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# WEAK SOLUTIONS TO THE COMPLEX HESSIAN EQUATION

by Zbigniew BŁOCKI

## 1. Introduction.

For a smooth function $u$ defined on an open subset of $\mathbb{C}^n$ and $m = 1, \dots, n$ the elementary complex Hessian operator is defined by

$$H_m(u) = \sum_{1 \leqslant j_1 < \ldots < j_m \leqslant n} \lambda_{j_1} \ldots \lambda_{j_m},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the complex Hessian $(\partial^2 u / \partial z_j \partial \overline{z}_k)$. We have $H_1 = \Delta/4$ and $H_n$ is the complex Monge-Ampère operator. Using the operators $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$, so that $dd^c = 2i\partial\overline{\partial}$, one gets

$$(dd^c u)^m \wedge \omega^{n-m} = 4^n m!(n-m)!\, H_m(u)\, d\lambda,$$

where $\omega = dd^c|z|^2$ is the fundamental Kähler form and $d\lambda$ is the volume form.

The class of smooth admissible functions for the operator $H_m$ is naturally defined by the condition $H_m(u + A|z|^2) \geqslant 0$ for every $A \geqslant 0$. Using for example approximation by smooth functions one can define this

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notion also for non-smooth functions. We will denote this class by $\mathcal{P}_m$ and call such functions *m-subharmonic*. We clearly have

$$PSH = \mathcal{P}_n \subset \ldots \subset \mathcal{P}_1 = SH.$$

The class $\mathcal{P}_m$ is essentially determined by the following property

$$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge \omega^{n-m} \geqslant 0, \quad u_1, \ldots, u_m \in \mathcal{P}_m \cap C^\infty.$$

It would be interesting to find a geometric characterization of such functions. A necessary condition is $dd^c u \wedge \omega^{m-n} \geqslant 0$, which means that $u$ is subharmonic on every complex $n-m+1$-dimensional affine subspace of $\mathbb{C}^n$, but this condition is not sufficient if $1 < m < n$.

The aim of this paper is to study basic, mostly local properties of $m$-subharmonic functions and the operator $H_m$. Similarly as in [4] or [5] one can introduce the domain of definition of the operator $H_m$: a function $u \in \mathcal{P}_m$ is said to belong to the class $\mathcal{D}_m$ if there is a regular Borel measure $\mu$ such that if $u_j$ is a decreasing sequence of smooth $m$-subharmonic functions converging to $u$ (we consider only germs of functions, $u_j$ may be defined on a smaller domain than $u$) then $H_m(u_j)$ tends weakly to $\mu$. For $u \in D_m$ we set $H_m(u) = \mu$ and one can easily show that $\mathcal{D}_m$ is the maximal subclass of $\mathcal{P}_m$ where the operator $H_m$ can be extended (the values of $H_m$ are regular Borel measures) so that it is continuous for decreasing sequences. Similarly as in [4] and [5] for $m = n$, we shall completely characterize the class $\mathcal{D}_m$. We will show in particular the following result.

THEOREM 1.1. — *If $K \Subset \Omega \subset \mathbb{C}^n$ and $u, v \in \mathcal{P}_m(\Omega)$ are such that $u \in \mathcal{D}_m(\Omega)$, $u \leqslant v$ in $\Omega \setminus K$, then $v \in \mathcal{D}_m(\Omega)$.*

Theorem 1.1 implies for example that the class $\mathcal{D}_m$ contains functions from $\mathcal{P}_m(\Omega)$ which are locally bounded away from a compact subset of $\Omega$.

One of the main results is the following natural characterization of $m$-maximal functions (a function $u \in \mathcal{P}_m(\Omega)$, $\Omega$ open in $\mathbb{C}^n$, is called *m-maximal* if $v \in \mathcal{P}_m(\Omega)$, $v \leqslant u$ outside a compact subset of $\Omega$ implies that $v \leqslant u$ in $\Omega$).

THEOREM 1.2. — *A function $u \in \mathcal{D}_m$ is m-maximal if and only if $H_m(u) = 0$.*

Theorem 1.2 implies in particular that $m$-maximality of locally bounded $m$-subharmonic functions is a local property. We conjecture that this is the case without the assumption of boundedness but this is an open problem even if $m = n = 2$.

One can check that the function

$$G(z) = \begin{cases} -|z|^{2-2n/m}, & m < n, \\ \log |z|, & m = n. \end{cases} \tag{1.1}$$

is a fundamental solution for the operator $H_m$ (note that $G \in \mathcal{D}_m$ by Theorem 1.1). We clearly have

$$G \in L^p_{loc} \Leftrightarrow p < \frac{nm}{n-m}. \tag{1.2}$$

This leads naturally to the conjecture that for every $p < nm/(n-m)$ one has $\mathcal{P}_m \subset L^p_{loc}$. We are only able to show the following partial result.

Proposition 1.3. — *For every $p < n/(n-m)$ we have $\mathcal{P}_m \subset L^p_{loc}$.*

Note that we get optimal exponent in the well known, extreme cases $m = 1$ and $m = n$.

For $m = n$ we deal with the complex Monge-Ampère operator and plurisubharmonic functions and the above results are of course known (see e.g. [1], [2], [5], [11], [18]). The aim of this paper is to concentrate on those problems related to the Hessian operator where the methods of the complex Monge-Ampère operator cannot be automatically repeated.

The real Hessian operator has also been studied quite extensively in the recent years – see e.g. [8], [15], [16], [19], [21], [22]. It is clear that if $u(z) = u(x + iy)$ is independent of $y$ then it is $m$-subharmonic if and only if it is $m$-convex (see [22]). This means that in a way the complex Hessian operator is a generalization of the real one and indeed, for example, some results of Section 2 are generalizations of some results from [22].

The real Hessian operator for functions on open subsets of $\mathbb{R}^n$ is defined in the same way, one only takes the real Hessian instead of the complex one. Denote the class of $m$-convex functions by $\Phi_m$. The fundamental solution for the real Hessian operator is

$$H(x) = \begin{cases} -|x|^{2-n/m}, & m < n/2, \\ \log |x|, & m = n/2, \\ |x|^{2-n/m}, & m > n/2. \end{cases}$$

One has

$$H \in W^{1,q}_{loc} \Leftrightarrow q < \frac{nm}{n-m}.$$

It was in fact proved in [22, Theorem 4.1] that $\Phi_m \subset W^{1,q}_{loc}$ for $q < nm/(n-m)$. From the Sobolev theorem it thus follows that for $m \leqslant n/2$ and $p < nm/(n-2m)$ we have $\Phi_m \subset L^p_{loc}$, which is an optimal result in terms of the exponent $p$.

In the complex case however it is not possible to prove optimal $L^p$ estimates for $m$-subharmonic functions via gradient estimates and the Sobolev theorem. On one hand we have

$$G \in W^{1,q}_{loc} \Leftrightarrow q < \frac{2nm}{2n-m}.$$

But the function $u(z) = \log|z_1|$ belongs to $\mathcal{P}_m$ for every $m$, $u \notin W^{1,2}_{loc}$ and $2 < 2nm/(2n-m)$ if $2 \leqslant m \leqslant n$.

Most of the results of this paper were contained in the first version of the author paper [5]. Later however, a simpler proof of the characterization of the domain of definition of the complex Monge-Ampère operator was found, not employing the complex Hessian operator.

The paper is organized as follows. In Section 2 certain basic facts on elementary symmetric functions are collected. In Section 3 we prove the main properties of $m$-subharmonic functions including a special case of Theorem 1.2 for continuous functions. The proof relies heavily on an a priori estimate for a special case of the complex Hessian equation which is presented in Section 4. We remark that a much more general estimate and a solution of the Dirichlet problem for non-degenerate equation was independently shown in [20] with essentially the same methods as below. Section 5 is devoted to the characterization of the class $\mathcal{D}_m$. Since most of the proofs are similar to those from [5], they are sketched only briefly. Finally, in Section 6 we prove Proposition 1.3.

We have concentrated here on the study of weak solutions of complex Hessian equations and therefore we restricted ourselves to the elementary Hessian operators $H_m$ - as [22] did for example in the real case. Of course many results could be generalized here to more general complex Hessian operators. The existence of strong solutions in domains in $\mathbb{C}^n$ for such equations was recently proved in [20]. In particular, one could study the operator $(dd^c u)^m \wedge \omega^{n-m}$ on manifolds, where $\omega$ is an arbitrary Kähler form. This would perhaps be interesting from a geometric point of view.

For the global Dirichlet problem on compact Kähler manifolds, in analogy with the case of the Calabi-Yau theorem, one would have to consider the operator $(\omega + dd^c\varphi)^m \wedge \omega^{n-m}$.

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## 2. Basic properties of elementary symmetric functions.

In this section we recall some basic facts from (multi-)linear algebra needed in the paper. We set

$$S_m(\lambda) = \sum_{1 \leqslant j_1 < \ldots < j_m \leqslant n} \lambda_{j_1} \ldots \lambda_{j_m}, \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.$$

The elementary symmetric function $S_m$ is determined by

$$(\lambda_1 + t) \ldots (\lambda_n + t) = \sum_{m=0}^{n} S_m(\lambda) t^{n-m}, \quad t \in \mathbb{R}.$$

By $\Gamma_m$ we denote the closure of the connected component of $\{S_m > 0\}$ containing $(1, \ldots, 1)$. One can show that

$$\Gamma_m = \{\lambda \in \mathbb{R}^n : S_m(\lambda_1 + t, \ldots, \lambda_n + t) \geqslant 0 \ \forall t \geqslant 0\}$$

and, since

$$S_m(\lambda_1 + t, \ldots, \lambda_n + t) = \sum_{p=0}^{m} S_p(\lambda) t^{m-p}, \quad t \in \mathbb{R},$$

we also have

$$\Gamma_m = \{S_1 \geqslant 0\} \cap \ldots \cap \{S_m \geqslant 0\}.$$

In particular

$$\Gamma_n \subset \ldots \subset \Gamma_1.$$

By Gårding [14] the set $\Gamma_m$ is a convex cone in $\mathbb{R}^n$ and $S_m^{1/m}$ is concave on $\Gamma_m$. By Maclaurin inequality on $\Gamma_m$ one also has

$$\binom{n}{m}^{-1/m} S_m^{1/m} \leqslant \binom{n}{p}^{-1/p} S_p^{1/p}, \quad 1 \leqslant p \leqslant m.$$

By $\mathcal{H}$ we will denote the vector space (over $\mathbb{R}$) of (complex) hermitian $n \times n$ matrices. For $A \in \mathcal{H}$ let $\lambda(A) = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be the eigenvalues of $A$. We set

$$\widetilde{S}_m(A) = S_m(\lambda(A)).$$

The function $\widetilde{S}_m$ is determined by

$$\det(A + tI) = \sum_{m=0}^{n} \widetilde{S}_m(A) t^{n-m}, \quad t \in \mathbb{R}.$$

Then $\widetilde{S}_m$ is a homogeneous polynomial of order $m$ on $\mathcal{H}$ which is hyperbolic with respect to $I$ (that is for every $A \in \mathcal{H}$ the equation $\widetilde{S}_m(A + tI) = 0$ has $m$ real roots; see [14]). As in [14] we define the cone

$$\widetilde{\Gamma}_m := \{A \in \mathcal{H} : \widetilde{S}_m(A + tI) \geqslant 0 \ \forall t \geqslant 0\}.$$

We have

$$\widetilde{\Gamma}_m = \{A \in \mathcal{H} : \lambda(A) \in \Gamma_m\} = \{\widetilde{S}_1 \geqslant 0\} \cap \dots \{\widetilde{S}_m \geqslant 0\}.$$

It was proved in [14] that the cone $\widetilde{\Gamma}_m$ is convex and the function $\widetilde{S}_m^{1/m}$ is concave on $\widetilde{\Gamma}_m$.

Let $M : \mathcal{H}^m \to \mathbb{R}$ be the polarized form of $\widetilde{S}_m$ - it is determined by the following three properties: $M$ is linear in every variable, symmetric and

$$M(A, \dots, A) = \widetilde{S}_m(A), \quad A \in \mathcal{H}.$$

The inequality due to Gårding [14, Theorem 5] asserts that

$$\text{(2.1)} \quad M(A_1, \dots, A_m) \geqslant \widetilde{S}_m(A_1)^{1/m} \dots \widetilde{S}_m(A_m)^{1/m}, \quad A_1, \dots, A_m \in \widetilde{\Gamma}_m.$$

Real $(1,1)$-forms $\beta$ we associate with hermitian matrices $(a_{j\overline{k}})$ by

$$\beta = 2 \sum_{j,k} a_{j\overline{k}} i dz_j \wedge d z_{\overline{k}}$$

(so that $\omega$ is associated with the identity matrix $I$). After diagonalizing $(a_{j\overline{k}})$, we see that

$$\beta^m \wedge \omega^{n-m} = m! \widetilde{S}_m((a_{j\overline{k}})) \omega^n.$$

It is also clear that $\beta_1 \wedge \dots \beta_m \wedge \omega^{n-m}$ is the polarized form of $\beta^m \wedge \omega^{n-m}$. Accordingly, we set

$$\widehat{\Gamma}_m := \{\beta \in \mathbb{C}_{(1,1)} : \beta \wedge \omega^{n-1} \geqslant 0, \beta^2 \wedge \omega^{n-2} \geqslant 0, \dots, \beta^m \wedge \omega^{n-m} \geqslant 0\}.$$

The crucial fact for us will be the following property.

PROPOSITION 2.1. — *For $\beta_1, \dots, \beta_p \in \widehat{\Gamma}_m$, $p \leqslant m$, we have*

$$\beta_1 \wedge \dots \wedge \beta_p \wedge \omega^{n-m} \geqslant 0.$$

*Proof.* — We need to show that for any $(1,0)$ forms $\alpha_1, \dots, \alpha_{m-p}$ we have

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \dots \wedge i\alpha_{m-p} \wedge \overline{\alpha}_{m-p} \wedge \beta_1 \wedge \dots \wedge \beta_q \wedge \omega^{n-m} \geqslant 0.$$

But since $i\alpha_j \wedge \overline{\alpha}_j \in \widehat{\Gamma}_n \subset \widehat{\Gamma}_m$ (this is because $(i\alpha_j \wedge \overline{\alpha}_j)^2 = 0$), we may assume that $p = m$. Then the proposition follows from the Gårding inequality (2.1). □

For $B \in \mathcal{H}$ we define

$$D_m(B) := \left( \frac{\partial \widetilde{S}_m}{\partial b_{p\overline{q}}}(B) \right) \in \mathcal{H}.$$

We then have

$$tr(AD_m(B)) = mM(A, B, \dots, B),$$

in particular

$$tr(BD_m(B)) = \widetilde{S}_m(B). \tag{2.2}$$

If $B$ is diagonal then so is $D_m(B)$. If $\lambda = \lambda(B)$ then

$$\lambda(D_m(B)) = (S_{m-1}(\lambda^{(1)}), \dots, S_{m-1}(\lambda^{(n)})),$$

where $\lambda^{(j)} = (\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n)$. If $B \in \widetilde{\Gamma}_m$ then for $t > 0$

$$S_{m-1}(\lambda^{(j)}) = t^{-1}\big(S_m(\lambda + (0, \dots, t, \dots, 0)) - S_m(\lambda)\big) \geqslant 0$$

so that $D_m(B) \geqslant 0$. By (2.1) we have

$$tr(AD_m(B)) \geqslant m\widetilde{S}_m(A)^{1/m}\widetilde{S}_m(B)^{(m-1)/m}, \quad A, B \in \widetilde{\Gamma}_m, \tag{2.3}$$

and

$$\begin{aligned}\widetilde{S}_m(A)^{1/m} &= \frac{1}{m}\inf\{M(A,B,\ldots,B) : B \in \widetilde{\Gamma}_m,\ \widetilde{S}_m(B) \geqslant 1\}\\ &= \frac{1}{m}\inf\{tr(AD_m(B)) : B \in \widetilde{\Gamma}_m,\ \widetilde{S}_m(B) \geqslant 1\}, \quad A \in \widetilde{\Gamma}_m.\end{aligned}$$

## 3. The $m$-subharmonic functions and the complex Hessian operator.

In this section we define the class of admissible functions for the complex Hessian operator $H_m$ and prove their basic properties. Most of the proofs are the same as in the case of plurisubharmonic functions and the Monge-Ampère operator (that is when $m = n$) and therefore we will present them only briefly.

A function $u$ is called *$m$-subharmonic* (we write $u \in \mathcal{P}_m$) if it is subharmonic and

$$dd^c u \wedge \beta_1 \wedge \ldots \wedge \beta_{m-1} \wedge \omega^{n-m} \geqslant 0, \quad \beta_1, \ldots, \beta_{m-1} \in \widehat{\Gamma}_m.$$

The following basic properties of $m$-subharmonic functions either follow immediately from Proposition 2.1 or can be proven in the same way as in the classical case, and therefore their proofs are left to the reader.

Proposition 3.1. — *i) If $u$ is $C^2$ smooth then it is $m$-subharmonic if and only if the form $dd^c u$ belongs pointwise to $\widetilde{\Gamma}_m$;*

*ii) If $u, v \in \mathcal{P}_m$ then $u + v \in \mathcal{P}_m$;*

*iii) If $u \in \mathcal{P}_m$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a convex, increasing function then $\gamma \circ u \in \mathcal{P}_m$;*

*iv) If $u$ is $m$-subharmonic then the standard regularizations $u * \rho_\varepsilon$ are also $m$-subharmonic ;*

*v) If $\{u_\iota\} \subset \mathcal{P}_m$ is locally uniformly bounded from above then $(\sup_\iota u_\iota)^* \in \mathcal{P}_m$, where $v^*$ denotes the upper regularization of $v$;*

*vi) $PSH = \mathcal{P}_n \subset \ldots \subset \mathcal{P}_1 = SH$.* □

The next result was proven in [23] for $m = n$.

Proposition 3.2. — *For a bounded domain $\Omega$ in $\mathbb{C}^n$ and $f \in C(\overline{\Omega})$ set*

$$u := \sup\{v \in \mathcal{P}_m(\Omega) : v \leqslant f\}.$$

*Assume moreover that $u^* = u_* = f$ on $\partial\Omega$. Then $u \in \mathcal{P}_m(\Omega) \cap C(\overline{\Omega})$.*

*Proof.* — It is clear that $u^* \in \mathcal{P}_m(\Omega)$ and $u^* \leqslant f$, thus $u = u^*$ is upper semi-continuous in $\Omega$. To show the lower semi-continuity in $\Omega$ fix $z_0 \in \Omega$ and $\varepsilon > 0$. From the uniform continuity of $f$ on $\overline{\Omega}$ and of $u = f$ on $\partial\Omega$ it follows that we can find $\delta > 0$ such that

$$\operatorname{dist}(z_0, \partial\Omega) \geqslant 2\delta, \tag{3.1}$$

$$z \in \overline{\Omega},\ w \in \partial\Omega,\ |z - w| \leqslant 3\delta \ \Rightarrow\ |u(z) - f(w)| \leqslant \varepsilon, \tag{3.2}$$

$$z, z' \in \overline{\Omega},\ |z - z'| \leqslant \delta \ \Rightarrow\ |f(z) - f(z')| \leqslant 2\varepsilon. \tag{3.3}$$

Fix $\widetilde{z}$ with $|z_0 - \widetilde{z}| \leqslant \delta$. For $z \in \Omega$ set

$$v(z) := \begin{cases} \max\{u(z + z_0 - \widetilde{z}) - 2\varepsilon, u(z)\}, & \operatorname{dist}(z, \partial\Omega) \geqslant \delta, \\ u(z), & \operatorname{dist}(z, \partial\Omega) < \delta. \end{cases}$$

If $\operatorname{dist}(z, \partial\Omega) \leqslant 2\delta$ then we can find $w \in \partial\Omega$ with $|w - z| \leqslant 2\delta$. Using (3.2) twice we get

$$u(z + z_0 - \widetilde{z}) \leqslant f(w) + \varepsilon \leqslant u(z) + 2\varepsilon.$$

This implies that $v(z) = u(z)$ if $\operatorname{dist}(z, \partial\Omega) \leqslant 2\delta$, and thus $v \in \mathcal{P}_m(\Omega)$. On the other hand, if $\operatorname{dist}(z, \partial\Omega) \geqslant \delta$, then by (3.3)

$$u(z + z_0 - \widetilde{z}) \leqslant f(z + z_0 - \widetilde{z}) \leqslant f(z) + 2\varepsilon$$

and it follows that $v \leqslant f$ in $\Omega$. Therefore $v \leqslant u$ in $\Omega$ and by (3.1)

$$u(\widetilde{z}) \geqslant v(\widetilde{z}) \geqslant u(z_0) - 2\varepsilon,$$

hence $u$ is lower semi-continuous. □

Proposition 3.2 will mostly be used in the situation when $\Omega$ is a regular domain (with respect to harmonic functions) and $f$ is harmonic in

$\Omega$. In such a case, from the maximum principle it follows that the condition $v \leqslant f$ in the definition of $u$ is equivalent to $v^* \leqslant f$ on $\partial\Omega$.

For continuous $m$-subharmonic functions we can inductively define a closed nonnegative current

$$(3.4)\quad \begin{aligned} dd^c u_1 \wedge \ldots dd^c u_p \wedge \omega^{n-m} := dd^c\big(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge \omega^{n-m}\big), \\ u_1, \ldots, u_p \in \mathcal{P}_m \cap C, \quad p \leqslant m. \end{aligned}$$

(We have used the fact that the coefficients of nonnegative currents are complex measures, see e.g. [13].) We can also define a nonnegative current

$$(3.5)\quad \begin{aligned} d(u_0 - u_1) \wedge d^c(u_0 - u_1) \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge \omega^{n-m} \\ u_0, u_1, \ldots, u_p \in \mathcal{P}_m \cap C, \quad p \leqslant m. \end{aligned}$$

as follows. We note that

$$d(u_0 - u_1) \wedge d^c(u_0 - u_1) = 2du_0 \wedge d^c u_0 + 2du_1 \wedge d^c u_1 - d(u_0 + u_1) \wedge d^c(u_0 + u_1)$$

and

$$du \wedge d^c u = \frac{1}{2} dd^c(u + C)^2 - (u + C) dd^c u, \quad u \in \mathcal{P}_m \cap C,$$

where $C$ is sufficiently big, and use the previous part.

The proofs of the following three results for $m = n$ can be essentially found in [1].

Proposition 3.3. — *The operators (3.4) and (3.5) are continuous for locally uniformly convergent sequences in $\mathcal{P}_m \cap C$.*

*Proof.* — It is enough to prove the continuity of the operator

$$(\mathcal{P}_m \cap C)^p \ni (u_1, \ldots, u_p) \longmapsto u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge \omega^{n-m}.$$

This follows inductively from the fact that the coefficients of a nonnegative current are complex measures and since the convergence is uniform. □

Proposition 3.4. — *For $u, v \in \mathcal{P}_m \cap C$ we have*

$$(dd^c \max\{u, v\})^m \wedge \omega^{n-m} \geqslant \chi_{\{u>v\}} (dd^c u)^m \wedge \omega^{n-m} + \chi_{\{u \leqslant v\}} (dd^c v)^m \wedge \omega^{n-m},$$

*where $\chi_A$ denotes the characteristic function of a set $A$.*

*Proof.* — For a compact $K \subset \{u = v\}$ by Proposition 3.3 we have

$$\int_K (dd^c \max\{u, v\})^m \wedge \omega^{n-m} \geqslant \overline{\lim_{\varepsilon \downarrow 0}} \int_K (dd^c \max\{u, v + \varepsilon\})^m \wedge \omega^{n-m}$$
$$= \int_K (dd^c v)^m \wedge \omega^{n-m}. \qquad \square$$

PROPOSITION 3.5. — *Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $u, v \in \mathcal{P}_m(\Omega) \cap C(\overline{\Omega})$ are such that $u \leqslant v$ on $\partial\Omega$ and $(dd^c u)^m \wedge \omega^{n-m} \geqslant (dd^c v)^m \wedge \omega^{n-m}$ in $\Omega$. Then $u \leqslant v$ in $\Omega$.*

*Proof.* — Suppose that $\{u > v\} \neq \emptyset$. We can then find $\varepsilon > 0$ such that $S := \{\widetilde{u} > v\} \neq \emptyset$, where $\widetilde{u} := u + \varepsilon\psi$ and $\psi(z) = |z|^2 - M$ is negative in $\overline{\Omega}$. We have

$$(dd^c(u + \varepsilon\psi))^m \wedge \omega^{n-m} \geqslant (dd^c u)^m \wedge \omega^{n-m} + \varepsilon^m \omega^n$$

and from Proposition 3.4 it follows that $(dd^c\widetilde{u})^m \wedge \omega^{n-m} \geqslant (dd^c v)^m \wedge \omega^{n-m}$ in $\Omega$. However, since $\widetilde{u} = v$ near $\partial\Omega$, regularizing $\widetilde{u}, v$ and using the Stokes theorem we get

$$\int_\Omega (dd^c\widetilde{u})^m \wedge \omega^{n-m} = \int_\Omega (dd^c v)^m \wedge \omega^{n-m}$$

and we must thus have $(dd^c\widetilde{u})^m \wedge \omega^{n-m} = (dd^c v)^m \wedge \omega^{n-m}$ in $\Omega$. On the other hand

$$\int_S (dd^c\widetilde{u})^m \wedge \omega^{n-m} \geqslant \int_S (dd^c u)^m \wedge \omega^{n-m} + \varepsilon^m \int_S \omega^n$$

and we get a contradiction. $\square$

A function $u \in \mathcal{P}_m(\Omega)$, $\Omega$ open in $\mathbb{C}^n$, is called *m-maximal* if $v \in \mathcal{P}_m(\Omega)$, $v \leqslant u$ outside a compact subset of $\Omega$ implies that $v \leqslant u$ in $\Omega$. We first prove Theorem 1.2 for continuous functions.

THEOREM 3.6. — *A function $u \in \mathcal{P}_m \cap C(\Omega)$ is $m$-maximal if and only if it solves $H_m(u) = 0$.*

Theorem 3.6 will easily follow from the comparison principle (Proposition 3.5) and the solution of the Dirichlet problem for the $m$-Hessian equation in a ball.

Theorem 3.7. — *Let $B$ be a ball in $\mathbb{C}^n$ and $\varphi$ a continuous function on $\partial B$. Then the following Dirichlet problem*

$$\begin{cases} u \in \mathcal{P}_m(B) \cap C(\overline{B}) \\ (dd^c u)^m \wedge \omega^{n-m} = 0 \quad in \quad B \\ u = \varphi \quad on \quad \partial B \end{cases}$$

*has a unique solution.*

*Proof.* — Uniqueness is a consequence of Proposition 3.5. To show the existence we first assume that $\varphi$ is smooth and for a constant $a > 0$ consider the Dirichlet problem

$$\begin{cases} u \in \mathcal{P}_m(B) \cap C^\infty(\overline{B}) \\ (dd^c u)^m \wedge \omega^{n-m} = a\omega^n \quad \text{in} \quad B \\ u = \varphi \quad \text{on} \quad \partial B. \end{cases} \tag{3.6}$$

By the Evans-Krylov theory (see e.g. [7, Theorem 1]) there exists a solution of (3.6) provided that we have an a priori bound

$$||u||_{C^{1,1}(\overline{B})} \leqslant C, \tag{3.7}$$

where $C$ depends only on $a$ and $\varphi$. The proof of this estimate is postponed to Section 4.

Assuming that (3.7) is proven, and thus that we can solve (3.6), let $\varphi$ be arbitrary continuous. Approximate it from below by $\varphi_j \in C^\infty(\partial B)$. Let $u_j$ be a solution of (3.6) with $\varphi_j$ and $a = 1/j$. Let $\psi(z) = |z - z_0|^2 - R^2$, where $z_0$ is the center and $R$ the radius of $B$. For $k \geqslant j$ Proposition 3.5 gives

$$u_k + j^{-1/m}\psi - ||\varphi_j - \varphi||_{L^\infty(\partial B)} \leqslant u_j \leqslant u_k.$$

This implies that $u_j$ converges uniformly on $\overline{B}$ to a certain $u$, which is a solution by Proposition 3.3. □

*Proof of Theorem 3.6.* — Proposition 3.5 directly implies that if $u$ satisfies $H_m(u) = 0$ then it is maximal. On the other hand, assume that $u$ is maximal and let $B \Subset \Omega$ be a ball. By Theorem 3.7 we find $\widetilde{u} \in C(\Omega)$ determined by $\widetilde{u} = u$ in $\Omega \setminus B$, $\widetilde{u} \in \mathcal{P}_m(B)$ and $(dd^c\widetilde{u})^m \wedge \omega^{n-m} = 0$ in $B$. By the comparison principle again we have $\widetilde{u} \geqslant u$ in $B$ and thus $\widetilde{u} \in \mathcal{P}_m(\Omega)$. Since $u$ is maximal, it follows that $\widetilde{u} = u$ and we get $H_m(u) = 0$. □

*Remark.*— For $m = n$ Theorem 3.7 was proved by Bedford and Taylor [1] with the help of an interior $C^{1,1}$ estimate ([1, Theorem 6.7]), which, together with later simplifications due to Demailly [11], gives an overall simpler and more elementary proof than the one presented here (not employing strong solutions at all and thus not using the Evans-Krylov theory and estimate (3.7)). It relied however on the following, rather rare, property: the group of smooth diffeomorphisms of the unit ball in $\mathbb{C}^n$ preserving plurisubharmonic functions is transitive. Note that this not true in the real case (where plurisubharmonic functions are replaced by the convex ones - then we only have the affine mappings) and one can also show that it is not true for $m = 1$, that is for subharmonic functions. One can namely check that in this case such a diffemorphism $F = (F^1, \dots, F^{2n})$ has to satisfy two properties: 1) the (real) Jacobian matrix of $F$ is orthogonal at every point; 2) $\Delta F^j = 0$, $j = 1, \dots, 2n$. By the Liouville theorem (see e.g. [6] or [17]) the mappings satisfying 1) are precisely the Möbius transformations. However, the Kelvin transformation $z \mapsto z/|z|^2$ is harmonic only in the real dimension 2. Thus, if $n > 1$ the mappings satisfying 1) and 2) are precisely linear Möbius transformations and the group in question is not transitive. We suspect that it is also not transitive if $1 < m < n$.

## 4. The a priori estimate.

In this section we will prove the estimate (3.7). We essentially follow [8] using some ideas from [7] and the simplification from [21]. We use the notation $u_j = \partial u/\partial z_j$, $u_{\bar{j}} = \partial u/\partial \bar{z}_j$. The real partial derivatives of $u$ will be denoted by $u_{x_j}$, $u_{y_j}$, and by $u_\zeta$ we mean the derivative of $u$ in direction $\zeta$.

It is no loss of generality to consider the equation

$$\widetilde{S}_m((u_{j\bar{k}})) = 1. \tag{4.1}$$

Computing the derivative of both sides of (4.1) in a direction $\zeta$ we get

$$a^{j\bar{k}} u_{\zeta j\bar{k}} = 0, \tag{4.2}$$

where

$$\left(a^{p\bar{q}}\right) = D_m((u_{j\bar{k}})) = \left(\frac{\partial \widetilde{S}_m}{\partial u_{p\bar{q}}}((u_{j\bar{k}}))\right).$$

By Section 2 we have $(a^{j\bar{k}}) > 0$. If $(u_{j\bar{k}})$ is diagonal then so is $(a^{j\bar{k}})$ which implies that the product of these is a hermitian matrix. This means that for every $p, q$

$$a^{p\bar{k}} u_{q\bar{k}} = a^{j\bar{q}} u_{j\bar{p}}, \tag{4.3}$$

and by (4.2)

$$a^{j\bar{k}} \left[z_p u_q - \bar{z}_q u_{\bar{p}}\right]_{j\bar{k}} = 0. \tag{4.4}$$

Since $\widetilde{S}_m^{1/m}$ is concave on $\widetilde{\Gamma}_m$ it follows that so is $G := \log \widetilde{S}_m$. Differentiating the logarithm of both sides of (4.1) twice in direction $\zeta$ we get

$$\sum_{j,k,p,q} \frac{\partial^2 G}{\partial u_{j\bar{k}} \partial u_{p\bar{q}}} u_{j\bar{k}\zeta} u_{p\bar{q}\zeta} + \sum_{j,k} \frac{\partial G}{\partial u_{j\bar{k}}} u_{j\bar{k}\zeta\zeta} = 0.$$

The concavity of $G$ implies that the first term is nonpositive and we get

$$a^{j\bar{k}} u_{\zeta\zeta j\bar{k}} \geqslant 0. \tag{4.5}$$

It is no loss of generality to assume that $B = B(0,1)$ is the unit ball in $\mathbb{C}^n$ and that $\varphi \in C^\infty(\overline{B})$ is harmonic in $B$. By $C$ we will denote possibly different constants depending only on $||\varphi||_{C^{3,1}(\overline{B})}$ and say that they are *under control*. We also set $\psi(z) := (|z|^2 - 1)/2$. From the comparison principle we get, for sufficiently big $C$, $\varphi + C\psi \leqslant u \leqslant \varphi$. This coupled with (4.2) gives

$$||u||_{C^{0,1}(\overline{B})} \leqslant C. \tag{4.6}$$

We now turn to the estimates of $D^2 u$ on $\partial B$. For $\zeta \in \partial B$ by $s, t$ we will denote the (real) tangential directions at $\zeta$ and by $N$ the outer normal direction. We clearly have

$$u_{st} = \varphi_{st} + (u - \varphi)_N \delta_{st}. \tag{4.7}$$

From (4.6) it follows therefore that

$$|u_{st}(\zeta)| \leqslant C, \quad \zeta \in \partial B. \tag{4.8}$$

Next we estimate the mixed tangential-normal derivative $u_{tN}(\zeta^0)$ for a fixed $\zeta^0 \in \partial B$. We may assume that $\zeta_0 = (0, \dots, 0, 1)$, so that at $\zeta^0$ we have $N = \partial/\partial x_n$. First assume that $t = \partial/\partial x_p$ for some $p \leqslant n-1$. Set

$$\begin{aligned} v : &= 2\,\mathrm{Re}\,[z_p(u-\varphi)_n - \overline{z}_n(u-\varphi)_{\overline{p}}] \\ &= x_p(u-\varphi)_{x_n} - x_n(u-\varphi)_{x_p} + y_p(u-\varphi)_{y_n} - y_n(u-\varphi)_{y_p}. \end{aligned}$$

Then $v = 0$ on $\partial B$, $|v| \leqslant C$ on $\partial B(\zeta^0, 1) \cap \overline{B}$ and by (4.4)

$$\pm a^{j\overline{k}} v_{j\overline{k}} \geqslant -C \sum_j a^{j\overline{j}}.$$

We now consider the barrier function $w := \pm v - C_1|z-\zeta_0|^2 + C_2\psi$. We can choose constants $0 \ll C_1 \ll C_2$ under control so that $w \leqslant 0$ on $\partial(B \cap B(\zeta^0, 1))$ and $a^{j\overline{k}} w_{j\overline{k}} \geqslant 0$ in $B \cap B(\zeta^0, 1)$. Therefore $w \leqslant 0$ in $B \cap B(\zeta^0, 1)$,

$$|v| \leqslant C_1|z-\zeta_0|^2 - C_2\psi$$

and it follows that $|v_{x_n}(\zeta^0)| \leqslant C$. At $\zeta^0$ we have however

$$v_{x_n} = -(u-\varphi)_{x_p} - (u-\varphi)_{x_p x_n}$$

and thus $|u_{x_p x_n}(\zeta^0)| \leqslant C$.

To estimate $u_{y_p x_n}(\zeta^0)$ we take

$$\begin{aligned} v : &= 2\,\mathrm{Im}\,[z_p(u-\varphi)_n - \overline{z}_n(u-\varphi)_{\overline{p}}] \\ &= y_p(u-\varphi)_{x_n} - x_n(u-\varphi)_{y_p} + y_n(u-\varphi)_{x_p} - x_p(u-\varphi)_{y_n}. \end{aligned}$$

and proceed similarly. Finally, for $t = \partial/\partial y_n$ one can check, using (4.2) and (4.3), that

$$a^{j\overline{k}}\,[y_n u_{x_n} - x_n u_{y_n}]_{j\overline{k}} = 2\,\mathrm{Im}(a^{n\overline{k}} u_{n\overline{k}}) = 0$$

and consider

$$v := y_n(u-\varphi)_{x_n} - x_n(u-\varphi)_{y_n}.$$

We will eventually obtain

$$|u_{tN}(\zeta)| \leqslant C \quad \zeta \in \partial B. \tag{4.9}$$

We claim that to get (3.7) it is now enough to estimate

$$u_{n\overline{n}}(\zeta^0) \leqslant C. \tag{4.10}$$

Indeed, this combined with (4.8), (4.9) and (4.5) implies that all the eigenvalues of the real Hessian matrix $D^2u$ are bounded from above by $C$ in $\overline{B}$. But since $u$ is in particular subharmonic, it follows that they must then be bounded from below by $-(2n-1)C$. It thus remains to show (4.10).

By (4.8) and (4.9) at $\zeta_0$ we may write

$$1 = u_{n\overline{n}} S'_{m-1} + O(1), \tag{4.11}$$

where $S'_{m-1}(\zeta^0) = \widetilde{S}_{m-1}((u_{j\overline{k}}(\zeta^0))')$ and if $A$ is an $n \times n$ matrix then by $A'$ we denote the $(n-1) \times (n-1)$ matrix created by deleting the $n$th row and $n$th column in $A$. We will now use an idea from [21]. By (4.11) we may assume that the quantity $S'_{m-1}(\zeta)$, $\zeta \in \partial B$, is minimized at $\zeta_0$. It is elementary to show that there exists a smooth mapping

$$\Phi : (\overline{B} \cap \overline{B}(\zeta^0, 1)) \times \mathbb{C}^n \to \mathbb{C}^n$$

such that for every $z \in \overline{B} \cap \overline{B}(\zeta^0, 1)$ the mapping $\Phi_z = \Phi(z, \cdot)$ is an orthogonal isomorphism of $\mathbb{C}^n$ (and of $B$), $\Phi_\zeta(\zeta) = \zeta^0$ for $\zeta \in \partial B \cap \overline{B}(\zeta^0, 1)$ and $\Phi_{\zeta^0}$ is the identity. For $\zeta \in \partial B \cap \overline{B}(\zeta^0, 1)$ we then have

$$S'_{m-1}(\zeta) = \widetilde{S}_{m-1}(U(\zeta)),$$

where by (4.7)

$$U(\zeta) = A(\zeta) + u_N(\zeta) I,$$
$$A(\zeta) = \Big( (\varphi \circ \Phi_\zeta)_{j\overline{k}}(\zeta^0) \Big)' - \varphi_N(\zeta) I.$$

It is clear that $||A||_{C^{1,1}(\overline{B}\cap\overline{B}(\zeta^0,1))} \leqslant C$. Define the $(n-1) \times (n-1)$ positive definite matrix

$$B_0 := D_{m-1}(U(\zeta^0)) = \left( \frac{\partial \widetilde{S}_{m-1}}{\partial a_{p\overline{q}}} (U(\zeta^0)) \right).$$

By (2.2) and (2.3)

$$tr\left[B_0(U(\zeta) - U(\zeta^0))\right] \geqslant S'_{m-1}(\zeta) - S'_{m-1}(\zeta^0) \geqslant 0.$$

We thus obtain

$$v(\zeta) := u_{x_n}(\zeta) - u_{x_n}(\zeta^0) + \langle \nabla u(\zeta), \zeta - \zeta^0 \rangle + (tr\, B_0)^{-1} tr\left[B_0(A(\zeta) - A(\zeta^0))\right] \geqslant 0$$

for $\zeta \in \partial B \cap \overline{B}(\zeta^0, 1)$.

Similarly as before, we define the barrier $w := v - C_1|z-\zeta^0|^2 + C_2\psi$, and choosing $C_1 \ll C_2$ under control we get $w \leqslant 0$ on $\partial(B \cap B(\zeta^0, 1))$ and $a^{j\overline{k}} w_{j\overline{k}} \geqslant 0$ in $B \cap B(\zeta^0, 1)$. Therefore $w \leqslant 0$ in $B \cap B(\zeta^0, 1)$ and

$$u_{x_n x_n}(\zeta^0) \leqslant C$$

which together with (4.8) gives (4.10). □

## 5. The class $\mathcal{D}_m$.

Essentially just repeating the proof of [5, Theorem 1.1] and using the necessary machinery from Section 3, we can get the following characterization of the class $\mathcal{D}_m$ (we consider the germs of functions).

THEOREM 5.1. — *For a negative $u \in \mathcal{P}_m$ the following are equivalent*

*i) $u \in \mathcal{D}_m$;*

*ii) For every sequence $u_j \in \mathcal{P}_m \cap C^\infty$ decreasing to $u$ the sequence $H_m(u_j)$ is locally weakly bounded;*

*iii) $u \in L^m_{loc}$ and for every sequence $u_j \in \mathcal{P}_m \cap C^\infty$ decreasing to $u$ the sequences*

$$|u_j|^{m-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{m-p-1}, \quad p = 0, 1, \ldots, m-2, \tag{5.1}$$

*are locally weakly bounded;*

*iv) $u \in L^m_{loc}$ and there exists a sequence $u_j \in \mathcal{P}_m \cap C^\infty$ decreasing to $u$ such that the sequences (5.1) are locally weakly bounded.*

For $m = 2$ it is clear that conditions iii) and iv) in Theorem 5.1 are equivalent, and they mean precisely that $u \in \mathcal{P}_2 \cap W^{1,2}_{loc}$.

We will now very briefly sketch the proof of Theorem 5.1. The crucial steps are the following two estimates.

PROPOSITION 5.2. — *Let $\Omega' \Subset \Omega$ be domains in $\mathbb{C}^n$. Assume that $2 \leqslant m \leqslant n$ and that either $r \leqslant 0$ or $r \geqslant 1$. Then for any $u \in \mathcal{P}_m \cap C(\Omega)$, $u < 0$, we have*

$$\int_{\Omega'} |u|^r (dd^c u)^m \wedge \omega^{n-m} \leqslant C \int_\Omega |u|^r du \wedge d^c u \wedge (dd^c u)^{m-2} \wedge \omega^{n-m+1},$$

*where $C$ is a positive constant depending only on $\Omega'$ and $\Omega$.*

*Proof.* — It is the same as the proof of [5, Proposition 2.1].

Theorem 5.3. — *Let $\Omega' \Subset \Omega$ be domains in $\mathbb{C}^n$. Assume that $2 \leqslant m \leqslant n$ and $r \geqslant 0$. Then for $u, v \in \mathcal{P}_m \cap C(\Omega)$ with $u \leqslant v < 0$ one has*

$$\int_{\Omega'} |v|^r dv \wedge d^c v \wedge (dd^c v)^{m-2} \wedge \omega^{n-m+1}$$
$$\leqslant C \left( \int_\Omega |u|^{m+r} \omega^n + \sum_{p=0}^{m-2} \int_\Omega |u|^{m-p+r-2} du \wedge d^c u \wedge (dd^c u)^p \wedge \omega^{n-p-1} \right),$$

*where $C$ is a constant depending only on $\Omega', \Omega$ and $r$.*

*Proof.* — One has to repeat the proof of [5, Theorem 2.2].

*Proof of Theorem 5.1 (sketch).* — It follows from Theorem 5.3 that the conditions iii) and iv) are equivalent. To show implication iii)⇒iv) one has to use Cegrell's arguments (see the proof of [9, Theorem 4.2], they are also presented in [5]). The implication i)⇒ii) is trivial and to show the remaining implication ii)⇒iii) we proceed the same way as in [5]. We have however to show in addition that $u$ is in $L^m_{loc}$ which is already guaranteed for $m = n$ (and it will follow from Proposition 1.3 if $m^2/(m-1) > n$).

Suppose that ii) is satisfied but $u \notin L^m_{loc}$. We can then find balls $B \Subset B'$ such that $u$ is defined in a neighborhood of $\overline{B'}$ and $u \notin L^m_{loc}(B)$. Let $v_j = u * \rho_{1/j}$ be the sequence of the regularizations of $u$. Then there exists an increasing sequence $k = k(j) \geqslant j$ such that

$$\int_B |v_j - v_k|^m d\lambda \geqslant j \tag{5.2}$$

We set

$$\begin{aligned} u_j :&= \sup\{w \in \mathcal{P}_m(B') : w \leqslant v_j \text{ in } B',\ w \leqslant v_k \text{ in } B\} \\ &= \sup\{w \in \mathcal{P}_m(B') : w \leqslant h_j\}, \end{aligned}$$

where $h_j \in C(\overline{B'})$ is defined by $h_j = v_k$ in $\overline{B}$, $h_j = v_j$ on $\partial B'$ and $h_j$ is harmonic in $B' \setminus \overline{B}$. By Proposition 3.2 $u_j \in \mathcal{P}_m(B') \cap C(\overline{B'})$. It is clear that $u_j$ is decreasing to $u$ in $B'$ and therefore by ii) we have

$$\sup_j \int_{\overline{B}} (dd^c u_j)^m \wedge \omega^{n-m} < \infty.$$

By Theorem 3.6 we have $(dd^c u_j)^m \wedge \omega^{n-m} = 0$ in $\{u_j < v_j\}$, and, since $u_j \leqslant v_j$, from Proposition 3.4 it follows that $(dd^c u_j)^m \wedge \omega^{n-m} \leqslant (dd^c v_j)^m \wedge \omega^{n-m}$ on $\{u_j = v_j\}$. By another application of the assumption, this time to the sequence $v_j$, we obtain therefore

$$\sup_j \int_{B'} (dd^c u_j)^m \wedge \omega^{n-m} < \infty.$$

However, integrating by parts in the same way as in the proof of [3, Theorem 2.1] or [5, Proposition 3.1], using (5.2) we obtain

$$j \leqslant \int_{B'} (v_j - u_j)^m d\lambda \leqslant C \int_{B'} (dd^c u_j)^m \wedge \omega^{n-m}$$

where $C$ is independent of $j$ - a contradiction. □

The proof of Theorem 1.1 is now the same as the proof of [5, Theorem 1.2], whereas to show Theorem 1.2 we have to proceed as in the proof of [4, Proposition 2.2] (using Theorem 3.6).

## 6. The $L^p$-estimate.

In this section we will prove Proposition 1.3. More precisely, we will show the following estimate.

Proposition 6.1. — *For $p < n/(n-m)$ and negative $u \in \mathcal{P}_m(B(0,2))$ one has*

$$||u||_{L^p(B(0,1/2))} \leqslant C||u||_{L^1(B(0,2))}, \tag{6.1}$$

*where $C$ is a positive constant depending only on $n, m$ and $p$.*

*Proof.* — We will use similar methods as for example in [10] and [24]. Thanks to regularization we may assume that $u$ is smooth. By $C_1, C_2, \dots$ we will denote constants depending only on $n, m$ and $p$.

For $\varepsilon > 0$ let $G_\varepsilon \in \mathcal{P}_m \cap C^\infty(\mathbb{C}^n)$ be such that $G_\varepsilon = G$ on $\mathbb{C}^n \setminus B(0,\varepsilon)$ and $G_\varepsilon \downarrow G$ as $\varepsilon \downarrow 0$, where $G$ is given by (1.1). For $w \in B(0,1/2)$ we have, denoting $B:=B(0,1)$ and $u_w:=u(w-\cdot)$,

$$\begin{aligned}C_1u(w)&=\int_B u_w(dd^cG)^m\wedge\omega^{n-m}=\lim_{\varepsilon\to 0}\int_B u_w(dd^cG_\varepsilon)^m\wedge\omega^{n-m}\\&=\lim_{\varepsilon\to 0}\bigg(\int_B G_\varepsilon dd^cu_w\wedge(dd^cG_\varepsilon)^{m-1}\wedge\omega^{n-m}\\&\qquad+\int_{\partial B}u_wd^cG\wedge(dd^cG)^{m-1}\wedge\omega^{n-m}\bigg)\\&=\int_B Gdd^cu_w\wedge(dd^cG)^{m-1}\wedge\omega^{n-m}\\&\qquad+\int_{\partial B}u_wd^cG\wedge(dd^cG)^{m-1}\wedge\omega^{n-m}=:u_1(w)+u_2(w).\end{aligned}$$

Since $u$ is in particular subharmonic,

$$|u_2(w)|\leqslant C_2\int_{\partial B}|u_w|d\sigma\leqslant C_3||u||_{L^1(B(0,2))},$$

it is thus enough to estimate $||u_1||_{L^p(B(0,1/2))}$.

Write $G=E\circ\psi$. Then by Proposition 2.1 we have

$$\begin{aligned}&0\leqslant dd^cu_w\wedge(dd^cG)^{m-1}\wedge\omega^{n-m}\\&\quad=(E'\circ\psi)^{m-2}(E'\circ\psi\,\omega+E''\circ\psi\,d\psi\wedge d^c\psi)\wedge dd^cu_w\wedge\omega^{n-2}\\&\quad\leqslant C_4|z|^{-2n(m-1)/m}\Delta u_w,\end{aligned}\tag{6.2}$$

since $E''<0$. Set $\widetilde{G}:=G_{1/2}$, so that $\widetilde{G}\in C^\infty(\overline{B})$ and $\widetilde{G}=G$ near $\partial B$, and let $\varphi\in C_0^\infty(B(0,3/2))$ be such that $\varphi=1$ in $B$ and $0\leqslant\varphi\leqslant 1$ elsewhere. Then

$$\begin{aligned}\int_B dd^cu_w\wedge(dd^cG)^{m-1}\wedge\omega^{n-m}&=\int_{\partial B}d^c\widetilde{G}\wedge dd^cu_w\wedge(dd^c\widetilde{G})^{m-2}\wedge\omega^{n-m}\\&=\int_B dd^cu_w\wedge(dd^c\widetilde{G})^{m-1}\wedge\omega^{n-m}\\&\leqslant\int_{B(0,3/2)}\varphi dd^cu_w\wedge(dd^c\widetilde{G})^{m-1}\wedge\omega^{n-m}\\&=\int_{B(0,3/2)}u_wdd^c\varphi\wedge(dd^c\widetilde{G})^{m-1}\wedge\omega^{n-m}\\&\leqslant C_5||u||_{L^1(B(0,2))}.\end{aligned}\tag{6.3}$$

The Jensen formula combined with (6.2) and (6.3) gives

$$|u_1(w)|^p \leqslant (C_5||u||_{L^1(B(0,2))})^{p-1} \int_B |G|^p dd^c u_w \wedge (dd^c G)^{m-1} \wedge \omega^{n-m}$$
$$\leqslant (C_5||u||_{L^1(B(0,2))})^{p-1} \int_{B(w,1)} |G_w|^p |z-w|^{-2n(m-1)/m} \Delta u \, d\lambda$$

from which (and (1.2)) (6.1) easily follows. □

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