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CALOGERO-MOSER SPACES AND AN ADELIC W-ALGEBRA

by Emil HOROZOV

0. Introduction.

In this paper we give an alternative description of the phase spaces of Calogero-Moser particle systems. In recent years they turned to be connected to several objects: rational solutions of KP-hierarchy [1, 26]; Wilson's adelic Grassmannian [38, 39]; isomorphism classes of right ideals of the Weyl algebra [11, 9], *etc.* to mention few of them.

The motion of Calogero-Moser particle [12, 31] systems is governed by the Hamiltonians

$$H_n = \frac{1}{2}\sum_{i}^{n} p_i^2 - \sum_{i<j}\frac{1}{(x_i - x_j)^2}.$$

In the original papers on the subject the variables are real and collisions $(x_i = x_j)$ are avoided; however here we work with complex variables and do allow collisions, which will be explained below (see [38, 25] for more details).

In [25] Kazhdan, Kostant and Sternberg used Hamiltonian reduction (in the opposite direction) to easily determine explicitly the Calogero-Moser flows. We briefly recall their construction. Consider the sets $\overline{C}_n 0, = 1, 2\ldots$ of all pairs of complex matrices X, Z subject to the condition:

$$[X, Z] + I$$

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has rank one. Here I is the identity matrix. LetC_n be the quotient space $C_n = \overline{C}_n/GL(n,\mathbb{C})$, where the group $GL(n,\mathbb{C})$ acts by simultaneous conjugation. Denote by C'_N the subspace of pairs,such that X is diagonalizable. In that case the pair can be conjugated to a pair of the form

$$X = \mathrm{diag}(x_1,\dots,x_n), \quad Z_{ii} = \alpha_i,\ Z_{ij}, \quad \text{for} \ \ i \neq j.$$

A matrix Z of that form is called *Calogero-Moser matrix*. The main result of [25] is that the Calogero-Moser flows are quotients of the simple flows $(X,Z) \to (X - tZ, Z)$. The last flows obviously make sense for any pair of matrices. For this reason the Calogero-Moser flows can be continued on the singular locus and the spaces C_n are completions of C'_N.

Our main objective in this paper will be to describe the spaces C_n in terms of representations of a suitable Lie algebra. This will be done by passing through an intermediate object - the so called *bispectral operators*.

Bispectral operators have been introduced by F. A. Grünbaum in his work on medical imaging [18] (see also [15]). An ordinary differential operator $L(x,\partial_x)$ is called bispectral if there exists an infinite-dimensional family of eigenfunctions $\psi(x,z)$, which are also eigenfunctions of another differential operator $\Lambda(z,\partial_z)$ in the spectral parameter z, *i.e.* for which the following identities hold

$$\begin{aligned} L(x,\partial_x)\psi(x,z) &= f(z)\psi(x,z), \\ \Lambda(z,\partial_z)\psi(x,z) &= \theta(x)\psi(x,z), \end{aligned}$$

with some non constant functions $f(z)$ and $\theta(x)$. G. Wilson [38] has classified all bispectral operators of rank one (see the next section for more details). Using slightly different terminology than in [38], they are all operators with rational coefficients that are Darboux transformations of operators with constant coefficients. Sato's theory associates with each operator (or rather with the maximal algebra of operators that commute with it) a plane in Sato's Grassmannian. The set of all planes corresponding to the rank one bispectral algebras of operators has been called by G.Wilson *an adelic Grassmannian* and denoted by $\mathrm{Gr}^{\mathrm{ad}}$. Originally G. Wilson has characterized the rank one bispectral algebras A as those whose spectral curve $SpecA$ is rational and its singularities are only cusps. Then in [39] he found an isomorphism between the disjoin union $\bigcup_{n\geqslant 0} C_n$ and $\mathrm{Gr}^{\mathrm{ad}}$.

In a different development [8] we have characterized those of bispectral algebras whose spectral curve has only one cusp in terms of representations of $W_{1+\infty}$-algebra. More precisely we have built certain bosonic highest weight modules of $W_{1+\infty}$. Denote the module corresponding to the rank one case by $\mathcal{M}_0$. Then the tau-functions of the bispectral operators (with the above restriction) lie in $\mathcal{M}_0$ and vice versa - all the tau functions in the module are tau-functions of bispectral operators.

A natural question (see [8]) is if a similar result holds for the entire set of rank one bispectral operators. The present paper gives an affirmative answer to this question. Obviously one has first to point out a suitable generalization of the $W_{1+\infty}$-algebra. The most natural candidate does the job - the algebra we look for is a central extension of the algebra of differential operators with rational coefficients. We call this new algebra *an adelic W-algebra*. Then we proceed as in [5, 8]. We construct a bosonic representation $\mathcal{M}^{\mathrm{ad}}$ which is similar to a highest weight representation. Then our main result is the following

THEOREM 0.1. — *If an element $\tau \in \mathcal{M}^{\mathrm{ad}}$ is a tau-function then the corresponding plane belongs to $\mathrm{Gr}^{\mathrm{ad}}$. Conversely, if $W \in \mathrm{Gr}^{\mathrm{ad}}$ then $\tau_W \in \mathcal{M}^{\mathrm{ad}}$.*

Returning to the realization of $\mathrm{Gr}^{\mathrm{ad}}$ as Calogero-Moser spaces we obtain immediately

THEOREM 0.2. — *The points of the Calogero-Moser spaces are in 1:1 correspondence with the tau-functions in $\mathcal{M}^{\mathrm{ad}}$.*

For other interpretations see [9, 10].

It would be interesting to find analogs of the present results for other bispectral operators - both continuous and discrete . For example there are particle systems connected to the bispectral operators from [34, 23]. Even more intriguing would be to consider particle systems comming from discrete bispectral operators [19, 20, 21]. In this respect it seems to me that the results of P. Iliev [24] will be very helpful.

Many of the constructions in the present paper are similar to those of [8]. We skip some of the auxiliary results but repeat (with less details) the main steps as there are some differences. The organization of the paper is the following. Section 1. contains preliminaries on Sato's Grassmannian, Darboux transformations, bispectral operators, $W_{1+\infty}$-algebra. In Section 2. we introduce the adelic W-algebra together with a bosonic representation $\mathcal{M}^{\mathrm{ad}}$. In Section 3. we show that the tau-functions in the module $\mathcal{M}^{\mathrm{ad}}$ correspond to planes in $\mathrm{Gr}^{\mathrm{ad}}$. In Section 4. we give the inverse result.

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1. Preliminaries.

Here we have collected some facts and notation needed throughout the paper. In particular we recall Sato's theory, Darboux transforms and the bispectral problem, $W_{1+\infty}$-algebra.

1.1. Sato's theory of KP-hierarchy.

In this subsection we recall some facts and notation from Sato's theory of KP-hierarchy [36, 13, 35] needed in the paper. We use the approach of V. Kac and D. Peterson based on infinite wedge products (see e.g. [27]) and the survey paper by P. van Moerbeke [36].

Consider the infinite-dimensional vector space of formal series

$$\mathbb{V} = \Big\{ \sum_{k\in\mathbb{Z}} a_k v_k \mid a_k = 0 \text{ for } k \gg 0 \Big\}.$$

Sato's Grassmannian Gr (more precisely - its big cell) [36, 13] consists of all subspaces ("planes") $W \subset \mathbb{V}$ which have an admissible basis

$$w_k = v_k + \sum_{i<k} w_{ik} v_i, \quad k = 0, 1, 2, \ldots$$

Then define the *fermionic Fock space* $F^{(0)}$ consisting of formal infinite sums of semi-infinite wedge monomials

$$v_{i_0} \wedge v_{i_1} \wedge \cdots$$

such that $i_0 < i_1 < \cdots$ and $i_k = k$ for $k \gg 0$. The wedge monomial

$$\psi_0 = v_0 \wedge v_1 \wedge \cdots$$

plays a special role and is called *the vacuum*. The plane that corresponds to it will be denoted by W_0. There exists a well known linear isomorphism, called a *boson-fermion correspondence*:

$$\sigma\colon F^{(0)} \to B, \tag{1}$$

(see [29]), where $B = \mathbb{C}\,[[t_1, t_2, \ldots]]$ is the bosonic Fock space.

To any plane $W \in \mathrm{Gr}$ one naturally associates a state $|W\rangle \in F^{(0)}$ as follows

$$|W\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \dots,$$

where $w_0, w_1, \dots,$ form an admissible basis. One of the main objects of Sato's theory is the *tau-function* of W defined as the image of $|W\rangle$ under the boson-fermion correspondence (1)

$$\tau_W(t) = \sigma(|W\rangle) = \sigma(w_0 \wedge w_{-1} \wedge w_{-2} \wedge \cdots).$$

It is a formal power series in the variables $t_1, t_2, \dots,$ *i.e.* an element of $B := \mathbb{C}\,[[t_1, t_2, \dots]]$. In particular the tau-function corresponding to the vacuum ψ_0 is $\tau_0 \equiv 1$. Using the tau-function one can define the other important function connected to W - the *Baker* or *wave function*

$$\Psi_W(t,z) = e^{\sum_{k=1}^{\infty} t_k z^k} \frac{\tau_W\left(t - [z^{-1}]\right)}{\tau_W(t)}, \tag{2}$$

where $[z^{-1}]$ is the vector $\left(z^{-1}, z^{-2}/2, \dots\right)$. Introducing the vertex operator

$$X(t,z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k}\right)$$

the above formula (2) can be written as

$$\Psi_W(t,z) = \frac{X(t,z)\tau(t)}{\tau(t)}.$$

We often use the formal series $\Psi_W(x,z) = \Psi_W(t,z)\big|_{t_1=x, t_2=t_3=\cdots=0}$, which we call again wave function. The wave function, corresponding to the vacuum is

$$\Psi_0(x,z) = e^{xz}.$$

The wave function $\Psi_W(x,z)$ contains the whole information about W and hence about τ_W, as the vectors $w_{-k} = \partial_x^k \Psi_W(x,z)\big|_{x=0}$ form an admissible basis of W (if we take $v_k = z^k$ as a basis of $\mathbb{V}$).

1.2. Darboux transforms and bispectral operators.

We shall recall a version of Darboux transform from [6]

Definition 1.1. — *We say that a plane W (or the corresponding wave function $\Psi_W(x,z)$, the tau-function τ_W) is a Darboux transformation of the vacuum (respectively - of the wave function $\Psi_0(x,z)$, the tau-function*

τ_0) iff there exist polynomials $f(z)$, $g(z)$ and differential operators $P(x,\partial_x)$, $Q(x,\partial_x)$ such that

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Psi_0(x,z),$$
$$\Psi_0(x,z) = \frac{1}{f(z)} Q(x,\partial_x)\Psi_W(x,z).$$

The Darboux transformation is called polynomial iff the operators $P(x,\partial_x)$ and $Q(x,\partial_x)$ have rational coefficients.

Obviously

$$Q(x,\partial_x)P(x,\partial_x)\Psi_0 = g(z)f(z)\Psi_0,$$

denoting the polynomial $g(z)f(z)$ by $h(z)$ and recalling that $\Psi_0 = e^{xz}$ we see that

$$Q(x,\partial_x)P(x,\partial_x) = h(\partial_x).$$

On the other hand the wave function Ψ_W is an eigen-function of the differential operator

$$L(x,\partial_x) = P(x,\partial_x)Q(x,\partial_x).$$

Notice that the operator L is a traditional Darboux transform of the operator $h(\partial_x)$, which justifies the terminology of the definition. We will also say that the operator L is *a polynomial Darboux transform* of the operator ∂_x.

We shall need a second definition of the polynomial Darboux transformation. In the above notation let the polynomial $h(\partial_x)$ factorize as:

$$h(\partial_x) = \prod_{j=1}^{m} (\partial_x - \lambda_j)^{d_j},$$

where λ_j are the different roots with multiplicities d_j. Then the kernel of $h(\partial_x)$ is given by

$$\ker h(\partial_x) = \bigoplus_{j=1}^{m} W_j,$$

where

$$W_j = \left\{e^{\lambda_j x}, xe^{\lambda_j x}, \ldots, x^{d_j - 1}e^{\lambda_j x}\right\}.$$

Definition 1.2. — *The Darboux transform is polynomial iff the kernel of P has the form*

$$\ker P = \bigoplus_{j=1}^{m} K_j,$$

where K_j is a linear subspace of W_j.

The equivalence of the two definitions can be found in [7]. Each nonzero element $f \in K_j$ will be called (after Wilson) *condition supported at λ_j*. The Darboux transform will be called *monomial* iff all the conditions are supported at one point. Finally we recall the bispectral involution b, which in this case maps the operators with polynomial coefficients in the x-variable into operators with polynomial coefficients in the z-variable by the formulas

$$b(\partial_x) = z, \qquad b(x) = \partial_z,$$

i.e. in this case b is the formal Fourier transform. It will be used when the differential operators are applied to Ψ_0 as follows:

$$\partial_x \Psi_0 = z\Psi_0, \qquad x\Psi_0 = \partial_z \Psi_0$$

We end this subsection with the following important result of G. Wilson [38]:

THEOREM 1.3. — *Any polynomial Darboux transform of ∂_x is a rank one bispectral operator and vice versa.*

This theorem is formulated by G. Wilson in a different terminology. See [22] for an exposition using Darboux transforms.

Following G. Wilson we will call the set of all planes $W \subset \mathrm{Gr}$ that are polynomial Darboux transforms of W_0 *the adelic Grassmannian* and denote it by $\mathrm{Gr}^{\mathrm{ad}}$. In another paper [39] G. Wilson proved that there is a bijection

$$\beta : \bigcup_{n \geqslant 0} C_n \to \mathrm{Gr}^{\mathrm{ad}}$$

between the union of all Calogero-Moser spaces and the adelic Grassmannian. Thus it is enough to prove Theorem 0.2.

1.3. $W_{1+\infty}$-algebra.

In this subsection we recall the definition of $W_{1+\infty}$, and some of its bosonic representations introduced in [5]. For more details see [28].

The algebra w_∞ of the additional symmetries of the KP-hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle

$$w_\infty \equiv \mathcal{D} = \mathrm{span}\{z^\alpha \partial_z^\beta \mid \alpha, \beta \in \mathbb{Z},\ \beta \geqslant 0\}.$$

It was introduced in [16, 33] and was extensively studied by many authors (see, e.g. [3, 14, 17, 32], *etc.*). Its unique central extension is denoted by $W_{1+\infty}$.

Denote by c the central element of $W_{1+\infty}$ and by $W(A)$ the image of $A \in \mathcal{D}$ under the natural embedding $\mathcal{D} \hookrightarrow W_{1+\infty}$ (as vector spaces). The algebra $W_{1+\infty}$ has a basis

$$c,\ J_k^l = W(-z^{l+k}\partial_z^l), \quad l, k \in \mathbb{Z},\ l \geqslant 0.$$

In [5] we constructed a family of highest weight modules of $W_{1+\infty}$. Here we need the most elementary one of them, for which the next theorem is an easy exercise.

Theorem 1.4. — *The function τ_0 satisfies the constraints*

$$J_k^l \tau_0 = 0, \quad k \geqslant 0,\ l \geqslant 0,$$

$$W\big(z^{-k} P_k(D_z) D_z^l\big)\tau_0 = 0, \quad k > 0,\ l \geqslant 0,$$

where $P_k(D_z) = \prod_{j=0}^{k-1}(D_z - j), \quad D_z = z\partial_z.$

The first constraint means that τ_0 is a highest weight vector with highest weight $\lambda(J_0^l) = 0$ of a representation of $W_{1+\infty}$ in the module

$$\mathcal{M}_0 = \mathrm{span}\Big\{ J_{k_1}^{l_1} \cdots J_{k_p}^{l_p} \tau_0 \mid k_1 \leqslant \cdots \leqslant k_p < 0 \Big\}.$$

One easily checks that the central charge $c = 1$. The second constraint yields that the module $\mathcal{M}_0$ is quasifinite, *i.e.* it is finite-dimensional in each level.

2. An adelic W-algebra.

The adelic W-algebra is a Lie algebra that we intend to introduce in analogy with $W_{1+\infty}$. Most of the definitions and constructions are similar to those of $W_{1+\infty}$. For that reason we list the facts but skip much of the arguments.

Instead of the Lie algebra w_∞ of regular operators on the circle we start with the Lie algebra $\mathcal{RD}$ of differential operators with rational coefficients on the complex line. We are going to use the following basis of $\mathcal{RD}$:

1) $z^{n+l}\partial_z^l, \quad n \in \mathbb{Z}, \quad l \geqslant 0;$

2) $(z-a)^{-n+l}\partial_z^l, \quad -n+l<0, \quad l \geqslant 0, \quad a \in \mathbb{C}-\{0\}.$

Usually we shall consider the elements from $\mathcal{RD}$ as differential operators with coefficients that are Laurent series in z^{-1} by expanding $(z-a)^{-n+l}$ around infinity. We would like to construct natural representations of $\mathcal{RD}$. We shall work with the space $\mathbb{V}$ where $v_k = z^k$. Obviously $\mathcal{RD}$ acts naturally on $\mathbb{V}$. Then we can associate with each operator $A \in \mathcal{RD}$ an infinite matrix having only finite number of diagonals below the principal one but having eventually infinite number above it. In other words the matrix $(a_{i,j})$, associated with A, has the property that $a_{i,j}=0$ for $i-j \gg 0$. The Lie algebra of such matrices will be denoted by a'_∞. It can be considered as a completion of the algebra a_∞ of matrices having only finite number of diagonals (see [29]). Now we explain how to construct representations in the fermionic Fock space $F^{(0)}$. We recall that in the case considered here $F^{(0)}$ consists of formal series of semi-infinite wedge monomials:

$$z^{i_0} \wedge z^{i_1} \wedge z^{i_2} \wedge \cdots,$$

with $i_0 < i_1 < \cdots$ and $i_k = k$ for $k \gg 0$. We can define the action of $A \in a'_\infty$ by the standard definition (see [29]. First for matrices with only finite number of entries define

$$\begin{aligned} r(A)(z^{i_0} \wedge z^{i_1} \wedge \cdots) &= Az^{i_0} \wedge z^{i_1} \wedge \cdots \\ &\quad + z^{i_0} \wedge Az^{i_1} \wedge \cdots \\ &\quad \cdots \end{aligned}$$

It is easy to check that if $A \in a'_\infty$ has no entries on the main diagonal $r(A)$ still makes sense, the image being infinite formal series. For matrices with infinite number of entries on the main diagonal the above definition is no longer meaningful. For that reason we need to modify it as follows. We put

$$\begin{aligned} &\hat{r}(E_{i,j}) = r(E_{i,j}) \quad \text{for} \quad i \neq j \quad \text{or} \quad i=j>0; \\ &\hat{r}(E_{i,i}) = r(E_{i,i}) - Id \quad \text{for} \quad i \leqslant 0. \end{aligned}$$

See [29] for more details.

This defines a representation of the central extension $a'_\infty \oplus \mathbb{C}c$. The corresponding central extension of the subalgebra $\mathcal{RD}$ of a'_∞ will be called *adelic W-algebra*. We will use the notation W^{ad}. The terminology and the notation are chosen to be similar to those of the adelic Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$. The main result of the present paper naturally connects the two objects.

We shall describe in some more details W^{ad}. By $W(A)$ we shall denote the image of the element $A \in \mathcal{RD}$ under the natural embedding $\mathcal{RD} \subset W^{\mathrm{ad}}$

(as vector spaces). Then for $a \in \mathbb{C}$, $l \geqslant 0$, $\quad n \in \mathbb{Z}$ put

$$J^l_n(a) = W(-(z-a)^{n+l}\partial^l_z). \tag{3}$$

For $a = 0$ we also shall use the notation $J^l_n = J^l_n(0)$. When a is fixed the above operators (3) together with the central charge c form a copy of $W_{1+\infty}$, which we shall denote by $W_{1+\infty}(a)$. Recall that $W_{1+\infty}(a)$ has a grading: the elements $J^l_n(a)$ have weight n. The elements with nonnegative grading are common for all a. In fact the common part is much larger: for all $n+l \geqslant 0$ the elements $J^l_n(a)$ are common. Thus we have the following basis for W^{ad}:

1) $J^l_n(0), \quad l \geqslant 0 \quad n+l \geqslant 0;$
2) $J^l_n(a), \quad n+l < 0, \quad a \neq 0;$
3) $c.$

In complete analogy to the case of $W_{1+\infty}$ we can construct a representation of W^{ad} in the Fock spaces using the vacuum. We formulate the needed properties in the bosonic picture.

Theorem 2.1. — *The tau-function τ_0 satisfies the following constrains*

1) $J^l_n\tau_0 = 0, \quad l \geqslant 0; \quad n \geqslant 0$
2) $W((z-a)^{-k}P_k((z-a)\partial_z))((z-a)\partial_z)^l\tau_0 = 0,$

where $P_k(u) = u(u-1)\cdots(u-k+1)$.

We set

$$W^{\mathrm{ad}}_- = span\{J^l_n, \quad a \in \mathbb{C}, \quad n < 0\}.$$

Then define the W^{ad}_--module $\mathcal{M}^{\mathrm{ad}}$ by

$$\mathcal{M}^{\mathrm{ad}} = span\{J^{l_1}_{n_1}(a_1)\cdots J^{l_m}_{n_m}(a_m)\tau_0, \}$$

where $n_j + l_j < 0$ for $a_j \neq 0$ and $n_j < 0$ for $a_j = 0$.

Corollary 2.2. — *The vector space $\mathcal{M}^{\mathrm{ad}}$ is a space of representation of the Lie algebra W^{ad}.*

Proof. — We need only to check that $J^l_n(J^{l_1}_{n_1}(a_1)\cdots J^{l_m}_{n_m}(a_m)\tau_0) \in \mathcal{M}^{\mathrm{ad}}$. We proceed by induction. Let $m = 1$. Then we have

$$J^l_nJ^{l_1}_{n_1}(a_1) = J^{l_1}_{n_1}(a_1)J^l_n + \sum b_kJ^{l_k}_{n_k}(a_1) + \sum J^{l_s}_{n_s},$$

where the sums are finite. Then this expression when applied to τ_0 by Corollary 2.2 obviously gives an element of $\mathcal{M}^{\mathrm{ad}}$. Proceeding by induction we can push the elements of the type J_n^l to the right and again use the theorem. $\square$

3. Tau-functions in the module $\mathcal{M}^{\mathrm{ad}}$.

In this section we prove the first part of Theorem 0.1. More precisely we will prove

THEOREM 3.1. — *If τ is a tau-function in the module $\mathcal{M}^{\mathrm{ad}}$ then it is polynomial Darboux transform of τ_0.*

Before giving the proof of the theorem we shall recall some results from [8] as well as their modifications for the case of the adelic W-algebra. We start with the commutation of the elements of W^{ad} with the vertex operator $X(t,z)$. First we consider elements from $W_{1+\infty}(0)$. The following result has been proved in [8]

LEMMA 3.2. — *Let $X(t,z)$ be the vertex operator. Then*

$$X(t,z)J_k^l = \Big(J_k^l + lJ_k^{l-1} + \delta_{k,0}\delta_{l,0} - z^{k+l}\partial_z^l\Big)X(t,z).$$

See [8] for the proof. In order to give the analog of the above relation for $J_n^l(a)$ when $a \neq 0$ we shall introduce some notation. Put

$$J_{n,-1}^l(a) = W(-z^{-1}(z-a)^{n+l}\partial_z^l). \tag{4}$$

The analog of Lemma 3.2 is

LEMMA 3.3. — *The commutation relation of the vertex operator $X(t,z)$ with $J_n^l(a)$ is*

$$X(t,z)J_n^l(a) = \Big(J_n^l(a) + lJ_{n+1,-1}^{l-1}(a) - (z-a)^{n+l}\partial_z^l\Big)X(t,z). \tag{5}$$

Proof. — The proof is a direct consequence of Lemma 3.2. We shall present $J_n^l(a)$ as infinite series:

$$J_n^l(a) = W(-(z-a)^{n+l}\partial_z^l) = W\Big(\sum_{j=0}^{\infty}\alpha_j z^{n+l-j}\partial_z^l\Big) = \sum_{j=0}^{\infty}\alpha_j J_{n-j}^l.$$

Now apply Lemma 3.2 to (3.4)and recall that $n+l<0$. We get

$$X(t,z)J^l_n(a)=\sum_{j=0}^{\infty}\alpha_j\big(J^l_{n-j}+lJ^{l-1}_{n-j}-z^{n-j+l}\partial^l_z\big)X(t,z).$$

Then using the notation (4) we can rewrite the last formula as (5). □

We also need the following result from [8] which we reformulate for the present situation.

Lemma 3.4. — *Let $\tau\in\mathcal{M}^{\mathrm{ad}}$. Then $\tau=u\tau_0$ where u is an element of the universal enveloping algebra of W_-^{ad} of the form:*

$$u=\sum b_{k,l,a}J^{l_1}_{-k_1}(a_1)\cdots J^{l_p}_{-k_p}(a_p)J^{l_{p+1}}_{-k_{p+1}}\cdots J^{l_{p+r}}_{-k_{p+r}},$$

where $l_j<k_j$.

Finally we need some facts about the so called *adjoint involution* [37]. It is an involution defined on the planes of Sato's Grassmannian. Introduce after [13] a non-degenerate bilinear form B on the space $\mathbb{V}$ of formal Laurent series in z^{-1} as follows

$$B(f,g)=-Res_{\infty}f(z)g(-z),\quad f,g\in\mathbb{V}.$$

If V is a plane in Gr then define aV to be the plane orthogonal to V with respect to the bilinear form B, *i.e.*

$$aV=\big\{g(z)\in\mathbb{V}\mid B(f,g)=0\quad\text{for all}\quad f\in\mathbb{V}\big\}.$$

Obviously the adjoint involution has meaning on related objects. On tau-functions it acts as

$$a(\tau_V(t_1,t_2,\ldots))=\tau_V(t_1,-t_2,\ldots(-1)^{n-1}t_n\ldots).$$

We can continue this action on the bosonic Fock space by linearity. We need to continue it also on elements of W^{ad}.

In [8] we proved that for $U\in W_{1+\infty}$ it is well defined demanding

$$a(U\tau)=a(U)a(\tau),\quad \tau\in B.$$

We will show that the same result holds for $U\in W^{\mathrm{ad}}$. The above definitions show that

$$a(J^0_k)=(-1)^{n-1}J^0_k.$$

Instead of working with the generators J^l_k of $W_{1+\infty}$ we will use fields (of dimension l) or generating functions:

$$J^l(z)=\sum_{k\in\mathbb{Z}}J^l_kz^{-k-l-1}.$$

In order to recall results from [8] we shall introduce other fields. Following [4] put

$$V^l(z) = \binom{2l}{l} \sum_{k=0}^{l} (-1)^k \binom{2l-k}{l} \partial^k J^{l-k}(z). \tag{7}$$

Then the modes V_k^l of the fields $V^l(z)$:

$$V^l(z) = \sum_{k \in \mathbb{Z}} V_k^l z^{-k-l-1}$$

obviously also form a basis (together with c) of $W_{1+\infty}$. In [8] we have proved the following:

Lemma 3.5. — *The adjoint involution a acts on V_k^l by the formula*

$$a(V^l(z)) = V^l(-z). \tag{8}$$

Now we want to continue the adjoint involution a to the adelic W-algebra. We have to define it only for $J_n^l(b)$ with $l+n<0$ and $b \neq 0$. But first we shall compute $a(J_n^l)$. We have

Lemma 3.6. — *The adjoint involution a acts on J_n^l by the formula*

$$a(J_n^l) = (-1)^{n+l+1}\Big(J_n^l + \sum_{s=1}^{l-1} \beta_{l,s}(n+l+1)\cdots(n+l-s+1)J_n^{l-s}\Big) \tag{9}$$

with some constants $\beta_{l,s}$ that do not depend on n.

Proof. — We will use induction on l. For $l=0$ we have

$$V^0(z) = J^0(z).$$

Hence in this case (9) holds. Suppose we have proved the formula for some l. From (7) we have

$$V_n^{l+1} = \sum_{s=0}^{l+1} \alpha_{l+1,s}(n+l-s+2)\cdots(n+l+1)J_n^{l+1-s} \tag{10}$$

with some constants $\alpha_{l+1,s}$ that do not depend on n and $\alpha_{l+1,0} \neq 0$. Also by (8) we have

$$a(V_n^{l+1}) = (-1)^{n+l+2} V_n^{l+1}. \tag{11}$$

Now plug in (11) the expression for V_n^{l+1} from (10). Then use linearity and the induction hypothesis. This proves (9). □

The above formula (9) allows us to write down an expression for $a(J^l_n(b))$ and hence to continue the involution on W^{ad}. Recall that $J^l_n(b)$ should be considered as infinite sum and for each summand a is already defined by (9).

Lemma 3.7. — *Let $n+l<0$ and $b\in\mathbb{C}$ be arbitrary. Then the adjoint involution is well defined for $a(J^l_n(b))$. More precisely we have*

(12)
$$a(J^l_n(b)) = (-1)^{k+l-1}\Big(J^l_n(-b)+\sum_{s=1}^{l}\gamma_{l,s}(n+l)\cdots(n+l-s+1)J^{l-s}_n(-b)\Big).$$

Proof. — We have

$$J^l_n(b) = \sum_{j=0}^{\infty}\mu_j W(-z^{n+l-j}\partial_z^l)$$

where μ_j are the coefficients of the expansion around infinity of $(z-b)^{n+l}$:

$$(z-b)^{n+l} = \sum_{j=0}^{\infty}\mu_j z^{n+l-j}.$$

Formula (9) gives

$$\begin{aligned} a(\sum_{j=0}^{\infty}\mu_j J^l_{n-j}) &= \sum_{j=0}^{\infty}\mu_j a(J^l_{n-j}) \\ &= \sum \mu_j(-1)^{n+l-j-1}\Big(J^l_{n-j}+\sum_{s=1}^{l}\beta_{l,s}(n+l) \\ &\qquad\qquad \cdots(n+l-s+1)J^{l-s}_{n-j}\Big). \end{aligned}$$

Let us consider the sum

$$\sum\mu_j(-1)^{n+l-j-1}(n+l)\cdots(n+l+s-1)J^{l-s}_{n-j}. \tag{13}$$

It is equivalent to

$$\sum\mu_j(-1)^{n+l-j-1}(n+l)\cdots(n+l+s-1)W(-z^{n-j+l-s}\partial_z^{l-s}).$$

In order to simplify the notation we drop $-\partial^{l-s}$ and the sign W. So we have

$$\begin{aligned} \sum\mu_j(-1)^{n+l-j-1}&(n+l)\cdots(n+l+s-1)z^{n-j+l-s} \\ &= (-1)^{n+l-1+s}\partial_z^s(z+b)^{k+l} \\ &= (-1)^{n+l+s-1}(n+l)\ldots(n+l-s+1)(z+b)^{n+l-s}. \end{aligned}$$

This shows that (13) is equivalent to

$$(-1)^{n+l+s-1} J_n^{l-s}(-b).$$

Thus we obtain (12). ☐

Now we are ready to give the proof of Theorem 3.1.

Proof. — of Theorem 3.1. Let $\tau_W = u\tau_0$ where u is an element of the universal enveloping algebra of the form from Lemma 3.4. We express the wave function $\Psi_W(t,z)$ in terms of u:

$$\Psi_W(t,z) = \frac{X(t,z)u\tau_0}{u\tau_0}\Big|_{t_1=x,t_2=\cdots=0}. \tag{14}$$

Using Lemma 3.2 and Lemma 3.3 we commute u and $X(t,z)$ to obtain

$$\Psi_W = \frac{U(t,z)X(t,z)\tau_0}{u\tau_0}\Big|_{t_1=x,t_2=\cdots=0},$$

where

$$\begin{aligned} U(t,z) = \sum b_{k,l,a} &\Big(J^{l_1}_{-k_1}(a_1) + l_1 J^{l_1-1}_{-k_1+1,-1} - (z-a_1)^{-k_1+l_1}\partial_z^{l_1}\Big) \\ &\cdots \Big(J^{l_p}_{-k_p}(a_p) + l_p J^{l_p-1}_{-k_p+1,-1} - (z-a_p)^{-k_p+l_p}\partial_z^{l_p}\Big) \\ &\cdots \Big(J^{l_{p+1}}_{-k_{p+1}} + l_{p+1} J^{l_p-1}_{-k_{p+1}} + \delta_{k_{p+1},0}\delta_{l_{p+1},0} - z^{-k_{p+1}+l_{p+1}}\partial_z^{l_{p+1}}\Big) \\ &\cdots \Big(J^{l_{p+r}}_{-k_{p+r}} + l_{p+r} J^{l_r-1}_{-k_{p+r}} + \delta_{k_{p+r},0}\delta_{l_{p+r},0} - z^{-k_{p+r}+l_{p+r}}\partial_z^{l_{p+r}}\Big). \end{aligned}$$

Now we use that

$$J^l_{-k}\big|_{t_1=x,t_2=t_3=\cdots=0} = x^k\delta_{l+1,k} \quad \text{if} \quad l<k.$$

The point is that there are no differentiations but only multiplications by powers of x. From the above formula we can derive for $J^l_{-k}(a)$ and for $J^l_{-k+1,-1}(a)$ the following ones:

$$\begin{aligned} J^l_{-k}(a)\big|_{t_1=x,t_2=t_3=\cdots=0} &= x^k\delta_{l+1,k}, \\ J^l_{-k+1,-1}(a)\big|_{t_1=x,t_2=t_3=0} &= 0. \end{aligned}$$

Both formulas follow from the expansion of the l.h.s. as infinite series and the fact that $l<k$.

In this way we get

$$\begin{aligned} &U(t,z)\big|_{t_1=x,t_2=t_3=\cdots=0} \\ &\quad = \sum b_{k,l,a}\Big(x^{k_1}\delta_{l_1+1,k_1} - (z-a_1)^{-k_1+l_1}\partial_z^{l_1}\Big) \\ &\qquad \cdots \Big(x^{k_{p+r}}(\delta_{l_{p+r}+1,k_{p+r}} + l_{p+r}\delta_{l_{p+r},k_{p+r}}) - z^{-k_{p+r}+l_{p+r}}\partial_z^{l_{p+r}}\Big) \\ &\quad = g(z)^{-1}P(x,z,\partial_z), \end{aligned}$$

where $g(z)$ is a polynomial with roots at a_j and 0 and P is a differential operator in z with polynomial coefficients in x and z. In the same way $u\big|_{t_1=x,t_2=t_3=\cdots=0} = p(x)$ is a polynomial in x. Thus (14) is equivalent to

$$\Psi_W = \frac{P(x,z,\partial_z)\Psi_0}{g(z)p(x)}$$

or using the bispectral involution we finally obtain:

$$\Psi_W = \frac{P_1(x,\partial_x)\Psi_0}{g(z)}, \tag{15}$$

where P_1 is an operator with rational coefficients.

We need also to express $\Psi_0(x,z)$ in terms of $\Psi_W(x,z)$. We shall use the adjoint involution a. From Lemma 3.7 we know that $a(\tau_W) = a(u)\tau_0$ is a tau-function in the module $\mathcal{M}^{\mathrm{ad}}$. The last formula (15) gives

$$\Psi_{aW}(x,z) = \frac{P_2(x,\partial_x)\Psi_0}{g_2(z)} \tag{16}$$

with some operator P_2 and a polynomial $g_2(z)$. We shall use the following simple lemma:

Lemma 3.8. — *If the wave functions Ψ_W and Ψ_V satisfy*

$$\Psi_W(x,z) = \frac{P(x,\partial_x)\Psi_V(x,z)}{g(z)}$$

then

$$\Psi_{aV}(x,z) = \frac{P^*(x,\partial_x)\Psi_{aW}(x,z)}{g(-z)}.$$

where P^* is the formal adjoint operator of P.

The lemma and (16) imply

$$\Psi_0(x,z) = \frac{P_2^*(x,\partial_x)\Psi_W}{g_2(z)},$$

which together with (15) gives the proof of the theorem. □

4. The planes of the adelic Grassmannian.

In this section we shall prove the inverse of Theorem 3.1.

Theorem 4.1. — *If a plane $W \in \mathrm{Gr}^{\mathrm{ad}}$ then the corresponding tau-function $\tau_W \in \mathcal{M}^{\mathrm{ad}}$.*

Proof. — To fix the notation let the Darboux transform be given by

$$\Psi_W(x,z) = \frac{P(x,\partial_x)\Psi_0(x,z)}{g(z)} \tag{17}$$

$$\Psi_0(x,z) = \frac{Q(x,\partial_x)\Psi_W(x,z)}{f(z)}, \tag{18}$$

where $Q \circ P = h(\partial_x)$ with some polynomial h. Let $\lambda_1,\dots,\lambda_m$ be the different points, where the conditions are supported. Then we can suppose that the polynomial h is:

$$h(z) = \prod_{j=0}^{m} (z-\lambda_j)^{d_j}.$$

Denote the degree of h by d. Let the number of the conditions supported at the point λ_j be r_j. Then

$$g(z) = \prod_{j=0}^{m} (z-\lambda_j)^{r_j}.$$

Put also $\deg g(z) = r = r_1 + \cdots r_m$. Let $\{\Phi_i\}\big|_{i=1,\dots,d}$ be the standard basis of $\ker h(z)$, *i.e.*

$$\{\Phi_i\} = \bigcup_{j=1}^{m} \{e^{\lambda_j x},\dots,x^{d_j-1}e^{\lambda_j x}\}.$$

Denote by $f_1,\dots,f_r$ the functions forming the kernel of the operator P, *i.e.* defining the Darboux transform (17)–(18). Then

$$f_l(x) = \sum_{i=1}^{d} a_{l,i}\Phi_i(x), \quad l = 1,\dots,r.$$

Denote by A the matrix formed by the above coefficients, *i.e.*

$$A = (a_{l,i}), \quad l = 1,\dots,r \quad , i = 1,\dots,d.$$

For any r-element subset $I\{i_1,\dots,i_r\} \subset \{1,\dots,d\}$ denote by A^I the following minor of A

$$A^I = (a_{l,i_k})\big|_{l,k=1,\dots,r}.$$

Put $\Phi = \{\Phi_{i_1,\dots,i_r}\}$ and

$$\Psi_I(x,z) = \frac{Wr(\Phi_I,\Psi_0)}{g(z)Wr(\Phi_I)}.$$

We need the following formula from [6]:

$$\Psi_W(x,z) = \frac{\sum_I detA^I Wr(\Phi_I)\Psi_I(x,z)}{\sum_I det^I Wr(\Phi_I)}.$$

Notice that for any I we have

$$Wr(\Phi_I, \Psi_0) = e^{x\sum \lambda_j r_j} P_I(x, \partial_x) = e^{x\sum \lambda_j r_j} \sum_{j=0}^{r} p_{I,j}(x)\partial_x^j e^{xz},$$

where $p_{I,j}(x)$ are polynomials. Also we have:

$$Wr(\Phi_I) = e^{x\sum \lambda_j r_j} q_I(x),$$

where the polynomial $q_I(x) = p_{I,r}(x)$. Notice that the exponential factor is the same everywhere. Then we have

$$\Psi_W = \frac{\sum_I detA^I P_I(x, \partial_x)e^{xz}}{g(z)\sum_I det^I q_I(x)}.$$

Among the subsets I there is one that corresponds to the set of following functions from the kernel of $h(\partial_x)$

$$\tilde{f}_1(x) = e^{\lambda_1 x}, \dots, \tilde{f}_{r_1}(x) = x^{r_1 - 1}e^{\lambda_1 x}$$

$$\vdots$$

$$\tilde{f}_{r-r_m+1} = e^{\lambda_m x}, \dots, \tilde{f}_r = x^{r_m - 1}e^{\lambda_r x}.$$

Denote this subset by I_0. Notice that $P_{I_0} = \sum_{j=0}^{r} \beta_j \partial_x^j$, where $b_j \in \mathbb{C}$, *i.e.* P_{I_0} is an operator with constant coefficients. It is easy to check that $P_{I_0} \equiv g(\partial_x)$. Introduce the matrix A_0 as follows. Let $I_0 = \left(i_1^0 < i_2^0, < \dots < i_r^0\ \right)$. Then let $A_0 = \left(a_{j,i}\delta_{j,i_k^0}\right)$. Now consider A as a deformation of A_0:

$$A(\epsilon) = \epsilon A + (1-\epsilon)A_0.$$

Obviously $a_{j,i}(\epsilon) = a_{j,i}$ if $i \in I_0$ and $a_{j,i} = \epsilon a_{j,i}$ otherwise.

Let us consider first the case when $detA_0^I \neq 0$. We can assume that $detA_0^I = 1$. Then the wave function $\Psi_{W(\epsilon)}$ reads:

$$\Psi_{W(\epsilon)} = \frac{(g(\partial_x) + \sum_{I \neq I_0} detA^I(\epsilon)P_I(x, \partial_x))\Psi_0}{g(z)(1 + \sum_{I \neq I_0} detA^I(\epsilon)q_I(x))}.$$

Using the bispectral involution and dividing by $g(z)$ we can write the above formula as

$$\Psi_{W(\epsilon)} = \frac{\left(1 + \sum detA^I \frac{\widetilde{P}_I(z,\partial_z)}{g(z)}\right)\Psi_0}{1 + \sum detA^I(\epsilon)q_I(x)}.$$

Notice that

$$\widetilde{P}_I(z, \partial_z) = \sum_{j=0}^{r} z^j p_{I,j}(\partial_z) = \sum \tilde{p}_{I,j}(z)\partial_z^j.$$

Here $\deg \tilde{p}_{I,j} \leqslant r$. We also have

$$\lim_{z\to\infty} \frac{\tilde{p}_{I,0}(z)}{g(z)} = p_{I,r}(0) = q_I(0).$$

So we can put

$$1 + \sum detA^I(\epsilon)\frac{\widetilde{P}_I(z,\partial_z)}{g(z)} = 1 + \sum detA^I(\epsilon)q_I(0) + \sum detA^I(\epsilon)Q_I(z,\partial_z),$$

where $W(Q_I(z,\partial_z)) \in W_-^{\mathrm{ad}}$.

Having in mind that $detA^I(\epsilon)$ is a polynomial in ϵ without a free term, we can write $\Psi_{W(\epsilon)}$ in the following form:

$$\Psi_{W(\epsilon)} = \frac{(1 + \sum_{j=1}^s \alpha_j\epsilon^j + \sum_{j=1}^s \beta_j\epsilon^j Q_j(z,\partial_z))\Psi_0(z,x)}{1 + \sum_{j=1}^s \alpha\epsilon^j + \sum_{j=1}^s \gamma_j\epsilon^j \tilde{q}_j(x)}$$

with $\tilde{q}_j(0) = 0$. Now expand the above expression around $\epsilon = 0$ and then use once again the bispectral involution to get rid of x-dependence. We get

$$\Psi_{W(\epsilon)} = (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z,\partial_z))\Psi_0.$$

The important fact here is that all the operators $P_j(z,\partial_z) \in W_-^{\mathrm{ad}}$. The standard basis of W_0 is given by $w_k = \partial_x^k\Psi_0 = z^k$, $k = 0, 1, \ldots$. We need to find expression for the basis of $W(\epsilon)$. We have

$$\partial_x^k\Psi_{W(\epsilon)} = (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z,\partial_z))w_k.$$

Using the boson-fermion correspondence σ we get the tau-function $\tau_{W(\epsilon)}$:

$$\begin{aligned}\tau_{W(\epsilon)} =& \sigma\Big((1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z,\partial_z))w_0 \wedge (1 + \sum_{j=1}^{\infty} \epsilon^j P_j(z,\partial_z))w_1 \wedge \cdots\Big)\\ &= \tau_0 + \epsilon r(P_1)\tau_0 + \epsilon^2\Big(r(P_2) + \frac{1}{2}r(P_1)^2 - \frac{1}{2}r(P_1^2)\Big)\tau_0 + \cdots\end{aligned}$$

Notice that the coefficients at the powers of ϵ are polynomials in $r(P_k^j)$, applied to τ_0. Hence all of them belong to the W^{ad}-module $\mathcal{M}^{\mathrm{ad}}$. We shall show that the entire series belong to it. Once again we need a formula from [6] - this time for the tau-function. We have

$$\tau_{W(\epsilon)} = \frac{1 + \sum_{I\neq I_0} detA^I(\epsilon)\Delta_I\tau_I}{1 + \sum_{I\neq I_0} detA^I(\epsilon)\Delta_I},$$

where $\Delta_I = Wr(\Phi_I)(0)$. We shall use the fact that the tau-function is defined up to a multiplicative constant. So multiplying $\tau_{W(\epsilon)}$ by the denominator, which is a polynomial in ϵ we get a polynomial in ϵ (the numerator), which, having a finite number of terms that belong to $\mathcal{M}^{\mathrm{ad}}$ by the above argument, itself belongs to $\mathcal{M}^{\mathrm{ad}}$ for all ϵ. In particular for $\epsilon = 1$ we get that $\tau_W \in \mathcal{M}^{\mathrm{ad}}$.

Next consider the case when $detA_0^I = 0$. Put $A(\zeta) = \zeta A + (1-\zeta)A_0$. For all but a finite number of values of ζ the expression $detA(\zeta) \neq 0$. The argument above shows that for this values of ζ the corresponding tau-function $\tau_{W(\zeta)} \in \mathcal{M}^{\mathrm{ad}}$. Again consider the numerator of $\tau_W(\zeta)$. It is a polynomial in ζ. Hence $\tau_{W(\zeta)} \in \mathcal{M}^{\mathrm{ad}}$ for all values of ζ. □

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