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Pavel M. BLEHER & Arno B.J. KUIJLAARS

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# INTEGRAL REPRESENTATIONS FOR MULTIPLE HERMITE AND MULTIPLE LAGUERRE POLYNOMIALS

by Pavel M. BLEHER & Arno B.J. KUIJLAARS (\*)

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## 1. Multiple orthogonal polynomials.

Multiple orthogonal polynomials are an extension of orthogonal polynomials that play a role in the random matrix ensemble with an external source

$$(1.1) \quad \frac{1}{Z_n} e^{-\text{Tr}(V(M)-AM)} dM$$

defined on  $n \times n$  Hermitian matrices, see [5, 6, 13]. Here  $A$  is a fixed  $n \times n$  Hermitian matrix and  $V : \mathbb{R} \rightarrow \mathbb{R}$  is a function with enough increase at  $\pm\infty$  so that the integral

$$Z_n = \int e^{-\text{Tr}(V(M)-AM)} dM$$

converges. Random matrices with external source were introduced and studied by Brézin and Hikami [8, 9, 10, 11, 12], and P. Zinn-Justin [21, 22]. Related recent work includes [2, 6, 15, 19].

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In what follows, we assume that  $A$  has  $m$  distinct eigenvalues  $a_1, \dots, a_m$  of multiplicities  $n_1, \dots, n_m$ . We consider  $m$  fixed and use multi-index notation  $\vec{n} = (n_1, \dots, n_m)$  and  $|\vec{n}| = n_1 + \dots + n_m$ .

The average characteristic polynomial  $P_{\vec{n}}(x) = \mathbb{E}[\det(\curvearrowright \mathbb{I} - \mathbb{M})]$  of the ensemble (1.1) is a monic polynomial of degree  $|\vec{n}|$  which satisfies for  $k = 1, \dots, m$ ,

$$(1.2) \quad \int_{-\infty}^{\infty} P_{\vec{n}}(x)x^j w_k(x)dx = 0, \quad j = 0, \dots, n_k - 1,$$

where

$$(1.3) \quad w_k(x) = e^{-iew(V(x)-a_kx)},$$

see [5]. The relations (1.2) characterize the polynomial  $P_{\vec{n}}$  uniquely. For  $A = 0$ , we have that  $P_{\vec{n}}$  is the usual orthogonal polynomial with respect to the weight  $e^{-V(x)}$ , which is a well-known fact from random matrix theory. For general  $m$ , the relations (1.2) are multiple orthogonality relations with respect to the weights (1.3) and the polynomial  $P_{\vec{n}}$  is called a multiple orthogonal polynomial of type II.

The multiple orthogonal polynomials of type I consist of a vector

$$(1.4) \quad (A_{\vec{n}}^{(1)}, A_{\vec{n}}^{(2)}, \dots, A_{\vec{n}}^{(m)}), \quad \deg A_{\vec{n}}^{(k)} \leq n_k - 1,$$

of polynomials such that the function

$$(1.5) \quad Q_{\vec{n}}(x) = \sum_{k=1}^m A_{\vec{n}}^{(k)}(x)w_k(x)$$

satisfies

$$(1.6) \quad \int_{-\infty}^{\infty} x^j Q_{\vec{n}}(x)dx = \begin{cases} 0, & j = 0, \dots, |\vec{n}| - 2, \\ 1, & j = |\vec{n}| - 1. \end{cases}$$

The polynomials  $A_{\vec{n}}^{(k)}$  are uniquely determined by the degree requirements (1.4) and the type I orthogonality relations (1.6).

By the Weyl integration formula, the random matrix ensemble (1.1) has the following joint eigenvalue distribution

$$\frac{1}{Z_n} \prod_{j=1}^n e^{-V(\lambda_j)} \left( \int e^{AU\Lambda U^*} dU \right) \prod_{j < k} (\lambda_j - \lambda_k)^2 d\lambda_1 d\lambda_2 \dots d\lambda_n$$

where  $dU$  is the normalized Haar measure on the unitary group  $U(n)$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Using the confluent form of the Harish-Chandra / Itzykson-Zuber formula [14, 16] to evaluate the integral  $\int e^{AU\Lambda U^*} dU$ , we find that the joint eigenvalue distribution can be written as

$$(1.7) \quad \frac{1}{Z'_n} \det(f_j(\lambda_k))_{1 \leq j, k \leq n} \det(g_j(\lambda_k))_{1 \leq j, k \leq n} d\lambda_1 d\lambda_2 \dots d\lambda_n,$$

where  $f_j(x) = x^{j-1}$  and  $g_1, \dots, g_n$  are the functions

$$(1.8) \quad x^j e^{-(V(x)-a_k x)}, \quad k = 1, \dots, m, \quad j = 0, 1, \dots, n_k - 1$$

taken in some arbitrary order. Then (1.7) is a biorthogonal ensemble in the sense of Borodin [7], which in particular implies that the eigenvalue point process is determinantal, that is, there is a kernel  $K(x, y)$  such that the  $k$  point correlation functions have determinantal form

$$(1.9) \quad \det (K(\lambda_i, \lambda_j))_{1 \leq i, j \leq k}.$$

The multiple orthogonal polynomials of types II and I can be used to biorthogonalize the functions  $f_j$  and  $g_j$  and to give an explicit formula for  $K$ . Choose a sequence of multi-indices  $\vec{n}_0, \vec{n}_1, \dots, \vec{n}_n = \vec{n}$  such that  $|\vec{n}_j| = j$  and  $\vec{n}_j \leq \vec{n}_{j+1}$  and define

$$p_j(x) = P_{\vec{n}_j}(x), \quad q_j(y) = Q_{\vec{n}_{j+1}}.$$

Then  $p_{j+1} \in \text{span}\{f_1, \dots, f_j\}$ , and  $q_{j+1} \in \text{span}\{g_1, \dots, g_j\}$  for a suitable ordering of the functions (1.8). In addition we have the biorthogonality

$$\int_{-\infty}^{\infty} p_i(x)q_j(x)dx = \delta_{i,j}.$$

As in [7] it then follows that

$$(1.10) \quad K(x, y) = \sum_{j=0}^{n-1} p_j(x)q_j(y).$$

By the Christoffel-Darboux formula for multiple orthogonal polynomials [13] the kernel  $K$  satisfies

$$(1.11) \quad (x - y)K(x, y) = P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{k=1}^m \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}} P_{\vec{n}-\vec{e}_k}(x)Q_{\vec{n}+\vec{e}_k}(y)$$

where

$$(1.12) \quad h_{\vec{n}}^{(k)} = \int_{-\infty}^{\infty} P_{\vec{n}}(x)x^{n_k}w_k(x)dx$$

and  $\vec{e}_k$  is the  $k$ th standard basis vector in  $\mathbb{R}^m$ .

In the following two sections we study two special cases related to multiple Hermite polynomials and multiple Laguerre polynomials. These cases correspond to the random matrix model (1.1) with  $V(M) = \frac{1}{2}M^2$  and  $V(M) = M$  respectively (in the latter case we restrict to positive definite matrices). Brézin and Hikami [9] and Baik, Ben Arous, and P  ch   [4] gave double integral representations for the correlation kernels for these cases. We will derive integral representations for the multiple Hermite and multiple Laguerre polynomials, and use that to show that the kernels agree with the multiple orthogonal polynomial kernel (1.11).

## 2. Multiple Hermite polynomials.

The special case  $V(M) = \frac{1}{2}M^2$  was considered in a series of papers of Brézin and Hikami, [8, 9, 10, 11, 12]. This case corresponds to  $M = H + A$  where  $H$  is a random matrix from the GUE ensemble  $(1/Z_n)e^{-\frac{1}{2}\text{Tr} H^2} dH$  and  $A$  is fixed as before. In [9] the following expression for the kernel was derived

$$(2.1) \quad K(x, y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k} \frac{1}{s-t}$$

where  $\Gamma$  is a closed contour encircling the points  $a_1, \dots, a_m$  once in the positive direction, and the path from  $-i\infty$  to  $i\infty$  does not intersect  $\Gamma$ , see also Johansson [17].

When  $V(x) = \frac{1}{2}x^2$ , the multiple orthogonal polynomials are called multiple Hermite polynomials, since they clearly generalize the usual Hermite polynomials [1, 3, 20]. We derive integral representations for the multiple Hermite polynomials of type I and type II, which resemble the integral representation (2.1) of the kernel.

### 2.1. Multiple Hermite polynomials of type II.

The multiple Hermite polynomial  $P_{\vec{n}}$  is the monic polynomial of degree  $|\vec{n}|$  that satisfies (1.2) with  $w_k(x) = e^{-\frac{1}{2}x^2 + a_k x}$ ,  $k = 1, \dots, m$ .

**THEOREM 2.1.** — *The multiple Hermite polynomials of type II has the integral representation*

$$(2.2) \quad P_{\vec{n}}(x) = \frac{1}{\sqrt{2\pi i}} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}(s-x)^2} \prod_{k=1}^m (s-a_k)^{n_k} ds.$$

*Proof.* — Let us denote the left-hand side of (2.2) by  $P(x)$ . After performing the change of variables  $s = t + x$ , we get

$$(2.3) \quad P(x) = \frac{1}{\sqrt{2\pi i}} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}t^2} \prod_{k=1}^m (t+x-a_k)^{n_k} dt,$$

which shows that  $P$  is a polynomial of degree  $|\vec{n}|$  with leading coefficient

$$\frac{1}{\sqrt{2\pi i}} \int_{-i\infty}^{i\infty} e^{\frac{1}{2}t^2} dt = 1.$$

So  $P$  is a monic polynomial.

Now we use (2.3) to compute for  $k = 1, \dots, m$ , and  $j = 0, 1, \dots$ ,

$$(2.4) \quad \int_{-\infty}^{\infty} P(x)x^j e^{-\frac{1}{2}x^2+a_kx} dx = \frac{e^{\frac{1}{2}a_k^2}}{\sqrt{2\pi i}} \int_{-\infty}^{i\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(t^2-(x-a_k)^2)} x^j \prod_{l=1}^m (t+x-a_l)^{n_l} dx dt.$$

Switching to polar coordinates  $x - a_k = r \cos \theta$ ,  $t = ir \sin \theta$ , we find that the right-hand side of (2.4) is equal to

$$(2.5) \quad \frac{e^{\frac{1}{2}a_k^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}r^2} r^{n_k+1} \left[ \int_0^{2\pi} (a_k+r \cos \theta)^j e^{in_k\theta} \prod_{l \neq k} (re^{i\theta}+a_k-a_l)^{n_l} d\theta \right] dr.$$

The  $\theta$ -integral vanishes for  $j = 0, \dots, n_k - 1$ , since the integrand can be written as a linear combination of  $e^{ip\theta}$  with integer  $p \geq n_k - j$ . Hence  $P$  is the multiple Hermite polynomial of type II and (2.2) follows.  $\square$

*Remark 2.2.* — Evaluating (2.5) for  $j = n_k$  we find that the  $\theta$ -integral is  $2\pi \left(\frac{r}{2}\right) \prod_{l \neq k} (a_k - a_l)^{n_l}$  and

$$(2.6) \quad \begin{aligned} h_{\vec{n}}^{(k)} &= \int_{-\infty}^{\infty} P_{\vec{n}}(x)x^{n_k} e^{-\frac{1}{2}x^2+a_kx} dx \\ &= \sqrt{2\pi}(n_k)! e^{\frac{1}{2}a_k^2} \prod_{l \neq k} (a_k - a_l)^{n_l}. \end{aligned}$$

### 2.2. Multiple Hermite polynomials of type I.

The multiple Hermite polynomials of type I are polynomials  $A_{\vec{n}}^{(k)}$  as in (1.4) such that the linear form

$$(2.7) \quad Q_{\vec{n}}(x) = \sum_{k=1}^m A_{\vec{n}}^{(k)}(x) e^{-\frac{1}{2}x^2+a_kx}$$

satisfies (1.6).

**THEOREM 2.3.** — *The multiple Hermite polynomials of type I have the integral representation*

$$(2.8) \quad A_{\vec{n}}^{(k)}(x) e^{-\frac{1}{2}x^2+a_kx} = \frac{1}{\sqrt{2\pi}2\pi i} \oint_{\Gamma_k} e^{-\frac{1}{2}(t-x)^2} \prod_{l=1}^m (t - a_l)^{-n_l} dt$$

where  $\Gamma_k$  is a closed contour encircling  $a_k$  once in the positive direction, but not enclosing any of the other points  $a_l, l \neq k$ .

In addition the function  $Q_{\vec{n}}$  of (2.7) has the integral representation

$$(2.9) \quad Q_{\vec{n}}(x) = \frac{1}{\sqrt{2\pi}2\pi i} \oint_{\Gamma} e^{-\frac{1}{2}(t-x)^2} \prod_{l=1}^m (t - a_l)^{-n_l} dt$$

where  $\Gamma$  is a closed contour encircling  $a_1, \dots, a_m$  once in the positive direction.

*Proof.* — By the residue theorem, we have that the right-hand side of (2.8) is equal to

$$(2.10) \quad \frac{1}{\sqrt{2\pi}} \frac{1}{(n_k - 1)!} \left( \frac{d}{dt} \right)^{n_k - 1} \left[ e^{-\frac{1}{2}(t-x)^2} \prod_{l \neq k} (t - a_l)^{-n_l} \right] \Big|_{t=a_k}.$$

It is easy to see that (2.10) has the form  $A_k(x)e^{-\frac{1}{2}x^2 + a_k x}$  where  $A_k$  is a polynomial of degree  $n_k - 1$ . Define the linear form

$$(2.11) \quad Q(x) = \sum_{k=1}^m A_k(x)e^{-\frac{1}{2}x^2 + a_k x} = \frac{1}{\sqrt{2\pi}2\pi i} \oint_{\Gamma} e^{-\frac{1}{2}(t-x)^2} \prod_{l=1}^m (t - a_l)^{-n_l} dt$$

where  $\Gamma$  encloses all the points  $a_j, j = 1, \dots, m$ , once in the positive direction.

Then

$$\int_{-\infty}^{\infty} x^j Q(x) dx = \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-\frac{1}{2}(t-x)^2} dx \right) \prod_{l=1}^m (t - a_l)^{-n_l} dt.$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-\frac{1}{2}(t-x)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (y + t)^j e^{-\frac{1}{2}y^2} dy$$

is a monic polynomial of degree  $j$  in the variable  $t$ , we find

$$(2.12) \quad \int_{-\infty}^{\infty} x^j Q(x) dx = \frac{1}{2\pi i} \oint_{\Gamma} \pi_j(t) \prod_{l=1}^m (t - a_l)^{-n_l} dt,$$

where  $\pi_j$  is a monic polynomial of degree  $j$ . Deforming the contour  $\Gamma$  to infinity, and using the fact that the integrand is  $s^{j - |\vec{n}|} + O(s^{j - |\vec{n}| - 1})$  as  $s \rightarrow \infty$ , we find that

$$\int_{-\infty}^{\infty} x^j Q(x) dx = 0 \quad \text{for } j = 0, \dots, |\vec{n}| - 2$$

and

$$\int_{-\infty}^{\infty} x^j Q(x) dx = 1 \quad \text{for } j = |\vec{n}| - 1.$$

This shows that  $Q = Q_{\vec{n}}$  and  $A_k = A_{\vec{n}}^{(k)}$  so that (2.8) and (2.9) follow.  $\square$

### 2.3. The multiple Hermite kernel.

Let us now show that the Brézin Hikami kernel (2.1) agrees with the multiple Hermite kernel (1.11). To that end we compute  $\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}$  for the kernel (2.1) in two ways.

First we have

$$\begin{aligned} & \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} \\ &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k} \frac{-s+x+t-y}{s-t} \\ &= (x-y)K(x,y) - \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k}. \end{aligned}$$

The last double integral factors into a product of two single integrals, which by (2.2) and (2.9) leads to

$$(2.13) \quad \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} = (x-y)K(x,y) - P_{\vec{n}}(x)Q_{\vec{n}}(y).$$

For the second way we evaluate  $\frac{\partial K}{\partial x}$  by noting that  $\partial_x e^{\frac{1}{2}(s-x)^2} = -\partial_s e^{\frac{1}{2}(s-x)^2}$ , and integrating by parts the  $s$ -integral

$$\begin{aligned} & \frac{\partial K}{\partial x} = -\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt \frac{\partial}{\partial s} e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k} \frac{1}{s-t} \\ (2.14) \quad &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \\ & \quad \times \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k} \frac{1}{s-t} \left\{ \sum_{k=1}^m \frac{n_k}{s-a_k} - \frac{1}{s-t} \right\}. \end{aligned}$$

Similarly, we use  $\partial_y e^{-\frac{1}{2}(t-y)^2} = -\partial_t e^{-\frac{1}{2}(t-y)^2}$ , and apply integration by parts to the  $t$ -integral, to obtain

$$\begin{aligned} (2.15) \quad \frac{\partial K}{\partial y} &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \\ & \quad \times \prod_{k=1}^m \left( \frac{s-a_k}{t-a_k} \right)^{n_k} \frac{1}{s-t} \left\{ -\sum_{k=1}^m \frac{n_k}{t-a_k} + \frac{1}{s-t} \right\}. \end{aligned}$$

We add (2.14) and (2.15)



$$\begin{aligned}
 \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} &= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \\
 &\quad \times \prod_{j=1}^m \left( \frac{s-a_j}{t-a_j} \right)^{n_j} \frac{1}{s-t} \sum_{k=1}^m \left( \frac{n_k}{s-a_k} - \frac{n_k}{t-a_k} \right) \\
 (2.16) \quad &= - \sum_{k=1}^m n_k \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_{\Gamma} dt e^{\frac{1}{2}(s-x)^2 - \frac{1}{2}(t-y)^2} \\
 &\quad \times \prod_{j \neq k} \left( \frac{s-a_j}{t-a_j} \right)^{n_j} \frac{(s-a_k)^{n_k-1}}{(t-a_k)^{n_k+1}}.
 \end{aligned}$$

For every  $k$ , the double integral in (2.16) factors into a product of two single integrals, which by (2.2) and (2.9) are given in terms of multiple Hermite polynomials. It leads to

$$(2.17) \quad \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} = - \sum_{k=1}^m n_k P_{\bar{n}-\bar{e}_k}(x) Q_{\bar{n}+\bar{e}_k}(y).$$

From (2.13) and (2.17) we get

$$(x-y)K(x,y) = P_{\bar{n}}(x)Q_{\bar{n}}(y) - \sum_{k=1}^m n_k P_{\bar{n}-\bar{e}_k}(x)Q_{\bar{n}+\bar{e}_k}(y),$$

which agrees with (1.11) since

$$(2.18) \quad n_k = \frac{h_{\bar{n}}^{(k)}}{h_{\bar{n}-\bar{e}_k}^{(k)}}$$

because of (2.6).

### 3. Multiple Laguerre polynomials.

Complex Gaussian sample covariance matrices have a distribution

$$(3.1) \quad \frac{1}{Z_n} e^{-\text{Tr}(\Sigma^{-1}M)} (\det M)^p dM$$

defined on  $n \times n$  positive definite Hermitian matrices  $M$ . The matrix  $M$  arises as  $M = XX^H$  where  $X$  is an  $n \times (n+p)$ , matrix whose independent columns are Gaussian distributed with covariance matrix  $\Sigma$ . Here  $p$  is a non-negative integer. The distribution (3.1) is also called a Wishart ensemble. We assume that  $\Sigma^{-1}$  has eigenvalues  $\beta_1, \dots, \beta_m > 0$  with respective multiplicities  $n_1, \dots, n_m$ .

Writing  $\Sigma^{-1} = I - A$ , we see that (3.1) takes the form (1.1) with  $V(M) = M$ , but restricted to positive definite Hermitian matrices. It follows that the ensemble (3.1) can be described with multiple orthogonal polynomials, which in this case are multiple Laguerre polynomials [1, 3, 18, 20]. (To be precise, they are called multiple Laguerre II in [3, 20] to distinguish them from another generalization of Laguerre polynomials, called multiple Laguerre I.)

The eigenvalues of  $M$  follow a determinantal point process on  $(0, \infty)$  with kernel  $K(x, y)$  given by (1.11)

$$(3.2) \quad (x - y)K(x, y) = P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{k=1}^m \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}} P_{\vec{n}-\vec{e}_k}(x)Q_{\vec{n}+\vec{e}_k}(y)$$

where now  $P_{\vec{n}}$  is the type II multiple Laguerre polynomial and  $Q_{\vec{n}}(x) = \sum_{k=1}^m A_{\vec{n}}^{(k)}(x)x^p e^{-\beta_k(x)}$  is the linear form involving the type I multiple Laguerre polynomials  $A_{\vec{n}}^{(k)}$ .

Baik, Ben Arous and P ech e [4] gave a double integral representation for the correlation kernel

$$(3.3) \quad K(x, y) = \frac{1}{(2\pi i)^2} \oint_{\Sigma} ds \oint_{\Gamma} dt e^{xs-yt} \left(\frac{t}{s}\right)^{|\vec{n}|+p} \prod_{k=1}^m \left(\frac{s-\beta_k}{t-\beta_k}\right)^{n_k} \frac{1}{s-t}$$

where  $\Sigma$  and  $\Gamma$  are disjoint closed contours both oriented counterclockwise such that  $\Sigma$  encloses 0 and lies in  $\{s \in \mathbb{C} \mid \Re s < \min \Re \beta_k\}$  and  $\Gamma$  encloses the points  $\beta_1, \dots, \beta_m$  and lies in the right half-plane.

In view of (3.3) and our experience with multiple Hermite polynomials we expect integral representations for the multiple Laguerre polynomials as well. We will see that this is indeed the case, and we use this to study the connection between the kernels (3.2) and (3.3). It will turn out that the two kernels are equal up to a multiplicative factor  $x^p y^{-p}$ . However, this difference does not affect the correlation functions (1.9).

### 3.1. Multiple Laguerre polynomials of type II.

The multiple Laguerre polynomial of type II is a monic polynomial  $P_{\vec{n}}$  of degree  $|\vec{n}|$  such that

$$(3.4) \quad \int_0^\infty P_{\vec{n}}(x)x^{j+p}e^{-\beta_k x} dx = 0, \quad k = 1, \dots, m, \quad j = 0, \dots, n_k - 1.$$

THEOREM 3.1. — *The multiple Laguerre polynomial of type II has the integral representation*

$$(3.5) \quad P_{\vec{n}}(x) = \frac{(|\vec{n}| + p)! x^{-p}}{2\pi i \prod_{k=1}^m (-\beta_k)^{n_k}} \oint_{\Sigma} e^{xs} s^{-|\vec{n}|-p-1} \prod_{k=1}^m (s - \beta_k)^{n_k} ds$$

where  $\Sigma$  is a closed contour around 0 oriented counterclockwise, which does not enclose any of the  $\beta_k$ 's.

*Proof.* — Denote the right-hand side of (3.5) by  $P(x)$  and write  $C = (|\vec{n}| + p)! / (2\pi i \prod_{k=1}^m (-\beta_k)^{n_k})$ . Then

$$(3.6) \quad P(x) = C x^{-p} \sum_{j=0}^{\infty} \frac{x^j}{j!} \oint_{\Sigma} s^{j-|\vec{n}|-p-1} \prod_{k=1}^m (s - \beta_k)^{n_k} ds.$$

All terms in (3.6) with  $j \geq |\vec{n}| + 1$  vanish by Cauchy's theorem, as well as all terms with  $j \leq p - 1$ , as we can see by deforming the contour  $\Sigma$  to infinity. It follows that  $P$  is a polynomial of degree  $|\vec{n}|$ , whose leading coefficient is

$$C \frac{1}{(|\vec{n}| + p)!} \oint_{\Sigma} s^{-1} \prod_{k=1}^m (s - \beta_k)^{n_k} ds = C \frac{2\pi i}{(|\vec{n}| + p)!} \prod_{k=1}^m (-\beta_k)^{n_k} = 1.$$

Hence  $P$  is a monic polynomial of degree  $|\vec{n}|$ .

We now verify the orthogonality conditions (3.4). Take  $k = 1, \dots, m$  and let  $j = 0, \dots, n_k - 1$ . Then

$$\int_0^{\infty} P(x) x^{j+p} e^{-\beta_k x} dx = C \oint_{\Sigma} \left( \int_0^{\infty} x^j e^{(s-\beta_k)x} dx \right) s^{-|\vec{n}|-p-1} \prod_{l=1}^m (s - \beta_l)^{n_l} ds$$

where we have assumed that  $\Sigma$  is so that  $\Re s < \beta_k$  for every  $s \in \Sigma$ . The inner integral is  $j!(\beta_k - s)^{-j-1}$ , so that

$$\int_0^{\infty} P(x) x^{j+p} e^{-\beta_k x} dx = C j! \oint_{\Sigma} s^{-|\vec{n}|-p-1} (s - \beta_k)^{n_k - j - 1} \prod_{l \neq k} (s - \beta_l)^{n_l} ds.$$

The integrand in the last integral has no pole at  $\beta_k$ , since  $j \leq n_k - 1$ . At infinity the integrand behaves like  $s^{-j-p-2}$ . So by deforming the contour  $\Sigma$  to infinity, we conclude that the integral is zero. The conclusion is that  $P$  is the multiple Laguerre polynomial of type II and (3.5) follows.  $\square$

### 3.2. Multiple Laguerre polynomials of type I.

The multiple Laguerre polynomials of type I  $A_{\vec{n}}^{(k)}$ ,  $k = 1, \dots, m$ , have degrees

$$(3.7) \quad \deg A_{\vec{n}}^{(k)} \leq n_k - 1, \quad k = 1, \dots, m$$

and are such that

$$Q_{\vec{n}}(x) = \sum_{k=1}^m A_{\vec{n}}^{(k)} x^p e^{-\beta_k x}$$

satisfies

$$\int_0^\infty x^j Q_{\vec{n}}(x) dx = \begin{cases} 0, & j = 0, \dots, |\vec{n}| - 2, \\ 1, & j = |\vec{n}| - 1. \end{cases}$$

There is also an integral representation for the multiple Laguerre polynomials of type I.

**THEOREM 3.2.** — *The multiple Laguerre polynomials of type I have the integral representation*

$$(3.9) \quad A_{\vec{n}}^{(k)}(x) x^p e^{-\beta_k x} = -\frac{\prod_{l=1}^m (-\beta_l)^{n_l} x^p}{2\pi i (|\vec{n}| + p - 1)!} \oint_{\Gamma_k} e^{-xt} t^{|\vec{n}|+p-1} \prod_{l=1}^m (t - \beta_l)^{-n_l} dt$$

where  $\Gamma_k$  is a closed contour around  $\beta_k$ , which does not enclose 0 nor any of the other points  $\beta_l$ ,  $l \neq k$ .

In addition the function  $Q_{\vec{n}}$  has the integral representation

$$(3.10) \quad Q_{\vec{n}}(x) = -\frac{\prod_{l=1}^m (-\beta_l)^{n_l} x^p}{2\pi i (|\vec{n}| + p - 1)!} \oint_{\Gamma} e^{-xt} t^{|\vec{n}|+p-1} \prod_{l=1}^m (t - \beta_l)^{-n_l} dt$$

where  $\Gamma$  is a closed contour around  $\beta_1, \dots, \beta_m$ , but which does not enclose 0.

*Proof.* — Only the pole at  $\beta_k$  contributes to the integral in (3.9). By the residue theorem, we have that the right-hand side of (3.9) is equal to

$$\text{const } x^p \left( \frac{d}{dt} \right)^{n_k - 1} \left[ e^{-xt} t^{|\vec{n}|+p-1} \prod_{l \neq k} (t - \beta_l)^{-n_l} \right] \Big|_{t=\beta_k}$$

which is easily seen to be of the form  $A_k(x) x^p e^{-\beta_k x}$  where  $A_k$  is a polynomial of degree  $n_k - 1$ .

Let

$$Q(x) = \sum_{k=1}^m A_k(x) x^p e^{-\beta_k x}$$

which is then equal to the right-hand side of (3.10). We may assume that  $\Gamma$  is entirely in the right half-plane. Then

$$\int_0^\infty x^j Q(x) dx = -\frac{\prod_{l=1}^m (-\beta_l)^{n_l}}{2\pi i (|\vec{n}| + p - 1)!} \oint_\Gamma dt \int_0^\infty dx x^{j+p} e^{-xt} t^{|\vec{n}|+p-1} \times \prod_{l=1}^m (t - \beta_l)^{-n_l}.$$

The  $x$ -integral is  $(j + p)! t^{-j-p-1}$ , so that

$$(3.11) \quad \int_0^\infty x^j Q(x) dx = -\frac{\prod_{l=1}^m (-\beta_l)^{n_l} (j + p)!}{2\pi i (|\vec{n}| + p - 1)!} \oint_\Gamma t^{|\vec{n}|-j-2} \prod_{l=1}^m (t - \beta_l)^{-n_l} dt.$$

Assuming  $j \leq |\vec{n}| - 2$ , we can deform  $\Gamma$  to infinity without picking up a residue contribution at  $t = 0$ . The integrand behaves like  $t^{-j-2}$  at infinity, and so (3.11) vanishes for  $j \leq |\vec{n}| - 2$ . For  $j = |\vec{n}| - 1$ , we pick up a residue contribution at  $t = 0$ , and the result is that

$$(3.12) \quad \int_0^\infty x^{|\vec{n}|-1} Q(x) dx = 1.$$

Thus  $Q$  satisfies the type I multiple orthogonality conditions (3.8) and the theorem follows. □

### 3.3. The multiple Laguerre kernel.

We finally compare the representations (3.2) and (3.3) of the kernel.

We start from the double integral (3.3) and evaluate  $xK(x, y)$  using an integration by parts on the  $s$ -integral. The result is

$$(3.13) \quad xK(x, y) = \frac{1}{(2\pi i)^2} \oint_\Sigma ds \oint_\Gamma dt e^{xs-yt} \left(\frac{t}{s}\right)^{|\vec{n}|+p} \prod_{l=1}^m \left(\frac{s - \beta_l}{t - \beta_l}\right)^{n_l} \times \frac{1}{s - t} \left\{ \frac{|\vec{n}| + p}{s} - \sum_{k=1}^m \frac{n_k}{s - \beta_k} + \frac{1}{s - t} \right\}.$$

Similarly, after integration by parts on the  $t$ -integral,

$$(3.14) \quad yK(x, y) = \frac{1}{(2\pi i)^2} \oint_\Sigma ds \oint_\Gamma dt e^{xs-yt} \left(\frac{t}{s}\right)^{|\vec{n}|+p} \prod_{l=1}^m \left(\frac{s - \beta_l}{t - \beta_l}\right)^{n_l} \times \frac{1}{s - t} \left\{ \frac{|\vec{n}| + p}{t} - \sum_{k=1}^m \frac{n_k}{t - \beta_k} + \frac{1}{s - t} \right\}.$$

Hence

$$\begin{aligned}
 &(x - y)K(x, y) \\
 &= \frac{1}{(2\pi i)^2} \oint_{\Sigma} ds \oint_{\Gamma} dt e^{xs-yt} \left(\frac{t}{s}\right)^{|\vec{n}|+p} \prod_{l=1}^m \left(\frac{s - \beta_l}{t - \beta_l}\right)^{n_l} \\
 &\qquad \qquad \qquad \times \left\{ -\frac{|\vec{n}| + p}{st} + \sum_{k=1}^m \frac{n_k}{(s - \beta_k)(t - \beta_k)} \right\} \\
 &= -\frac{|\vec{n}| + p}{(2\pi i)^2} \oint_{\Sigma} ds \oint_{\Gamma} dt e^{xs-yt} \frac{t^{|\vec{n}|+p-1}}{s^{|\vec{n}|+p+1}} \prod_{l=1}^m \left(\frac{s - \beta_l}{t - \beta_l}\right)^{n_l} \\
 &\quad + \sum_{k=1}^m \frac{n_k}{(2\pi i)^2} \oint_{\Sigma} ds \oint_{\Gamma} dt e^{xs-yt} \left(\frac{t}{s}\right)^{|\vec{n}|+p} \prod_{l \neq k} \left(\frac{s - \beta_l}{t - \beta_l}\right)^{n_l} \frac{(s - \beta_k)^{n_k-1}}{(t - \beta_k)^{n_k+1}}.
 \end{aligned}$$

Now we have  $m + 1$  double integrals and they all factor into products of two single integrals of the forms (3.5) and (3.10). The result is that

(3.15)

$$(x - y)K(x, y) = x^p y^{-p} \left( P_{\vec{n}}(x)Q_{\vec{n}}(y) - \sum_{k=1}^m n_k \frac{|\vec{n}| + p}{\beta_k^2} P_{\vec{n}-\vec{e}_k}(x)Q_{\vec{n}+\vec{e}_k}(y) \right).$$

It can be shown that

$$n_k \frac{|\vec{n}| + p}{\beta_k^2} = \frac{h_{\vec{n}}^{(k)}}{h_{\vec{n}-\vec{e}_k}^{(k)}}$$

so that (3.15) agrees with (3.2) up to the factor  $x^p y^{-p}$ . However, this factor is not essential since it does not change the correlation functions (1.9). Hence (3.2) and (3.3) are essentially the same.

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Pavel M. BLEHER,  
Indiana University-Purdue University Indianapolis  
Department of Mathematical Sciences  
402 N. Blackford St.  
Indianapolis, IN 46202 (USA)  
bleher@math.iupui.edu  
Arno B.J. KUIJLAARS,  
Katholieke Universiteit Leuven  
Department of mathematics  
Celestijnenlaan 200 B  
3001 Leuven (Belgium)  
arno@wis.kuleuven.ac.be