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CRAMÉR'S FORMULA FOR HEISENBERG MANIFOLDS

by Mahta KHOSRAVI & John A. TOTH

1. Introduction.

Let (M, g) be a closed n-dimensional Riemannian manifold with metric g and Laplace-Beltrami operator Δ. We denote its spectral counting function by $N(t)$, defined as the number of the eigenvalues of Δ not exceeding t. The local Weyl law [Hö] asserts that as $t \to \infty$,

$$N(t) = \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n} t^{n/2} + O(t^{(n-1)/2}), \tag{1}$$

where $\text{vol}(B_n)$ is the volume of the n-dimensional unit ball.

By considering the unit sphere, it is straightforward to show that the estimate for the error term in (1) defined by

$$R(t) = N(t) - \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n} t^{n/2}, \tag{2}$$

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is in general sharp. However, the question of determining the optimal bound for this error term in any given example is a difficult one and depends on the properties of the associated geodesic flow. In many cases, this is an open problem. Nevertheless, for certain types of manifolds some improvements have been obtained and in a few cases the conjectured optimal bound has been attained (see [BG], [Be], [Bl], [Fr], [Go], [Hu], [Iv], [KP] and [Vo]).

The results obtained in this direction can be put into three categories: (i) The first type of results deal with the upper bound for the rate of the growth of the error term (*i.e.* the O-results). (ii) The second type deal with finding a lower bound for this growth (*i.e.* the Ω-results). (iii) Finally, the third type are results about various averages and moments of the error term.

For manifolds with completely integrable geodesics, Duistermaat and Guillemin [DG] have proved that $R(t) = o(t^{(n-1)/2})$.

For convex surfaces of revolution, Colin de Verdière [Co] showed that $R(t) = O(t^{1/3})$.

Hardy's conjecture [Ha] for 2-dimensional tori, $\mathbb{T}^2$, says that

$$R(t) = O_\delta(t^{\frac{1}{4}+\delta}) \ \text{ for } \forall\delta > 0.$$

Hardy proved that for $\mathbb{T}^2$ this is the best possible upper bound.

There is a classical result of Cramér [Cr] which states that for $\mathbb{T}^2$:

$$\lim_{T\to\infty} \frac{1}{T^{\frac{3}{2}}} \int_1^T |R(t)|^2 dt = C,$$

where $C = \frac{1}{6\pi^3} \sum_1^\infty \frac{r(n)^2}{n^{3/2}}$ with $r(n) = \#\{(a,b) \in \mathbb{Z}^2; n = a^2 + b^2\}$. This result is consistent with Hardy's conjecture.

In this paper, we address the following:

Question 1.1. — *In what degree of generality is Cramér's result true?*

As the first, natural, non-commutative generalization of $\mathbb{T}^2$ consider H_1, a 3-dimensional Heisenberg manifold which has a completely integrable geodesic flow [Bu]. Petridis and Toth [PT] have proved that for certain 'arithmetic' Heisenberg metrics on H_1, $R(t) = O(t^{5/6+\epsilon})$. Later in [PT2] the exponent was improved and the result extended to all left-invariant

Heisenberg metrics. It was conjectured in [PT] that for H_1,

$$R(t) = O_\delta(t^{\frac{3}{4}+\delta}). \tag{3}$$

Moreover, as evidence for this conjecture, Petridis and Toth [PT] proved the following L^2 result for H_1 by averaging over the moduli space of left-invariant metrics:

$$\int_{I^3} |N(t;u) - \frac{1}{6\pi^2}\mathrm{vol}(M(u))t^{3/2}|^2 du \leqslant C_\delta t^{3/2+\delta},$$

where $I = [1-\epsilon, 1+\epsilon]$. They also proved that for sufficiently large T,

$$\frac{1}{T}\int_T^{2T} |N(t) - \frac{1}{6\pi^2}\mathrm{vol}(M)t^{3/2}|dt \geqslant C_\delta T^{3/4}.$$

The conjecture (3) would follow from the standard conjectures on the growth of exponential sums (see [CPT]).

In higher dimensions, *i.e.* $(H_n/\Gamma, g)$ where $n > 1$, Khosravi and Petridis [KP] proved that:

$$R(t) = O_\delta(t^{n-\frac{1}{4}+\delta}).$$

Moreover, they showed that this bound is sharp.

The main purpose of this paper is to prove that $\int_1^T |R(t)|^2 dt$ has meaningful asymptotics for H_1.

Theorem 1.2. — *Let $M = (H_1/\Gamma, g)$ be the 3-dimensional Heisenberg manifold where the metric g is in the form $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\pi \end{pmatrix}$. Then, there exists a positive constant c such that*

$$\int_1^T |R(t)|^2 dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{4}$$

for every $\delta > 0$.

Remark 1. — Theorem 1.2 holds for all left-invariant Riemannian metrics on H_1/Γ satisfying the clean intersection condition (see [DG]). The proof is similar.

For Heisenberg manifolds of dimension $n > 3$ we prove:

THEOREM 1.3. — *For* $(2n+1)$*-dimensional Heisenberg manifold with the metric* $g = \begin{pmatrix} I_{2n\times 2n} & 0 \\ 0 & 2\pi \end{pmatrix}$*, where* $I_{2n\times 2n}$ *is the identity matrix, one can similarly prove that there exists a positive constant* c *such that*

$$\int_1^T |R(t)|^2 dt = cT^{2n+\frac{1}{2}} + O_\delta(T^{2n+\frac{1}{4}+\delta})R, \tag{5}$$

for every $\delta > 0$.

Remark 2. — The proof of Theorem (1.3) is very similar to the case $n = 3$ and we include it in section 5. We are currently unable to extend Theorem 1.3 to all left-invariant Riemannian metrics on H_n/Γ, but we hope to return to this question elsewhere.

2. Background on Heisenberg manifolds.

We review here some basic material on Heisenberg manifolds. The reader should consult [GW], [St] or [Fo] for further details.

2.1. Basic Definitions and notation.

For any two real numbers x and y let

$$\gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;,\; X(x,y,t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The real 3-dimensional Heisenberg group H_1 is the Lie subgroup of $Gl_3(\mathbb{R})$ consisting of all matrices of the form $\gamma(x,y,t)$:

$$H_1 = \{\gamma(x,y,t) : x, y \in \mathbb{R}, t \in \mathbb{R}\}.$$

The Lie algebra of H_1 is:

$$\mathfrak{h}_1 = \{X(x,y,t) : x, y \in \mathbb{R}, t \in \mathbb{R}\}.$$

The matrix exponential maps $\mathfrak{h}_1$ diffeomorphically onto H_1 and is given by the formula

$$\begin{cases} \exp : \mathfrak{h}_1 \mapsto H_1, \\ X(x,y,t) \mapsto \gamma(x,y,t+\frac{1}{2}x.y). \end{cases}$$

The product operation in H_1 and Lie bracket in $\mathfrak{h}_1$ are given by

$$\gamma(x,y,t).\gamma(x',y',t') = \gamma(x+x',y+y',t+t'+x.y'),$$
$$[X(x,y,t),X(x',y',t')] = X(0,0,x.y'-x'.y).$$

The algebra $\mathfrak{z}_1 = \{X(0,0,t), t \in \mathbb{R}\}$ is both the center and the derived subalgebra of $\mathfrak{h}_1$. It is also convenient to identify the subspace $\{X(x,y,0), x,y \in \mathbb{R}\}$ of $\mathfrak{h}_1$ with $\mathbb{R}^2$ and so, $\mathfrak{h}_1 = \mathbb{R}^2 \oplus \mathfrak{z}_1$.

The standard basis of $\mathfrak{h}_1$ is the set $\delta = \{X_1, Y_1, Z\}$, where the first 2 elements are the standard basis of $\mathbb{R}^2$ and $Z = X(0,0,1)$. The only nonzero bracket among the elements of δ is given by $[X_1, Y_1] = Z$.

DEFINITION 2.1. — *A Riemannian Heisenberg manifold is a pair $(H_1/\Gamma, g)$ where Γ is a uniform discrete subgroup of H_1 ("uniform" means that the quotient H_1/Γ is compact), and g is a Riemannian metric on H_1/Γ whose lift to H_1 is left H_1-invariant.*

2.2. Classification of the uniform discrete subgroups of H_1.

For every positive integer r, define

$$\Gamma_r = \{\gamma(x,y,t) : x \in r\mathbb{Z}, y \in r\mathbb{Z}, t \in \mathbb{Z}\}.$$

It is clear that Γ_r is a uniform discrete subgroup of H_1.

THEOREM 2.2. — *([GW], Theorem 2.4) The subgroups Γ_r classify the uniform discrete subgroups of H_1 up to automorphisms. In other words, for every uniform discrete subgroup of H_1 there exists a unique $r \in \mathbb{Z}_+$ and an automorphism of H_1 which maps Γ to Γ_r. Also, if two subgroups Γ_r and Γ_s are isomorphic then r and s are equal.*

COROLLARY 2.3. — *([GW], Corollary 2.5) Given any Riemannian Heisenberg manifold $M = (H_1/\Gamma, g)$, there exists a unique positive integer r and a left-invariant metric $\tilde{g}$ on H_1 such that M is isometric to $(H_1/\Gamma_r, \tilde{g})$.*

Since every left-invariant metric g on H_1 is uniquely determined by an inner product on $\mathfrak{h}_1$, the left-invariant metrics can be identified with their matrices relative to the standard basis of $\mathfrak{h}_1$. For any g we can choose an inner automorphism φ of H_1 such that $\mathbb{R}^2$ is orthogonal to $\mathfrak{z}_1$ with respect to φ^*g. Therefore, $(H_1/\Gamma, g)$ will be isometric to $(H_1/\Gamma, \varphi^*g)$ and we can replace every left-invariant metric g by $\varphi^* g$ and always assume that the metric g has the form $g = \begin{pmatrix} h & 0 \\ 0 & g_3 \end{pmatrix}$, where h is a positive-definite 2×2 matrix and g_3 is a positive real number. The volume of the Heisenberg manifold is given by the formula $\mathrm{vol}(H_1/\Gamma_r, g) = r\sqrt{det(g)}$.

2.3. The spectrum of Heisenberg manifolds.

Let $M = (H_1/\Gamma, g)$ be a Heisenberg manifold and $C^\infty(M)$, the set of the smooth functions on M. We can view functions on M as left Γ-invariant functions on H_1. The Laplace-Beltrami operator on $C^\infty(M)$ is given by:

$$\Delta f = -\sum_{i=1}^{3} U_i^2 f,$$

where U_1, U_2, U_3 is any g-orthonormal basis of $\mathfrak{h}_1$ and the action of U_i is defined by

$$U_i f(\gamma) = \left(\frac{d}{dt}\right)_{t=0} f(\gamma \exp(tU_i)) = (R_* U_i) f(\gamma),$$

where R is the quasi-regular representation of H_1 on $L^2(H_1/\Gamma)$, that is $R(\gamma')f(\gamma) := f(\gamma\gamma')$. Thus, the extension of Δ to an unbounded operator on $L^2(H_1/\Gamma)$ is given by,

$$\Delta f = -\sum_{i=1}^{3} (R_* U_i)^2 f.$$

Let Σ be the spectrum of the Laplacian on $M = (H_1/\Gamma, g)$, where the eigenvalues are counted with multiplicities. Then, $\Sigma = \Sigma_1 \cup \Sigma_2$ (see [GW] page 258) where,

$$\Sigma_1 = \{\lambda(m,n) = 4\pi^2(m^2+n^2); (m,n) \in \mathbb{Z}^2\},$$

such that $\lambda(m,n)$ is counted once for each pair $(m,n) \in \mathbb{Z}^2$ such that $\lambda = \lambda(m,n)$.

The second part of the spectrum, Σ_2, is the set:

$$\Sigma_2 = \{\mu(c,k) = 2\pi c(c + (2k+1)); c \in \mathbb{Z}_+, k \in (\mathbb{Z}_+ \cup \{0\})\},$$

where every $\mu(c,k)$ is counted with multiplicity $2c$.

3. Estimates for regularized spectral counting function.

The idea of the proof is to use the Poisson summation formula to write the error term corresponding to type II eigenvalues, in a form which can be estimated by the method of the stationary phase.

The spectral counting function can be written in the form

$$N(t) = N_T(t) + N_H(t), \tag{6}$$

where $N_T(t)$ is the spectral counting function of the torus, defined by:

$$N_T(t) = \#\{\lambda \in \Sigma_1; \lambda \leqslant t\},$$

and $N_H(t)$ is defined by

$$N_H(t) = \#\{\lambda \in \Sigma_2;\ \lambda \leqslant t\}.$$

The estimates for $N_T(t)$ are well-known. For example,

$$N_T(t) = \frac{t}{4\pi} + O(t^{\frac{1}{2}}), \tag{7}$$

will suffice for our purposes. This bound was known to Gauss. To evaluate $N_H(t)$, we write:

$$N_H(t) = \sum_{c(c+(2k+1))\leqslant t/2\pi} 2c. \tag{8}$$

To estimate (8) we split the sum, into two pieces: Define $A_t = \{(x,y); x > 0, y > 0, x(x+y) \leqslant t\}$ and $B_t = \{(x,y); x > 0, y > 0, x(x+2y) \leqslant t\}$. Then, we have

$$N_H(2\pi t) = N_A(2\pi t) - N_B(2\pi t), \tag{9}$$

where

$$N_A(2\pi t) = \sum_{(c,k)\in\mathbb{Z}^2} (2c)\chi_{A_t}(c,k), \tag{10}$$

and

$$N_B(2\pi t) = \sum_{(c,k)\in\mathbb{Z}^2} (2c)\chi_{B_t}(c,k). \tag{11}$$

In order to apply the Poisson summation formula for $N_A(2\pi t)$ and $N_B(2\pi t)$, we need to regularize the characteristic functions χ_{A_t} and χ_{B_t}. Take ρ to be a smooth symmetric positive function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \rho(x,y)dxdy = 1$ and $\operatorname{supp}(\rho) \subseteq [-1,1]^2$. Let $\rho_\epsilon(x,y) = \epsilon^{-2}\rho(\frac{x}{\epsilon}, \frac{y}{\epsilon})$, where we make an explicit choice of $\epsilon > 0$ later on. Consider the mollified counting functions:

$$N_A^\epsilon(t) := \sum_{(c,k)\in\mathbb{Z}^2} (2c)\chi_{A_t}(c,k) * \rho_\epsilon(c,k), \tag{12}$$

and

$$N_B^\epsilon(t) := \sum_{(c,k)\in\mathbb{Z}^2} (2c)\chi_{B_t}(c,k) * \rho_\epsilon(c,k). \tag{13}$$

Lemma 3.1. — *Let T be an arbitrarily large number and put $\epsilon = T^{-1}$. Then, for $1 < t < T$ and $C > 2$ we have,*

$$N_A^\epsilon(t - C) \leqslant N_A(2\pi t) \leqslant N_A^\epsilon(t + C),$$

and

$$N_B^\epsilon(t - C) \leqslant N_B(2\pi t) \leqslant N_B^\epsilon(t + C).$$

Proof . — We prove the first series of inequalities in 3.1. The second series follows in the same way. Given $A_t = \{(x,y); x > 0, y > 0, x(x+y) \leqslant t\}$, let ∂A_t to be the hyperbola $x(x+y) = t$. If a point $X = (x,y) \in {\mathbb{Z}_+}^2$ lies at a distance greater than $\sqrt{2}\epsilon$ from ∂A_t, then $\chi_{A_t} * \rho_\epsilon(X) = \chi_{A_t}(X)$.

Therefore, by taking $\Omega_1 = \{(c,k) \in \mathbb{Z}^2; \operatorname{dist}((c,k), \partial A_{t+K\epsilon}) > \sqrt{2}\epsilon\}$, we have,

$$\begin{aligned} N_A^\epsilon(t + K\epsilon) &= \sum_{(c,k)\in\mathbb{Z}^2} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c,k) \\ &= \sum_{(c,k)\in\Omega_1} (2c)\chi_{A_{t+K\epsilon}}(c,k) + \sum_{(c,k)\in\mathbb{Z}^2\setminus\Omega_1} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c,k). \end{aligned}$$

On the other hand,

$$N_A(2\pi t) = \sum_{(c,k)\in\mathbb{Z}^2} (2c)\chi_{A_t}(c,k).$$

So, to get $N_A^\epsilon(t+K\epsilon) \geqslant N_A(2\pi t)$, it suffices to choose ϵ and K so that $\mathbb{Z}^2 \cap A_t \subseteq \Omega_1$. Since the closest point of $\mathbb{Z}^2 \cap A_t$ to $\partial A_{t+K\epsilon}$ is $(1,[t-1])$, it suffices to require that:

$$\operatorname{dist}((1,t),(\frac{-t+\sqrt{t^2+4t+4K\epsilon}}{2},t)) > \sqrt{2}\epsilon. \tag{14}$$

Equation (14) is equivalent to $4K\epsilon > 4\epsilon^2+4+4\epsilon t+8\epsilon$. So, it is enough to choose $K \geqslant 2T$ and $\epsilon \leqslant \frac{1}{T}$. The inequality $N_A^\epsilon(t-C) \leqslant N_A(2\pi t)$ can be proved in the same way and we are done. □

Lemma 3.1 will help us to convert our results on $N_A^\epsilon(t)$ and $N_B^\epsilon(t)$ back to $N_H(t)$.

Remark 3. —

(1) Henceforth, we always assume $\epsilon = T^{-1}$ for a fixed large T and $t \in [1,T]$. Also, we assume that δ is an arbitrary small positive number independent of T.

(2) By the notation $f(x) \ll g(x)$, we mean that there exists a positive constant C such that $|f(x)| \leqslant C|g(x)|$ for every x.

Proposition 3.2. — *The following asymptotic expansion holds for N_A^ϵ:*

$$N_A^\epsilon(t) = \frac{4}{3}t^{\frac{3}{2}} - \frac{3}{2}t + R_A^\epsilon(t) + O(t^{\frac{1}{2}+\delta}), \tag{15}$$

where,

$$\begin{aligned} R_A^\epsilon(t) &= \frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leqslant\mu} t^{\frac{3}{4}}\cos\left(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4}\right)\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{1}{\sqrt{2}\pi}\sum_{0<\nu<\mu} t^{\frac{3}{4}}\cos\left(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4}\right)\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu). \end{aligned} \tag{16}$$

Proof. — Applying the Poisson summation formula to $N_A^\epsilon(t)$ in (12) gives:

$$N_A^\epsilon(t) = \sum_{\lambda,\nu} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu)$$

$$\begin{aligned}&= \widehat{2x\chi_A}(0,0)\widehat{\rho_\epsilon}(0,0) + \sum_{\lambda\neq 0,\nu=0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu)\\ &+ \sum_{\lambda=0,\nu\neq 0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) + \sum_{\lambda\neq 0,\nu\neq 0} \widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu). \qquad (17)\end{aligned}$$

We first estimate each term on the right-hand side of (17). For the first term, we get:

$$\widehat{2x\chi_A}(0,0) = \int\int_A 2xdydx = \int_0^{\sqrt{t}}\int_0^{\frac{t}{x}-x} 2xdydx = \frac{4}{3}t^{\frac{3}{2}}.$$

Since $\widehat{\rho_\epsilon}(0,0) = 1$,

$$\widehat{2x\chi_A}(0,0).\widehat{\rho_\epsilon}(0,0) = \frac{4}{3}t^{\frac{3}{2}}. \qquad (18)$$

To evaluate the second term in (17), we write

$$\begin{aligned}\widehat{2x\chi_A}(\lambda,0) &= \int\int_A 2xe^{2\pi i\lambda x}dydx = \int_0^{\sqrt{t}}\int_0^{\frac{t}{x}-x} 2xe^{2\pi i\lambda x}dydx\\ &= -\frac{t}{\pi i\lambda} + \frac{4}{(2\pi i\lambda)^3} + \left(\frac{4\sqrt{t}}{(2\pi i\lambda)^2} - \frac{4}{(2\pi i\lambda)^3}\right)e^{2\pi i\lambda\sqrt{t}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{\lambda\neq 0}\widehat{2x\chi_A}(\lambda,0).\widehat{\rho_\epsilon}(\lambda,0)\\ &\qquad= \sum_{\lambda\neq 0}\left(\frac{-t}{\pi i\lambda} + \frac{4}{(2\pi i\lambda)^3} + \frac{8\sqrt{t}\pi i\lambda - 4}{(2\pi i\lambda)^3}e^{2\pi i\lambda\sqrt{t}}\right).\widehat{\rho_\epsilon}(\lambda,0)\\ &\qquad= \sum_{\lambda\neq 0}\frac{-t}{\pi i\lambda}\widehat{\rho_\epsilon}(\lambda,0) + O(\sqrt{t}). \qquad (19)\end{aligned}$$

Without loss of generality, we can assume that $\rho(x,y) = \varrho(x)\varrho(y)$, where, $\varrho\in C_0^\infty((0,1))$ such that $\int\varrho(x)dx = 1$. Then,

$$\begin{aligned}\sum_{\lambda\neq 0}\frac{\widehat{\rho}(\epsilon\lambda,0)}{\lambda} = \sum_{\lambda\neq 0}\frac{\widehat{\varrho}(\epsilon\lambda)}{\lambda} &= \sum_{\lambda\neq 0}\frac{1}{\lambda}\int_{\mathbb{R}} e^{2\pi i\epsilon\lambda x}\varrho(x)dx\\ &= \int_{\mathbb{R}}\left(\sum_{\lambda\neq 0}\frac{e^{2\pi i\epsilon\lambda x}}{\lambda}\right)\varrho(x)dx,\end{aligned}$$

where, for the last equality we have used the absolutely convergence of the summation $\sum_{\lambda\neq 0}\frac{1}{\lambda}\int_{\mathbb{R}} e^{2\pi i\epsilon\lambda x}\varrho(x)dx$. This follows by noticing that $\frac{1}{\lambda}\int_{\mathbb{R}} e^{2\pi i\epsilon\lambda x}\varrho(x)dx$
$= -\frac{1}{2\pi i\epsilon\lambda^2}\int_{\mathbb{R}} e^{2\pi i\epsilon\lambda x}\varrho'(x)dx$.

Using the formula $[z]-z+\frac{1}{2}=\sum_{n\neq 0}\frac{e^{2\pi i n z}}{2\pi i n}$, which holds for every $z\notin\mathbb{Z}$, we get:

$$\begin{aligned}\sum_{\lambda\neq 0}\frac{\widehat{\rho}(\epsilon\lambda,0)}{\lambda} &= \int_{\mathbb{R}} 2\pi i([\epsilon x]-\epsilon x+\frac{1}{2})\varrho(x)dx\\ &= \int_{\mathbb{R}} 2\pi i(\frac{1}{2}-\epsilon x)\varrho(x)dx = \pi i + O(\epsilon),\end{aligned}\tag{20}$$

since $[\epsilon x]=0$, because $\varrho\in C_0^\infty((0,1))$.

Therefore, substituting (20) into (19) gives the following result for the second term on the right-hand side of (17):

$$\sum_{\lambda\neq 0}\widehat{2x\chi_A}(\lambda,0)\widehat{\rho}_\epsilon(\lambda,0) = -t+O(\epsilon t)+O(\sqrt{t}) = -t+O(\sqrt{t}),\tag{21}$$

since $\epsilon = T^{-1}$.

For the third term on the right-hand side of (17), we have:

$$\begin{aligned}\widehat{2x\chi_A}(0,\nu) &= \int\int_A 2xe^{2\pi i\nu y}dydx\\ &= \int_0^{\sqrt{t}}\frac{2x}{2\pi i\nu}e^{2\pi i\nu(t/x-x)}dx - \int_0^{\sqrt{t}}\frac{2x}{2\pi i\nu}dx\\ &= \int_0^1\frac{2tx}{2\pi i\nu}e^{2\pi i\sqrt{t}\nu(1/x-x)}dx - \int_0^1\frac{2tx}{2\pi i\nu}dx.\end{aligned}\tag{22}$$

We claim that the first integral on the right-hand side of (22) is $\ll\frac{\sqrt{t}}{\nu^2}$. To prove this, put $f(x)=\frac{1}{x}-x$. Since f has no critical point, we integrate by parts to get:

$$\begin{aligned}\int_0^1 xe^{2\pi i\sqrt{t}\nu f(x)}dx &= \left[\frac{xe^{2\pi i\sqrt{t}\nu f(x)}}{\sqrt{t}\nu f'(x)}\right]_0^1\\ &\qquad - \int_0^1 e^{2\pi i\sqrt{t}\nu f(x)}\frac{\sqrt{t}\nu f'(x)-x\sqrt{t}\nu f''(x)}{(\sqrt{t}\nu f'(x))^2}dx\\ &\ll \frac{1}{\sqrt{t}\nu}.\end{aligned}$$

Therefore,

$$\sum_{\nu\neq 0}\widehat{2x\chi_A}(0,\nu)\widehat{\rho}_\epsilon(0,\nu)=\sum_{\nu\neq 0}O\left(\frac{\sqrt{t}}{\nu^2}\right)-t\sum_{\nu\neq 0}\frac{\widehat{\rho}_\epsilon(0,\nu)}{2\pi i\nu}=O(\sqrt{t})-\frac{t}{2}+O(\epsilon t)$$

$$=-\frac{t}{2}+O(\sqrt{t}),\tag{23}$$

since by symmetry $\sum_{\nu\neq 0}\frac{\widehat{\rho}_\epsilon(0,\nu)}{2\pi i\nu}=\sum_{\lambda\neq 0}\frac{\widehat{\rho}_\epsilon(\lambda,0)}{2\pi i\lambda}=\frac{1}{2}+O(\epsilon)$ (see (20)) and $\epsilon=T^{-1}$.

Finally, for the fourth term on the right-hand side of (17), we need the following proposition:

Proposition 3.3. — *The sum*

$$\sum_{\lambda\neq 0}\sum_{\nu\neq 0}\widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu)=R_A^\epsilon(t)+O(t^{\frac{1}{2}+\delta}),\tag{24}$$

where,

$$R_A^\epsilon(t)=\frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leqslant\mu}t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$

$$+\frac{1}{\sqrt{2}\pi}\sum_{0<\nu<\mu}t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu).\tag{25}$$

Proof. — See appendix A. □

So, combining the results (18), (21), (23) and (24) for the four terms in (17) proves Proposition 3.2 and we are done. □

Remark 4. — The following similar estimate holds for N_B^ϵ:

$$N_B^\epsilon(t)=\frac{2}{3}t^{\frac{3}{2}}-t+R_B^\epsilon(t)+O(t^{\frac{1}{2}+\delta}),\tag{26}$$

where

$$R_B^\epsilon(t)=\frac{1}{\pi}t^{\frac{3}{4}}\sum_{0<\nu\leqslant\mu}\cos(2\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon((\mu+\nu)/2,\nu)$$

$$+\frac{1}{\pi}t^{\frac{3}{4}}\sum_{0<\nu<\mu}\cos(2\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon((\mu+\nu)/2,\nu).\tag{27}$$

4. Proof of theorem 1.2.

Given the formulas for the regularized counting functions in Proposition 3.2 and Remark 4, we prove Theorem 1.2 in three steps: First, we find a new expression for R^ϵ_A in Proposition 4.1 so as to effectively estimate averages over short spectral intervals. Then, we evaluate L^2-estimates for $R^\epsilon_A(t)$ and $R^\epsilon_A(t+C)-R^\epsilon_B(t-C)$. Finally, using Lemma 3.1, we get rid of the mollifier ϵ and prove Theorem 1.2.

4.1. Step 1: A new expression for R^ϵ_A.

Following an argument of Cramér [Cr], we claim:

PROPOSITION 4.1. — *One can rewrite $R^\epsilon_A(t)$ in the form:*

$$\begin{aligned}R^\epsilon_A(t) &= \frac{1}{2\sqrt{2}\pi^2}\sum_{0<\nu\leqslant\mu}\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &+\frac{1}{2\sqrt{2}\pi^2}\sum_{0<\nu<\mu}\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &+O(t^{\frac{1}{2}+\delta}),\end{aligned} \tag{28}$$

where $\theta(f(x))=f(x+1)-f(x)$.

Proof. — Let $F^\epsilon_A(t)$ be the first summation on the right-hand side of (25), that is:

$$F^\epsilon_A(t)=\frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leqslant\mu}t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu).$$

Then,

$$\begin{aligned}\int_0^t F^\epsilon_A(u)du &= \frac{t^{\frac{5}{4}}}{2\sqrt{2}\pi^2}\sum_{0<\nu\leqslant\mu}\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &+\frac{5t^{\frac{3}{4}}}{16\sqrt{2}\pi^3}\sum_{0<\nu\leqslant\mu}\mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &+O(t^{\frac{1}{4}}).\end{aligned}$$

Writing $\int_t^{t+1} F_A^\epsilon(u)du = \int_0^{t+1} F_A^\epsilon(u)du - \int_0^t F_A^\epsilon(u)du$, we get:

$$\begin{aligned}\int_t^{t+1} & F_A^\epsilon(u)du \\ &= \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu\leqslant\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{5}{16\sqrt{2}\pi^3} \sum_{0<\nu\leqslant\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ O(t^{\frac{1}{4}}), \end{aligned} \tag{29}$$

where $\theta(f(x)) = f(x+1) - f(x)$.

Next, we need:

Lemma 4.2. — *The following holds,*

$$R_A^\epsilon(t) = \int_t^{t+1} R_A^\epsilon(u)du + O(t^{\frac{1}{2}+\delta}). \tag{30}$$

Proof. — Write

$$\int_t^{t+1} R_A^\epsilon(u)du = R_A^\epsilon(t) + \int_t^{t+1} (R_A^\epsilon(u) - R_A^\epsilon(t))du. \tag{31}$$

For $t \leqslant u \leqslant t+1$,

$$R_A^\epsilon(u) - R_A^\epsilon(t) = N_A^\epsilon(u) - N_A^\epsilon(t) + O(\sqrt{t}).$$

So, using Lemma 3.1,

$$\begin{aligned} R_A^\epsilon(u) - R_A^\epsilon(t) &= O\left(N_A(2\pi(u+C)) - N_A\left(2\pi(t-C)\right)\right) + O(\sqrt{t}) \\ &= O(N_A(2\pi u) - N_A(2\pi t)) + O(\sqrt{t}). \end{aligned} \tag{32}$$

From the definition of $N_A(2\pi t)$ (see (10)),

$$N_A(2\pi u) - N_A(2\pi t) \leqslant \sum_{c|[t+1], c\leqslant\sqrt{[t+1]}} 2c = O_\delta(t^{\frac{1}{2}+\delta}), \tag{33$$

for any $\delta > 0$. The lemma follows from (31), (32) and (33). □

Thus, from (29) and Lemma 4.2, it follows that:

$$\begin{aligned} R_A^{\epsilon}(t) &= \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu\leqslant\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{5}{16\sqrt{2}\pi^3} \sum_{0<\nu\leqslant\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu<\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ \frac{5}{16\sqrt{2}\pi^3} \sum_{0<\nu<\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &+ O(t^{\frac{1}{2}+\delta}). \end{aligned} \tag{34}$$

We claim that the second and the fourth terms on the right-hand side of (34) are $O(t^{\frac{1}{4}})$. Indeed, to bound the second sum on the right-hand side of (34), use that $\theta(f(t)) = \int_t^{t+1} f'(u)du$ to get

$$\begin{aligned} &\sum_{0<\nu\leqslant\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}\theta\left(t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) \\ &\qquad \ll \sum_{0<\nu\leqslant\mu} \mu^{-\frac{9}{4}}\nu^{-\frac{5}{4}}t^{\frac{1}{4}}\sqrt{\mu\nu} = O(t^{\frac{1}{4}}). \end{aligned} \tag{35}$$

The estimate for the fourth sum on the right-hand side of (34) is the same as in (35).

Consequently, from (34) and (35), Proposition 4.1 follows. □

4.2. Step 2: L^2-estimate for R_A^{ϵ}.

We now show that for any $\delta > 0$,

$$\int_1^T |R_A^{\epsilon}(t)|^2 dt = c_1 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}),$$

where c_1 is a positive constant.

For simplicity, we do the computations for $E_A^{\epsilon}(t)$, which is the first summation on the right-hand side of (28) in Proposition 4.1; that is,

$$E_A^{\epsilon}(t) = \frac{1}{2\sqrt{2}\pi^2} \sum_{0<\nu\leqslant\mu} \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu).$$

Then,

$$\int_1^T |E_A^\epsilon(t)|^2 dt = \frac{1}{8\pi^4} \sum_{\substack{0<\nu\leqslant\mu,\\ 0<\nu'\leqslant\mu'}} \int_1^T \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right)$$
$$\theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu'\nu'} - \frac{\pi}{4})\right) dt$$
$$\times \mu^{-\frac{7}{4}} \nu^{-\frac{3}{4}} \mu'^{-\frac{7}{4}} \nu'^{-\frac{3}{4}} \widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')}. \tag{36}$$

Let $n = \mu\nu$, $m = \mu'\nu'$, $\theta_n(t) = \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{n} - \frac{\pi}{4})\right)$ and $\theta_m(t) = \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{m} - \frac{\pi}{4})\right)$. It follows that,

$$|\int_1^T \theta_n(t)\theta_m(t)dt| \ll |\int_1^T \theta\left(t^{\frac{5}{4}} e^{4\pi i\sqrt{t}\sqrt{n}}\right) \theta\left(t^{\frac{5}{4}} e^{4\pi i\sqrt{t}\sqrt{m}}\right) dt|$$
$$+ |\int_1^T \theta\left(t^{\frac{5}{4}} e^{-4\pi i\sqrt{t}\sqrt{n}}\right) \theta\left(t^{\frac{5}{4}} e^{4\pi i\sqrt{t}\sqrt{m}}\right) dt|. \tag{37}$$

For $m > n$, both integrals on the right-hand side of (37) are bounded by:

$$\left|\int_1^T G(t)\ \ d\left(\frac{e^{4\pi i\sqrt{t}(\sqrt{m}-\sqrt{n})}}{\sqrt{m}-\sqrt{n}}\right)\right| < \frac{|G(T) + |G(1)| + \int_1^T |G'(t)|dt}{\sqrt{m}-\sqrt{n}}, \tag{38}$$

where

$$G(t) = \frac{t^3}{2\pi i}\left((1+\frac{1}{t})^{\frac{5}{4}} e^{4\pi i\sqrt{m}(\sqrt{t+1}-\sqrt{t})} - 1\right)$$
$$\times \left((1+\frac{1}{t})^{\frac{5}{4}} e^{-4\pi i\sqrt{n}(\sqrt{t+1}-\sqrt{t})} - 1\right).$$

By Taylor expansion, one can show that $G(t) \ll \min\{t^3, t^2\sqrt{mn}\}$ and $G'(t) \ll \min\{t^2 + t^{\frac{3}{2}} m^{\frac{1}{2}}, t\sqrt{mn}\}$. So,

$$\left|\int_1^T \theta_n(t)\theta_m(t)dt\right| \ll \frac{\min\{T^3 + T^{\frac{5}{2}} m^{\frac{1}{2}}, T^2 m^{\frac{1}{2}} n^{\frac{1}{2}}\}}{\sqrt{m}-\sqrt{n}}. \tag{39}$$

Next, we recall that:

$$\sum_{\substack{0<\nu\leqslant\mu,\\ 0<\nu'\leqslant\mu',\\ \mu'\nu'\neq\mu\nu}} \left(\int_1^T \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\right.$$

$$\times\,\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}}$$

$$= 2\sum_{m>0}\sum_{\mu'|m,\mu'\geqslant\sqrt{m}}\sum_{0<n<m}\sum_{\mu|n,\mu\geqslant\sqrt{n}}\left(\int_1^T \theta_n(t)\theta_m(t)dt\right)n^{-\frac{3}{4}}\mu^{-1}m^{-\frac{3}{4}}\mu'^{-1}$$

$$\ll \sum_{0<m}\sum_{0<n<m}\left(\int_1^T \theta_n(t)\theta_m(t)dt\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}. \tag{40}$$

Therefore, substituting the estimate (39) in (40), gives:

$$\sum_{\substack{0<\nu\leqslant\mu,\\ 0<\nu'\leqslant\mu',\\ \mu'\nu'\neq\mu\nu}} \left(\int_1^T \theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\right.$$

$$\times\,\theta\left(t^{\frac{5}{4}}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt\mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}}$$

$$\ll \sum_{0<m}\sum_{0<n<m}\left(\frac{\min\{T^3+T^{\frac{5}{2}}m^{\frac{1}{2}},T^2m^{\frac{1}{2}}n^{\frac{1}{2}}\}}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}$$

$$\ll \sum_{0<m\leqslant T,0<n<m}\left(\frac{T^2m^{\frac{1}{2}}n^{\frac{1}{2}}}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}$$

$$+\sum_{\substack{m>T\\ 0<n<m}}\left(\frac{T^{\frac{5}{2}}m^{\frac{1}{2}}+T^3}{\sqrt{m}-\sqrt{n}}\right)n^{-\frac{5}{4}+\delta}m^{-\frac{5}{4}+\delta}.$$

$$= O(T^{2+3\delta})+O(T^{\frac{9}{4}+\delta}) = O(T^{\frac{9}{4}+\delta}). \tag{41}$$

Thus, we are left with the case where $m=n$, that is $\mu\nu=\mu'\nu'$. This diagonal case will give the leading term in (36). We have,

$$(\theta_n(t))^2 = \frac{i}{4}\theta^2\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)+\frac{i}{4}\theta^2\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right)$$

$$+\frac{1}{2}\theta\left(t^{\frac{5}{4}}e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac{5}{4}}e^{-4\pi i\sqrt{t}\sqrt{n}}\right). \tag{42}$$

The same argument used to prove (39) shows that:

$$\sum_{n>0}\int_1^T \theta^2\left(t^{\frac54}\,e^{4\pi i\sqrt{t}\sqrt{n}}\right)\times n^{-\frac52+\delta}\ll\sum_{n>0}\frac{T^2 n^{\frac12}n^{\frac12}}{\sqrt{n}+\sqrt{n}}\times n^{-\frac52+\delta}=O(T^2),$$

and the same estimate holds for $\theta^2\left(t^{\frac54}\,e^{-4\pi i\sqrt{t}\sqrt{n}}\right)$. So, we just continue with $\frac12\theta\left(t^{\frac54}\,e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac54}\,e^{-4\pi i\sqrt{t}\sqrt{n}}\right)$.

Now, for $n<T$, using the fact that $\theta(f(t))=f(t+1)-f(t)=f'(t)+\int_t^{t+1}du\int_t^u f''(s)ds$, we have:

$$\begin{aligned}&\theta\left(t^{\frac54}\,e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac54}\,e^{-4\pi i\sqrt{t}\sqrt{n}}\right)\\&=\left(2\pi i\sqrt{n}t^{\frac34}\,e^{4\pi i\sqrt{t}\sqrt{n}}+O(nt^{\frac14})\right)\left(-2\pi i\sqrt{n}t^{\frac34}\,e^{-4\pi i\sqrt{t}\sqrt{n}}+O(nt^{\frac14})\right)\\&=4\pi^2 nt^{\frac32}+O(n^{\frac32}t+n^2t^{\frac12}).\end{aligned}$$

On the other hand, for $n\geqslant T$,

$$\theta\left(t^{\frac54}\,e^{4\pi i\sqrt{t}\sqrt{n}}\right)\theta\left(t^{\frac54}\,e^{-4\pi i\sqrt{t}\sqrt{n}}\right)=O(t^{\frac52}).$$

Therefore,

$$\begin{aligned}&\sum_{0<\nu\leqslant\mu}\ \sum_{\substack{0<\nu'\leqslant\mu',\\ \mu'\nu'=\mu\nu}}\left(\int_1^T\theta\left(t^{\frac54}\sin(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\right)\right.\\&\cdot\theta\left(t^{\frac54}\sin(4\pi\sqrt{t}\sqrt{\mu'\nu'}-\frac{\pi}{4})\right)dt\times\mu^{-\frac74}\nu^{-\frac34}\mu'^{-\frac74}\nu'^{-\frac34}\widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')}\\&=\sum_{0<n<T}\ \sum_{\mu|n,\mu\geqslant\sqrt{n}}\ \sum_{\mu'|n,\mu'\geqslant\sqrt{n}}\left(\frac{8\pi^2}{5}nT^{\frac52}+O(n^{\frac32}T^2+n^2T^{\frac32})\right)n^{\frac{-3}{2}}\mu^{-1}\mu'^{-1}\\&\times\widehat{\rho}_\epsilon(\mu+\frac{n}{\mu},\frac{n}{\mu})\overline{\widehat{\rho}_\epsilon(\mu'+\frac{n}{\mu'},\frac{n}{\mu'})}+\sum_{n\geqslant T}\ \sum_{\substack{\mu|n,\mu\geqslant\sqrt{n},\\ \mu'|n,\mu'\geqslant\sqrt{n}}}O(T^{\frac52})n^{\frac{-3}{2}}\mu^{-1}\mu'^{-1}+O(T^2)\\&=T^{\frac52}\sum_{0<n<T}\ \sum_{\substack{\mu|n,\mu\geqslant\sqrt{n},\\ \mu'|n,\mu'\geqslant\sqrt{n}}}\frac{8\pi^2}{5}n^{\frac{-1}{2}}\mu^{-1}\mu'^{-1}\widehat{\rho}_\epsilon(\mu+\frac{n}{\mu},\frac{n}{\mu})\overline{\widehat{\rho}_\epsilon(\mu'+\frac{n}{\mu'},\frac{n}{\mu'})}\\&+O(T^{2+\delta}).\end{aligned}\tag{43}$$

We split the sum in (43) into the pieces where $\mu \leqslant T^{1/4}$ and $\mu > T^{1/4}$. We claim that the piece where $\mu > T^{1/4}$ is residual. To see this, note that:

$$\begin{aligned} T^{\frac{5}{2}} \sum_{\substack{0<\nu\leqslant\mu,\\ \mu>T^{1/4}}} \sum_{\substack{0<\nu'\leqslant\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} \\ &= T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-\frac{3}{2}} \sum_{0<\nu\leqslant\mu} \nu^{-\frac{1}{2}} \sum_{\mu'|\mu\nu,\mu'\geqslant\sqrt{\mu\nu}} \mu'^{-1} \\ &\leqslant T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-\frac{3}{2}} \sum_{0<\nu\leqslant\mu} \nu^{-\frac{1}{2}} \frac{1}{\sqrt{\mu\nu}} d(\mu\nu) \\ &= T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-2} d(\mu) \sum_{0<\nu\leqslant\mu} \nu^{-1} d(\nu) \\ &= T^{\frac{5}{2}} \sum_{\mu>T^{1/4}} \mu^{-2} d(\mu) \log^2(\mu) = O_\delta(T^{\frac{9}{4}+\delta}). \end{aligned} \tag{44}$$

So, if $\mu \leqslant T^{1/4}$ then $\mu' \leqslant \mu'\nu' = \mu\nu \leqslant \mu^2 \leqslant T^{\frac{1}{2}}$. Since $\epsilon = T^{-1}$, we have $\epsilon\nu \leqslant \epsilon\mu \leqslant T^{-\frac{3}{4}}$ and $\epsilon\nu' \leqslant \epsilon\mu' \leqslant T^{-\frac{1}{2}}$. Therefore, by Taylor expanding the functions $\widehat{\rho}(\epsilon\mu + \epsilon\nu, \epsilon\nu)$ and $\widehat{\rho}(\epsilon\mu' + \epsilon\nu', \epsilon\nu')$ around the point $(0,0)$ and using (44), we can evaluate the summation in (43) as follows:

$$\begin{aligned} T^{\frac{5}{2}} \sum_{0<\nu\leqslant\mu} \sum_{\substack{0<\nu'\leqslant\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} \widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')} \\ = T^{\frac{5}{2}} \sum_{0<\nu\leqslant\mu} \sum_{\substack{0<\nu'\leqslant\mu',\\ \mu'\nu'=\mu\nu}} \mu^{-1}\mu'^{-1}(\mu\nu)^{-\frac{1}{2}} + O_\delta(T^{\frac{9}{4}+\delta}). \end{aligned} \tag{45}$$

Therefore, substituting (45) in (43), we get:

$$\begin{aligned} &\sum_{0<\nu\leqslant\mu} \sum_{\substack{0<\nu'\leqslant\mu',\\ \mu'\nu'=\mu\nu}} \left(\int_1^T \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4})\right) \right. \\ &. \left. \theta\left(t^{\frac{5}{4}} \sin(4\pi\sqrt{t}\sqrt{\mu'\nu'} - \frac{\pi}{4})\right) dt \right. \\ &\qquad \times \mu^{-\frac{7}{4}}\nu^{-\frac{3}{4}}\mu'^{-\frac{7}{4}}\nu'^{-\frac{3}{4}} \widehat{\rho}_\epsilon(\mu+\nu,\nu)\overline{\widehat{\rho}_\epsilon(\mu'+\nu',\nu')} \\ &\qquad = T^{\frac{5}{2}} \sum_{0<n<T} \sum_{\mu|n,\mu\geqslant\sqrt{n}} \sum_{\mu'|n,\mu'\geqslant\sqrt{n}} \frac{8\pi^2}{5} n^{-\frac{1}{2}} \mu^{-1}\mu'^{-1} + O_\delta(T^{\frac{9}{4}+\delta}). \end{aligned} \tag{46}$$

Finally, combining the results from (41) and (46), gives:

$$\int_1^T |E_A^\epsilon(t)|^2 dt = c_{11} T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}),$$

where,

$$c_{11} := \frac{1}{10\pi^2} \sum_{\mu=1}^{\infty} \mu^{-\frac{3}{2}} \sum_{0<\nu\leqslant\mu} \nu^{-\frac{1}{2}} \sum_{\mu'|\mu\nu, \mu'\geqslant\sqrt{\mu\nu}} \mu'^{-1}$$
$$= \frac{1}{10\pi^2} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geqslant\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'\geqslant\sqrt{n}} \mu'^{-1}.$$

The argument for $R_A^\epsilon(t)$ follows in the same way and one gets:

$$\int_1^T |R_A^\epsilon(t)|^2 dt = c_1 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{47}$$

where,

$$c_1 = \frac{1}{10\pi^2} \Big(\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geqslant\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'\geqslant\sqrt{n}} \mu'^{-1}$$
$$+ \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'>\sqrt{n}} \mu'^{-1}$$
$$+ 2 \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n, \mu\geqslant\sqrt{n}} \mu^{-1} \sum_{\mu'|n, \mu'>\sqrt{n}} \mu'^{-1} \Big)$$

Remark 5. — The argument for $R_B^\epsilon(t)$ is the same as for $R_A^\epsilon(t)$. The result is that:

$$\int_1^T |R_B^\epsilon(t)|^2 dt = c_2 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{48}$$

where $c_2 = 2c_1$, and

$$\int_1^T |R_A^\epsilon(t+C) - R_B^\epsilon(t-C)|^2 dt = c T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{49}$$

for $c = c_1 + c_2 - 2c_3$, where c_3 is a positive constant defined by

$$c_3 = \frac{1}{10\pi^2}\left(\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu\geqslant\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'\geqslant 2\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'>2\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu\geqslant\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'>2\sqrt{n}} \mu'^{-1} + \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \sum_{\mu|n,\mu>\sqrt{n}} \mu^{-1} \sum_{\mu'|4n,\mu'\geqslant 2\sqrt{n}} \mu'^{-1}\right).$$

Also the same result is true for $\int_1^T |R_A^\epsilon(t-C) - R_B^\epsilon(t+C)|^2 dt$.

Remark 6. — One can rewrite c as the following:

$$c = \frac{1}{5\pi^2}\sum_{n=1}^{\infty} n^{-\frac{5}{4}}\delta(n)\left(6\delta(n) - \delta(4n)\right) + \frac{2}{5\pi^2}\sum_{n=1}^{\infty} n^{-4}\delta(n^2) + \frac{1}{10\pi^2}\sum_{n=1}^{\infty} n^{-4}\left(6\delta(n^2) - \delta(4n^2)\right) + \frac{1}{4\pi^2}\sum_{n=1}^{\infty} n^{-3},$$

where $\delta(n) = \sum_{d|n,d<\sqrt{n}} d.$

4.3. Step 3: Eliminating the mollification.

The last step in the proof of the Theorem 1.2 is to use Lemma 3.1 to get rid of the mollification in ϵ and prove the L^2-estimate for $R_H(t)$, which is the error term corresponding to type II eigenvalues. From Lemma 3.1, by choosing $\epsilon = T^{-1}$ and $t \in [1, T]$, we get,

$$(N_A^\epsilon(t-C) - N_B^\epsilon(t+C))^2 \leqslant (N_2(2\pi t))^2 \leqslant (N_A^\epsilon(t+C) - N_B^\epsilon(t-C))^2. \tag{50}$$

For simplicity we do the calculations for the second inequality in (50), the other should be proceeded similarly. Taking L^2-norms in (50) gives:

$$\int_1^T \left(\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + R_H(t)\right)^2 dt \leqslant \int_1^T \left(\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}) + R_A^\epsilon(t+C) - R_B^\epsilon(t-C)\right)^2 dt. \tag{51}$$

Thus, by expanding both sides in (51) we get:

$$\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})^2 dt + \int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt \leqslant$$
$$\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})^2 dt + \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C) + O(t^{\frac{1}{2}+\delta}))^2 dt$$
$$+ 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt.$$

Thus,

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt \leqslant \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C))^2 dt$$
$$+ 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt.$$

We claim that

$$(52) \quad \int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2} + O(t^{\frac{1}{2}+\delta}))(R_A^\epsilon(t+C) - R_B^\epsilon(t-C)) dt = O(T^{\frac{9}{4}}).$$

To see this, note that

$$\int_1^T t^{\frac{3}{2}} E_A^\epsilon(t) dt = \frac{1}{2\pi} \sum_{0<\nu\leqslant\mu\leqslant T^{1+\alpha}}$$
$$\left(\int_1^T t^{\frac{9}{4}} e^{2\pi i\sqrt{t}\sqrt{\mu\nu}} dt\right) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} \widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(T^{-\infty})$$
$$\ll \sum_{0<\nu\leqslant\mu\leqslant T^{1+\alpha}} T^{\frac{11}{4}} \mu^{-7/4} \nu^{-\frac{3}{4}} = O(T^{\frac{9}{4}}).$$

Similarly, we have $\int_1^T t^{\frac{3}{2}} R_A^\epsilon(t) dt = O(T^{\frac{9}{4}})$ and $\int_1^T t^{\frac{3}{2}} R_B^\epsilon(t) dt = O(T^{\frac{9}{4}})$, which proves our claim in (52).

Hence,

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}) R_H(t) dt$$
$$\leqslant \int_1^T (R_A^\epsilon(t+C) - R_B^\epsilon(t-C))^2 dt + O(T^{\frac{9}{4}}),$$

which implies that

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt \leqslant cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

On the other hand, from the leftmost inequality in (50), we also have

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt \geqslant cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

Hence,

$$\int_1^T (R_H(t))^2 dt + 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}). \tag{53}$$

Similarly, it is also true that

$$\int_1^T (R_H(t))^2 dt - 2\int_1^T (\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2})R_H(t)dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}), \tag{54}$$

since

$$\begin{aligned} R_B^\epsilon(t-C) - R_A^\epsilon(t+C) + O(t^{\frac{1}{2}+\delta}) &\leqslant -R_H(t) \\ &\leqslant R_B^\epsilon(t+C) - R_A^\epsilon(t-C) + O(t^{\frac{1}{2}+\delta}). \end{aligned}$$

Therefore, by adding a term $\frac{2}{3}t^{\frac{3}{2}} - \frac{t}{2}$ to both sides of this inequality and taking L^2-norms we are done.

Combining (53) and (54), proves that

$$\int_1^T (R_H(t))^2 dt = cT^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

Now $R_H(t)$ is the error term corresponding to $N_H(2\pi t)$ and we know that it differs with $R(2\pi t)$ which is the error term corresponding to $N(2\pi t)$ only by a term of order $O(\sqrt{t})$. Therefore,

$$\int_1^T (R(t))^2 dt = c(2\pi)^{7/2}T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}).$$

This proves Theorem 1.2. □

5. Proof of theorem 1.3.

Let $N(t)$ be the spectral counting function of the $(2n+1)$-dimensional Heisenberg manifold. Therefore,

$$N(t) = N_T(t) + N_H(t),$$

where $N_T(t)$ is the spectral counting function of $2n$-dimensional torus with the metric $h = I_{2n\times 2n}$ and $N_H(t)$ is defined by

$$N_H(t) = \#\{(c, k_1, k_2, ..., k_n); c > 0, k_j \geqslant 0, 2\pi c(c+2k_1+...+2k_n+n) \leqslant t\}.$$

For $N_T(t)$ we use the trivial estimate resulted from Hörmander's theorem:

$$N_T(t) = \frac{1}{n!2^n}\left(\frac{t}{2\pi}\right)^n + O\left(t^{n-\frac{1}{2}}\right),$$

and we continue with computing $N_H(t)$:

$$\begin{aligned} N_H(2\pi t) &= \sum_{c(c+2\sum k_j+n)\leqslant t} 2c^n = \sum_{c(c+2k+n)\leqslant t} 2c^n \sum_{k_1+...+k_n=k} 1 \\ &= \sum_{c(c+2k+n)\leqslant t} 2c^n \binom{k+n-1}{n-1} \\ &= \sum \frac{2}{(n-1)!} c^n k^{n-1} + \sum \frac{n}{(n-2)!} c^n k^{n-2} + O\left(t^{n-\frac{1}{2}}\right). \end{aligned}$$

Let $A_t = \{(x, y); x(x+2y+n) \leqslant t, x > 0, y > 0\}$ and $\rho_\epsilon(x, y)$ be as defined in third section. We define the mollified counting function $N_\epsilon(t)$ as:

$$\begin{aligned} N_\epsilon(t) :=& \frac{2}{(n-1)!} \sum_{(c,k)\in\mathbb{Z}^2} (c^n k^{n-1})\chi_{A_t}(c, k) * \rho_\epsilon(c, k) \\ &+ \frac{n}{(n-2)!} \sum_{(c,k)\in\mathbb{Z}^2} (c^n k^{n-2})\chi_{A_t}(c, k) * \rho_\epsilon(c, k). \end{aligned} \tag{55}$$

Proposition 5.1. — *The following asymptotic expansion holds for $N_\epsilon(t)$:*

$$N_\epsilon(t) = \frac{2^{n+1}n!}{(2n+1)!} t^{n+\frac{1}{2}} - \frac{1}{n!2^n} t^n + R_\epsilon(t) + O\left(t^{n-\frac{1}{2}+\delta}\right), \tag{56}$$

where,

$$R_\epsilon(t) = \frac{t^{n-\frac{1}{4}}}{2^{n-1}(n-1)!\pi} \sum_{0<\nu<\mu} (-1)^{\nu n} \cos(2\pi\sqrt{t\mu\nu} - \frac{\pi}{4}) \\ \mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}(1-\frac{\nu}{\mu})^{n-1}\widehat{\rho}_\epsilon(\frac{\mu+\nu}{2},\nu).$$

Proof. — Applying the Poisson summation formula to the first sum in $N_\epsilon(t)$ (defined by (55)) gives:

$$\sum_{(c,k)\in\mathbb{Z}^2} (c^n k^{n-1})\chi_{A_t}(c,k) * \rho_\epsilon(c,k) = \sum_{\lambda,\nu} x^n \widehat{y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu)$$
$$= x^n\widehat{y^{n-1}\chi_A}(0,0)\widehat{\rho_\epsilon}(0,0) + \sum_{\lambda\neq 0,\nu=0} x^n\widehat{y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu)$$
$$(57)\quad + \sum_{\lambda=0,\nu\neq 0} x^n\widehat{y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) + \sum_{\lambda\neq 0,\nu\neq 0} x^n\widehat{y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu).$$

We first estimate each term on the right-hand side of (57). For the first term, we get:

$$x^n\widehat{y^{n-1}\chi_A}(0,0) = \int_0^{\sqrt{t}} \int_0^{\frac{t}{2x}-\frac{x}{2}-\frac{n}{2}} x^n y^{n-1} dy dx$$
$$= \int_0^{\sqrt{t}} \frac{1}{n2^n}(t-x^2-nx)^n dx$$
$$(58)\quad = \frac{(n!)^2 2^n}{(2n+1)!n} t^{n+\frac{1}{2}} - \frac{1}{2^{n+1}} t^n + O\left(t^{n-\frac{1}{2}}\right).$$

Also, similar computations to the ones in section 3 show that:

$$(59)\quad \sum_{\lambda\neq 0} x^n\widehat{y^{n-1}\chi_A}(\lambda,0).\widehat{\rho_\epsilon}(\lambda,0) = \frac{-1}{n2^{n+1}} t^n + O\left(t^{n-\frac{1}{2}}\right),$$

and

$$(60)\quad \sum_{\nu\neq 0} x^n\widehat{y^{n-1}\chi_A}(0,\nu)\widehat{\rho_\epsilon}(0,\nu) = O\left(t^{n-\frac{1}{2}}\right).$$

For the fourth term on the right-hand side of (57),

$$\begin{aligned}
\widehat{x^n y^{n-1}\chi_A}(\lambda,\nu) &= \int_0^{\sqrt{t}} \int_0^{\frac{t}{2x}-\frac{x}{2}-\frac{n}{2}} x^n y^{n-1} e^{2\pi i(\lambda x+\nu y)} dydx \\
&= \frac{(-1)^{\nu n}}{2^n \pi i \nu} \int_0^{\sqrt{t}} x^n (\frac{t}{x} - x - n)^{n-1} e^{\pi i((2\lambda-\nu)x+\frac{\nu t}{x})} dx \\
&- \frac{(-1)^{\nu n}(n-1)}{2^n(\pi i \nu)^2} \int_0^{\sqrt{t}} x^n (\frac{t}{x} - x - n)^{n-2} e^{\pi i((2\lambda-\nu)x+\frac{\nu t}{x})} dx \\
&= \frac{(-1)^{\nu n} t^n}{2^n \pi i \nu} \int_0^1 x(1 - x^2 - \frac{nx}{\sqrt{t}})^{n-1} e^{\pi i\sqrt{t}((2\lambda-\nu)x+\frac{\nu}{x})} dx \\
&- \frac{(-1)^{\nu n}(n-1)t^{n-1}}{2^n(\pi i \nu)^2} \int_0^1 x^n(1 - x^2 - \frac{nx}{\sqrt{t}})^{n-2} e^{\pi i\sqrt{t}((2\lambda-\nu)x+\frac{\nu}{x})} dx
\end{aligned}$$

Now, using the method of the stationary phase and following the same argument as in the appendix, we get:

$$\begin{aligned}
&\sum_{\lambda\neq 0,\nu\neq 0} \widehat{x^n y^{n-1}\chi_A}(\lambda,\nu)\widehat{\rho_\epsilon}(\lambda,\nu) \\
&= \frac{t^{n-\frac{1}{4}}}{2^{n-1}\pi} \sum_{0<\nu<\mu} (-1)^{\nu n} \cos(2\pi\sqrt{t\mu\nu} - \frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}(1-\frac{\nu}{\mu})^{n-1}\widehat{\rho_\epsilon}(\frac{\mu+\nu}{2},\nu) \\
&+ O\left(t^{n-\frac{1}{2}+\delta}\right) \qquad (61)
\end{aligned}$$

Combining the results from (58),..., (61), we have proved that:

$$\begin{aligned}
&\frac{2}{(n-1)!} \sum_{(c,k)\in\mathbb{Z}^2} (c^n k^{n-1})\chi_{A_t}(c,k) * \rho_\epsilon(c,k) \\
&= \frac{2^{n+1} n!}{(2n+1)!} t^{n+\frac{1}{2}} - \frac{t^n}{(n-1)! 2^n} - \frac{t^n}{n! 2^n} + R_\epsilon(t) + O\left(t^{n-\frac{1}{2}+\delta}\right).
\end{aligned}$$

Finally applying the Poisson summation formula to the second sum in $N_\epsilon(t)$ (defined by (55)) and using the same argument that we used for the first sum, we get:

$$\frac{n}{(n-2)!} \sum_{(c,k)\in\mathbb{Z}^2} (c^n k^{n-2})\chi_{A_t}(c,k) * \rho_\epsilon(c,k) = \frac{1}{(n-1)! 2^n} t^n + O\left(t^{n-\frac{1}{2}}\right).$$

This completes the proof of proposition 5.1. □

Given the estimate (56) for $N_\epsilon(t)$, the rest of the proof for theorem 1.3 follows exactly like the proof of theorem 1.2.

APPENDIX A

5.1. Proof of Proposition 3.3.

After a simple integration, we get for $\nu \neq 0$:

$$\widehat{2x\chi_A}(\lambda,\nu) = \int\int_A 2xe^{2\pi i(\lambda x+\nu y)}dydx = \int_0^{\sqrt{t}}\int_0^{t/x-x} 2xe^{2\pi i\lambda x}e^{2\pi i\nu y}dydx$$

$$= \int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}e^{2\pi i\nu(t/x-x)}dx - \int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}dx. \tag{62}$$

The summation over the second integral in (62) leads to a term of order $O(t^{\frac{1}{2}+\delta})$ for every positive δ. To see this, for $\nu \neq 0$,

$$\int_0^{\sqrt{t}} \frac{x}{\nu}e^{2\pi i\lambda x}dx = \frac{1}{\nu}\left[\frac{xe^{2\pi i\lambda x}}{2\pi i\lambda} - \frac{e^{2\pi i\lambda x}}{2\pi i\lambda^2}\right]_0^{\sqrt{t}} \ll \frac{\sqrt{t}}{\nu\lambda} + \frac{1}{\nu\lambda^2}.$$

Therefore,

$$\sum_{\lambda\neq 0,\nu\neq 0}\left(\int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}dx\right)\widehat{\rho}_\epsilon(\lambda,\nu)$$
$$\ll \sum_{0<\lambda\leqslant t^{1+\alpha}}\ \sum_{0<\nu\leqslant t^{1+\alpha}}\left(\frac{\sqrt{t}}{\nu\lambda} + \frac{1}{\nu\lambda^2}\right) + O(t^{-\infty})$$
$$= O(t^{\frac{1}{2}}\ln^2(t)) = O(t^{\frac{1}{2}+\delta}), \tag{63}$$

where α and δ are arbitrarily small positive numbers.

To evaluate the first integral on the right-hand side of (62), make the change of variable $y = \frac{x}{\sqrt{t}}$,

$$\int_0^{\sqrt{t}} \frac{2x}{2\pi i\nu}e^{2\pi i\lambda x}e^{2\pi i\nu(t/x-x)}dx = \int_0^1 \frac{2ty}{2\pi i\nu}e^{2\pi i\sqrt{t}((\lambda-\nu)y+\frac{\nu}{y})}dy. \tag{64}$$

It is convenient to introduce the new variable $\mu = \lambda-\nu$. Let $f(y) = \mu y + \frac{\nu}{y}$. Then, the phase, $f(y)$, has no critical point iff $\mu = 0$ or $\frac{\nu}{\mu} < 0$ or $\frac{\nu}{\mu} > 1$.

We show that in any of these cases, the summation over the integral in (64) leads to a term of order $O(t^{\frac{1}{2}+\delta})$ for every positive δ. To see this, note that

$$\int_0^1 ye^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy$$
$$= \left[\frac{ye^{2\pi i\sqrt{t}f(y)}}{\sqrt{t}f'(y)}\right]_0^1 - \frac{1}{\sqrt{t}}\int_0^1 \left(\frac{f'(y)-yf''(y)}{f'^2(y)}\right)e^{2\pi i\sqrt{t}f(y)}dy$$
$$\ll \frac{1}{\sqrt{t}|\mu-\nu|} + \frac{1}{\sqrt{t}}\max_{0\leqslant y\leqslant 1}|\frac{1}{f'(y)}| + \frac{1}{\sqrt{t}}\int_0^1 |\frac{f''(y)}{f'^2(y)}|dy = \frac{3}{\sqrt{t}|\mu-\nu|}.$$

Hence,

$$\sum_{\mu=0 \text{ or } 1<\frac{\nu}{\mu} \text{ or } \frac{\nu}{\mu}<0} \left(\int_0^1 \frac{ty}{\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)$$
$$\ll \sqrt{t}\sum_{0<\nu}\frac{1}{\nu^2} + \sqrt{t}\sum_{0<\mu<\nu\leqslant t^{1+\alpha}}\frac{1}{(\nu-\mu)\nu} + \sqrt{t}\sum_{\substack{\frac{\nu}{\mu}<0,0<|\mu|\leqslant t^{1+\alpha},\\ 0<|\nu|\leqslant t^{1+\alpha}}}\frac{1}{(\nu-\mu)\nu} + O(t^{-\infty})$$
$$= O(t^{\frac{1}{2}+\delta}). \tag{65}$$

Therefore, combining the results form (62), (63), (65) and (66), we have

$$\sum_{\lambda\neq 0}\sum_{\nu\neq 0}\widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu)$$
$$= \sum_{0<\frac{\nu}{\mu}\leqslant 1,\mu\neq 0}\left(\int_0^1 \frac{ty}{\pi i\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}). \tag{66}$$

If $0<\frac{\nu}{\mu}\leqslant 1$, then the phase has a critical point $\sqrt{\frac{\nu}{\mu}}$. Without loss of generality assume that $0<\nu\leqslant\mu$. After making a change of variable $z=\sqrt{\frac{\mu}{\nu}}y$, we get:

$$\int_0^1 \frac{ty}{\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy = \int_0^{\sqrt{\frac{\mu}{\nu}}}\frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz$$
$$= \int_0^{\frac{1}{2}}\frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz + \int_{\frac{1}{2}}^1\frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz$$
$$+ \int_1^{\sqrt{\frac{\mu}{\nu}}}\frac{tz}{\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz. \tag{67}$$

Using a standard integration by parts, one can see that $\int_0^{\frac{1}{2}} \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz$ $= O(\frac{t}{\mu\sqrt{t\mu\nu}})$. Therefore the the summation over this integral leads to a term of order $O(t^{\frac{1}{2}+\delta})$, that is:

$$\sum_{0<\nu\leqslant\mu}\left(\int_0^{\frac{1}{2}} \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu) = O(t^{\frac{1}{2}+\delta}). \tag{68}$$

Consider the second integral on the right-hand side of (67). Applying the method of stationary phase (see [Cop]), we get:

$$\int_{\frac{1}{2}}^1 \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz = \frac{t}{2\sqrt{2}\mu\sqrt[4]{t\mu\nu}} e^{4\pi i\sqrt{t\mu\nu}+\frac{i\pi}{4}} + O(\frac{t}{\mu\sqrt{t\mu\nu}}), \tag{69}$$

and therefore, taking the summation we have:

$$\begin{aligned}&\sum_{0<\nu\leqslant\mu}\left(\int_{\frac{1}{2}}^1 \frac{tz}{\pi i\mu} e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\ &= \frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}}\sum_{0<\nu\leqslant\mu} e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}}\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu) + O(t^{\frac{1}{2}+\delta}).\end{aligned} \tag{70}$$

To evaluate the third integral on the right-hand side of (67), we use the following lemma (for proof, see [Cop] pages 29–33):

Lemma 5.2. — *Suppose f and ϕ are analytic functions, regular in a simply connected open region D in the complex plane, containing the interval $[1,a]$ from the real axis. Also, suppose that f is real on the real axis and has exactly one stationary point $x=1$ in $[1,a]$ where $f''(1)>0$. Then,*

$$\int_1^a \phi(x)e^{isf(x)}dx = \sqrt{\frac{\pi}{2sf''(1)}}\phi(1)e^{isf(1)+\frac{i\pi}{4}} + O(\frac{1}{\varepsilon s}), \tag{71}$$

where $\varepsilon := \sqrt{f(a)-f(1)}$.

Therefore, from (71) we get that:

$$\int_1^{\sqrt{\frac{\mu}{\nu}}} \frac{tz}{\mu} e^{2\pi i\sqrt{t\mu\nu}(\frac{1}{z}+z)}dz = \frac{t}{2\sqrt{2}\mu\sqrt[4]{t\mu\nu}} e^{4\pi i\sqrt{t\mu\nu}+\frac{i\pi}{4}} + O(\frac{t}{\mu\varepsilon\sqrt{t\mu\nu}}), \tag{72}$$

where $\varepsilon = \frac{\sqrt{\mu}-\sqrt{\nu}}{\sqrt[4]{\mu\nu}}$. Hence, taking the summation gives:

$$\begin{aligned}
&\sum_{0<\nu\leqslant\mu}\left(\int_1^{\sqrt{\frac{\mu}{\nu}}}\frac{tz}{\pi i\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\
&=\frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}}\sum_{0<\nu<\mu}e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}}\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)+O(t^{\frac{1}{2}+\delta}).
\end{aligned}\tag{73}$$

Combining (68), (70) and (73), we find that:

$$\begin{aligned}
&\sum_{0<\nu\leqslant\mu}\left(\int_0^{\sqrt{\frac{\mu}{\nu}}}\frac{tz}{\pi i\mu}e^{2\pi i\sqrt{t\mu\nu}(z+\frac{1}{z})}dz\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\
&=\frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}}\sum_{0<\nu\leqslant\mu}e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}}\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\
&+\frac{t^{\frac{3}{4}}}{2\pi\sqrt{2}}\sum_{0<\nu<\mu}e^{4\pi i\sqrt{t\mu\nu}-\frac{i\pi}{4}}\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)+O(t^{\frac{1}{2}+\delta}).
\end{aligned}\tag{74}$$

Given a similar result as the one in (74) for the case $\mu\leqslant\nu<0$, we have proved that:

$$\begin{aligned}
&\sum_{\lambda\neq0}\sum_{\nu\neq0}\widehat{2x\chi_A}(\lambda,\nu)\widehat{\rho}_\epsilon(\lambda,\nu)\\
&=\sum_{0<\frac{\nu}{\mu}\leqslant1,\mu\neq0}\left(\int_0^1\frac{ty}{\pi i\nu}e^{2\pi i\sqrt{t}(\mu y+\frac{\nu}{y})}dy\right)\widehat{\rho}_\epsilon(\mu+\nu,\nu)+O(t^{\frac{1}{2}+\delta})\\
&=\frac{1}{\sqrt{2}\pi}\sum_{0<\nu\leqslant\mu}t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)\\
&+\frac{1}{\sqrt{2}\pi}\sum_{0<\nu<\mu}t^{\frac{3}{4}}\cos(4\pi\sqrt{t}\sqrt{\mu\nu}-\frac{\pi}{4})\mu^{-\frac{5}{4}}\nu^{-\frac{1}{4}}\widehat{\rho}_\epsilon(\mu+\nu,\nu)+O(t^{\frac{1}{2}+\delta}),
\end{aligned}$$

which proves the proposition. □

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