu} |u|^{2\lambda} \Big[\Pi_{r\neq j} \frac{1}{\tau_{k_r}^{\alpha_r \ell}} \bar\partial \frac{1}{\tau_{k_j}^{\alpha_j \ell}}\Big] \wedge \alpha \wedge \tilde\xi\rho. \tag{2.4}$$

If one of the m_i is equal to k_j, then clearly this integral vanishes for $\operatorname{Re}\lambda >> 0$, and trivially therefore it has an analytic continuation to $\lambda > -\epsilon$, with the value 0 at $\lambda = 0$. However, since τ_{k_j} is a factor in both h and f_0, and $\operatorname{codim}\{h = 0\} \cap Z = m-1$, for degree reasons it follows that ξ vanishes on this set, and therefore, cf. e.g., [7], [25] or [3], each term in $\tilde\xi$ contains either a factor $\bar\tau_{k_j}$ or $d\bar\tau_{k_j}$. In any case, this implies that already the integral (2.3) vanishes. On the other hand, if no m_i is equal to τ_{k_j}, it is easy to see that (2.4) has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ and takes the value (2.3) at $\lambda = 0$. In fact, this follows easily since if $[1/s^\ell]$ is the usual principal value distribution in $\mathbb{C}$ and $v > 0$ is smooth and strictly positive, then

$$|s|^{2\lambda} v^\lambda [1/s^\ell]$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ and takes the value $[1/s^\ell]$ at $\lambda = 0$. Thus the proposition is proved. □

Proof of Proposition 2.1. — By assumption ϕR^f is a current with support on $\{h = 0\}$, and hence (locally) $|h|^{2\lambda}\phi R^f = 0$ if $\operatorname{Re}\lambda >> 0$. From Lemma 2.2 it follows that

$$\phi R^f = |h|^{2\lambda}\phi R^f|_{\lambda=0} = 0.$$

□

Let $L \to X$ be a holomorphic line bundle and let ϕ be a holomorphic section of $\Lambda^k E \otimes L$.

Theorem 2.3. — *Let $\ell \geqslant 0$ and suppose that $H^{0,s}(X, \Lambda^{s+\ell+1}E \otimes L) = 0$ for all $1 \leqslant s \leqslant m - \ell - 1$. Moreover, let $\phi \in \mathcal{O}(X, \Lambda^\ell E \otimes L)$. Then $\delta_f \psi = \phi$ has a solution $\psi \in \mathcal{O}((X, \Lambda^{\ell+1}E \otimes L))$ if and only if there is a smooth solution w, defined in a neighborhood of Z^f, to*

$$\nabla_f(w \wedge R^f) = \phi \wedge R^f. \tag{2.5}$$

In view of (2.2), the condition on ϕ is void if $\ell > m - p$. Moreover, since $w = w_{\ell+1,0} + w_{\ell+2,1} + \cdots$ the condition means precisely that $\phi \wedge R^f = 0$ if $\ell = m - p$. In the case $\ell = 0$ and $p = m$, i.e., f defines a complete intersection, we get back the well-known duality theorem, first proved in [15] and [24].

It was also proved in [3] that $h^{\min(m,n)} R^f = 0$ if h is holomorphic and $\|h\| \leqslant C\|f\|$. The local version of Theorem 2.3 therefore immediately implies the Briançon-Skoda theorem, [11]: *If $\|\phi\| \leqslant C\|f\|$, then locally $\phi^{\min(m,n)}$ belongs to the ideal (f).*

Proof of Theorem 2.3. — First suppose that the holomorphic solution ψ exists. Then $\nabla_f \psi = \phi$ and hence $\nabla_f(\psi \wedge R^f) = \phi \wedge R^f$ since $\nabla_f R^f = 0$. Conversely, if (2.5) holds for some smooth w, we claim that $\nabla_f v = \phi$, if

$$v = (-1)^\ell \phi \wedge U^f + w \wedge R^f.$$

In fact, since $\nabla_f \phi = 0$,

$$\nabla v = \phi \wedge \nabla_f U^f + \nabla_f(w \wedge R^f) = \phi \wedge (1 - R^f) + \phi \wedge R^f = \phi.$$

This means that

$$\bar\partial v_{m,m-\ell-1} = 0 \quad \text{and} \quad \delta_f v_{k+1,k-\ell} = \bar\partial v_{k,k-\ell-1}.$$

By the assumption on the Dolbeault cohomology, we can successively solve the equations

$$\bar\partial \eta_{m,m-\ell-2} = v_{m,m-\ell-1}, \quad \bar\partial \eta_{k,k-\ell-2} = v_{k,k-\ell-1} + \delta_f \eta_{k+1,k-\ell-1}, \quad k \geqslant \ell,$$

and then finally $\psi = v_{\ell,0} + \delta_f \eta_{\ell+1,0}$ is the desired holomorphic solution. □

Example 2.4. — Suppose that X is a compact and L is a strictly positive line bundle. Then there is an $r_0 >$ such that $H^{0,k}(X, \Lambda E^\bullet \otimes L^r) = 0$ for all $k \geqslant 1$ if $r \geqslant r_0$. If f is a holomorphic section of E^*, then a holomorphic section $\phi \in \mathcal{O}(\Lambda^\ell E \otimes L^r)$, $r \geqslant r_0$, is in the image of the morphism

$$\mathcal{O}(X, \Lambda^{\ell+1}E \otimes L^r) \to \mathcal{O}(X, \Lambda^\ell E \otimes L^r) \tag{2.6}$$

if $\phi \wedge R^f = 0$. If $\ell = m - p$ the condition is necessary.

We shall now focus on the case where $X = \mathbb{P}^n$ and E is the Hermitian vector bundle from Section 1. Let $E_1, \dots, E_m$ be trivial line bundles over $\mathbb{P}^n$ with basis elements $\epsilon_1, \dots, \epsilon_m$, and let E_j^* be the dual bundles, with bases ϵ_j^*. Then we have that

$$E^* = (L^{d_1} \otimes E_1^*) \oplus \cdots \oplus (L^{d_m} \otimes E_m^*),$$
$$E = (L^{-d_1} \otimes E_1) \oplus \cdots \oplus (L^{-d_m} \otimes E_m),$$

and for instance our section f can be written

$$f = \sum_1^m f_j \epsilon_j^*.$$

Its dual section s is then, cf., (1.4),

$$s = \sum_j \frac{\overline{f_j(z)}}{|z|^{2d_j}} \epsilon_j,$$

so

$$R^f = \bar\partial \|f\|^{2\lambda} \wedge \sum_{\ell+1}^m \frac{s \wedge (\bar\partial s)^{\ell-1}}{\|f\|^{2\ell}}\Big|_{\lambda=0}.$$

In $\mathbb{C}^n = \{z_0 \neq 0\} \subset \mathbb{P}^n$ we have the coordinates z' and the natural holomorphic frame $e_j = z_0^{-d_j} \epsilon_j$ and its dual $e_j^* = z_0^{d_j} \epsilon_j^*$. If $f_j'(z') = f_j(1, z')$ then

$$f = \sum_1^m f_j' e_j^*$$

and

$$s = \sum_1^m \frac{\overline{f_j'(z')}}{(1+|z'|)^{d_j}} e_j.$$

When $\operatorname{codim} Z^f = m$, the residue current R^f is independent of the metric, it just contains the top degree term $R^f_{m,m}$, and in fact, see [3],

$$R^f = \big[\bar\partial \frac{1}{f_m'} \wedge \ldots \wedge \bar\partial \frac{1}{f_1'}\big] \wedge e_1 \wedge \ldots \wedge e_m,$$

where the expression in brackets is a Coleff-Herrera residue current. Choosing the local coordinates $z_0, \zeta_1, \dots, \zeta_n$ in $\mathbb{C}^{n+1} \setminus \{0\}$, where $\zeta_j = z_j/z_0$, it is easy to see that

$$\pi^* \big[\bar\partial \frac{1}{f_m'} \wedge \ldots \wedge \bar\partial \frac{1}{f_1'}\big] = z_0^{\sum d_j} \big[\bar\partial \frac{1}{f_m} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\big],$$

and hence we can identify R^f with the Coleff-Herrera current

$$\big[\bar\partial \frac{1}{f_m} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\big] \wedge \epsilon_1 \wedge \ldots \wedge \epsilon_m$$

in $\mathbb{C}^{n+1} \setminus \{0\}$.

Proof of Theorem 1.13. — It is well-known, see, e.g., [14], that $H^{0,k}(\mathbb{P}^n, L^\nu) = 0$ for all ν if $1 \leqslant k \leqslant n-1$ and that $H^{0,n}(\mathbb{P}^n, L^\nu) = 0$ if (and only if) $\nu \geqslant -n$. Since $E = L^{-d_1} \oplus \cdots \oplus L^{-d_m}$ we have that

$$\Lambda^\nu E \otimes L^r = \sideset{}{'}\bigoplus_{|J|=\nu} L^{-d_{J_1}} \otimes \cdots \otimes L^{-d_{J_\nu}} \otimes L^r = \sideset{}{'}\bigoplus_{|J|=\nu} L^{r-d_{J_1}\cdots-d_{J_\nu}}.$$

Thus $H^{0,s}(\mathbb{P}^n, \Lambda^{s+\ell+1}E \otimes L^r) = 0$ for $1 \leqslant s \leqslant m-\ell-1$ if either $m-\ell-1 \leqslant n-1$ or

$$r - \sum_1^{n+\ell+1} d_j \geqslant -n.$$

Now Theorem 1.13 follows from Theorem 2.3. □

3. Integral representation

The aim of this section is to present an explicit integral representation of the solution Q_j to the division problem in Theorem 1.1. We have

THEOREM 3.1. — *Let $F_1, \ldots, F_m, \Phi$ be polynomials in $\mathbb{C}^n$, let f and R^f be as before, and let ϕ be the r-homogenization of Φ (deg $\Phi \leqslant r$). Then there is an explicit decomposition*

$$\Phi(z') = \sum_1^m F_j(z') \int_{\mathbb{P}^n} T^j(\zeta, z')\phi(\zeta) + \int_{\mathbb{P}^n} S(\zeta, z') \wedge R^f(\zeta)\phi(\zeta), \tag{3.1}$$

where $T^j(\zeta, z'), S(\zeta, z')$ are smooth forms (in $[\zeta]$) on $\mathbb{P}^n$ and holomorphic polynomials in z', such that

$$\deg_{z'}\big(F_j(z')T^j(\zeta, z')\big) \leqslant d_1 + d_2 + \cdots + d_{\mu+1} + r,$$

if $\mu = \min(n, m-1)$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$.

Thus, if $\phi R^f = 0$ we get back the conclusion of Theorem 1.1 but with the extra term $d_1 + \cdots + d_{\mu+1}$ in the estimate of the degree.

For fixed $z \in \mathbb{C}^n$,

$$\eta = 2\pi i \sum_0^n z_j \frac{\partial}{\partial \zeta_j}$$

is an $L_z \otimes L_\zeta^{-1}$-valued $(1,0)$-form on $\mathbb{P}^n$, and if δ_η denotes interior multiplication with η, then

$$\delta_\eta \colon \mathcal{D}'_{\ell+1,0}(\mathbb{P}^n, L^{r+1}) \to \mathcal{D}'_{\ell,0}(\mathbb{P}^n, L^r).$$

Remark 3.2. — When we say that η is a section of $L_z \otimes L_\zeta^{-1}$ rather than $L^{-1} = L_\zeta^{-1}$, we just indicate that it is 1-homogeneous in z; it would be more correct, but less convenient, to consider η as a section of the bundle $L_z \otimes L_\zeta^{-1} \otimes (T_\zeta^*)_{0,1}$ over $\mathbb{P}_z^n \times \mathbb{P}_\zeta^n$.

Let $\nabla_\eta = \delta_\eta - \bar\partial$. Notice that if

$$\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar\zeta}{|\zeta|^2} - \bar\partial \frac{\bar\zeta \cdot d\zeta}{2\pi i |\zeta|^2},$$

then the first term, α_0, is a section of $L_z \otimes L_\zeta^{-1}$ and the second term, α_1, is a projective form (since $\delta_\zeta \alpha_1 = 0$); moreover

$$\nabla_\eta \alpha = 0. \tag{3.2}$$

We have the following basic integral representation of global holomorphic sections of L^r.

Proposition 3.3. — *Assume that $r \geqslant 0$ and that $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$. Then*

$$\phi(z) = \int_{\mathbb{P}^n} \alpha^{n+r} \phi.$$

For degree reasons, actually

$$\phi(z) = \frac{(n+r)!}{n!r!} \int_{\mathbb{P}^n} \alpha_0^r \wedge \alpha_1^n \phi;$$

this formula appeared already in [9]; expressed in affine coordinates it is the well-known weighted Bergman representation formula for polynomials in $\mathbb{C}^n$. However, we prefer to supply a direct proof on $\mathbb{P}^n$, following the ideas in [1].

Proof. — Let σ be the $L_z^{-1} \otimes L_\zeta \otimes T_{1,0}^*(\mathbb{P}_\zeta^n)$ valued $(1,0)$-form on $\mathbb{P}^n$ that is dual, with respect to the natural metric, to η. Then, since η has a first order zero at $[z]$ (and no others), it follows (see [1]) that

$$\nabla_\eta \frac{\sigma}{\nabla_\eta \sigma} = 1 - \big[[z]\big].$$

The rightmost term is the $L_z^{-n} \otimes L_\zeta^n$-valued (n,n)-current point evaluation at $[z]$ for sections of L^{-n}. If ϕ is a global holomorphic section of L^r it follows by (3.2) that

$$\nabla_\eta \Big(\frac{\sigma}{\nabla_\eta \sigma} \wedge \alpha^{n+r} \phi \Big) = \phi \alpha^{n+r} - \phi \big[[z]\big],$$

where this time the last term is ϕ times the $L_z^r \otimes L_\zeta^{-r}$-valued current point evaluation at $[z]$. If we integrate this equality over $\mathbb{P}^n$ we get the desired representation formula. $\square$

Let $E_1, \dots, E_m$ be the trivial line bundles over $\mathbb{P}^n$ with basis elements $\epsilon_1, \dots, \epsilon_m$, so that $E = (L^{-d_1} \otimes E_1) \oplus \cdots \oplus (L^{-d_m} \otimes E_m)$ as in Section 2. We also introduce disjoint copies $\widetilde{E}_j$ of E_j with bases $\tilde{\epsilon}_j$ and the bundle

$$\widetilde{E} = (L^{-d_1} \otimes \widetilde{E}_1) \oplus \cdots \oplus (L^{-d_m} \otimes \widetilde{E}_m).$$

Let Λ be the exterior algebra bundle over the direct sum of all the bundles E, $\widetilde{E}$, E^*, and $T^*(\mathbb{P}^n)$. Any form γ with values in Λ can be written uniquely as $\gamma = \gamma' \wedge (\sum \epsilon_j^* \wedge \epsilon_j)^m/m! + \gamma''$ where γ'' denotes terms that do not contain a factor $(\sum \epsilon_j^* \wedge \epsilon_j)^m/m!$, and we define

$$\int_\epsilon \gamma = \gamma'.$$

We have a globally defined form

$$\tau = \sum_1^m \epsilon_j^* \wedge (\epsilon_j - \tilde{\epsilon}_j).$$

From now on we consider $[z]$ as a fixed arbitrary point in $\mathbb{C}^n \subset \mathbb{P}^n$, and let $z = (1, z')$. We also introduce the section

$$f_z = \sum_j \zeta_0^{d_j} f_j(1, z)\epsilon_j^* = \sum \zeta_0^{d_j} F_j(z')\epsilon_j^*$$

of E^* and let $\tilde{f}_z$ be the corresponding section of $\widetilde{E}^*$.

Lemma 3.4. — *There is a holomorphic section $H = \sum H_j \wedge \epsilon_j$ of $E^* \otimes L \otimes T^*_{1,0}$, thus H_j are sections of $L^{d_j} \otimes L \otimes T^*_{1,0}$, such that*

$$\delta_\eta H = f - f_z,$$

and such that the coefficients in H_j are polynomials in z'/z_0 of degrees (at most) $d_j - 1$.

Proof. — For each $F_j(z')$ we can find Hefer functions $h_j^k(\zeta', z')$, polynomials of degree $d_j - 1$ in (ζ', z'), such that

$$\sum_{k=1}^n h_j^k(\zeta', z')(\zeta_k - z_k) = F_j(\zeta') - F_j(z').$$

If we then take

$$H_j = \frac{\zeta_0^{d_j+1}}{2\pi i} \sum_1^n h_j^k(\zeta'/\zeta_0, z')d(\zeta_k/\zeta_0),$$

then clearly H_j is a projective $(1,0)$-form, and moreover,

$$\delta_\eta H_j = f_j(\zeta) - \zeta_0^{d_j} F_j(z')$$

as wanted. $\square$

Let δ_F denote interior multiplication with the section $F = f + \tilde{f}_z$ of $E^* \oplus \tilde{E}^*$. Then $\delta_F \tau = f - f_z = -\delta_\eta H$. If

$$\nabla = \delta_F + \delta_\eta - \bar{\partial},$$

thus

$$\nabla(\tau + H) = 0. \tag{3.3}$$

We are now ready to define the explicit division formula.

Proof of Theorem 3.1. — From (3.3) it follows that

$$(\nabla_\eta + \delta_F)(e^{\tau+H} \wedge U^f) = e^{\tau+H} \wedge (1 - R^f). \tag{3.4}$$

We can rewrite this as

$$\delta_F(e^{\tau+H} \wedge U^f) + e^{\tau+H} \wedge R^f = e^{\tau+H} - \nabla_\eta(e^{\tau+H} \wedge U^f). \tag{3.5}$$

We claim that the component of full bidegree (n,n) of

$$\int_\epsilon \left[e^{\tau+H} - \nabla_\eta(e^{\tau+H} \wedge U^f)\right] \wedge \alpha^{n+r}\phi \tag{3.6}$$

is equal to

$$\frac{(n+r)!}{n!r!}\alpha_1^n \alpha_0^r \phi + \bar{\partial}(\cdots)$$

where $(\cdots)$ is a scalar-valued $(n, n-1)$-form. In fact, since α^{n+r} has bidegree $(*,*)$ the factor $U_{\ell,\ell-1}$ must be combined with H_ℓ, and then it follows that τ can be replaced by $\omega = \sum_j \epsilon_j^* \wedge \epsilon_j$. Observe that the component of $U_{\ell,\ell-1}$ with basis element $\epsilon_{J_1} \wedge \ldots \wedge \epsilon_{J_\ell}$ takes values in $L^{-(d_{J_1}+\cdots+d_{J_\ell})}$, whereas the component of H_ℓ with basis element $\epsilon_{J_1}^* \wedge \ldots \wedge \epsilon_{J_\ell}^*$ takes values in $L^{d_{J_1}+\cdots+d_{J_\ell}} \otimes L^\ell$. The product of these two factors must be combined with $\alpha_1^{n-\ell}\alpha_0^{\ell+r}\phi$ which gives a scalar-valued (n,n)-form as claimed. Thus we can integrate (3.6) over $\mathbb{P}^n$, and by Proposition 3.3 and Stokes' theorem it is equal to $\phi(z)$.

We now consider the left hand side of (3.5) multiplied with $\alpha^{n+r}\phi$. To begin with,

$$\int_{\mathbb{P}^n}\int_\epsilon e^{\tau+H} \wedge R^f \wedge \alpha^{n+r}\phi$$

is well defined with the same argument as above, and again one can replace τ by ω. Moreover, since $\alpha^{n+r}\phi$ contains no ϵ_j,

$$\int_\epsilon \delta_f(e^{\tau+H} \wedge U^f) \wedge \alpha^{n+r}\phi = \int_\epsilon \delta_f(e^{\tau+H} \wedge U^f \wedge \alpha^{n+r}\phi) = 0.$$

Since

$$\delta_{\tilde{f}_z} \sum_j \tilde{\epsilon}_j \wedge \epsilon_j^* = \sum_j F(z')\zeta_0^{d_j}\epsilon_j^* = f_z,$$

another computation shows that the component of bidegree (n,n) of

$$\int_\epsilon \delta_{\tilde{f}_z}(e^{\tau+H}\wedge U^f)\wedge\alpha^{n+r}\phi$$

is equal to

$$\int_\epsilon f_z\wedge\sum_{k=0}^{m-1}\omega_{m-k-1}\wedge H_k\wedge U_{k+1,k}\wedge\alpha_1^{n-k}\alpha_0^{k+r}\phi.$$

Again one can check that this form is scalar-valued. Summing up we have the desired decomposition (3.1) with

$$\begin{aligned}S(\zeta,z')\wedge R^f(\zeta)&=\int_\epsilon e^{\omega+H}\wedge R^f\wedge\alpha^{n+r}\\&=\sum_{k=\operatorname{codim}Z^f}^{m}\int_\epsilon\frac{(n+r)!}{(n-k)!(k+r)!}\omega_{m-k}\wedge H_k\wedge R^f_{k,k}\alpha_1^{n-k}\alpha_0^{k+r},\end{aligned}$$

and

$$\begin{aligned}&T^j(\zeta,z')\\&=\int_\epsilon\epsilon_j^*\zeta_0^{d_j}\wedge\sum_{k=1}^{m-1}\frac{(n+r)!}{(n-k)!(k+r)!}\tilde{I}_{n-k-1}\wedge H_k\wedge U_{k+1,k}\wedge\alpha_1^{n-k}\alpha_0^{k+r}\phi,\end{aligned}$$

Both α and H are polynomials in z' so it just remains to check the degrees of T^j. The worst case occur when k is as large as possible which is $k=\mu=\min(m-1,n)$. Then the factor α_0^{k+r} has degree $k+r$. Recall that $H=\sum H_\ell\wedge\epsilon_\ell^*$ and that $\deg H_\ell=d_\ell-1$. The term H_j cannot occur, because of the presence of ϵ_j^*, and thus we get that $d_j+\deg Q_j$ is at most $d_1-1+d_2-1+\cdots d_{\mu+1}-1+1+\mu+r=d_1+\cdots+d_{\mu+1}+r$. □

The division formula constructed here, Theorem 3.1, is a generalization to $\mathbb{P}^n$ of the formula in [3], which was used to give an explicit representation of the solutions in the local version of Theorem 2.3; in particular it provided the first known explicit proof of the Briançon-Skoda theorem. This division formula is based on the ideas in [1] and it differs from Berndtsson's classical formula, [8], in some respects. To begin with our formula works also for sections with values in $\Lambda^\ell E$, although in this paper we have only generalized the scalar-valued part to $\mathbb{P}^n$. The more interesting novelty with regard to this paper, is that the residue term contains precisely the factor ϕR^f, so that our formula provides a solution of the division problem as soon as $\phi R^f=0$ (or $\phi R^f=\nabla_f(w\wedge R^f)$ for some smooth w). One can obtain a similar formula involving residues (but not precisely R^f except for the complete intersection case) from Berndtsson's formula; this was first done by Passare in [24], and various variants have been used by several authors since then, see [4] and the references given there. These formulas all go back to the construction of weighted integral formulas in [10]. However, the division formula in [3], even in the simplest case, when f is nonvanishing, could not have been obtained from [10], because the required choice of weight, see formula (2.12) in Remark 3 in [2], is not encompassed by the method in [10], but the more general construction in [1] is needed.

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