



ANNALES

DE

L'INSTITUT FOURIER

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Tome 55, n° 5 (2005), p. 1685-1734.

http://aif.cedram.org/item?id=AIF_2005__55_5_1685_0

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NON ABELIAN REIDEMEISTER TORSION AND VOLUME FORM ON THE $SU(2)$ -REPRESENTATION SPACE OF KNOT GROUPS

by Jérôme DUBOIS

Introduction: motivation and main ideas.

The space of conjugacy classes of irreducible $SU(2)$ -representations of a knot group is a real semi-algebraic set and its regular part is a 1-dimensional manifold (for more details see *infra*). The aim of this paper is to compare two *a priori* different constructions on this 1-dimensional manifold: one by means of Reidemeister torsion and another using Casson's original construction. Firstly, we associate to any regular $SU(2)$ -representation ρ of the knot group a non abelian Reidemeister torsion form on the first cohomology group of the knot exterior with coefficients in the adjoint representation associated to ρ . Secondly, we construct "à la Casson" a volume form on the 1-dimensional manifold consisting of conjugacy classes of regular representations, which appears to be a knot invariant. Finally, we prove that these *a priori* two different points of view are indeed equivalent (see Theorem 7.1).

The Reidemeister torsion was introduced for the first time in 1935 by K. Reidemeister in his work [Re] on the combinatorial classification of 3-dimensional lens spaces. The torsion is a combinatorial invariant which is not a homotopy invariant. It has become a fundamental tool in low-dimensional topology, see for example Turaev's monograph [Tu3]. Informally speaking, the Reidemeister torsion is a graded version of the

Keywords: Knot groups, representation space, volume form, Reidemeister torsion, Casson invariant, adjoint representation, $SU(2)$.

Math. classification: 57M25, 57Q10, 57M27.

determinant in the same way that the Euler characteristic is a graded version of the dimension.

In 1985, A. Casson constructed an integer valued invariant of integral homology 3-spheres which appeared extremely useful. The original definition of the Casson invariant is based on $SU(2)$ -representation spaces. Informally speaking, the Casson invariant of the homology sphere M counts algebraically the number of conjugacy classes of irreducible $SU(2)$ -representations of $\pi_1(M)$ in the same sense that the Lefschetz number of a map counts the number of fixed points.

In 1992, X.-S. Lin used Casson's construction to define an integer knot invariant. He indirectly proved that this invariant is equal to half the signature of knots (see [Li]). At first sight, the equality between these two apparently different quantities seems mysterious. In 2003, M. Heusener explained Lin's result using an orientation on the $SU(2)$ -representation space of knot groups. More precisely, given a knot $K \subset S^3$ let M_K denote its exterior and let $G_K = \pi_1(M_K)$ denote its group. In general the representation space of G_K in $SU(2)$ has singularities; to avoid this difficulty Heusener and Klassen introduced in [HK] the notion of regular representation. An irreducible $SU(2)$ -representation ρ of G_K is called regular if $\dim H_\rho^1(M_K) = 1$, where $H_\rho^*(M_K)$ denotes the $(\text{Ad} \circ \rho)$ -twisted cohomology of M_K . Let $\mathcal{R}eg(K)$ denote the set of conjugacy classes of regular representations of G_K . Heusener proved that $\mathcal{R}eg(K)$ is a canonically oriented 1-dimensional manifold (see [He, Section 1]).

In this article, we investigate a volume form on $\mathcal{R}eg(K)$ which is an invariant of K . In particular, we explain an intimate connection between this volume form and a non abelian Reidemeister torsion form on $H_\rho^1(M_K)$. Observe that our torsion is different from the one previously studied by E. Witten for surfaces in [Wi]. This difference is precisely due to the non-triviality of the second twisted cohomology group $H_\rho^2(M_K)$ of M_K . The definition of the Reidemeister torsion form for the exterior of K needs a reference basis for $H_\rho^2(M_K)$. Our construction of this basis is inspired by the work [Po, Cor. 3.23] of J. Porti. Let us describe the main ideas of the construction. For a regular representation ρ of G_K in $SU(2)$, the twisted cohomology groups of M_K satisfy $H_\rho^0(M_K) = 0$, $H_\rho^1(M_K) \cong \mathbb{R}$ and $H_\rho^2(M_K) \cong \mathbb{R}$. We produce a distinguished basis for $H_\rho^2(M_K)$ using the fundamental class of ∂M_K and the isomorphisms $H_\rho^2(M_K) \cong H_\rho^2(\partial M_K) \cong H^2(\partial M_K; \mathbb{Z}) \otimes \mathbb{R}$. With this reference basis, we construct a linear form τ_ρ^K on the 1-dimensional vector space $H_\rho^1(M_K)$.

This form is called the Reidemeister torsion form associated to K and ρ . It is a combinatorial invariant and maps any generator of $H_\rho^1(M_K)$ to the sign-determined $(\text{Ad} \circ \rho)$ -twisted Reidemeister torsion of M_K corresponding to this choice of bases for $H_\rho^*(M_K)$ and to the canonical cohomology orientation of M_K .

In another way, we construct according to Casson a canonical volume form on $\text{Reg}(K)$. This volume form is denoted ω^K . The construction of ω^K uses a plat presentation of the knot K to obtain a splitting of its exterior. This particular decomposition replaces the Heegaard splitting used in the original construction of the Casson invariant for homology 3-spheres (see [GM]).

Let $\widehat{R}(M_K)$ denote the space of conjugacy classes of irreducible $\text{SU}(2)$ -representations of G_K and assume that ρ is a regular representation. We show that the tangent space to $\widehat{R}(M_K)$ at the conjugacy class of ρ is isomorphic to $H_\rho^1(M_K)$. This isomorphism allows us to formulate the main result of the present work.

MAIN THEOREM (Theorem 7.1). — *If $\rho: G_K \rightarrow \text{SU}(2)$ is regular, then*

$$\tau_\rho^K = \omega_{[\rho]}^K.$$

The present paper is organized as follows. Sections 1 to 3 are reviews on $\text{SU}(2)$ -representation spaces, sign-determined Reidemeister torsion and twisted derivations respectively. In Section 4, we discuss the canonical cohomology orientation and the twisted cohomology of knot exteriors and also the notion of regular representations. In Section 5, we investigate the non abelian Reidemeister torsion form. Section 6 deals with the Casson-type construction of the natural volume form on the $\text{SU}(2)$ -representation space of knot groups. In Section 7, we explain how to interpret this volume form in terms of the Reidemeister torsion form. The rest of the article is devoted to the proof of the Main Theorem (Theorem 7.1), see Sections 9 and 10.

This paper is part of the author Ph.D. thesis [Du2].

1. Representation spaces.

Here we collect some well-known results about $\text{SU}(2)$ -representation spaces and introduce the notation used throughout this paper.

1.1. Some notation.

In all this article “connected” means “arc-connected”.

The fundamental group $\pi_1(X)$ of a connected topological space X is considered without specifying a base point since all the constructions we do are invariant by conjugation, see [Po, p. 9] for details.

The Lie group $SU(2)$ acts on its Lie algebra $\mathfrak{su}(2)$ via the adjoint representation $\text{Ad}_A : \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ defined by $\text{Ad}_A(x) = Ax A^{-1}$, where $A \in SU(2)$. As a manifold $SU(2)$ is identified with the 3-sphere S^3 and we identify the 2-sphere S^2 with the set of zero-trace matrices of $SU(2)$: $S^2 = \{A \in SU(2) \mid \text{Tr}(A) = 0\}$. The Lie algebra $\mathfrak{su}(2)$ is identified with the pure quaternions i.e. with the quaternions of the form $q = ai + bj + ck$. Recall that for all $A \in SU(2)$ there exist $\theta \in [0, \pi]$ and $P \in S^2$ such that $A = \cos(\theta) + \sin(\theta)P$. Moreover, the pair (θ, P) is unique if and only if $A \neq \pm 1$. Note that Ad_A is the rotation of angle 2θ which fixes P . The Lie algebra $\mathfrak{su}(2)$ is equipped with the usual scalar product defined by $\langle x, y \rangle = -\frac{1}{2} \text{Tr}(x \cdot y)$. This scalar product coincides with the Killing form of $\mathfrak{su}(2)$ multiplied by -2 . We think of $SO(3)$ as the base space of the 2-fold covering $\text{Ad} : SU(2) \rightarrow SO(3)$ given by $A \mapsto \text{Ad}_A$.

1.2. Representation spaces.

Given a finitely generated group G we let

$$R(G) = \text{Hom}(G; SU(2))$$

denote the space of $SU(2)$ -representations of G . This space is endowed with the compact-open topology. Here G is assumed to have the discrete topology and $SU(2)$ the usual one. A representation ρ of G in $SU(2)$ is called abelian or reducible (resp. central) if its image $\rho(G)$ is an abelian subgroup of $SU(2)$ (resp. is contained in the center $\{\pm 1\}$ of $SU(2)$). A representation is called irreducible if it is not abelian. We let $\tilde{R}(G)$ (resp. $C(G)$) denote the subspace of irreducible (resp. central) representations.

The compact Lie group $SU(2)$ acts on $R(G)$ by conjugation. We write $[\rho]$ for the conjugacy class of the representation $\rho \in R(G)$ and we let $SU(2)(\rho)$ denote its orbit. The action by conjugation factors through $SO(3) = SU(2)/\{\pm 1\}$ as a free action on the open subspace $\tilde{R}(G)$ and we set $\hat{R}(G) = \tilde{R}(G)/SO(3)$. In this way, we can think of $\hat{R}(G)$ as the base space of a principal $SO(3)$ -bundle with total space $\tilde{R}(G)$, see [GM, Section 3.A]. Furthermore, we know that $\hat{R}(G)$ admits the structure of a real affine semi-algebraic set (see [He, Section 2]). Here a subset of \mathbb{R}^m

is called a semi-algebraic set if it is a finite union of finite intersections of sets defined by polynomial equalities and inequalities.

NOTATION. — For a connected CW-complex W we write

$$R(W) = R(\pi_1(W)), \quad \tilde{R}(W) = \tilde{R}(\pi_1(W)), \quad \widehat{R}(W) = \widehat{R}(\pi_1(W)), \text{ etc.}$$

For a knot K in S^3 let $M_K = S^3 \setminus N(K)$ denote its exterior and let $G_K = \pi_1(M_K)$ denote its group. Here $N(K)$ is an open tubular neighbourhood of K . Recall that M_K is a compact 3-manifold bounded by the 2-torus ∂M_K .

Let G'_K be the subgroup generated by the commutators of G_K . The abelianization $G_K/G'_K \cong H_1(M_K; \mathbb{Z})$ is generated by the meridian m of K . So each abelian representation of G_K in $SU(2)$ is conjugate to one and only one of the $\varphi_\theta: G_K \rightarrow SU(2)$ defined by $\varphi_\theta(m) = \cos(\theta) + \sin(\theta)\mathbf{i}$, with $0 \leq \theta \leq \pi$.

NOTATION. — Let $\rho \in R(G)$ and let $g \in G$ be such that $\rho(g) \neq \pm \mathbf{1}$. There exists a unique pair $(\theta^\rho, P^\rho) \in (0, \pi) \times S^2$ such that

$$\rho(g) = \cos(\theta^\rho(g)) + \sin(\theta^\rho(g))P^\rho(g).$$

2. Review on Reidemeister torsion.

This section reviews the definitions of a volume form, of the notion of compatibility and of the sign-determined Reidemeister torsion of a CW-complex to set up the conventions which will be used. For more details, we refer to Milnor’s survey [Mi2] and to Turaev’s monographs [Tu2], [Tu3].

2.1. Volume forms and compatibility.

Let E be a n -dimensional real vector space. Let $E^* = \text{Hom}_{\mathbb{R}}(E, \mathbb{R})$ denote its dual space. A volume form v on E is a generator of the n th exterior power $\Lambda^n E^*$. Let E', E'' be two real finite dimensional vector spaces and let v', v'' be volume forms on E', E'' respectively. The direct sum $E' \oplus E''$ has a canonical volume form denoted $v' \wedge v''$.

Consider now a short exact sequence $0 \rightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \rightarrow 0$ of finite dimensional real vector spaces. Let v', v and v'' be volume forms on E', E and E'' respectively; let s denote a section of j , so that $i \oplus s: E' \oplus E'' \rightarrow E$ is an isomorphism. We say that the previous three

volume forms are *compatible* with each other if $v' \wedge v'' = (i \oplus s)^*(v)$. It is easy to verify that the notion of compatibility is independent of the chosen section s . Furthermore, if any two of the vector spaces E' , E and E'' are endowed with a volume form, then the third is endowed with a unique well-defined volume form which is compatible with the two others. In particular, if v, v'' are volume forms on E, E'' respectively, we write $v' = v/v''$ the unique compatible volume form on E' to indicate its dependence on v and v'' (cf. [Mi1, Section 1]). Compatibility will be used to set up the Multiplicativity Lemma (see Subsection 7.3) as well as in order to build up “new” volume forms (see Subsection 6.4).

2.2. Algebraic preliminaries.

Keep the notation of the previous subsection. For two ordered bases $\mathbf{a} = \{a_1, \dots, a_n\}$ and $\mathbf{b} = \{b_1, \dots, b_n\}$ of E we write

$$[\mathbf{a}/\mathbf{b}] = \det(p_{i,j})_{i,j},$$

where $a_i = \sum_{j=1}^n p_{i,j} b_j$, for all i .

The bases \mathbf{a} and \mathbf{b} are equivalent if $[\mathbf{a}/\mathbf{b}] = +1$. Two equivalent bases define the same volume form on E .

Let $C_* = (0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0 \rightarrow 0)$ be a chain complex of finite dimensional vector spaces over \mathbb{R} . For each i , consider $B_i = \text{im}(d_{i+1}: C_{i+1} \rightarrow C_i)$, $Z_i = \text{ker}(d_i: C_i \rightarrow C_{i-1})$ and the homology group $H_i = Z_i/B_i$. We suppose that C_i (resp. H_i) is endowed with a reference basis \mathbf{c}^i (resp. \mathbf{h}^i) for each i . In this case, we say that C_* is based and homology based. The Reidemeister torsion of the based and homology based chain complex C_* , with reference bases \mathbf{c}^* and \mathbf{h}^* , is defined as follows. Let \mathbf{b}^i denote a basis of B_i . We have the two short exact sequences:

$$(1) \quad 0 \rightarrow Z_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0,$$

$$(2) \quad 0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0.$$

Choose a lift $\tilde{\mathbf{b}}^i$ of \mathbf{b}^i in C_i and a lift $\tilde{\mathbf{h}}^i$ of \mathbf{h}^i in Z_i . Many choices of such lifts are of course possible. Using the exactness of (1) and (2) the sequences of vectors \mathbf{b}^i , $\tilde{\mathbf{h}}^i$ and $\tilde{\mathbf{b}}^{i-1}$ combine to yield a new basis $\mathbf{b}^i \tilde{\mathbf{h}}^i \tilde{\mathbf{b}}^{i-1}$ of C_i . With this notation, the Reidemeister torsion of C_* , with reference bases \mathbf{c}^* and \mathbf{h}^* , is (see [Tu2, Def. 3.1]):

$$(3) \quad \text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = \prod_{i=0}^n [\mathbf{b}^i \tilde{\mathbf{h}}^i \tilde{\mathbf{b}}^{i-1} / \mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{R}^*.$$

The torsion $\text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choice of the bases \mathbf{b}^i of B_i nor on the lifts $\tilde{\mathbf{b}}^i$ and $\tilde{\mathbf{h}}^i$.

We say that the based chain complex C_* is acyclic if $H_i(C_*)$ vanishes for all i . In this case, if $\Gamma_* = (C_* \rightarrow 0)$ denotes the left-shift chain complex obtained from C_* , i.e. $\Gamma_0 = 0$ and $\Gamma_i = C_{i-1}$ for $i \geq 1$, then

$$(4) \quad \text{tor}(\Gamma_*) = (\text{tor}(C_*))^{-1}.$$

The Reidemeister torsion of C_* does only depend on the equivalence classes of the reference bases \mathbf{c}^i and \mathbf{h}^i . More precisely, if \mathbf{c}'^i is a different basis of C_i and \mathbf{h}'^i a different one of H_i , then we have the *basis change formula*

$$(5) \quad \frac{\text{tor}(C_*, \mathbf{c}'^*, \mathbf{h}'^*)}{\text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)} = \prod_{i=0}^n \left(\frac{[\mathbf{c}'^i / \mathbf{c}^i]}{[\mathbf{h}'^i / \mathbf{h}^i]} \right)^{(-1)^i}.$$

CONVENTION. — Suppose that the chain complex C_* is not based and not homology based but is such that each C_i and each H_i is endowed with distinguished volume forms. The Reidemeister torsion of such C_* is computed—according to formula (5)—with respect to any reference bases of C_i and H_i which have volume 1.

2.3. The Reidemeister torsion of a CW-complex.

If formula (3) is used to define the Reidemeister torsion of a CW-complex, then we will fall into the well-known “up-to-sign ambiguity” of the Reidemeister torsion. To solve this problem V. Turaev has introduced a sign-determined Reidemeister torsion.

The sign-determined torsion. — Set

$$\alpha_i(C_*) = \sum_{k=0}^i \dim C_k \in \mathbb{Z}/2\mathbb{Z}, \quad \beta_i(C_*) = \sum_{k=0}^i \dim H_k \in \mathbb{Z}/2\mathbb{Z},$$

$$|C_*| = \sum_{k \geq 0} \alpha_k(C_*)\beta_k(C_*) \in \mathbb{Z}/2\mathbb{Z}.$$

The sign-determined Reidemeister torsion of C_* is the “sign-corrected” torsion

$$(6) \quad \text{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \text{tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{R}^*,$$

see [Tu1, Section 3.1] or [Tu3, formula (1.a)].

The Reidemeister torsion of a CW-complex. — Let W be a finite CW-complex; consider a representation $\rho: \pi_1(W) \rightarrow \text{SU}(2)$. The universal covering \widetilde{W} of W is endowed with the induced CW-complex structure and the group $\pi_1(W)$ acts on \widetilde{W} by the covering transformations. This action turns $C_*(\widetilde{W}; \mathbb{Z})$ into a chain complex of left $\mathbb{Z}[\pi_1(W)]$ -modules. Further observe that the Lie algebra $\mathfrak{su}(2)$ can be viewed as a left $\mathbb{Z}[\pi_1(W)]$ -module via the representation $\text{Ad} \circ \rho$. The $(\text{Ad} \circ \rho)$ -twisted cochain complex of W is

$$C^*(W; \text{Ad} \circ \rho) = \text{Hom}_{\pi_1(X)}(C_*(\widetilde{W}; \mathbb{Z}), \mathfrak{su}(2)).$$

The cochain complex $C^*(W; \text{Ad} \circ \rho)$ computes the $(\text{Ad} \circ \rho)$ -twisted cohomology of W . This cohomology is denoted $H_\rho^*(W)$. When $H_\rho^*(W) = 0$ we say that ρ is acyclic.

Let $\{e_1^{(i)}, \dots, e_{n_i}^{(i)}\}$ denote the set of i -dimensional cells of W . Choose a lift $\tilde{e}_j^{(i)}$ of the cell $e_j^{(i)}$ in \widetilde{W} and choose an arbitrary order and an arbitrary orientation for the cells $\tilde{e}_j^{(i)}$. Thus, for each i , $\mathbf{c}^i = \{\tilde{e}_1^{(i)}, \dots, \tilde{e}_{n_i}^{(i)}\}$ is a $\mathbb{Z}[\pi_1(W)]$ -basis of $C_i(\widetilde{W}; \mathbb{Z})$ and we consider the corresponding “dual” basis over \mathbb{R}

$$\mathbf{c}_{\mathfrak{su}(2)}^i = \{\tilde{e}_{1,i}^{(i)}, \tilde{e}_{1,j}^{(i)}, \tilde{e}_{1,k}^{(i)}, \dots, \tilde{e}_{n_i,i}^{(i)}, \tilde{e}_{n_i,j}^{(i)}, \tilde{e}_{n_i,k}^{(i)}\}$$

of $C^i(W; \text{Ad} \circ \rho) = \text{Hom}_{\pi_1(X)}(C_i(\widetilde{W}; \mathbb{Z}), \mathfrak{su}(2))$.

If \mathbf{h}^i is a basis of $H_\rho^i(W)$ then $\text{Tor}(C^*(W; \text{Ad} \circ \rho), \mathbf{c}_{\mathfrak{su}(2)}^*, \mathbf{h}^*) \in \mathbb{R}^*$ is well defined. The cells $\{\tilde{e}_j^{(i)}\}_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$ are in one-to-one correspondence with the cells of W and their order and orientation induce an order and an orientation for the cells $\{e_j^{(i)}\}_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$. We thus produce a basis over \mathbb{R} for $C^*(W; \mathbb{R})$ which is denoted \mathbf{c}^* .

Choose a *cohomology orientation* of W i.e. an orientation of the real vector space $H^*(W; \mathbb{R}) = \bigoplus_{i \geq 0} H^i(W; \mathbb{R})$; let \mathfrak{o} denote such an orientation.

Provide each vector space $H^i(W; \mathbb{R})$ with a reference basis h^i such that the basis $\mathbf{h}^* = \{h^0, \dots, h^{\dim W}\}$ of $H^*(W; \mathbb{R})$ is positively oriented with respect to the cohomology orientation \mathfrak{o} . Compute the sign-determined Reidemeister torsion $\text{Tor}(C^*(W; \mathbb{R}), \mathbf{c}^*, \mathbf{h}^*) \in \mathbb{R}^*$ of the resulting based and cohomology based chain complex $C^*(W; \mathbb{R})$ and consider its sign $\tau_0 = \text{sgn}(\text{Tor}(C^*(W; \mathbb{R}), \mathbf{c}^*, \mathbf{h}^*)) \in \{\pm 1\}$. Further observe that τ_0 does not depend on the choice of the positively oriented basis \mathbf{h}^* .

The sign-determined Reidemeister torsion of the cohomology oriented CW-complex W twisted by the representation $\text{Ad} \circ \rho$ is the product

$$\text{TOR}(W; \text{Ad} \circ \rho, \mathbf{h}^*, \mathfrak{o}) = \tau_0 \cdot \text{Tor}(C^*(W; \text{Ad} \circ \rho), \mathbf{c}_{\mathfrak{su}(2)}^*, \mathbf{h}^*) \in \mathbb{R}^*.$$

The torsion $\text{TOR}(W; \text{Ad} \circ \rho, \mathfrak{h}^*, \mathfrak{o})$ is called the $(\text{Ad} \circ \rho)$ -twisted Reidemeister torsion of W . It is well-defined. It does not depend on the choice of the lifts $\tilde{e}_j^{(i)}$ nor on the order and orientation of the cells (because they appear twice). Finally, it just depends on the conjugacy class of ρ .

One can prove that TOR is invariant under cellular subdivision, homeomorphism class and simple homotopy type. In fact, it is precisely the sign $(-1)^{|C^*|}$ in (6) which ensures all these particularly important properties of invariance (see [Du2, Chap. 2] for detailed proofs).

3. Twisted derivations and tangent bundle to representation spaces.

This section reviews the useful concept of twisted derivations.

3.1. Twisted derivations.

Let G be a finitely generated group and consider a representation $\rho: G \rightarrow \text{SU}(2)$.

An $(\text{Ad} \circ \rho)$ -twisted derivation is a map $d: G \rightarrow \mathfrak{su}(2)$ satisfying the cocycle condition $d(g_1 g_2) = d(g_1) + \text{Ad}_{\rho(g_1)} d(g_2)$, for all $g_1, g_2 \in G$. We let $\text{Der}_\rho(G)$ denote the set of $(\text{Ad} \circ \rho)$ -twisted derivations. Among twisted derivations we distinguish the inner ones. A map $\delta: G \rightarrow \mathfrak{su}(2)$ is an inner derivation if there exists $a \in \mathfrak{su}(2)$ such that $\delta(g) = a - \text{Ad}_{\rho(g)} a$, for all $g \in G$. We let $\text{Inn}_\rho(G)$ denote the set of inner derivations. Observe that $\text{Inn}_\rho(G) \cong \mathfrak{su}(2)$ for any irreducible representation ρ . Recall that

$$Z_\rho^1(G) \cong \text{Der}_\rho(G), \quad B_\rho^1(G) \cong \text{Inn}_\rho(G), \quad H_\rho^1(G) \cong \text{Der}_\rho(G)/\text{Inn}_\rho(G),$$

$$H_\rho^0(G) = \mathfrak{su}(2)^{\text{Ad} \circ \rho(G)} = \{v \in \mathfrak{su}(2) \mid v = \text{Ad}_{\rho(g)} v, \forall g \in G\}.$$

For each irreducible representation ρ of G we thus have the short exact sequence

$$(7) \quad 0 \rightarrow \mathfrak{su}(2) \longrightarrow \text{Der}_\rho(G) \longrightarrow H_\rho^1(G) \rightarrow 0.$$

3.2. Derivations and tangent spaces.

We begin by a digression on $\text{SL}_2(\mathbb{C})$ -representations. Let

$$R(G, \text{SL}_2(\mathbb{C})) = \text{Hom}(G; \text{SL}_2(\mathbb{C}))$$

denote the space of $\text{SL}_2(\mathbb{C})$ -representations of G endowed with the compact-open topology. To each representation $\rho \in R(G, \text{SL}_2(\mathbb{C}))$ we associate its

character $\chi_\rho : G \rightarrow \mathbb{C}$ defined by $\chi_\rho(g) = \text{Tr}(\rho(g))$ for all $g \in G$. The set of characters of G is called the representation variety and is denoted $X(G)$. In a way $X(G)$ is the “algebraic quotient” of $R(G, \text{SL}_2(\mathbb{C}))$ by the action by conjugation of $\text{PSL}_2(\mathbb{C})$ because the naive quotient $R(G, \text{SL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C})$ is not Hausdorff in general. It is well known that $R(G, \text{SL}_2(\mathbb{C}))$ and $X(G)$ have the structure of complex algebraic affine sets (see [CS]).

In [We], A. Weil proved that the Zariski tangent space to $R(G, \text{SL}_2(\mathbb{C}))$ at an irreducible representation ρ can be identified to a subspace of $Z^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho})$. Here $\mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}$ denotes the left $\mathbb{Z}[G]$ -module structure on $\mathfrak{sl}_2(\mathbb{C})$ induced by the adjoint representation. This inclusion is explicitly given by

$$(8) \quad T_\rho^{\text{Zar}} R(G, \text{SL}_2(\mathbb{C})) \longrightarrow Z^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}),$$

$$\frac{d\rho_t}{dt} \Big|_{t=0} \longmapsto \left\{ \begin{array}{l} G \rightarrow \mathfrak{sl}_2(\mathbb{C}), \\ g \mapsto \frac{d\rho_t(g)\rho(g^{-1})}{dt} \Big|_{t=0} \end{array} \right.$$

where $\rho_0 = \rho$ (cf. [Po, §3.1.3]). For an irreducible representation $\rho : G \rightarrow \text{SL}_2(\mathbb{C})$ the orbit $\text{SL}_2(\mathbb{C})(\rho) = \{\text{Ad}_A \circ \rho \mid A \in \text{PSL}_2(\mathbb{C})\}$ is isomorphic to $B_\rho^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho})$. So we get the natural inclusion

$$T_{\chi_\rho}^{\text{Zar}} X(G) \rightarrow Z^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}) / B_\rho^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}) = H^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}).$$

After this general digression we turn back to the case of $\text{SU}(2)$ -representations. Consider the involution $\sigma : \text{SL}_2(\mathbb{C}) \rightarrow \text{SL}_2(\mathbb{C})$ defined by $\sigma(A) = (\bar{A}^T)^{-1}$. Let $\text{SL}_2(\mathbb{C})^\sigma$ denote its set of fixed points. With this notation we have $\text{SL}_2(\mathbb{C})^\sigma = \text{SU}(2)$. Let σ_* be the linear map $D_1\sigma : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{sl}_2(\mathbb{C})$. We have $\sigma_*(x) = -\bar{x}^T$ and $\mathfrak{sl}_2(\mathbb{C})^{\sigma_*} = \mathfrak{su}(2)$. For every irreducible representation $\rho : G \rightarrow \text{SU}(2)$ we define the tangent space to $R(G) = (R(G, \text{SL}_2(\mathbb{C})))^\sigma$ at ρ to be

$$T_\rho R(G) = (T_\rho^{\text{Zar}} R(G, \text{SL}_2(\mathbb{C})))^{\sigma_*} \subset T_\rho^{\text{Zar}} R(G, \text{SL}_2(\mathbb{C})).$$

The equality $Z^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}) = Z_\rho^1(G) \otimes_{\mathbb{R}} \mathbb{C}$ provides the inclusion

$$(9) \quad T_\rho R(G) = (T_\rho^{\text{Zar}} R(G, \text{SL}_2(\mathbb{C})))^{\sigma_*} \hookrightarrow (Z^1(G; \mathfrak{sl}_2(\mathbb{C})_{\text{Ad} \circ \rho}))^{\sigma_*} = Z_\rho^1(G).$$

We also have

$$(10) \quad (\text{SL}_2(\mathbb{C})(\rho))^\sigma = \text{SU}(2)(\rho) \quad \text{for all } \rho \in \tilde{R}(G).$$

Equality (10) follows from the two following facts:

- 1) two irreducible representations of G in $SL_2(\mathbb{C})$ with the same character are conjugate by an element of $SL_2(\mathbb{C})$, see [CS, Prop. 1.5.2];
- 2) if two irreducible (i.e. non abelian) representations in $SU(2)$ are conjugate by an element of $SL_2(\mathbb{C})$, then they are conjugate by an element of $SU(2)$, see [Kl, proof of Prop. 15].

Inclusion (9) and equality (10) combine to yield

PROPOSITION 3.1. — *If G is the free group F_n (resp. the fundamental group of a punctured 2-sphere), then inclusion (9) gives rise to an isomorphism $T_{[\rho]}\widehat{R}(G) \xrightarrow{\sim} H^1_\rho(G)$ for every irreducible representation $\rho: G \rightarrow SU(2)$.*

4. Twisted cohomology for knot exteriors.

In this section, we turn to the geometric application of the previous algebraic preliminaries.

Assume that S^3 and K are oriented. Let ρ be any non boundary central representation of G_K in $SU(2)$, i.e. such that $\rho(\pi_1(\partial M_K)) \not\subset \{\pm \mathbf{1}\}$, and fix a presentation of the group of K of the form

$$(11) \quad G_K = \langle S_1, \dots, S_r \mid R_1, \dots, R_{r-1} \rangle.$$

The meridian m of K is only defined up to conjugation and is oriented by the convention $lk(K, m) = +1$, where lk denotes the linking number.

We begin by reviewing the notions of regularity and of μ -regularity for a representation (see [BZ], [HK], [Po, Def. 3.21] and [He, Section 1]).

4.1. Regular representations.

The long exact sequence in $(\text{Ad} \circ \rho)$ -twisted cohomology corresponding to the pair $(M_K, \partial M_K)$ and the Poincaré duality imply $\dim H^1_\rho(M_K) \geq 1$. So a $SU(2)$ -representation of a knot group is never acyclic.

If $\rho: G_K \rightarrow SU(2)$ is irreducible, then $H^0_\rho(M_K) = 0$. Moreover, we have $\dim H^2_\rho(M_K) = \dim H^1_\rho(M_K)$ because $\sum_i (-1)^i \dim H^i_\rho(M_K) = 3\chi(M_K) = 0$.

Regular representations. — Among irreducible representations we focus on the regular ones. An irreducible $SU(2)$ -representation ρ of G_K

is called regular if $\dim H_\rho^1(M_K) = 1$. It is easy to see that this notion is invariant under conjugation. We let $\mathcal{R}eg(K)$ denote the set of conjugacy classes of regular representations of G_K in $SU(2)$.

Example 1. — If K denotes a torus knot or the figure eight knot, then any irreducible representation of G_K in $SU(2)$ is regular (see the proof of Proposition 5.4).

M. Heusener and E. Klassen proved that if ρ is regular, then its conjugacy class $[\rho]$ is a smooth point of $\widehat{R}(M_K)$ and $\dim \widehat{R}(M_K) = 1$ in a neighbourhood of $[\rho]$ (cf. [HK, Prop. 1]). So, $\mathcal{R}eg(K)$ is a 1-dimensional manifold. Using properties of Subsection 3.2, we can establish

PROPOSITION 4.1. — *If $\rho: G_K \rightarrow SU(2)$ is regular, then inclusion (9) gives rise to an isomorphism $\varphi_{[\rho]}: T_{[\rho]} \widehat{R}(M_K) \xrightarrow{\sim} H_\rho^1(M_K)$.*

μ -regular representations. — Consider $\rho \in \widetilde{R}(M_K)$ and fix a simple closed oriented curve μ in ∂M_K . Let $N_{[\rho]}$ be a neighbourhood of $[\rho]$ in $\widehat{R}(M_K)$. Introduce the map $\theta_\mu: N_{[\rho]} \subset \widehat{R}(M_K) \rightarrow S^1$ which associates to $[\varrho] \in N_{[\rho]}$ half the angle of the rotation $\text{Ad} \circ \varrho(\mu)$. As we only consider knot exteriors remark that the map θ_μ is never zero and always a well-defined analytic map.

A representation ρ is called μ -regular if ρ is regular and if $\theta_\mu: \widehat{R}(M_K) \rightarrow S^1$ is a submersion at $[\rho]$ (see [Po, Prop. 3.26]). Observe that the notion of μ -regularity is invariant under conjugation and does not depend on the orientation of μ .

Example 2. — If K denotes a torus knot, then any irreducible representation of G_K in $SU(2)$ is m -regular (see [Du2, Example 1.43] and the proof of Proposition 5.4).

We use in the sequel the following alternative cohomology formulation of μ -regularity. Consider the linear form

$$f_\mu^\rho: H_\rho^1(M_K) \longrightarrow \mathbb{R}, \quad f_\mu^\rho(v) = \langle v(\mu), P^\rho \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product of $\mathfrak{su}(2)$.

PROPOSITION 4.2. — *The representation ρ is μ -regular if and only if f_μ^ρ is an isomorphism.*

It follows that if ρ is μ -regular, then we have a reference generator for $H_\rho^1(M_K) \cong \mathbb{R}$ denoted $h_\rho^{(1)}(\mu)$ which satisfies $f_\mu^\rho(h_\rho^{(1)}(\mu)) = +1$.

Proposition 4.2 is a consequence of the following claim.

CLAIM 4.3. — *If $\rho \in R(M_K)$ is regular, then $f_\mu^\rho(v) = D_{[\rho]}\theta_\mu(\varphi_{[\rho]}^{-1}(v))$, for all $v \in H_\rho^1(M_K)$.*

Proof. — Fix $v \in H_\rho^1(M_K)$ and choose a germ ρ_s in a neighbourhood of the origin such that $\varphi_{[\rho]}^{-1}(v) = \frac{d\rho_s}{ds}|_{s=0}$ with $\rho = \rho_0$. We can assume that ρ_s is regular because ρ is itself regular. There exists a family of matrices A_s such that

$$\rho_s(\mu) = A_s \begin{pmatrix} e^{i\theta_\mu(\rho_s)} & 0 \\ 0 & e^{-i\theta_\mu(\rho_s)} \end{pmatrix} A_s^{-1}.$$

Observe that $P^\rho = \text{Ad}_{A_0}(\mathbf{i})$ and

$$\frac{dA_s^{-1}}{ds} \Big|_{s=0} = -A_0^{-1} \left(\frac{dA_s}{ds} \Big|_{s=0} \right) A_0^{-1}.$$

Thus,

$$(12) \quad \frac{d\rho_s(g)\rho(g)^{-1}}{ds} \Big|_{s=0} = (\text{Id} - \text{Ad}_{\rho(\mu)}) \frac{dA_s}{ds} \Big|_{s=0} A_0^{-1} + \frac{d\theta_\mu(\rho_s)}{ds} \Big|_{s=0} P^\rho.$$

Taking into account the fact that $(\text{Id} - \text{Ad}_{\rho(\mu)})P^\rho = 0$ we deduce that the term $(\text{Id} - \text{Ad}_{\rho(\mu)}) \frac{dA_s}{ds} \Big|_{s=0} A_0^{-1}$ in equation (12) is orthogonal to P^ρ . Thus,

$$f_\mu^\rho(v) = \left\langle \frac{d\rho_s(g)\rho(g)^{-1}}{ds} \Big|_{s=0}, P^\rho \right\rangle = \frac{d\theta_\mu(\rho_s)}{ds} \Big|_{s=0}. \quad \square$$

4.2. Cohomology orientation of knot exteriors.

In this subsection, we equip knot exteriors with the canonical cohomology orientation (see [Tu3, Chap. V.3] for more details).

We associate to the presentation (11) of G_K the 2-dimensional CW-complex X_K in the usual way. The 0-skeleton of X_K consists of a single point $e^{(0)}$, the 1-skeleton X_K^1 is a wedge of r oriented circles $e_1^{(1)}, \dots, e_r^{(1)}$ corresponding to the generators S_i . Finally, X_K is obtained from X_K^1 by gluing $r - 1$ closed 2-cells $e_1^{(2)}, \dots, e_{r-1}^{(2)}$ attached using the relations R_j . We have $\pi_1(X_K) = G_K$. In [Wa], F. Waldhausen proved that the Whitehead group of a knot group is trivial. As a result X_K has the same simple homotopy type as M_K . So, the CW-complex X_K can be used to compute the Reidemeister torsion of M_K .

Set $\mathfrak{X}_i = C^{2-i}(X_K; \mathbb{R})$ for $i = 0, 1, 2$. For consistency reasons, we think of \mathfrak{X}_* as a chain complex of real vector spaces. Explicitly

$$(13) \quad \mathfrak{X}_* = 0 \rightarrow \mathbb{R}e^{(0)} \longrightarrow \bigoplus_{j=1}^r \mathbb{R}e_j^{(1)} \longrightarrow \bigoplus_{k=1}^{r-1} \mathbb{R}e_k^{(2)} \rightarrow 0.$$

Observe that $\chi(\mathfrak{X}_*) = 0$ and that \mathfrak{X}_* is not acyclic: $H_i(\mathfrak{X}_*) \cong H^{2-i}(M_K; \mathbb{R})$, for $i = 0, 1, 2$.

Let $[[\text{pt}]] \in H^0(M_K; \mathbb{Z})$ be the cohomology class of a point and let $m^* : m \mapsto 1$ be the dual of m . We base $H_*(\mathfrak{X}_*) = H^0(M_K; \mathbb{R}) \oplus H^1(M_K; \mathbb{R})$ with $\{[[\text{pt}]], m^*\}$. In the sequel, we assume that \mathfrak{X}_* endows the cohomology orientation induced by this basis. Moreover we always compute the $(\text{Ad} \circ \rho)$ -twisted Reidemeister torsion of knot exteriors with respect to this canonical cohomology orientation.

4.3. The knot exterior twisted chain complex.

Let ρ be a non boundary central representation of G_K . The *knot exterior twisted chain complex* is the chain complex \mathcal{X}_*^ρ defined by $\mathcal{X}_i^\rho = C^{2-i}(X_K; \text{Ad} \circ \rho)$ for $i = 0, 1, 2$. As in Subsection 4.2 we think of \mathcal{X}_*^ρ as a chain complex of real vector spaces. The presentation (11) of G_K provides

$$(14) \quad \mathcal{X}_*^\rho = 0 \rightarrow \mathfrak{su}(2) \xrightarrow{d_2^\rho} \mathfrak{su}(2)^r \xrightarrow{d_1^\rho} \mathfrak{su}(2)^{r-1} \rightarrow 0.$$

The boundary operators d_1^ρ and d_2^ρ are induced by the usual ones of $C_*(\widetilde{M}_K, \mathbb{Z})$ obtained using the Fox differential calculus (see [CF, Chap. 7]). If we write $g \circ x = \text{Ad}_{\rho(g)}(x)$, for $g \in G_K$ and $x \in \mathfrak{su}(2)$, then

$$d_2^\rho(x) = ((1 - S_1) \circ x, \dots, (1 - S_r) \circ x) \quad \text{for all } x \in \mathfrak{su}(2),$$

$$d_1^\rho((x_j)_{1 \leq j \leq r}) = \left(\sum_{j=1}^r \frac{\partial R_i}{\partial S_j} \circ x_j \right)_{1 \leq i \leq r-1} \quad \text{for all } (x_j)_{1 \leq j \leq r} \in \mathfrak{su}(2)^r.$$

4.4. The Reidemeister torsion and the Alexander polynomial.

In the previous subsections, we focus on non abelian representations. In the present one, let us make a digression on abelian representations. The aim of this subsection is to compute the Reidemeister torsion of the exterior of K twisted by the adjoint representation associated to an abelian representation of G_K in terms of the Alexander polynomial of K .

Recall that an abelian representation of G_K in $SU(2)$ is entirely determined by its value on the meridian m of K . Let $\varphi_\theta: G_K \rightarrow SU(2)$ be the abelian representation defined by $\varphi_\theta(m) = \cos(\theta) + \sin(\theta)\mathbf{i}$ with $0 < \theta < \pi$ (i.e. φ_θ is supposed to be non boundary central). When $e^{2i\theta}$ is not a zero of the Alexander polynomial Δ_K of K we say that φ_θ is regular. In this case, E. Klassen proved in [Kl, Theorem 19] that $H_{\varphi_\theta}^i(M_K) \cong H^i(M_K; \mathbb{Z}) \otimes \mathbb{R}$, for all i . We can also remark that $H_{\varphi_\theta}^0(M_K) = \ker d_2^{\varphi_\theta}$ is generated by $h^{(0)} = \mathbf{i}$ (the common fixed axis of the rotations $\text{Ad} \circ \varphi_\theta(S_j)$, $1 \leq j \leq r$) and $H_{\varphi_\theta}^1(M_K)$ is generated by $h^{(1)} = \mathbf{i}_1 + \dots + \mathbf{i}_{2n}$, where \mathbf{i}_k is the vector in $\mathfrak{su}(2)^{2n}$ of which all entries are zero except the one of index k which is equal to \mathbf{i} .

With this notation we have (see [Mil, Theorem 4], [Tu1, §1.2] and [Du2, §2.5])

PROPOSITION 4.4. — *Let φ_θ be a regular abelian representation with $0 < \theta < \pi$. The $(\text{Ad} \circ \varphi_\theta)$ -twisted Reidemeister torsion of M_K calculated in the basis $\{h^{(0)}, h^{(1)}\}$ of $H_{\varphi_\theta}^*(M_K)$ and with respect to the canonical cohomology orientation of M_K satisfies*

$$(15) \quad \text{TOR}(M_K; \text{Ad} \circ \varphi_\theta, \{h^{(0)}, h^{(1)}\}) = \frac{4 \sin^2(\theta)}{|\Delta_K(e^{2i\theta})|^2}.$$

Here Δ_K denotes the Alexander polynomial of K .

Remark 1. — If K is the trivial knot, then

$$\text{TOR}(M_K; \text{Ad} \circ \varphi_\theta, \{h^{(0)}, h^{(1)}\}) = 4 \sin^2(\theta)$$

is the twisted Reidemeister torsion of the solid torus M_K .

Ideas of the proof. — We just give the main ideas and leave the detailed computations. For all j , in the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, we have

$$\text{Ad} \circ \varphi_\theta(S_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta) \end{pmatrix} \in \text{Aut}(\mathfrak{su}(2)).$$

Take the complexification of the Lie algebra $\mathfrak{su}(2)$. The three subspaces respectively generated by \mathbf{i} , $\mathbf{j} - i\mathbf{k}$ and $\mathbf{j} + i\mathbf{k}$ define a splitting of $\mathfrak{su}(2)$. The action of $\text{Ad} \circ \varphi_\theta$ leaves these subspaces invariant: it acts trivially on the first factor, as the multiplication by $e^{2i\theta}$ on the second one and as the

multiplication by $e^{-2i\theta}$ on the third one. The factor corresponding to the subspace generated by \mathbf{i} contributes trivially because $H_1(M_K)$ is torsion free. To compute the other contributions recall that the Reidemeister torsion twisted by the abelianization representation is (with our conventions for the Reidemeister torsion) equal to $(t - 1)/\Delta_K(t)$, see [Mil]. This result implies that the factor corresponding to the subspace generated by $\mathbf{j} - i\mathbf{k}$ contributes by $(e^{2i\theta} - 1)/\Delta_K(e^{2i\theta})$ and that the factor corresponding to the subspace generated by $\mathbf{j} + i\mathbf{k}$ contributes by $(e^{-2i\theta} - 1)/\Delta_K(e^{-2i\theta})$. The product of these three contributions gives equation (15). \square

5. Non abelian Reidemeister torsion.

In this section, we investigate the non abelian Reidemeister torsion for knots in S^3 . For this purpose, we define a reference basis for the second twisted cohomology group $H_\rho^2(M_K)$. The main idea of the construction is to look at the restriction of the $SU(2)$ -representations of G_K to the peripheral subgroup of K (see Subsections 5.1–5.2). We assume that S^3 and $K \subset S^3$ are oriented. Later we shall see that the definition of the torsion does not depend on the orientation of K but depends on the one of S^3 (see Proposition 5.3).

5.1. Twisted cohomology of the torus.

The exterior of K is oriented and is bounded by the 2-torus ∂M_K . This boundary inherits an orientation by the convention “*the inward pointing normal vector in the last position*”. Let $\text{int}(\cdot, \cdot)$ be the intersection form on ∂M_K induced by this orientation. The peripheral subgroup $\pi_1(\partial M_K)$ is generated by the meridian-longitude system m, ℓ of K . Here m is oriented by the convention $lk(K, m) = +1$ and ℓ is oriented using the rule $\text{int}(m, \ell) = +1$.

Observe that if $\rho \in \tilde{R}(G_K)$ then $\rho(m) \neq \pm 1$ (because m normally generates G_K). So there exists a unique pair $(\theta, P^\rho) \in (0, \pi) \times S^2$ such that $\rho(m) = \cos(\theta) + \sin(\theta)P^\rho$. Notice that P^ρ generates the common fixed axis of the rotations $\text{Ad} \circ \rho(\pi_1(\partial M_K))$ and thus generates the 1-dimensional space $H_\rho^0(\partial M_K)$.

The usual scalar product of $\mathfrak{su}(2)$ induces a cup-product

$$\cup : H_\rho^p(\partial M_K) \times H_\rho^q(\partial M_K) \longrightarrow H^{p+q}(\partial M_K; \mathbb{R}).$$

This cup-product is non-degenerated and gives an explicit isomorphism between $H_\rho^*(\partial M_K)$ and $H^*(\partial M_K; \mathbb{R})$ (see [Po, Prop. 3.18] for a proof).

We summarize this in the following result.

LEMMA 5.1. — *The map $\phi_{P\rho}^* : H_\rho^*(\partial M_K) \rightarrow H^*(\partial M_K; \mathbb{R})$, defined by $\phi_{P\rho}^*(z) = P\rho \cup z$, is a natural isomorphism.*

5.2. Reference basis of the second twisted cohomology group of M_K .

Recall that $H_\rho^1(M_K) \cong \mathbb{R}$ and $H_\rho^2(M_K) \cong \mathbb{R}$ for all regular representation ρ of G_K in $SU(2)$. The construction of the reference generator of $H_\rho^2(M_K)$ is based on the following lemma (see [Po, Cor. 3.23]).

LEMMA 5.2. — *Let $\rho : G_K \rightarrow SU(2)$ be a regular representation. The inclusion $\partial M_K \hookrightarrow M_K$ induces a natural isomorphism*

$$i^* : H_\rho^2(M_K) \longrightarrow H_\rho^2(\partial M_K).$$

Proof. — Associated to the pair $(M_K, \partial M_K)$ is the long exact sequence in $(\text{Ad} \circ \rho)$ -twisted cohomology

$$\dots \rightarrow H_\rho^1(\partial M_K) \longrightarrow H_\rho^2(M_K, \partial M_K) \longrightarrow H_\rho^2(M_K) \xrightarrow{i^*} H_\rho^2(\partial M_K) \rightarrow 0.$$

Thus $\text{rk } i^* = \dim H_\rho^2(\partial M_K) = \dim H^2(\partial M_K; \mathbb{R}) = 1$ (see Lemma 5.1). Furthermore $\dim H_\rho^2(M_K) = 1$. So i^* is an isomorphism. □

Joining together Lemmas 5.1 and 5.2 we conclude that the composition

$$\phi_{P\rho}^{(2)} \circ i^* : H_\rho^2(M_K) \longrightarrow H_\rho^2(\partial M_K) \longrightarrow H^2(\partial M_K; \mathbb{R})$$

is an isomorphism. Let c denote the generator of $H^2(\partial M_K; \mathbb{Z})$ corresponding to the fundamental class $[[\partial M_K]] \in H_2(\partial M_K; \mathbb{Z})$ induced by the orientation of ∂M_K . The reference generator $h_\rho^{(2)}$ of $H_\rho^2(M_K)$ is defined by

$$(16) \quad h_\rho^{(2)} = (\phi_{P\rho}^{(2)} \circ i^*)^{-1}(c).$$

5.3. Definition of the volume form τ^K .

Let $\rho \in R(G_K)$ be regular. The Reidemeister torsion form τ^K at ρ is the linear form $\tau_\rho^K : T_{[\rho]} \widehat{R}(M_K) \rightarrow \mathbb{R}$ defined by

$$\tau_\rho^K(v) = \text{TOR}(M_K; \text{Ad} \circ \rho, \{\varphi_{[\rho]}(v), h_\rho^{(2)}\}), \quad \text{for all } v \neq 0.$$

Here $\varphi_{[\rho]} : T_{[\rho]} \widehat{R}(M_K) \xrightarrow{\sim} H_\rho^1(M_K)$ (see Proposition 4.1) and $h_\rho^{(2)}$ is the reference generator of $H_\rho^2(M_K)$ (see Equation (16)).

With the notation of Subsections 4.2–4.3 we observe

$$\tau_\rho^K(v) = \text{sgn}(\text{Tor}(\mathfrak{X}_*)) \cdot \text{Tor}(\mathcal{X}_*^\rho, \{\varphi_{[\rho]}(v), h_\rho^{(2)}\}).$$

The linear form τ_ρ^K just depends on the $\text{SO}(3)$ -conjugacy class $[\rho]$ of ρ , thus $\tau^K: [\rho] \mapsto \tau_\rho^K$ is a well-defined 1-volume form on $\text{Reg}(K)$.

Here are some remarks.

Remark 2. — Note that $\text{Tor}(\mathfrak{X}_*)$ is ± 1 , because the homology of M_K with integral coefficients is torsion free (see [Tu1]). So that $\text{sgn}(\text{Tor}(\mathfrak{X}_*)) = \text{Tor}(\mathfrak{X}_*)$.

Remark 3. — For a regular representation ρ , there exist a unique $P^\rho \in S^2$ and a unique $\bar{P}^\rho \in S^2$ such that $\rho(m) = \cos(\theta) + \sin(\theta)P^\rho$ and $\rho(m) = \cos(2\pi - \theta) + \sin(2\pi - \theta)\bar{P}^\rho$ where $0 < \theta < \pi$. If we repeat the construction of the torsion form replacing the vector P^ρ by \bar{P}^ρ , then we obtain the opposite volume form $-\tau_\rho^K$ (because $h_\rho^{(2)}$ is replaced by $-h_\rho^{(2)}$, see Lemma 5.1).

PROPOSITION 5.4. — *If $K \subset S^3$ is a knot, then τ^K has the following properties:*

- 1) τ^K does not depend on the orientation of K .
- 2) If K^* denotes the mirror image of K , then we have

$$(\text{Reg}(K^*), \tau^{K^*}) = (-\text{Reg}(K), -\tau^K).$$

Proof. — It requires two steps.

1) If we reverse the orientation of K , then the meridian m is changed in m^{-1} and the orientation of M_K is not affected. So that the cohomology orientation is reversed and the reference generator $h_\rho^{(2)}$ defined by (16) is replaced by $-h_\rho^{(2)}$. Formula (5) implies that $\text{Tor}(\mathfrak{X}_*)$ and $\text{Tor}(\mathcal{X}_*^\rho)$ change their sign at the same time.

2) If we reverse the orientation of S^3 , then the orientation of M_K and the one of ∂M_K change at the same time. Hence the meridian m is changed in m^{-1} . So that the cohomology orientation is reversed. Thus $\text{Tor}(\mathcal{X}_*^\rho)$ changes its sign. Moreover the reference generator $h_\rho^{(2)}$ defined by (16) is unchanged. Thus $\text{Tor}(\mathcal{X}_*^\rho)$ does not change.

Thus $\text{Tor}(\mathfrak{X}_*)$ changes its sign. □

5.4. An explicit computation.

Let $q > 0$ be an odd integer. Let K_q denote the torus knot of type $(2, q)$ and let M_q denote its exterior. Recall that the group G_q of K_q admits the presentation $G_q = \langle x, y \mid x^2 = y^q \rangle$. In G_q the meridian m of K_q can be written as $m = xy^{\frac{1}{2}(1-q)}$ and the longitude l as $l = x^2m^{-2q}$.

Before making the explicit computation of τ^{K_q} we give a parametrization of $\widehat{R}(M_q)$ (cf. [K1, Theorem 1]). Consider the map $\rho_{\ell,t}: G_q \rightarrow \text{SU}(2)$ defined by $\rho_{\ell,t}(x) = \mathbf{i}$ and

$$\rho_{\ell,t}(y) = \cos\left(\frac{(2\ell - 1)\pi}{q}\right) + \sin\left(\frac{(2\ell - 1)\pi}{q}\right)(\cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j}),$$

for $0 < t < 1$ and $\ell \in \{1, \dots, \frac{1}{2}(q - 1)\}$. For all t and ℓ , $\rho_{\ell,t}$ defines an irreducible representation of G_q . In fact, all irreducible representations of G_q into $\text{SU}(2)$ arise in this way; more precisely, any $\rho \in \widetilde{R}(M_q)$ is conjugate to one and only one of the $\rho_{\ell,t}$. With this notation we show

PROPOSITION 5.4. — We have $\mathcal{R}eg(K_q) = \widehat{R}(M_q)$. The Reidemeister torsion form associated to the right-hand torus knot K_q of type $(2, q)$ satisfies

$$\tau_{[\rho_{\ell,t}]}^{K_q}\left(\frac{d\rho_{\ell,t}}{ds}\Big|_{s=t}\right) = -\frac{8}{q} \sin^2\left(\frac{(2\ell - 1)\pi}{q}\right) \frac{d\theta_m^{\rho_{\ell,t}}}{ds}\Big|_{s=t}$$

where $\theta_m^{\rho_{\ell,t}} = \theta_m(\rho_{\ell,t}) = \arccos((-1)^{\ell-1} \cos((2\ell - 1)\pi/2q) \cos(\pi t))$.

Proof. — The proof is a direct computation. Essentially it consists in the explicit determination of the generator $h_{\rho_{\ell,t}}^{(2)}$ of $H_{\rho_{\ell,t}}^2(M_q)$ defined by equation (16).

1) For all t and ℓ , observe that $\rho_{\ell,t}(m) = \cos(\theta_m^{\rho_{\ell,t}}) + \sin(\theta_m^{\rho_{\ell,t}})P^{\rho_{\ell,t}}$ with

$$P^{\rho_{\ell,t}} = (-1)^{\ell-1} \frac{\sin((2\ell - 1)\pi/2q) \mathbf{i} - \cos((2\ell - 1)\pi/2q) \sin(\pi t) \mathbf{k}}{\sqrt{\sin^2((2\ell - 1)\pi/2q) + \cos^2((2\ell - 1)\pi/2q) \sin^2(\pi t)}}.$$

2) Let X_q denote the 2-dimensional CW-complex associated to the presentation $G_q = \langle x, y \mid r \rangle$, where $r = x^2y^{-q}$. The corresponding twisted chain complex $\mathcal{Z}_*^{\rho_{\ell,t}} = C^{2-*}(X_q; \text{Ad} \circ \rho_{\ell,t})$ is

$$\mathcal{Z}_*^{\rho_{\ell,t}} = 0 \rightarrow \mathfrak{su}(2) \xrightarrow{d_2^{\rho_{\ell,t}}} \mathfrak{su}(2) \oplus \mathfrak{su}(2) \xrightarrow{d_1^{\rho_{\ell,t}}} \mathfrak{su}(2) \rightarrow 0$$

and the coboundary operators are

$$d_2^{\rho_{\ell,t}} = \begin{pmatrix} \text{Ad}_X - \text{Id} \\ \text{Ad}_Y - \text{Id} \end{pmatrix}, \quad d_1^{\rho_{\ell,t}} = (\text{Id} + \text{Ad}_X, -(\text{Id} + \text{Ad}_Y + \dots + \text{Ad}_{Y^{q-1}})).$$

Here $X = \rho_{\ell,t}(x)$, $Y = \rho_{\ell,t}(y)$. Observe that $\text{coker } d_1^{\rho_{\ell,t}} \cong \mathbb{R}$ is generated by \mathbf{k} . So $\dim H_{\rho_{\ell,t}}^2(M_q) = 1$ and thus $\rho_{\ell,t}$ is regular.

Now, we compute the homomorphism $i^* : H_{\rho_{\ell,t}}^2(M_q) \rightarrow H_{\rho_{\ell,t}}^2(\partial M_q)$ induced by $i : \partial M_q \hookrightarrow M_q$. For this purpose, we write the relation of $\pi_1(\partial M_q)$ as follows:

$$R = \ell m \ell^{-1} m^{-1} = x r x^{-1} (x y^{\frac{1}{2}(1-q)}) r^{-1} (x y^{\frac{1}{2}(1-q)})^{-1}.$$

So that the inclusion i induces, at the level of 2-cocycles, the map $i^* : \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ defined by $i^*(z) = \text{Ad}_X(z) - \text{Ad}_W(z)$, where $W = \rho_{\ell,t}(m)$. Thus, the reference generator $h_{\rho_{\ell,t}}^{(2)} \in H_{\rho_{\ell,t}}^2(M_q)$ is the equivalence class of the element $u^{(2)} = \langle \text{Ad}_X(\mathbf{k}) - \text{Ad}_W(\mathbf{k}), P^{\rho_{\ell,t}} \rangle^{-1} \mathbf{k} \in \mathfrak{su}(2)$.

3) The computation of the torsion for the torus knot K_q requires a reference generator for $H_{\rho_{\ell,t}}^1(M_q)$. Observe that $v = \begin{pmatrix} \mathbf{k} \\ 0 \end{pmatrix} \in \ker d_1^{\rho_{\ell,t}} \setminus \text{im } d_2^{\rho_{\ell,t}}$. Furthermore $v(m) = v(x) = \mathbf{k}$, so that the equivalence class of the element $u^{(1)} = \langle \mathbf{k}, P^{\rho_{\ell,t}} \rangle^{-1} v$ generates $H_{\rho_{\ell,t}}^1(M_q)$.

4) A direct determinant computation gives

$$\text{Tor}(\mathcal{Z}_*^{\rho_{\ell,t}}, \{u^{(1)}, u^{(2)}\}) = -\frac{8}{q} \sin^2 \left(\frac{(2\ell - 1)\pi}{q} \right).$$

For the right-hand torus knot we have

$$\tau_0 = \text{sign}(\text{Tor}(C^*(X_q; \mathbb{R}))) = +1.$$

Thus

$$\begin{aligned} \text{TOR}(M_q; \text{Ad} \circ \rho_{\ell,t}, \{u^{(1)}, u^{(2)}\}) &= \tau_0 \text{Tor}(\mathcal{Z}_*^{\rho_{\ell,t}}, \{u^{(1)}, u^{(2)}\}) \\ &= -\frac{8}{q} \sin^2 \left(\frac{(2\ell - 1)\pi}{q} \right). \end{aligned}$$

The basis change formula (5) provides

$$\tau_{[\rho_{\ell,t}]}^{K_q} \left(\frac{d\rho_{\ell,s}}{ds} \Big|_{s=t} / u^{(1)} \right) = \left[\frac{d\rho_{\ell,s}}{ds} \Big|_{s=t} \right] \cdot \text{TOR}(M_q; \text{Ad} \circ \rho_{\ell,t}, \{u^{(1)}, u^{(2)}\}),$$

and because

$$\left[\frac{d\rho_{\ell,s}}{ds} \Big|_{s=t} / u^{(1)} \right] = \left\langle \frac{d\rho_{\ell,s}(m)\rho_{\ell,t}(m)^{-1}}{ds} \Big|_{s=t}, P^{\rho_{\ell,t}} \right\rangle = \frac{d\theta_m^{\rho_{\ell,s}}}{ds} \Big|_{s=t}$$

we obtain the required formula of the proposition. □

Remark 4. — Proposition 5.4 can be deduced from the computations made by J. Park for graph manifolds in [Pa]. However the distinguished

generator of $H^2_{\rho_{\ell,t}}(M_q)$ must be explicitly known which is the main difficult part of the preceding proof.

6. A volume form associated to a knot.

In this section, we review in detail the “natural” Casson-type construction of the volume form on $\mathcal{R}eg(K)$, see [Du1] and [Du3].

6.1. The “base \wedge fiber” condition.

A volume form v on a n -dimensional manifold is a nowhere vanishing differential n -form. In the sequel, we will make a great use of the “base \wedge fiber” condition which is the following. Given two volume forms v and w on the manifolds M^m and N^n respectively, a submersion $f: M \rightarrow N$ and a point $y \in N$, then the subspace $f^{-1}(y) \subset M$ is a submanifold of dimension $m - n$, the tangent space $T_x f^{-1}(y)$ is the kernel of $D_x f$ and we have the short exact sequence

$$0 \rightarrow T_x f^{-1}(y) \xrightarrow{i} T_x M \xrightarrow{D_x f} T_y N \rightarrow 0.$$

The submanifold $f^{-1}(y)$ is endowed with the unique volume form ω such that, for all $x \in f^{-1}(y)$, $\omega_x = v_x/w_y$ i.e. $\omega_x \wedge w_y = (i \oplus s)^*(v_x)$, s being a section of $D_x f$.

6.2. Plat decomposition and splitting of knot exteriors.

Each knot $K \subset S^3$ can be presented as a $2n$ -plat $\hat{\zeta}$. Here $\hat{\zeta}$ is obtained from the $2n$ -braid $\zeta \in B_{2n}$ by closing it with $2n$ half circles as on Figure 1. Such a presentation of K as plat gives rise to a splitting of its exterior of the form $M_K = B_1 \cup_S B_2$, where B_1, B_2 are two handlebodies of genus n and $S = B_1 \cap B_2 = S^2 \setminus N(K)$ is a $2n$ -punctured 2-sphere (cf. [He, Section 3] and [Du3]). This decomposition is similar to the Heegaard splitting used in the construction of the Casson invariant. It also gives rise to special systems of generators for $\pi_1(B_i)$, $i = 1, 2$, and for $\pi_1(S)$ respectively denoted $\mathcal{T}_i = \{t_j^{(i)} \mid 1 \leq j \leq n\}$ and $\mathcal{S} = \{s_j \mid 1 \leq j \leq 2n\}$ (see Figure 1). In fact, these systems depend on the orientation of S^3 .

The group $\pi_1(B_i)$ is the free group with basis

$$\mathcal{T}_i = \{t_j^{(i)} \mid 1 \leq j \leq n\}, \quad i = 1, 2.$$

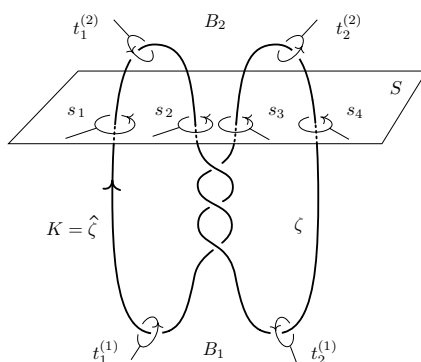


Figure 1. Special systems of generators

The group $\pi_1(S)$ admits the finitely presentation

$$\pi_1(S) = \langle s_1, \dots, s_{2n} \mid s_1 \cdots s_{2n} \rangle.$$

Furthermore, each element of \mathcal{T}_i and S is a meridian of K . In particular, all these elements are conjugate in G_K . This obvious remark will be essential in the sequel.

The inclusions $S \hookrightarrow B_i$ and $B_i \hookrightarrow M_K$, $i = 1, 2$, give rise to the commutative diagram

$$(17) \quad \begin{array}{ccc} & \pi_1(B_1) & \\ \nearrow \kappa_1 & & \searrow p_1 \\ \pi_1(S) & & \pi_1(M_K) = G_K \\ \searrow \kappa_2 & & \nearrow p_2 \\ & \pi_1(B_2) & \end{array}$$

Each homomorphism of the previous diagram is onto. The Seifert-Van Kampen Theorem and diagram (17) combine to yield the following presentation of G_K

$$(18) \quad G_K = \langle t_i^{(1)}, t_i^{(2)}, 1 \leq i \leq n \mid p_1 \circ \kappa_1(s_j) = p_2 \circ \kappa_2(s_j), 1 \leq j \leq 2n-1 \rangle.$$

Observe that (18) is a particular Wirtinger presentation of G_K .

6.3. Representation subspaces.

Corresponding to a plat decomposition of K —which gives rise to the splitting $M_K = B_1 \cup_S B_2$ —we introduce certain special representation subspaces of $R(B_i)$, $i = 1, 2$, and $R(S)$, see [He, Section 3] and [Du3].

Consider a representation ρ of G_K in $SU(2)$ and look at its restrictions $\rho_i = \rho \circ p_i$ and $\rho_S = \rho \circ p_i \circ \kappa_i$. In this way we do not obtain all the representations of $\pi_1(B_i)$ or $\pi_1(S)$ because all the generators of these groups are conjugate to the meridian of K . For this reason we not only consider groups but marked groups. A marked group is a pair (G, \mathcal{G}) where G is a finitely generated group and \mathcal{G} is a fixed finite set of generators of G . For the marked group (G, \mathcal{G}) let $R^{\mathcal{G}}(G)$ denote the subset of $R(G) \setminus C(G)$ defined by

$$(19) \quad R^{\mathcal{G}}(G) = \{ \rho \in R(G) \setminus C(G) \mid \text{Tr}\rho(s) = \text{Tr}\rho(t), \forall s, t \in \mathcal{G} \}.$$

It is obvious that the action by conjugation of $SO(3)$ leaves $R^{\mathcal{G}}(G)$ invariant. Thus we set $\widehat{R}^{\mathcal{G}}(G) = \widetilde{R}^{\mathcal{G}}(G)/SO(3)$ where $\widetilde{R}^{\mathcal{G}}(G) = R^{\mathcal{G}}(G) \cap \widetilde{R}(G)$. We observe that $\widetilde{R}^{\mathcal{G}}(G)$ can be identified with the total space of a principal $SO(3)$ -bundle with $\widehat{R}^{\mathcal{G}}(G)$ as base space.

NOTATION. — We write $\widehat{R}^{\mathcal{T}_i}(B_i) = \widehat{R}^{\mathcal{T}_i}(\pi_1(B_i))$, $i = 1, 2$, $\widehat{R}^{\mathcal{S}}(S) = \widehat{R}^{\mathcal{S}}(\pi_1(S))$ etc.

M. Heusener proved in [He] that $\widehat{R}^{\mathcal{T}_i}(B_i)$ is a $(2n - 2)$ -manifold and that $\widehat{R}^{\mathcal{S}}(S)$ is a $(4n - 5)$ -manifold. Corresponding to diagram (17) we obtain the commutative diagram

$$(20) \quad \begin{array}{ccc} & \widehat{R}^{\mathcal{T}_1}(B_1) & \\ \widehat{\kappa}_1 \swarrow & & \nwarrow \widehat{p}_1 \\ \widehat{R}^{\mathcal{S}}(S) & & \widehat{R}(M_K) \\ \widehat{\kappa}_2 \searrow & & \swarrow \widehat{p}_2 \\ & \widehat{R}^{\mathcal{T}_2}(B_2) & \end{array}$$

In diagram (20) all arrows are inclusions. Therefore we can see $\widehat{R}(M_K)$ as the intersection of the images of $\widehat{R}^{\mathcal{T}_1}(B_1)$ and $\widehat{R}^{\mathcal{T}_2}(B_2)$ inside $\widehat{R}^{\mathcal{S}}(S)$. The commutative diagram (20) is the main ingredient to define (generically) a volume form on the $SU(2)$ -representation space of G_K .

We can prove that a representation ρ of G_K in $SU(2)$ is regular if and only if the images in $\widehat{R}^{\mathcal{S}}(S)$ of the manifolds $\widehat{R}^{\mathcal{T}_1}(B_1)$ and $\widehat{R}^{\mathcal{T}_2}(B_2)$ intersect transversally at $[\rho]$ (see [He, Prop. 3.3]).

6.4. Construction of the volume form.

The construction of our volume form on $\mathcal{R}eg(K)$ is based on the two following facts:

- If ρ is a regular representation, then

$$(21) \quad 0 \rightarrow T_{[\rho]} \widehat{R}(M_K) \xrightarrow{D_{[\rho] \widehat{p}}} T_{[\rho_1]} \widehat{R}^{\mathcal{T}_1}(B_1) \oplus T_{[\rho_2]} \widehat{R}^{\mathcal{T}_2}(B_2) \xrightarrow{D_{[\rho] \widehat{\kappa}}} T_{[\rho_S]} \widehat{R}^{\mathcal{S}}(S) \rightarrow 0,$$

is a short exact sequence (cf. diagram (20) and the fact that $\widehat{R}^{\mathcal{T}_1}(B_1)$ and $\widehat{R}^{\mathcal{T}_2}(B_2)$ intersect transversally at $[\rho]$).

- The representation subspace $\widehat{R}^{\mathcal{T}_i}(B_i)$ (resp. $\widehat{R}^{\mathcal{S}}(S)$) has a natural $(2n-2)$ -volume form $v_{\widehat{R}^{\mathcal{T}_i}(B_i)}$ (resp. a natural $(4n-5)$ -volume form $v_{\widehat{R}^{\mathcal{S}}(S)}$). The constructions of these volume forms are postponed to Subsection 6.5.

If we assume these two facts, then the 1-volume form $\omega_{[\zeta]}^{\hat{\zeta}}$ is defined using the notation of Subsection 2.1 and the exactness of (21) by

$$(22) \quad \omega_{[\zeta]}^{\hat{\zeta}} = (-1)^n (v_{[\rho_1]}^{\widehat{R}^{\mathcal{T}_1}(B_1)} \wedge v_{[\rho_2]}^{\widehat{R}^{\mathcal{T}_2}(B_2)}) / v_{[\rho_S]}^{\widehat{R}^{\mathcal{S}}(S)}.$$

In this way we locally construct a 1-volume form $\omega^{\hat{\zeta}} : [\zeta] \mapsto \omega_{[\zeta]}^{\hat{\zeta}}$ on the 1-dimensional manifold $\mathcal{R}eg(\hat{\zeta})$.

As it stands, $\omega^{\hat{\zeta}}$ is defined in terms of the plat decomposition of the knot and apparently depends on it. In fact, we can prove that $\omega^{\hat{\zeta}}$ is independent on $\hat{\zeta}$ and is moreover an invariant of knots, but this result will not be essential in the sequel. The proof of the invariance is based on a theorem of Birman-Reidemeister, which is similar for the plats to the Markov Theorem for the closed braids (see [Du3] or the author Ph.D. [Du2] for details). Further observe that it is precisely the sign $(-1)^n$ which ensures the invariance of ω^K . Thus $\mathcal{R}eg(K)$ is endowed with a well-defined 1-volume form denoted ω^K which is constructed “à la Casson”.

Remark 5. — The orientation induced by ω^K is precisely the orientation on $\mathcal{R}eg(K)$ defined in [He], see [Du3].

6.5. Volume forms on the representation subspaces.

The construction is based on the following steps (see [Du3] for more details).

- The Lie group $SU(2)$ is endowed with the 3-volume form η induced by the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Similarly η denotes the 3-volume form on $SO(3) = SU(2)/\{\pm \mathbf{1}\}$ deduced from the one of $SU(2)$. Considering the

fact that the trace-function $\text{Tr}: \text{SU}(2) \setminus \{\pm \mathbf{1}\} \rightarrow (-2, 2)$ is a submersion, we define a 2-volume form ν on the 2-sphere $S^2 = \{A \in \text{SU}(2) \mid \text{Tr}(A) = 0\}$ exploiting the “base \wedge fiber” condition. Explicitly, for all $A \in S^2$, we have the short exact sequence

$$0 \rightarrow T_A S^2 \longrightarrow T_A \text{SU}(2) \longrightarrow T_0(-2, 2) \rightarrow 0,$$

in which $\text{SU}(2)$ is endowed with the 3-volume form induced by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $(-2, 2)$ is endowed with the usual 1-volume form. The 2-volume form ν on S^2 is the unique compatible volume form with the two others.

- The map $R(B_i) \rightarrow \text{SU}(2)^n$ defined by $\rho \mapsto (\rho(t_1^{(i)}), \dots, \rho(t_n^{(i)}))$ is an isomorphism which allows us to identify $R(B_i)$ with $\text{SU}(2)^n$. Using the inclusion $(-2, 2) \times (S^2)^n \rightarrow \text{SU}(2)^n$, defined by

$$(2 \cos(\theta), P_1, \dots, P_n) \longmapsto (\cos(\theta) + \sin(\theta)P_i)_{1 \leq i \leq n},$$

we identify $R^{T_i}(B_i)$ with the product $(-2, 2) \times (S^2)^n$. As a consequence, the $(2n + 1)$ -manifold $R^{T_i}(B_i)$ is endowed with a natural $(2n + 1)$ -volume form $v^{R^{T_i}(B_i)}$, namely the one induced by the product volume form on $(-2, 2) \times (S^2)^n$.

Next, we define a $(4n - 2)$ -volume form on $R^S(S)$ as follows. Let D^* be the $2n$ -punctured disk $S \setminus \{\infty\}$. The fundamental group of D^* is the free group of rank $2n$ generated by $\mathcal{S} = \{s_1, \dots, s_{2n}\}$ (see Fig. 1). Let U be the subgroup normally generated by the product $s_1 \cdots s_{2n}$, we have $\pi_1(S) = \pi_1(D^*)/U$. In [He, Lemma 3.1], Heusener proved that the map $\psi: R^S(D^*) \rightarrow \text{SU}(2)$ defined by $\psi(\rho) = \rho(s_1 \cdots s_{2n})$ is surjective, that $R^S(S) = \psi^{-1}(\mathbf{1})$ and that the set of critical points of ψ coincides with the set of abelian $\text{SU}(2)$ -representations of $\pi_1(D^*)$. Thus, we have the short exact sequence

$$0 \rightarrow T_\rho \tilde{R}^S(S) \longrightarrow T_\rho \tilde{R}^S(D^*) \longrightarrow \mathfrak{su}(2) \rightarrow 0.$$

In this sequence, $\mathfrak{su}(2)$ is endowed with the 3-volume form induced by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $\tilde{R}^S(D^*)$ is endowed with the natural $(4n + 1)$ -volume form (because $\tilde{R}^S(D^*)$ is an open subset of $R^S(D^*) \cong (-2, 2) \times (S^2)^{2n}$ which is endowed with the product volume form). Then $\tilde{R}^S(S)$ is endowed with the unique $(4n - 2)$ -volume form $v^{\tilde{R}^S(S)}$ which is compatible with the two others.

- Finally, if (G, \mathcal{G}) is one of the marked groups $(\pi_1(B_1), \mathcal{T}_1)$, $(\pi_1(B_2), \mathcal{T}_2)$ or $(\pi_1(S), \mathcal{S})$, then $\tilde{R}^G(G) \rightarrow \hat{R}^G(G)$ is a submersion. So we define a volume form on $\hat{R}^G(G)$ exploiting the “base \wedge fiber” condition.

7. Comparison.

This section is devoted to the comparison of the two constructions we have done in Sections 5 and 6. We explain how we can interpret the volume form ω^K on $\mathcal{R}eg(K)$ in terms of the Reidemeister torsion form τ^K and finally prove that they produce the same topological invariant.

7.1. Comparison of torsion and volume forms.

The main result of the article is

THEOREM 7.1. — *Let K be a knot in S^3 . For any regular representation $\rho: G_K \rightarrow \text{SU}(2)$ we have*

$$\tau_\rho^K = \omega_{[\rho]}^K.$$

The proof of Theorem 7.1 needs several steps (and is postponed to Sections 9 and 10). The key idea to compare the two volume forms τ_ρ^K and $\omega_{[\rho]}^K$ on $T_{[\rho]}\widehat{R}(M_K)$ is to interpret them as one and the same Reidemeister torsion of an auxiliary chain complex. We shall discuss this point in Subsection 7.3.

7.2. Some remarks and consequences.

1) According to Remark 5 the induced orientation on $\mathcal{R}eg(K)$ deduced from the volume form τ^K is exactly the one defined on $\mathcal{R}eg(K)$ by Heusener in [He].

2) The Casson-Lin invariant $\lambda_{CL}(K)$, defined in [Li], can be interpreted via the sign-determined Reidemeister torsion as follows (cf. [He, Prop. 5.4 and Lemma 5.5]). Let (G, \mathcal{G}) be a marked group. Consider

$R_0^{\mathcal{G}}(G) = \{\rho \in R^{\mathcal{G}} \mid \text{Tr}(\rho(t)) = 0, \forall t \in \mathcal{G}\}$ and $\widehat{R}_0^{\mathcal{G}}(G) = \widetilde{R}_0^{\mathcal{G}}(G)/\text{SO}(3)$ (with $\widetilde{R}_0^{\mathcal{G}}(G) = R_0^{\mathcal{G}}(G) \cap \widetilde{R}(G)$). Set $\widehat{Q}_i^{(0)} = \widehat{\kappa}_i(\widehat{R}_0^{\mathcal{G}}(B_i))$ for $i = 1, 2$. The intersection $\widehat{Q}_1^{(0)} \cap \widehat{Q}_2^{(0)}$ is compact and the Casson-Lin invariant of the closed braid σ^\wedge , $\sigma \in B_n$, is the intersection number

$$\lambda_{CL}(\sigma^\wedge) = (-1)^n \langle \widehat{Q}_1^{(0)}, \widehat{Q}_2^{(0)} \rangle_{\widehat{R}_0^{\mathcal{G}}(S)}$$

see [Li, Theorem 18] and [He, Section 5.3]. Keep in mind the notation of Subsection 4.1. If all the representations in $\widehat{Q}_1^{(0)} \cap \widehat{Q}_2^{(0)}$ are m -regular, i.e. if $\widehat{Q}_1^{(0)}$ and $\widehat{Q}_2^{(0)}$ intersect transversally, then

$$\lambda_{CL}(\sigma^\wedge) = \sum_{[\rho] \in \widehat{Q}_1^{(0)} \cap \widehat{Q}_2^{(0)}} \text{sgn}(\tau_\rho^{\sigma^\wedge}(h_\rho^{(1)}(m))).$$

7.3. Main ideas of the proof.

Let K be an oriented knot in S^3 presented as a $2n$ -plat $\hat{\zeta}$ as in Section 6 and let $\rho: G_K \rightarrow \text{SU}(2)$ be a regular representation. The plat presentation $\hat{\zeta}$ of K induces the Wirtinger presentation (18) of G_K .

We want to compare the Reidemeister torsion of the acyclic chain complex (21)

$$T_*^\rho = 0 \rightarrow T_{[\rho]} \widehat{R}(M_K) \longrightarrow T_{[\rho_1]} \widehat{R}^{T_1}(B_1) \oplus T_{[\rho_2]} \widehat{R}^{T_2}(B_2) \longrightarrow T_{[\rho_S]} \widehat{R}^S(S) \rightarrow 0,$$

which allows us to define the volume form $\omega_{[\rho]}^K$, to the Reidemeister torsion of the non-acyclic $(\text{Ad} \circ \rho)$ -twisted chain complex (14) of the knot exterior

$$\mathcal{X}_*^\rho = 0 \rightarrow \mathfrak{su}(2) \xrightarrow{d_2^\rho} \mathfrak{su}(2)^{2n} \xrightarrow{d_1^\rho} \mathfrak{su}(2)^{2n-1} \rightarrow 0.$$

For this purpose we introduce and use the *auxiliary chain complex*

$$\mathcal{D}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \xrightarrow{p^*} \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}(S) \rightarrow 0.$$

Applying Propositions 3.1 and 4.1 we get

$$\begin{aligned} \text{Der}_\rho(G_K) &\cong T_\rho \widetilde{R}(M_K), & \text{Der}_{\rho_i}(\pi_1(B_i)) &\cong T_{\rho_i} \widetilde{R}(B_i), \\ & & \text{and } \text{Der}_{\rho_S}(\pi_1(S)) &\cong T_{\rho_S} \widetilde{R}(S). \end{aligned}$$

The chain complex \mathcal{D}_*^ρ is not acyclic and can be consider as a sort of “Mayer-Vietoris sequence”. The advantage in using \mathcal{D}_*^ρ belongs to the fact that, in opposition to \mathcal{X}_*^ρ , its homology in dimension 1 vanishes and it has just homology in dimension 0: $H_0(\mathcal{D}_*^\rho) \cong H_\rho^2(M_K) \cong H_0(\mathcal{X}_*^\rho)$. We will discuss the chain complex \mathcal{D}_*^ρ in detail in Section 8.

The main ingredient to perform the comparison of the torsions is the multiplicativity of the sign-determined Reidemeister torsion. Let us recall this property.

MULTIPLICATIVITY LEMMA. — *Let*

$$(23) \quad 0 \rightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \rightarrow 0$$

be an exact sequence of chain complexes. For each i , fix volume forms on $C'_i, C_i, C''_i, H_i(C'_), H_i(C_*)$ and $H_i(C''_*)$. Associated to (23) is the long exact sequence in homology*

$$\dots \rightarrow H_i(C'_*) \longrightarrow H_i(C_*) \longrightarrow H_i(C''_*) \longrightarrow H_{i-1}(C'_*) \rightarrow \dots$$

Let \mathcal{H}_ denote this acyclic chain complex and endow*

$$\mathcal{H}_{3i+2} = H_i(C'_*), \quad \mathcal{H}_{3i+1} = H_i(C_*), \quad \mathcal{H}_{3i} = H_i(C''_*)$$

with the volume form on $H_i(C'_), H_i(C_*)$ and $H_i(C''_*)$ respectively.*

If, for all i , the volume forms on the components of the short exact sequence $0 \rightarrow C'_i \rightarrow C_i \rightarrow C''_i \rightarrow 0$ are compatible, then

$$\text{Tor}(C_*) = (-1)^{\alpha(C'_*, C''_*) + \varepsilon(C'_*, C_*, C''_*)} \text{Tor}(C'_*) \cdot \text{Tor}(C''_*) \cdot \text{tor}(\mathcal{H}_*),$$

where

$$\begin{aligned} \alpha(C'_*, C''_*) &= \sum_{i \geq 0} \alpha_{i-1}(C'_*) \alpha_i(C''_*) \in \mathbb{Z}/2\mathbb{Z}, \\ \varepsilon(C'_*, C_*, C''_*) &= \sum_{i \geq 0} [(\beta_i(C'_*) + 1)(\beta_i(C'_*) + \beta_i(C''_*)) \\ &\quad + \beta_{i-1}(C'_*) \beta_i(C''_*)] \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The proof of the Multiplicativity Lemma is a careful computation based on linear algebra, see [Tu1, Lemma 3.4.2] and [Mi2, Theorem 3.2].

Keeping in mind this property, the proof of Theorem 7.1 is done in Sections 9 and 10; it is quite natural and essentially consists of two parts. Firstly (see Section 9) we compare the sign-determined Reidemeister torsions of the chain complexes \mathcal{T}_*^ρ and \mathcal{D}_*^ρ . Secondly (see Section 10) we compare the sign-determined Reidemeister torsions of the chain complexes \mathcal{D}_*^ρ and \mathcal{X}_*^ρ to obtain the required equality between the torsion form τ^K and the volume form ω^K .

8. The auxiliary chain complex.

The proof of Theorem 7.1 uses some more properties about the notion of twisted derivations; this material is described in the following subsections. The notion of twisted derivations gives a method to identify the tangent bundles of the $SU(2)$ -representation subspaces $\tilde{R}^{T_i}(B_i)$ and $\tilde{R}^S(S)$. These identifications are used to effectively compare the volume forms ω^K and τ^K .

8.1. Volume forms on the twisted derivation spaces.

Suppose that K is presented as a $2n$ -plat and keep the notation of Section 6. The plat presentation of K gives the splitting $M_K = B_1 \cup_S B_2$, where B_1, B_2 are handlebodies and S is a $2n$ -punctured 2-sphere.

NOTATION. — We write $\text{Der}_\rho(M_K) = \text{Der}_\rho(G_K)$, $\text{Der}_{\rho_i}(B_i) = \text{Der}_{\rho_i}(\pi_1(B_i))$ and $\text{Der}_{\rho_S}(S) = \text{Der}_{\rho_S}(\pi_1(S))$, here $\rho \in R(M_K)$.

The choice of the system of generators \mathcal{T}_i (resp. \mathcal{S}) for the group $\pi_1(B_i)$, $i = 1, 2$, (resp. $\pi_1(S)$) gives

FACT 8.1. — For $i = 1, 2$,

$$\text{Der}_{\rho_i}(B_i) \cong \mathfrak{su}(2)^n \quad \text{and} \quad \text{Der}_{\rho_S}(S) \cong \mathfrak{su}(2)^{2n-1}.$$

Proof. — We use the fact that $\pi_1(B_i)$ (resp. $\pi_1(S)$) is the free group of rank n (resp. is isomorphic to the free group of rank $2n - 1$). Explicitly the isomorphisms are:

- $\varphi_i : \mathfrak{su}(2)^n \rightarrow \text{Der}_{\rho_i}(B_i)$ defined by $\varphi_i : \mathbf{x} = (x_1, \dots, x_n) \mapsto d^{\mathbf{x}}$, where $d^{\mathbf{x}} \in \text{Der}_{\rho_i}(B_i)$ is such that $d^{\mathbf{x}}(t_j^{(i)}) = x_j$, for all $j = 1, \dots, n$;
- $\varphi_S : \mathfrak{su}(2)^{2n-1} \rightarrow \text{Der}_{\rho_S}(S)$ defined by $\varphi_S : \mathbf{x} = (x_1, \dots, x_{2n-1}) \mapsto d^{\mathbf{x}}$, where $d^{\mathbf{x}} \in \text{Der}_{\rho_S}(S)$ is such that $d^{\mathbf{x}}(s_j) = x_j$, for all $j = 1, \dots, 2n - 1$, and

$$d^{\mathbf{x}}(s_{2n}) = -\text{Ad}_{\rho_S(s_1 \cdots s_{2n-1})^{-1}}(x_1) - \cdots - \text{Ad}_{\rho_S(s_{2n-1})^{-1}}(x_{2n-1}). \quad \square$$

The isomorphism $\text{Der}_{\rho_i}(B_i) \cong \mathfrak{su}(2)^n$ allows us to define a volume form on $\text{Der}_{\rho_i}(B_i)$ namely the one induced by the product volume form on $\mathfrak{su}(2)^n$. Similarly, the isomorphism $\text{Der}_{\rho_S}(S) \cong \mathfrak{su}(2)^{2n-1}$ allows us to define a volume form on $\text{Der}_{\rho_S}(S)$ in the same way. But it is possible to define an *a priori* different volume form on $\text{Der}_{\rho_S}(S)$ in another “natural” way. Consider the $2n$ -punctured disc $D^* = S \setminus \{\infty\}$ and the map $\psi : \text{Der}_{\rho_S}(D^*) \rightarrow \mathfrak{su}(2)$ defined by $\psi(d) = d(s_1 \cdots s_{2n})$. Thus, we have the short exact sequence

$$(24) \quad 0 \rightarrow \text{Der}_{\rho_S}(S) \longrightarrow \text{Der}_{\rho_S}(D^*) \xrightarrow{\psi} \mathfrak{su}(2) \rightarrow 0.$$

Recall that $\text{Der}_{\rho_S}(D^*) \cong \mathfrak{su}(2)^{2n}$ is endowed with the volume form induced by the product volume form.

CLAIM 8.2. — *The volume forms deduced from the product volume forms on $\mathfrak{su}(2)^m$ on each component of the exact sequence (24) are compatible.*

Proof. — Let $d^{\mathbf{x}} \in \text{Der}_{\rho_S}(D^*)$ be defined by $d^{\mathbf{x}}(s_j) = 0$, for all $j = 1, \dots, 2n - 1$, and $d^{\mathbf{x}}(s_{2n}) = \text{Ad}_{\rho_S(s_{2n})}(x)$. We consider the section $s : \mathfrak{su}(2) \rightarrow \text{Der}_{\rho_S}(D^*)$ of ψ defined by $s(x) = d^{\mathbf{x}}$. The previous identifications for $\text{Der}_{\rho_S}(S)$ and $\text{Der}_{\rho_S}(D^*)$ gives us the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \text{Der}_{\rho_S}(S) & \longrightarrow & \text{Der}_{\rho_S}(D^*) & \xrightarrow{\psi} & \mathfrak{su}(2) & \rightarrow 0 \\ & \downarrow \cong & & \downarrow \cong & & \parallel & \\ 0 \rightarrow & \mathfrak{su}(2)^{2n-1} & \xrightarrow{i} & \mathfrak{su}(2)^{2n} & \xleftarrow[\substack{j \\ s}]{} & \mathfrak{su}(2) & \rightarrow 0 \end{array}$$

in which

$$\begin{aligned}
 i(x_1, \dots, x_{2n-1}) &= (x_1, \dots, x_{2n-1}, -\text{Ad}_{\rho_S(s_1 \dots s_{2n-1})^{-1}}(x_1) \\
 &\quad - \dots - \text{Ad}_{\rho_S(s_{2n-1})^{-1}}(x_{2n-1})), \\
 j(x_1, \dots, x_{2n}) &= x_1 + \text{Ad}_{\rho_S(s_1)}(d(s_2)) + \dots + \text{Ad}_{\rho_S(s_1 \dots s_{2n-1})}(d(s_{2n})).
 \end{aligned}$$

Moreover the section s of ψ can be identified with the one of j also denoted $s: \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)^{2n}$ and defined by $s(x) = (0, \dots, 0, \text{Ad}_{\rho_S(s_{2n})}(x))$.

The determinant of the transition matrix from the canonical product basis $\{\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1, \dots, \mathbf{i}_{2n}, \mathbf{j}_{2n}, \mathbf{k}_{2n}\}$ of $\mathfrak{su}(2)^{2n}$ to the basis

$$\{i(\mathbf{i}_1), i(\mathbf{j}_1), i(\mathbf{k}_1), \dots, i(\mathbf{i}_{2n-1}), i(\mathbf{j}_{2n-1}), i(\mathbf{k}_{2n-1}), s(\mathbf{i}), s(\mathbf{j}), s(\mathbf{k})\}$$

of $\mathfrak{su}(2)^{2n}$ is $+1$. So, the volume form on $\text{Der}_{\rho_S}(S)$ which is compatible with the two others on the components of the exact sequence (24) can be identified with the product volume form on $\mathfrak{su}(2)^{2n-1}$. □

8.2. The auxiliary derivation chain complex.

The Mayer-Vietoris sequence for the twisted cohomology associated to the splitting $M_K = B_1 \cup_S B_2$ and to the irreducible representation $\rho: G_K \rightarrow \text{SU}(2)$ reduces to

$$0 \rightarrow H^1_{\rho}(M_K) \longrightarrow H^1_{\rho_1}(B_1) \oplus H^1_{\rho_2}(B_2) \longrightarrow H^1_{\rho_S}(S) \longrightarrow H^2_{\rho}(M_K) \rightarrow 0.$$

From the splitting $M_K = B_1 \cup_S B_2$ we deduce the *auxiliary derivation chain complex*

$$\mathcal{D}^{\rho}_* = 0 \rightarrow \text{Der}_{\rho}(M_K) \xrightarrow{p^*} \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}(S) \rightarrow 0.$$

Here

$$\begin{aligned}
 p^*: \text{Der}_{\rho}(M_K) &\rightarrow \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2), \quad d \mapsto (d \circ p_1, -d \circ p_2), \\
 \kappa^*: \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) &\rightarrow \text{Der}_{\rho_S}(S), \quad (d_1, d_2) \mapsto d_1 \circ \kappa_1 + d_2 \circ \kappa_2.
 \end{aligned}$$

The equalities $\ker p^* = 0$, $\text{im } p^* = \ker \kappa^*$ and $\text{coker } \kappa^* = H^2_{\rho}(M_K)$ give us

FACT 8.3. — *The homology of \mathcal{D}^{ρ}_* reduces to*

$$H_*(\mathcal{D}^{\rho}_*) = H_0(\mathcal{D}^{\rho}_*) \cong H^2_{\rho}(M_K).$$

We now describe the reference volume form we choose on each component of the chain complex \mathcal{D}_*^ρ . Fact 8.1 tells us that $\varphi_i : \text{Der}_{\rho_i}(B_i) \rightarrow \mathfrak{su}(2)^n$ (resp. $\varphi_S : \text{Der}_{\rho_S}(S) \rightarrow \mathfrak{su}(2)^{2n-1}$) is an isomorphism. So $\text{Der}_{\rho_i}(B_i)$ (resp. $\text{Der}_{\rho_S}(S)$) is endowed with the volume form induced by φ_i (resp. φ_S) and by the product volume form on $\mathfrak{su}(2)^n$ (resp. $\mathfrak{su}(2)^{2n-1}$). We endow $\text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2)$ with the product volume form corrected by the factor $(-1)^n$ (as in equation (22)). To produce a reference volume form on $\text{Der}_\rho(M_K)$ we use the exactness of sequence (7)

$$0 \rightarrow \mathfrak{su}(2) \longrightarrow \text{Der}_\rho(M_K) \longrightarrow H_\rho^1(M_K) \rightarrow 0.$$

Here $\mathfrak{su}(2)$ is endowed with the 3-volume form $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ and $H_\rho^1(M_K) \cong T_{[\rho]} \widehat{R}(M_K)$ is endowed with the 1-volume form $\omega_{[\rho]}^K$. Using compatibility we endow $\text{Der}_\rho(M_K)$ with the 4-volume form w_ρ such that $w_\rho / \omega_{[\rho]}^K = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ (we use the notation described in Subsection 2.1).

The complex \mathcal{D}_*^ρ is based using the convention of Subsection 2.2; that is, we endow each \mathcal{D}_i^ρ with any basis which has volume 1.

8.3. Tangent bundles of $\widetilde{R}^{\mathcal{T}_i}(B_i)$, $i = 1, 2$, and $\widetilde{R}^{\mathcal{S}}(S)$.

Let (G, \mathcal{G}) be a marked group. Set $\mathcal{G} = \{g_1, \dots, g_m\}$ and fix $\rho \in \widetilde{R}^{\mathcal{G}}(G)$. Just as we are interested in the representation subspace $R^{\mathcal{G}}(G)$ we introduce the subspace

$$\begin{aligned} \text{Der}_\rho^{\mathcal{G}}(G) &= \{d \in \text{Der}_\rho(G) \mid \langle d(g_j), P^\rho(g_j) \rangle \\ &= \langle d(g_k), P^\rho(g_k) \rangle, 1 \leq j, k \leq m\} \end{aligned}$$

of $\text{Der}_\rho(G)$. We also consider the subspace

$$\text{Der}_\rho^{\mathcal{G}}(G)_0 = \{d \in \text{Der}_\rho(G) \mid \langle d(g_j), P^\rho(g_j) \rangle = 0, 1 \leq j \leq m\}$$

of $\text{Der}_\rho^{\mathcal{G}}(G)$. The equalities between the scalar products can be considered as an “infinitesimal” counterpart of the equalities between traces in (19).

NOTATION. — We write $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i) = \text{Der}_{\rho_i}^{\mathcal{T}_i}(\pi_1(B_i))$ and $\text{Der}_{\rho_S}^{\mathcal{S}}(S) = \text{Der}_{\rho_S}^{\mathcal{S}}(\pi_1(S))$, etc.

The sequel of this subsection is devoted to a careful analysis of the subspaces $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)$ and $\text{Der}_{\rho_S}^{\mathcal{S}}(S)$ where $\rho \in \widetilde{R}(M_K)$. In fact, we shall prove the two isomorphisms $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i) \cong T_{\rho_i} \widetilde{R}^{\mathcal{T}_i}(B_i)$ and $\text{Der}_{\rho_S}^{\mathcal{S}}(S) \cong T_{\rho_S} \widetilde{R}^{\mathcal{S}}(S)$. Let us begin with a lemma.

LEMMA 8.4. — We have $\text{Inn}_{\rho_i}(B_i) \subset \text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)_0$, for $i = 1, 2$, and $\text{Inn}_{\rho_S}(S) \subset \text{Der}_{\rho_S}^S(S)_0$. Moreover if $d \in \text{Der}_{\rho}(M_K)$, then $d_i = d \circ p_i$ belongs to $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)$ for $i = 1, 2$.

Proof. — The proof is a straightforward consequence of the well-known identity $\langle x, P \rangle = \langle \text{Ad}_A(x), P \rangle$, where $A \in \text{SU}(2) \setminus \{\pm 1\}$, Ad_A is a rotation which fixes $P \in S^2$ and $x \in \mathfrak{su}(2)$. In fact, we prove the following more general result: if G is a group, if $g \in G$ and if $\rho \in \tilde{R}(G)$ is such that $\rho(g) \neq \pm 1$, then we have the properties

- 1) if $\delta \in \text{Inn}_{\rho}(G)$, then $\langle \delta(g), P^{\rho}(g) \rangle = 0$;
- 2) if $d \in \text{Der}_{\rho}(G)$ and g, g' are conjugate, then $\langle d(g), P^{\rho}(g) \rangle = \langle d(g'), P^{\rho}(g') \rangle$.

For this purpose,

1) If $\delta \in \text{Inn}_{\rho}(G)$, then there exists $a \in \mathfrak{su}(2)$ such that $\delta(g) = a - \text{Ad}_{\rho(g)} a$ for all $g \in G$; so that $\langle \delta(g), P^{\rho}(g) \rangle = \langle a, P^{\rho}(g) \rangle - \langle \text{Ad}_{\rho(g)} a, P^{\rho}(g) \rangle = 0$.

2) If we write g' as $g' = hgh^{-1}$, then

$$d(g') = d(h) - \text{Ad}_{\rho(g')} d(h) + \text{Ad}_{\rho(h)} d(g), \quad P^{\rho}(g') = \text{Ad}_{\rho(h)} P^{\rho}(g).$$

We thus have

$$\begin{aligned} \langle d(g'), P^{\rho}(g') \rangle &= \langle d(h), P^{\rho}(g') \rangle - \langle \text{Ad}_{\rho(g')} d(h), P^{\rho}(g') \rangle \\ &\quad + \langle \text{Ad}_{\rho(h)} d(g), \text{Ad}_{\rho(h)} P^{\rho}(g) \rangle, \end{aligned}$$

which gives $\langle d(g'), P^{\rho}(g') \rangle = \langle d(g), P^{\rho}(g) \rangle$.

The inclusions stated in the lemma straight follow from the first property; the second part arises from the second property (because $t_j^{(i)}$ is conjugate to m). □

The same arguments give us

LEMMA 8.5. — If $\rho \in \tilde{R}(M_K)$, then

$$\text{Inn}_{\rho}(M_K) \subset \text{Der}_{\rho}(M_K)_0 \subset \text{Der}_{\rho}(M_K).$$

Here we write $\text{Der}_{\rho}(M_K)_0 = \{d \in \text{Der}_{\rho}(M_K) \mid \langle d(m), P^{\rho}(m) \rangle = 0\}$.

Remark 6. — Recall that ρ is regular if and only if $\dim \text{Der}_{\rho}(M_K) = 4$. Further observe that for any regular representation ρ we have either $\text{Der}_{\rho}(M_K)_0 = \text{Inn}_{\rho}(M_K)$ when ρ is m -regular, or $\text{Der}_{\rho}(M_K)_0 = \text{Der}_{\rho}(M_K)$ when ρ is regular but not m -regular.

If (G, \mathcal{G}) is one of the marked groups $(\pi_1(B_1), \mathcal{T}_1)$, $(\pi_1(B_2), \mathcal{T}_2)$, or $(\pi_1(S), \mathcal{S})$, then inclusion (9) induces an inclusion from $T_\rho \widetilde{R}^{\mathcal{G}}(G)$ into $\text{Der}_\rho^{\mathcal{G}}(G)$. Using a dimensional argument we obtain

LEMMA 8.6. — *If $\rho \in \widetilde{R}(M_K)$, then the inclusions*

$$T_\rho \widetilde{R}^{\mathcal{T}_i}(B_i) \hookrightarrow \text{Der}_\rho^{\mathcal{T}_i}(B_i) \quad (i = 1, 2) \quad \text{and} \quad T_\rho \widetilde{R}^{\mathcal{S}}(S) \hookrightarrow \text{Der}_\rho^{\mathcal{S}}(S)$$

are isomorphisms.

The space $\text{Der}_\rho^{\mathcal{T}_i}(B_i) \cong T_\rho \widetilde{R}^{\mathcal{T}_i}(B_i)$, $i = 1, 2$, (resp. $\text{Der}_\rho^{\mathcal{S}}(S) \cong T_\rho \widetilde{R}^{\mathcal{S}}(S)$) is endowed with the volume form induced by the natural one $v_{\rho_i}^{\widetilde{R}^{\mathcal{T}_i}(B_i)}$ on $T_\rho \widetilde{R}^{\mathcal{T}_i}(B_i)$ (resp. $v_{\rho_S}^{\widetilde{R}^{\mathcal{S}}(S)}$ on $T_\rho \widetilde{R}^{\mathcal{S}}(S)$), see Subsection 6.5. These isomorphisms between tangent subspaces and derivation subspaces are the main tool in the proof of Theorem 7.1.

For any regular representation ρ of G_K we have the acyclic chain complex

$$(25) \quad \mathcal{C}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \xrightarrow{p^*} \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}^{\mathcal{S}}(S) \rightarrow 0,$$

and the isomorphisms $\text{Der}_\rho(M_K) \cong T_\rho \widetilde{R}(M_K)$, $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i) \cong T_{\rho_i} \widetilde{R}^{\mathcal{T}_i}(B_i)$, $i = 1, 2$, and $\text{Der}_{\rho_S}^{\mathcal{S}}(S) \cong T_{\rho_S} \widetilde{R}^{\mathcal{S}}(S)$. We endow the components of \mathcal{C}_*^ρ with reference volume forms as follows:

- $\mathcal{C}_2^\rho = \text{Der}_\rho(M_K)$ is endowed with the 4-volume form w_ρ ,
- $\mathcal{C}_1^\rho = \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)$ is endowed with the sign-corrected product volume form $(-1)^n (v_{\rho_1}^{\widetilde{R}^{\mathcal{T}_1}(B_1)} \wedge v_{\rho_2}^{\widetilde{R}^{\mathcal{T}_2}(B_2)})$ as in equation (22),
- $\mathcal{C}_0^\rho = \text{Der}_{\rho_S}^{\mathcal{S}}(S)$ is endowed with $v_{\rho_S}^{\widetilde{R}^{\mathcal{S}}(S)}$.

The complex \mathcal{C}_*^ρ is based using the convention of Subsection 2.2; that is, we endow each \mathcal{C}_i^ρ with any basis which has volume 1.

We finally introduce the linear forms

$$(26) \quad \pi_i : \text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i) \longrightarrow \mathbb{R}, \quad d \mapsto \langle d(t_1^{(i)}), P^{\rho_i}(t_1^{(i)}) \rangle, \quad i = 1, 2,$$

$$(27) \quad \pi_S : \text{Der}_{\rho_S}^{\mathcal{S}}(S) \longrightarrow \mathbb{R}, \quad d \mapsto \langle d(s_1), P^{\rho_S}(s_1) \rangle.$$

It is quite obvious that π_i and π_S are surjective, that $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)_0 = \ker \pi_i$, for $i = 1, 2$, and that $\text{Der}_{\rho_S}^{\mathcal{S}}(S)_0 = \ker \pi_S$. These equalities allow us to define a natural volume form on each kernel—namely the one induced by the volume form on $\text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)$ and $\text{Der}_{\rho_S}^{\mathcal{S}}(S)$ respectively.

9. Proof of Theorem 7.1: Part 1.

We turn now to the proof of Theorem 7.1, which will be discussed in this section and the next. To prove the equality between the volume form ω^K and torsion form τ^K we interpret them as the Reidemeister torsion of the auxiliary chain complex

$$\mathcal{D}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \xrightarrow{p^*} \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}(S) \rightarrow 0.$$

The proof splits into two parts. In the first one (this section) we explicitly compute the sign-determined Reidemeister torsion of \mathcal{D}_*^ρ . In the next one (Section 10) we achieve the proof by comparing the sign-determined Reidemeister torsion of \mathcal{D}_*^ρ to the one of the twisted complex (14) of the knot exterior

$$\mathcal{X}_*^\rho = 0 \rightarrow \mathfrak{su}(2) \xrightarrow{d_2^\rho} \mathfrak{su}(2)^{2n} \xrightarrow{d_1^\rho} \mathfrak{su}(2)^{2n-1} \rightarrow 0.$$

The main purpose of the first part of the proof is to establish

LEMMA 9.1. — *If $H_0(\mathcal{D}_*^\rho) \cong H_\rho^2(M_K)$ is based with the generator $h_\rho^{(2)}$ (see equation (16)), then*

$$\text{Tor}(\mathcal{D}_*^\rho) = \text{sgn}(\text{Tor}(\mathcal{X}_*^\rho)).$$

Here \mathcal{X}_* is the chain complex (13).

The proof of Lemma 9.1 requires three steps. It is based on Lemma 8.6.

9.1. First Step.

The aim of the first step is to compare the sign-determined Reidemeister torsion of the acyclic based chain complex (see Subsection 8.3)

$$\mathcal{C}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \longrightarrow \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) \longrightarrow \text{Der}_{\rho_S}^S(S) \rightarrow 0$$

with the one of the acyclic chain complex

$$\begin{aligned} \mathcal{T}_*^\rho = 0 \rightarrow T_{[\rho]} \widehat{R}(M_K) &\longrightarrow T_{[\rho_1]} \widehat{R}^{\mathcal{T}_1}(B_1) \oplus T_{[\rho_2]} \widehat{R}^{\mathcal{T}_2}(B_2) \\ &\longrightarrow T_{[\rho_S]} \widehat{R}^S(S) \rightarrow 0. \end{aligned}$$

The components of this last complex are endowed with the compatible volume forms described in Subsections 6.4–6.5:

- $\mathcal{T}_0^\rho = T_{[\rho_S]} \widehat{R}^S(S)$ is endowed with $v_{[\rho_S]}^{\widehat{R}^S(S)}$,
- $\mathcal{T}_1^\rho = T_{[\rho_1]} \widehat{R}^{\mathcal{T}_1}(B_1) \oplus T_{[\rho_2]} \widehat{R}^{\mathcal{T}_2}(B_2)$ is endowed with $(-1)^n (v_{[\rho_1]}^{\widehat{R}^{\mathcal{T}_1}(B_1)} \wedge v_{[\rho_2]}^{\widehat{R}^{\mathcal{T}_2}(B_2)})$,
- $\mathcal{T}_2^\rho = T_{[\rho]} \widehat{R}(M_K)$ is endowed with the 1-volume form $\omega_{[\rho]}^K$ defined by equation (22).

We based \mathcal{T}_*^ρ using the convention of Subsection 2.2.

First, the compatibility of the volume forms on the components of \mathcal{T}_*^ρ implies $\text{Tor}(\mathcal{T}_*^\rho) = 1$.

Next, to compare the Reidemeister torsion of \mathcal{C}_*^ρ with the one of \mathcal{T}_*^ρ we use the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{S}_* = 0 & \longrightarrow & \mathfrak{su}(2) & \longrightarrow & \mathfrak{su}(2) \oplus \mathfrak{su}(2) & \longrightarrow & \mathfrak{su}(2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}_*^\rho = 0 & \longrightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) & \longrightarrow & \text{Der}_{\rho_S}^S(S) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{T}_*^\rho = 0 & \longrightarrow & T_{[\rho]} \widehat{R}(M_K) & \longrightarrow & T_{[\rho_1]} \widehat{R}^{\mathcal{T}_1}(B_1) \oplus T_{[\rho_2]} \widehat{R}^{\mathcal{T}_2}(B_2) & \longrightarrow & T_{[\rho_S]} \widehat{R}^S(S) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Each row and each column of the previous diagram is an acyclic chain complex.

LEMMA 9.2. — *The Reidemeister torsion of \mathcal{C}_*^ρ satisfies $\text{Tor}(\mathcal{C}_*^\rho) = -1$.*

Proof. — The spaces \mathcal{S}_0 and \mathcal{S}_2 are endowed with the 3-volume form $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$, and the space $\mathcal{S}_1 = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is endowed with the product volume form. By construction the volume forms on the components of the short exact sequence $0 \rightarrow \mathcal{S}_i \rightarrow \mathcal{C}_i^\rho \rightarrow \mathcal{T}_i^\rho \rightarrow 0$ are compatible, for $i = 0, 1, 2$ (see Subsection 6.5).

The Multiplicativity Lemma gives $\text{Tor}(\mathcal{C}_*^\rho) = (-1)^\alpha \text{Tor}(\mathcal{S}_*) \cdot \text{Tor}(\mathcal{T}_*^\rho)$, where $\alpha = \alpha(\mathcal{S}_*, \mathcal{T}_*^\rho) = 1 \in \mathbb{Z}/2\mathbb{Z}$. It is easy to see that $\text{Tor}(\mathcal{S}_*) = 1$, which gives $\text{Tor}(\mathcal{C}_*^\rho) = -1$. □

9.2. Second Step.

We compare the sign-determined Reidemeister torsion of the chain complex

$$\mathcal{C}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \longrightarrow \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) \longrightarrow \text{Der}_{\rho_S}^{\mathcal{S}}(S) \rightarrow 0$$

with the one of

$$\mathcal{B}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K)_0 \longrightarrow \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0 \longrightarrow \text{Der}_{\rho_S}^{\mathcal{S}}(S)_0 \rightarrow 0.$$

The chain complex \mathcal{B}_*^ρ is acyclic if and only if $\text{Der}_\rho(M_K)_0 = \text{Inn}_\rho(M_K)$, i.e. if and only if ρ is m -regular (see Remark 6). If not, the homology of \mathcal{B}_*^ρ reduces to $H_*(\mathcal{B}_*^\rho) = H_0(\mathcal{B}_*^\rho) \cong H_\rho^2(M_K) \cong \mathbb{R}$. We thus split the discussion into two cases to make it clearer.

1) *m-regular case.* — We suppose that ρ is m -regular.

To compare the Reidemeister torsion of the acyclic chain complex \mathcal{C}_*^ρ to the one of the acyclic chain complex \mathcal{B}_*^ρ we use the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K)_0 & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0 & \longrightarrow & \text{Der}_{\rho_S}^{\mathcal{S}}(S)_0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) & \longrightarrow & \text{Der}_{\rho_S}^{\mathcal{S}}(S) \rightarrow 0 \\
 & & \downarrow \beta & & \downarrow \pi_1 \oplus \pi_2 & & \downarrow \pi_S \\
 \mathcal{R}_* = 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\sigma} & \mathbb{R} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Here π_i is defined by equation (26), $i = 1, 2$, π_S by equation (27),

$$\beta: \text{Der}_\rho(M_K) \longrightarrow \mathbb{R}, \quad d \mapsto \langle d(m), P^\rho(m) \rangle,$$

$$\alpha: \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R}, \quad x \mapsto (x, -x), \quad \sigma: \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto x + y.$$

Each row and each column of the previous diagram is an acyclic chain complex.

The commutativity of the previous diagram comes from the fact that all elements of the systems of generators \mathcal{T}_i , $i = 1, 2$, and \mathcal{S} are conjugate to the meridian of K .

Recall that

- $\mathcal{B}_2^\rho = \text{Der}_\rho(M_K)_0 = \text{Inn}_\rho(M_K) \cong \mathfrak{su}(2)$ is endowed with the 3-volume form induced by the one on $\mathfrak{su}(2)$,
- $\mathcal{B}_1^\rho = \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0$ is endowed with the $4n$ -volume form induced by the one on $\mathcal{C}_1^\rho = \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)$ and
- $\mathcal{B}_0^\rho = \text{Der}_{\rho_S}^{\mathcal{S}}(S)_0$ is endowed with the $(4n - 3)$ -volume form induced by the one on $\mathcal{C}_0^\rho = \text{Der}_{\rho_S}^{\mathcal{S}}(S)$ (see Subsection 8.3).

As ρ is supposed to be m -regular there exist a unique vector $h_\rho^{(1)}(m) \in H_\rho^1(M_K)$ which satisfies $f_m^\rho(h_\rho^{(1)}(m)) = \langle h_\rho^{(1)}(m) \cdot m, P^\rho(m) \rangle = 1$. Set

$$(28) \quad \tau = \omega_{[\rho]}^K(\varphi_{[\rho]}^{-1}(h_\rho^{(1)}(m)))$$

which is expected to be the $(\text{Ad} \circ \rho)$ -twisted Reidemeister torsion of M_K with respect to the basis $\{h_\rho^{(1)}(m), h_\rho^{(2)}\}$ of $H_\rho^*(M_K)$. With this notation we compute the Reidemeister torsion of \mathcal{B}_*^ρ in this case.

LEMMA 9.3. — We have $\text{Tor}(\mathcal{B}_*^\rho) = \tau$.

Proof. — The volume forms on the components of the short exact sequence $0 \rightarrow \mathcal{B}_i^\rho \rightarrow \mathcal{C}_i^\rho \rightarrow \mathcal{R}_i \rightarrow 0$ are compatible for $i = 0, 1$. We easily show that the volume form on $\mathcal{R}_2 = \mathbb{R}$ which is compatible with $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ on $\mathcal{B}_2^\rho = \text{Der}_\rho(M_K)_0 = \text{Inn}_\rho(M_K) \cong \mathfrak{su}(2)$ and with w_ρ on $\mathcal{C}_2^\rho = \text{Der}_\rho(M_K)$ is the usual 1-volume form on \mathbb{R} corrected by the factor τ^{-1} . The Multiplicativity Lemma applies and gives

$$\text{Tor}(\mathcal{C}_*^\rho) = (-1)^\alpha \text{Tor}(\mathcal{B}_*^\rho) \cdot \text{Tor}(\mathcal{R}_*),$$

where $\alpha = \alpha(\mathcal{B}_*^\rho, \mathcal{R}_*) = 1 \in \mathbb{Z}/2\mathbb{Z}$. It is easy to see that $\text{Tor}(\mathcal{R}_*) = \tau^{-1}$. Thus $\text{Tor}(\mathcal{C}_*^\rho) = -\tau^{-1} \cdot \text{Tor}(\mathcal{B}_*^\rho)$. We conclude referring ourselves to Lemma 9.2. □

2) *Non m-regular case.* — Now we suppose that ρ is regular but not m -regular.

To compare the Reidemeister torsion of \mathcal{C}_*^ρ with the sign-determined one of the non acyclic based and homology based chain complex \mathcal{B}_*^ρ we use the commutative diagram of Figure 2.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0 & \longrightarrow & \text{Der}_{\rho_S}^{\mathcal{S}}(S)_0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2) & \longrightarrow & \text{Der}_{\rho_S}^{\mathcal{S}}(S) \rightarrow 0 \\
 & & \downarrow & & \downarrow \pi_1 \oplus \pi_2 & & \downarrow \pi_S \\
 \mathcal{R}'_* = 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\sigma} & \mathbb{R} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Figure 2

In the diagram of Figure 2, $\sigma : \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\sigma(x, y) = x + y$ and we recall that $\mathcal{B}_2^\rho = \text{Der}_\rho(M_K) = \text{Der}_\rho(M_K)_0$ in this case.

Let \mathcal{H}_* denote the long exact sequence in homology associated to the exact sequence of chain complexes $0 \rightarrow \mathcal{B}_*^\rho \rightarrow \mathcal{C}_*^\rho \rightarrow \mathcal{R}'_* \rightarrow 0$. Here \mathcal{H}_* reduces to

$$(29) \quad \mathcal{H}_* = 0 \rightarrow H_1(\mathcal{R}'_*) \xrightarrow{\sim} H_0(\mathcal{B}_*^\rho) \rightarrow 0.$$

We have $H_1(\mathcal{R}'_*) = \ker \sigma = \{(x, -x) \mid x \in \mathbb{R}\}$. Choose $(1, -1)$ as generator for $H_1(\mathcal{R}'_*) = \ker \sigma$. This choice will not affect our computation as we will see in Claim 9.6. With this choice we have (for the same reason as in the previous case) $\text{Tor}(\mathcal{R}'_*) = 1$ (because $|\mathcal{R}'_*| = 0 \in \mathbb{Z}/2\mathbb{Z}$).

LEMMA 9.4. — *If $H_0(\mathcal{B}_*^\rho)$ is based with any generator, then the sign-determined Reidemeister torsion of \mathcal{B}_*^ρ satisfies $\text{Tor}(\mathcal{B}_*^\rho) = (\text{tor}(\mathcal{H}_*))^{-1}$.*

Proof. — As in the proof of Lemma 9.3 we verify that the volume forms on the components of the exact sequence $0 \rightarrow \mathcal{B}_i^\rho \rightarrow \mathcal{C}_i^\rho \rightarrow \mathcal{R}'_i \rightarrow 0$ are compatible for $i = 0, 1, 2$. The Multiplicativity Lemma applies and gives

$$\text{Tor}(\mathcal{C}_*^\rho) = (-1)^{\varepsilon + \alpha} \text{Tor}(\mathcal{B}_*^\rho) \cdot \text{Tor}(\mathcal{R}'_*) \cdot \text{tor}(\mathcal{H}_*),$$

where $\varepsilon = \varepsilon(\mathcal{B}_*^\rho, \mathcal{C}_*^\rho, \mathcal{R}'_*) = 1 \in \mathbb{Z}/2\mathbb{Z}$ and $\alpha = \alpha(\mathcal{B}_*^\rho, \mathcal{R}'_*) = 0 \in \mathbb{Z}/2\mathbb{Z}$.

Finally, Lemma 9.2 provides $\text{Tor}(\mathcal{B}_*^\rho) = (\text{tor}(\mathcal{H}_*))^{-1}$. □

9.3. Third Step.

This last step of the proof of Lemma 9.1 is the more difficult one. We compare the sign-determined Reidemeister torsion of

$$\mathcal{B}_*^\rho = 0 \longrightarrow \text{Der}_\rho(M_K)_0 \rightarrow \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0 \longrightarrow \text{Der}_{\rho_S}^S(S)_0 \rightarrow 0$$

with the one of the based and homology based chain complex (see Subsection 8.2)

$$\mathcal{D}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \xrightarrow{P^*} \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}(S) \rightarrow 0.$$

With the notation and results obtained in the last two steps we can achieve the computation of $\text{Tor}(\mathcal{D}_*^\rho)$.

Proof of Lemma 9.1. — It requires several steps. We must repeat the splitting used in the second step to use the computations we made for $\text{Tor}(\mathcal{B}_*^\rho)$.

1) *m-regular case.* — We suppose that ρ is *m-regular*.

Consider the linear maps

$$\beta: \text{Der}_\rho(M_K) \longrightarrow \mathbb{R}, \quad d \mapsto \langle d(m), P^\rho(m) \rangle,$$

$$\beta_i: \text{Der}_{\rho_i}(B_i) \longrightarrow \mathbb{R}^n,$$

$$d \mapsto (\langle d(t_1^{(i)}), P^{\rho_i}(t_1^{(i)}) \rangle, \dots, \langle d(t_n^{(i)}), P^{\rho_i}(t_n^{(i)}) \rangle) \quad (i = 1, 2),$$

$$\beta_S: \text{Der}_{\rho_S}(S) \longrightarrow \mathbb{R}, \quad d \mapsto (\langle d(s_1), P^{\rho_S}(s_1) \rangle, \dots, \langle d(s_{2n}), P^{\rho_S}(s_{2n}) \rangle).$$

We can easily see that these three maps are surjective and that

$$\text{Der}_\rho(M_K)_0 = \ker \beta, \quad \text{Der}_{\rho_i}^{\mathcal{T}_i}(B_i)_0 = \ker \beta_i, \quad \text{Der}_{\rho_S}^S(S)_0 = \ker \beta_S.$$

If $\psi: H_\rho^2(M_K) \rightarrow \mathbb{R}$ denotes the isomorphism defined by $\psi(h_\rho^{(2)}) = 1$, then the diagram of Figure 3 is commutative.

Each row of the diagram of Figure 3 is an acyclic chain complex. The volume forms induced on the components of $\overline{\mathcal{R}}_*$ which are compatible along columns are the following ones:

- $\overline{\mathcal{R}}_3 = \mathbb{R}$ is endowed with the usual 1-volume form,
- $\overline{\mathcal{R}}_2 = \mathbb{R}^n \oplus \mathbb{R}^n$ is endowed with the usual product volume form on $\mathbb{R}^n \oplus \mathbb{R}^n$,

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \bar{\mathcal{B}}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K)_0 & \rightarrow & \text{Der}_{\rho_1}^{\tau_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\tau_2}(B_2)_0 & \rightarrow & \text{Der}_{\rho_S}^S(S)_0 & \rightarrow & 0 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \bar{\mathcal{D}}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \rightarrow & \text{Der}_{\rho_2}(B_2) \oplus \text{Der}_{\rho_2}(B_2) & \rightarrow & \text{Der}_{\rho_S}(S) & \rightarrow & H_\rho^2(M_K) & \rightarrow & 0 \\
 & & \downarrow \beta & & \downarrow \beta_1 \oplus \beta_2 & & \downarrow \beta_S & & \downarrow \psi & & \\
 \bar{\mathcal{R}}_* = 0 & \rightarrow & \mathbb{R} & \rightarrow & \mathbb{R}^n \oplus \mathbb{R}^n & \rightarrow & \mathbb{R}^{2n} & \rightarrow & \mathbb{R} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

Figure 3

- $\bar{\mathcal{R}}_1 = \mathbb{R}^{2n}$ is endowed with the opposite of the usual product volume form,
- $\bar{\mathcal{R}}_0 = \mathbb{R}$ is endowed with the usual 1-volume form corrected by the factor τ^{-1} (see equation (28)).

First, observe that $\text{Tor}(\mathcal{D}_*)^\rho = -(\text{Tor}(\bar{\mathcal{D}}_*^\rho))^{-1}$ and $\text{Tor}(\bar{\mathcal{B}}_*^\rho) = (\text{Tor}(\mathcal{B}_*)^\rho)^{-1}$. Moreover, Lemma 9.3 implies $\text{Tor}(\mathcal{B}_*)^\rho = \tau$, thus $\text{Tor}(\bar{\mathcal{B}}_*^\rho) = \tau^{-1}$.

Next, the Multiplicativity Lemma gives

$$\text{Tor}(\bar{\mathcal{D}}_*^\rho) = -\tau^{-1} \cdot \text{Tor}(\bar{\mathcal{R}}_*).$$

It remains to compute $\text{Tor}(\bar{\mathcal{R}}_*)$. For this purpose, let \mathcal{M}_* be the chain complex equal to $\bar{\mathcal{R}}_*$ but differently based. We suppose that each component of \mathcal{M}_* is endowed with the usual product volume form on \mathbb{R}^p . If we apply the basis change formula (5) we obtain $\text{Tor}(\bar{\mathcal{R}}_*) = -\tau \cdot \text{Tor}(\mathcal{M}_*)$. The aim of the following claim is to compute the sign-determined Reidemeister torsion of \mathcal{M}_* .

CLAIM 9.5. — We have

$$(30) \quad \text{Tor}(\mathcal{M}_*) = -\text{sgn}(\text{Tor}(\mathfrak{X}_*)).$$

Here we recall that \mathfrak{X}_* is the chain complex (13).

Proof of Claim 9.5. — Several steps are required to accomplish it.

(a) We begin by interpreting the acyclic chain complex \mathcal{M}_* as a Mayer-Vietoris sequence as follows. Consider a 3-dimensional ball D^3

which contain the knot K . We write $\widehat{M}_K = M_K \cap D^3$, $\widehat{B}_1 = B_1 \cap D^3$ and $\widehat{B}_2 = B_2 \cap D^3$. We can easily see that

$$H^1(\widehat{M}_K; \mathbb{R}) \cong H^1(M_K; \mathbb{R}) \cong \mathbb{R}, \quad H^1(\widehat{B}_i; \mathbb{R}) \cong \mathbb{R}^n, \quad i = 1, 2.$$

Using the long exact sequence in cohomology associated to the pair $(\partial M_K, \widehat{M}_K)$, we further observe that $H^2(\widehat{M}_K; \mathbb{R}) \cong H^2(\partial M_K; \mathbb{R}) \cong \mathbb{R}$. Thus the Mayer-Vietoris sequence with real coefficients corresponding to the splitting $\widehat{M}_K = \widehat{B}_1 \cup_{D^2 \setminus K} \widehat{B}_2$ reduces to

$$(31) \quad 0 \rightarrow H^1(\widehat{M}_K; \mathbb{R}) \xrightarrow{d^*} H^1(\widehat{B}_1; \mathbb{R}) \oplus H^1(\widehat{B}_2; \mathbb{R}) \xrightarrow{s^*} H^1(D^2 \setminus K; \mathbb{R}) \xrightarrow{\partial} H^2(\widehat{M}_K; \mathbb{R}) \rightarrow 0.$$

To prove that the Mayer-Vietoris sequence (31) is equal to \mathcal{M}_* we introduce some more notation. The boundary of B_i , $i = 1, 2$, is the union of n disjoint cylinders denoted $T_j^{(i)}$. The boundary of S is the disjoint union of $2n$ circles denoted S_j^1 . The basis of $T_i^{(2)}$ are the circles S_{2i-1}^1 and S_{2i}^1 and the basis of $T_i^{(1)}$ are the circles $S_{j_{i,1}}^1$ and $S_{j_{i,2}}^1$, where $j_{i,1}$ and $j_{i,2}$ are the two integers whose images by ζ are in the i th-handle (see Figures 1 and 4).

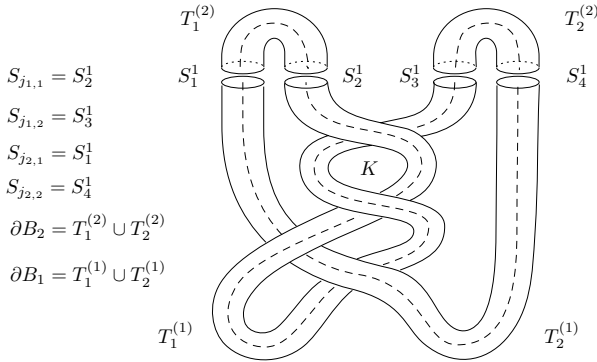


Figure 4. Boundary of the exterior of the figure eight knot and the tubes $T_j^{(i)}$

Let us furthermore consider the canonical inclusions $i: \partial M_K \hookrightarrow M_K$, $j_i: S_i^1 \hookrightarrow S$, $k_i^{(1)}: T_i^{(1)} \hookrightarrow \widehat{B}_1$, $k_i^{(2)}: T_i^{(2)} \hookrightarrow \widehat{B}_2$, $l_i: S_i^1 \hookrightarrow D^2 \setminus K$ and $\iota: \widehat{M}_K \hookrightarrow \partial M_K$. Write the Mayer-Vietoris sequences for cohomology associated to $M_K = B_1 \cup_S B_2$, to $\partial M_K = \partial B_1 \cup_{\partial S} \partial B_2$ and to $\widehat{M}_K = \widehat{B}_1 \cup_{D^2 \setminus K} \widehat{B}_2$. Taking into account the naturality of the constructions and the fact that $H_{\rho_S}^1(S_i^1) = \mathbb{R}P^{\rho_S}(s_i)$ and $H_\rho^1(\partial M_K) = \mathbb{R}P^\rho(m)$, these sequences combine to yield the commutative diagram of Figure 5.

$$\begin{array}{ccccccc}
 H^1_{\rho_1}(B_1) \oplus H^1_{\rho_2}(B_2) & \longrightarrow & H^1_{\rho_S}(S) & \xrightarrow{\delta} & H^2(M_K) & \longrightarrow & 0 \\
 \downarrow f_1 \oplus f_2 & & \downarrow \oplus_{i=1}^{2n} J_i^* & & \downarrow i^* & & \\
 \bigoplus_{i=1}^n H^1_{\rho_1}(T_i^{(1)}) \oplus \bigoplus_{i=1}^n H^1_{\rho_2}(T_i^{(2)}) & \longrightarrow & \bigoplus_{i=1}^{2n} H^1(S_i^1) & \xrightarrow{\Delta_1} & H^2(\partial M_K) & \longrightarrow & 0 \\
 \downarrow r_1 \oplus r_2 & & \downarrow \oplus_{i=1}^{2n} P^{\rho_S}(s_i) \cup \cdot & & \downarrow P^{\rho}(m) \cup \cdot & & \\
 \bigoplus_{i=1}^n H^1(T_i^{(1)}; \mathbb{R}) \oplus \bigoplus_{i=1}^n H^1(T_i^{(2)}; \mathbb{R}) & \longrightarrow & \bigoplus_{i=1}^{2n} H^1(S_i^1; \mathbb{R}) & \xrightarrow{\Delta_2} & H^2(\partial M_K; \mathbb{R}) & \longrightarrow & 0 \\
 \uparrow \iota_1 \oplus \iota_2 & & \uparrow \oplus_{i=1}^{2n} \ell_i^* & & \uparrow \iota^* & & \\
 H^1(\widehat{B}_1) \oplus H^1(\widehat{B}_2) & \longrightarrow & H^1(D^2 \setminus K; \mathbb{R}) & \xrightarrow{\partial} & H^2(\widehat{M}_K; \mathbb{R}) & \longrightarrow & 0
 \end{array}$$

Figure 5

In the diagram of Figure 5 each map δ , Δ_1 and Δ_2 , and ∂ is the connecting operator of the Mayer-Vietoris sequence associated to $M_K = B_1 \cup_S B_2$, $\partial M_K = \partial B_1 \cup_{\partial S} \partial B_2$ and $\widehat{M}_K = \widehat{B}_1 \cup_{D^2 \setminus K} \widehat{B}_2$ respectively. We also use the following notation:

$$\begin{aligned}
 f_1 &= \bigoplus_{i=1}^n (k_i^{(1)})^*, & f_2 &= \bigoplus_{i=1}^n (k_i^{(2)})^*, \\
 r_1 &= \bigoplus_{i=1}^n P^{\rho_1}(t_i^{(1)}) \cup \cdot, & r_2 &= \bigoplus_{i=1}^n P^{\rho_2}(t_i^{(2)}) \cup \cdot, \\
 \iota_1 &= \bigoplus_{i=1}^n (k_i^{(1)})^*, & \iota_2 &= \bigoplus_{i=1}^n (k_i^{(2)})^*.
 \end{aligned}$$

and the following identifications: $H^2(\partial M_K; \mathbb{R}) \cong \mathbb{R}$,

$$\bigoplus_{i=1}^n H^1(T_i^{(1)}; \mathbb{R}) \oplus \bigoplus_{i=1}^n H^1(T_i^{(2)}; \mathbb{R}) \cong \mathbb{R}^n \oplus \mathbb{R}^n,$$

and $\bigoplus_{i=1}^{2n} H^1(S_i^1; \mathbb{R}) \cong \mathbb{R}^{2n}$. All this suffices to establish the identification of \mathcal{M}_* with the chain complex (31).

(b) Having the notation of part (a) in mind we are ready to compute $\text{Tor}(\mathcal{M}_*)$. In this part we explicitly describe the homomorphisms d^* and s^* of the chain complex (31).

Consider the following matrix corresponding to d^*

$$P_{d^*} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 1 & 0 & & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Let A denote the matrix of the homomorphism s^* ; A is a presentation matrix of $H^2(\widehat{M}_K; \mathbb{Z}) \cong \mathbb{Z}$. Let P_{s^*} be the matrix deduced from

$$A' = \left(A \mid \begin{array}{c} 0 \\ \vdots \\ 1 \end{array} \right).$$

by deleting its first column. As we easily see, we have

$$(32) \quad \text{Tor}(\mathcal{M}_*) = \det P_{s^*} \cdot (\det P_{d^*})^{-1}.$$

We give an example to make the notation and the proof clearer.

Example. — For the trefoil knot (see Fig. 1) we have

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_{s^*} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

For the figure eight knot (see Fig. 3) we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad P_{s^*} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

We can obtain the matrix A in the following way.

- The i th column of A , with $1 \leq i \leq n$, is the vector with all entries of which are zero except the one of index $2i - 1$ and $2i$ which is equal to $(-1)^{\varepsilon_{2i-1}}$ and $(-1)^{\varepsilon_{2i}}$ respectively. Here $\varepsilon_k \in \{\pm 1\}$ depends on the orientation of K and is defined by the rule $i_*(s_{2k-1}^{(2)}) = (t_k^{(2)})^{\varepsilon_k}$ (see Fig. 1).

- The $(n + i)$ th column of A , with $1 \leq i \leq n$, is the vector with all entries of which are zero except the ones of index $j_{i,1}$ and $j_{i,2}$ which are equal to $(-1)^{\varepsilon_{j_{i,1}}}$ and $(-1)^{\varepsilon_{j_{i,2}}}$ respectively. Here $j_{i,1}$ and $j_{i,2}$ are the two integers whose image by ζ are in the i th-handle (see Fig. 1).

We moreover observe that each row of A contains exactly two coefficients non zero, equal to ± 1 ; and that each column contains exactly two coefficients non zero, opposed and equal to ± 1 . Thus, the sum of all rows' entries is zero. Seeing that A is a presentation matrix of \mathbb{Z} , the determinant of the matrix $A_{n,n}$, deduced from A by deleting the n th row and the n th column, is equal to ± 1 . This remark will be used in the sequel.

(c) In this last step we prove that $\text{Tor}(\mathcal{M}_*) = -(\text{Tor}(\mathcal{X}_*))^{-1}$.

The chain complex (13) denoted \mathfrak{X}_* reduces to

$$0 \rightarrow \mathbb{R} \xrightarrow{\delta_2} \mathbb{R}^{2n} \xrightarrow{\delta_1} \mathbb{R}^{2n-1} \rightarrow 0,$$

where $\delta_2 = 0$. We have $|\mathfrak{X}_*| = 1$ and $H_*(\mathfrak{X}_*)$ is based with $\{[[pt]], m^*\}$, see Subsection 4.2. To compute $\text{Tor}(\mathfrak{X}_*)$ in terms of $\text{Tor}(\mathcal{M}_*)$, we associate to \mathfrak{X}_* the acyclic chain complex

$$\mathfrak{X}'_* = 0 \rightarrow H^1(M_K; \mathbb{R}) \xrightarrow{\sigma} \mathbb{R}^{2n} \xrightarrow{\delta_1} \mathbb{R}^{2n-1} \oplus \mathbb{R} \xrightarrow{\text{Pf}_{2n}} \mathbb{R} \rightarrow 0.$$

Here, $H^1(M_K; \mathbb{R})$ is based with m^* , where $\sigma(m^*) = (1, \dots, 1, -1, \dots, -1)$ belongs to \mathbb{R}^{2n} , and the matrix of δ_1 is obtained from A by replacing the last row by $(0, \dots, 0)$.

Equation (4) implies $\text{tor}(\mathfrak{X}'_*) = -(\text{Tor}(\mathfrak{X}_*))^{-1} = -\text{Tor}(\mathfrak{X}_*) \in \{\pm 1\}$, see Remark 2. With obvious notation we have

$$\text{tor}(\mathfrak{X}'_*) = [b^0/c^0]^{-1} \cdot [b^1\tilde{b}^0/c^1] \cdot [b^2\tilde{b}^1/c^2]^{-1} \cdot [\tilde{b}^2/c^3],$$

where $[b^0/c^0] = [\tilde{b}^2/c^3] = 1$, $[b^1\tilde{b}^0/c^1] = \det A_{n,n} = \det P_{s^*}$ and

$$[b^2\tilde{b}^1/c^2] = \begin{vmatrix} 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 \\ -1 & 0 & \dots & 0 \end{vmatrix} = -(-1)^{2n+1} = 1.$$

It follows $\text{tor}(\mathfrak{X}'_*) = \text{Tor}(\mathcal{M}_*)$, which complete the proof of Claim 9.5 keeping in mind Remark 2. □

Claim 9.5 provides $\text{Tor}(\mathcal{D}_*^\rho) = -(\text{Tor}(\mathcal{M}_*))^{-1} = \text{sgn}(\text{Tor}(\mathfrak{X}_*))$ which achieve the proof of Lemma 9.1 in the m -regular case.

2) *Non m -regular case.* — Now we suppose that ρ is regular but not m -regular.

The comparison of the Reidemeister torsion of the based and homology based chain complex \mathcal{B}_*^ρ with the one of the homology based chain complex \mathcal{D}_*^ρ uses the commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1)_0 \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)_0 & \longrightarrow & \text{Der}_{\rho_S}^{\mathcal{S}}(S)_0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{D}_*^\rho = 0 & \rightarrow & \text{Der}_\rho(M_K) & \longrightarrow & \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) & \longrightarrow & \text{Der}_{\rho_S}(S) \rightarrow 0 \\
 & & \downarrow & & \downarrow \beta_1 \oplus \beta_2 & & \downarrow \beta_S \\
 \mathcal{R}_*'' = 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{R}^n \oplus \mathbb{R}^n & \longrightarrow & \mathbb{R}^{2n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Let \mathcal{H}'_* denote the long exact sequence in homology associated to the previous diagram; \mathcal{H}'_* reduces to

$$(33) \quad \mathcal{H}'_* = 0 \rightarrow H_1(\mathcal{R}''_*) \xrightarrow{\sim} H_0(\mathcal{B}^\rho_*) \longrightarrow H_0(\mathcal{D}^\rho_*) \xrightarrow{\sim} H_0(\mathcal{R}''_*) \rightarrow 0.$$

Remark that $H_2(\mathcal{R}''_*) = \ker s = \{(x, \dots, x, -x, \dots, -x) \mid x \in \mathbb{R}\} \cong \mathbb{R}$ and choose $(1, \dots, 1, -1, \dots, -1)$ as generator.

CLAIM 9.6. — *If $H_0(\mathcal{D}^\rho_*) = H_\rho^2(M_K)$ is based with the generator $h_\rho^{(2)}$ (see Equation (16)) and if \mathcal{H}_* denotes the chain complex (29), then $\text{tor}(\mathcal{H}'_*) = \text{tor}(\mathcal{H}_*)$.*

Proof. — We begin by the following observation

$$H_1(\mathcal{R}''_*) = \ker s = \Delta^{(n)} \oplus \Delta^{(n)}(\ker \sigma) = \Delta^{(n)} \oplus \Delta^{(n)}(H_1(\mathcal{R}'_*)),$$

where $\Delta^{(n)}: \mathbb{R} \ni x \mapsto (x, \dots, x) \in \mathbb{R}^n$. Thus the isomorphism

$$\Delta^{(n)} \oplus \Delta^{(n)}: \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}^n \oplus \mathbb{R}^n$$

maps the generator $(1, -1)$ of $H_1(\mathcal{R}'_*)$ to the generator $(1, \dots, 1, -1, \dots, -1)$ of $H_1(\mathcal{R}''_*)$.

Elsewhere, the Reidemeister torsion of the chain complex \mathcal{H}'_* is the inverse of the product of two changes of basis: the one corresponding to the isomorphism $H_0(\mathcal{D}^\rho_*) \cong H_0(\mathcal{R}''_*)$ and the other to the isomorphism $H_1(\mathcal{R}''_*) \cong H_0(\mathcal{B}^\rho_*)$. Keeping in mind the choices of bases, the determinant

of the transition matrix of the change of basis corresponding to the first isomorphism $H_0(\mathcal{D}_*^\rho) \cong H_0(\mathcal{R}_*'')$ is $+1$. We remark that

$$\mathbf{x} = (x, \dots, x, -x, \dots, -x) = \Delta^{(n)} \oplus \Delta^{(n)}(x, -x) \in \ker s$$

lifts to an element in $\text{Der}_{\rho_1}^{\mathcal{T}_1}(B_1) \oplus \text{Der}_{\rho_2}^{\mathcal{T}_2}(B_2)$. Thus the determinant of the transition matrix of the change of basis corresponding to the second isomorphism $H_1(\mathcal{R}_*'') \cong H_0(\mathcal{B}_*^\rho)$ is equal to the determinant of the transition matrix of the change of basis corresponding to the isomorphism $H_1(\mathcal{R}_*') \cong H_0(\mathcal{B}_*^\rho)$. This is sufficient to prove the required equality. \square

We can now finish the computation of $\text{Tor}(\mathcal{D}_*^\rho)$. We establish the compatibility along each column of the previous diagram using the same technique as in the case (1) where ρ is supposed m -regular. The Multiplicativity Lemma gives

$$\text{Tor}(\mathcal{D}_*^\rho) = (-1)^{\varepsilon+\alpha} \text{Tor}(\mathcal{B}_*^\rho) \cdot \text{Tor}(\mathcal{R}_*'') \cdot \text{tor}(\mathcal{H}_*')$$

with $\varepsilon = \varepsilon(\mathcal{B}_*^\rho, \mathcal{D}_*^\rho, \mathcal{R}_*'') = 0 \in \mathbb{Z}/2\mathbb{Z}$ and $\alpha = \alpha(\mathcal{B}_*^\rho, \mathcal{R}_*'') = 0 \in \mathbb{Z}/2\mathbb{Z}$.

Claim 9.6 and Lemma 9.4 respectively furnish $\text{tor}(\mathcal{H}_*') = \text{tor}(\mathcal{H}_*)$ and $\text{Tor}(\mathcal{B}_*^\rho) = (\text{tor}(\mathcal{H}_*))^{-1}$. Thus $\text{Tor}(\mathcal{D}_*^\rho) = \text{Tor}(\mathcal{R}_*'')$. It just remains to compute $\text{Tor}(\mathcal{R}_*'')$. Using Claim 9.5 it is easy to see that $\text{Tor}(\mathcal{R}_*'') = \text{sgn}(\text{Tor}(\mathcal{X}_*))$, which completes the proof of Lemma 9.1. \square

10. Proof of Theorem 7.1: Part 2.

The second part of the proof of Theorem 7.1 consists in the comparison of the Reidemeister torsion of the chain complex

$$\mathcal{D}_*^\rho = 0 \rightarrow \text{Der}_\rho(M_K) \xrightarrow{p^*} \text{Der}_{\rho_1}(B_1) \oplus \text{Der}_{\rho_2}(B_2) \xrightarrow{\kappa^*} \text{Der}_{\rho_S}(S) \rightarrow 0$$

to the one of the chain complex (14) of the knot exterior

$$\mathcal{X}_*^\rho = 0 \rightarrow \mathfrak{su}(2) \xrightarrow{d_2^\rho} \mathfrak{su}(2)^{2n} \xrightarrow{d_1^\rho} \mathfrak{su}(2)^{2n-1} \rightarrow 0.$$

It splits into two steps.

10.1. First step.

We compare the sign-determined Reidemeister torsion of the non acyclic homology based chain complex \mathcal{X}_*^ρ with the one of \mathcal{D}_*^ρ using the diagram of Figure 6.

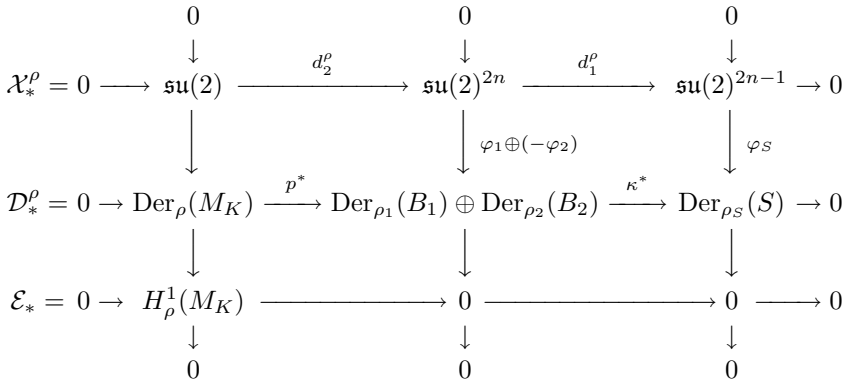


Figure 6

Each component of the rows is endowed with a reference volume form. These reference volume forms are compatible along each column (see Subsection 8.1). With the notation above, we prove

LEMMA 10.1. — *For all non zero vector v in $H_\rho^1(M_K)$, we have*

$$\tau_\rho^K(\varphi_{[\rho]}^{-1}(v)) = -(\text{Tor}(\mathcal{E}_*))^{-1}.$$

Before proving Lemma 10.1 we establish

CLAIM 10.2. — *The diagram of Figure 6 is commutative.*

Proof. — To establish the commutativity of the diagram we use an explicit description of the boundary operators of the chain complex \mathcal{X}_*^ρ . Using the Wirtinger presentation (18) of G_K we recall that

$$d_2^\rho(x) = \left((1 - t_1^{(1)}) \circ x, \dots, (1 - t_n^{(1)}) \circ x, (1 - t_1^{(2)}) \circ x, \dots, (1 - t_n^{(2)}) \circ x \right)$$

for all $x \in \mathfrak{su}(2)$, and

$$d_1^\rho((x_j)_{1 \leq j \leq 2n}) = \left(\sum_{j=1}^n \frac{\partial r_i}{\partial t_j^{(1)}} \circ x_{2j-1} + \sum_{j=1}^n \frac{\partial r_i}{\partial t_j^{(2)}} \circ x_{2j} \right)_{1 \leq i \leq 2n-1}$$

for all $(x_j)_{1 \leq j \leq 2n} \in \mathfrak{su}(2)^{2n}$, where $r_i = \kappa_1(s_i)\kappa_2(s_i^{-1})$. Essentially the commutativity comes from the two following equalities:

$$\frac{\partial r_i}{\partial t_k^{(1)}} = \frac{\partial \kappa_1(s_i)}{\partial t_k^{(1)}} \quad \text{and} \quad \frac{\partial r_i}{\partial t_k^{(2)}} = -\frac{\partial \kappa_2(s_i)}{\partial t_k^{(2)}}. \quad \square$$

Next, the twisted homology groups of \mathcal{X}_*^ρ , \mathcal{D}_*^ρ and \mathcal{E}_* respectively satisfies:

- $H_0(\mathcal{X}_*^\rho) \cong H_\rho^2(M_K)$, $H_1(\mathcal{X}_*^\rho) \cong H_\rho^1(M_K)$ and $H_i(\mathcal{X}_*^\rho) = 0$ if $i \geq 2$ (see § 4.3),
- $H_0(\mathcal{D}_*^\rho) \cong H_\rho^2(M_K)$ and $H_i(\mathcal{D}_*^\rho) = 0$ if $i \geq 1$ (Fact 8.3),
- $H_2(\mathcal{E}_*) \cong H_\rho^1(M_K)$ and $H_i(\mathcal{E}_*) = 0$ if $i \neq 2$.

Now we turn to the proof of Lemma 10.1.

Proof of Lemma 10.1. — Let \mathcal{H}_*'' denote the long exact sequence in homology associated to $0 \rightarrow \mathcal{X}_*^\rho \rightarrow \mathcal{D}_*^\rho \rightarrow \mathcal{E}_* \rightarrow 0$. The Multiplicativity Lemma gives

$$\text{Tor}(\mathcal{D}_*^\rho) = (-1)^{\varepsilon+\alpha} \text{Tor}(\mathcal{X}_*^\rho) \cdot \text{Tor}(\mathcal{E}_*) \cdot \text{Tor}(\mathcal{H}_*''),$$

where $\varepsilon = \varepsilon(\mathcal{X}_*^\rho, \mathcal{D}_*^\rho, \mathcal{E}_*) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and $\alpha = \alpha(\mathcal{X}_*^\rho, \mathcal{E}_*) = 1 \in \mathbb{Z}/2\mathbb{Z}$.

The sequence \mathcal{H}_*'' reduces to the two isomorphisms

$$H_2(\mathcal{E}_*) \xrightarrow{\sim} H_1(\mathcal{X}_*^\rho) \quad \text{and} \quad H_0(\mathcal{X}_*^\rho) \xrightarrow{\sim} H_0(\mathcal{D}_*^\rho).$$

Thus $\text{Tor}(\mathcal{H}_*'')$ is the inverse of the product of two basis changes, the one corresponding to $H_\rho^2(M_K)$ and the other to $H_\rho^1(M_K)$. Because of the choice of the bases of the twisted cohomology groups in dimension 1 and 2, we have $\text{tor}(\mathcal{H}_*'') = 1$. So $\text{Tor}(\mathcal{D}_*^\rho) = -\text{Tor}(\mathcal{X}_*^\rho) \cdot \text{Tor}(\mathcal{E}_*)$. Finally, using Lemma 9.1, we conclude that

$$(34) \quad \text{sgn}(\text{Tor}(\mathfrak{X}_*)) \cdot \text{Tor}(\mathcal{X}_*^\rho) = -(\text{Tor}(\mathcal{E}_*))^{-1}.$$

□

10.2. Second step.

It remains to bring together all the computations of Reidemeister torsions we have done before.

End of the proof of Theorem 7.1. — Fix a non zero vector $v \in H_\rho^1(M_K)$; we will prove

$$\omega_{[\rho]}^K(\varphi_{[\rho]}^{-1}(v)) = \tau_\rho^K(\varphi_{[\rho]}^{-1}(v)).$$

To this end, we must compute the sign-determined Reidemeister torsion of the based and homology based chain complex \mathcal{E}_* . Let z denote the generator of $H_\rho^1(M_K)$ such that $\omega_{[\rho]}^K(\varphi_{[\rho]}^{-1}(z)) = 1$. The basis of $H_\rho^1(M_K)$

formed by the single vector v and the basis of $H_\rho^2(M_K)$ formed by the single vector $h_\rho^{(2)}$ are the bases of the twisted cohomology groups of M_K in which we compute the sign-determined Reidemeister torsion $\text{Tor}(\mathcal{X}_*^\rho)$. We observe that $\mathcal{E}_2 = H_\rho^1(M_K)$ is based with $\{z\}$ and $H_2(\mathcal{E}_*) \cong H_\rho^1(M_K)$ is based with $\{v\}$. With our conventions, we thus have $\text{Tor}(\mathcal{E}_*) = -[z/v]$ (because $|\mathcal{E}_*| = 1$).

Lemma 10.1 provides

$$(35) \quad \tau_\rho^K(\varphi_{[\rho]}^{-1}(v)) = -(\text{Tor}(\mathcal{E}_*))^{-1} = [v/z].$$

So, by using formula (35) and the definition of z we get

$$\omega_{[\rho]}^K(\varphi_{[\rho]}^{-1}(v)) = [v/z] = \tau_\rho^K(\varphi_{[\rho]}^{-1}(v)),$$

which completes the proof of Theorem 7.1. \square

Acknowledgements. — The author wishes to express his gratitude to Michael Heusener, Joan Porti, Daniel Lines, Vladimir Turaev, Michel Boileau, Jean-Yves LeDimet, Louis Funar, Rinat Kashaev and Pierre de la Harpe for helpful discussions related to this paper.

The author also would like to thanks the referee for his careful reading of the paper and for his remarks which contributed to make it clearer.

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Manuscrit reçu le 19 avril 2004,
accepté le 11 septembre 2004.

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