



ANNALES

DE

L'INSTITUT FOURIER

Giorgio OTTAVIANI & Elena RUBEI

Resolutions of homogeneous bundles on \mathbb{P}^2

Tome 55, n° 3 (2005), p. 973-1015.

http://aif.cedram.org/item?id=AIF_2005__55_3_973_0

© Association des Annales de l'institut Fourier, 2005, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

RESOLUTIONS OF HOMOGENEOUS BUNDLES ON \mathbb{P}^2

by Giorgio OTTAVIANI & Elena RUBEI

1. Introduction.

Homogeneous bundles on $\mathbb{P}^2 = \mathrm{SL}(3)/P$ can be described by representations of the parabolic subgroup P . In 1966 Ramanan [Ra] proved that if ρ is an irreducible representation of P then the induced bundle E_ρ on \mathbb{P}^2 is simple and even stable. Since P is not a reductive group, there is a lot of indecomposable reducible representations of P and to classify homogeneous bundles on \mathbb{P}^2 and among them the simple ones, the stable ones, etc. by means of the study of the representations of the parabolic subgroup P seems difficult.

In this paper our point of view is to consider the minimal free resolution of the bundle. Our aim is to classify homogeneous vector bundles on \mathbb{P}^2 by means of their minimal resolutions. Precisely we observe that if E is a homogeneous vector bundle on $\mathbb{P}^2 = \mathbb{P}(V)$ (V complex vector space of dimension 3) there exists a minimal free resolution of E

$$0 \rightarrow \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} A_q \longrightarrow \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} B_q \longrightarrow E \rightarrow 0$$

with $\mathrm{SL}(V)$ -invariant maps (A_q and B_q are $\mathrm{SL}(V)$ -representations) and we characterize completely the representations that can occur as A_q and B_q and the maps $A \rightarrow B$ that can occur ($A := \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} A_q$, $B := \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} B_q$). To state the theorem we need some notation.

Keywords : Homogeneous bundles, minimal resolutions, quivers.
Math. classification : 14M17, 14F05, 16G20.

NOTATION 1. — Let $q, r \in \mathbb{N}$; for every $\rho \geq p$, let $\varphi_{\rho,p}$ be a fixed $\mathrm{SL}(V)$ -invariant nonzero map $S^{p,q,r}V \otimes \mathcal{O}_{\mathbb{P}(V)}(p) \rightarrow S^{\rho,q,r}V \otimes \mathcal{O}_{\mathbb{P}(V)}(\rho)$ (it is unique up to multiples) such that $\varphi_{\rho,p} = \varphi_{\rho,p'} \circ \varphi_{p',p}$ for all $\rho \geq p' \geq p$ (where $S^{p,q,r}$ denotes the Schur functor associated to (p, q, r) , see §2).

Let $\mathcal{P}, \mathcal{R} \subset \mathbb{N}$, $c \in \mathbb{Z}$; for any $\mathrm{SL}(V)$ -invariant map

$$\gamma: \bigoplus_{p \in \mathcal{P}} A^p \otimes S^{p,q,r}V(c+p) \longrightarrow \bigoplus_{\rho \in \mathcal{R}} B^\rho \otimes S^{\rho,q,r}V(c+\rho)$$

(A^p and B^ρ vector spaces) we define $M(\gamma)$ to be the map

$$M(\gamma): \bigoplus_{p \in \mathcal{P}} A^p \longrightarrow \bigoplus_{\rho \in \mathcal{R}} B^\rho$$

such that

$$\gamma_{\rho,p} = \begin{cases} M(\gamma)_{\rho,p} \otimes \varphi_{\rho,p} & \text{for all } \rho, p \text{ with } \rho \geq p, \\ 0 & \text{for all } \rho, p \text{ with } \rho < p, \end{cases}$$

where

$\gamma_{\rho,p}: A^p \otimes S^{p,q,r}V(c+p) \longrightarrow B^\rho \otimes S^{\rho,q,r}V(c+\rho)$, $M(\gamma)_{\rho,p}: A^p \longrightarrow B^\rho$ are the maps induced respectively by γ and $M(\gamma)$ (by restricting and projecting).

THEOREM 2. — (i) On $\mathbb{P}^2 = \mathbb{P}(V)$ let

$$A = \bigoplus_{p,q,i} A_i^{p,q} \otimes S^{p,q}V(i), \quad B = \bigoplus_{p,q,i} B_i^{p,q} \otimes S^{p,q}V(i)$$

with p, q, i varying in a finite subset of \mathbb{N} , $A_i^{p,q}$ and $B_i^{p,q}$ finite dimensional vector spaces. Then A and B are the first two terms of a minimal free resolution of a homogeneous bundle on \mathbb{P}^2 if and only if for all $c \in \mathbb{Z}$, $q, \tilde{p} \in \mathbb{N}$

$$\dim \left(\bigoplus_{p \geq \tilde{p}} A_{c+p}^{p,q} \right) \leq \dim \left(\bigoplus_{\rho > \tilde{p}} B_{c+\rho}^{p,q} \right).$$

(ii) Let

$$A = \bigoplus_{p,q,r} A^{p,q,r} \otimes S^{p,q,r}V(p+q+r), \quad B = \bigoplus_{p,q,r} B^{p,q,r} \otimes S^{p,q,r}V(p+q+r),$$

p, q, r varying in a finite subset of \mathbb{N} , $A^{p,q,r}$ and $B^{p,q,r}$ finite dimensional vector spaces; let α be an $\mathrm{SL}(V)$ -invariant map $A \rightarrow B$. Then there exists a homogeneous bundle E on \mathbb{P}^2 such that

$$0 \rightarrow A \xrightarrow{\alpha} B \rightarrow E \rightarrow 0$$

is a minimal free resolution of E if and only if:

- $M(\alpha_{p,q,r}): A^{p,q,r} \rightarrow B^{p,q,r}$ is zero for all p,q,r

and

- $M(\alpha_{q,r}): \bigoplus_p A^{p,q,r} \rightarrow \bigoplus_\rho B^{\rho,q,r}$ is injective for all q,r ,

where

$$\alpha_{p,q,r}: A^{p,q,r} \otimes S^{p,q,r}V(p) \longrightarrow B^{p,q,r} \otimes S^{p,q,r}V(p),$$

$$\alpha_{q,r}: \bigoplus_p A^{p,q,r} \otimes S^{p,q,r}V(p) \longrightarrow \bigoplus_\rho B^{\rho,q,r} \otimes S^{\rho,q,r}V(p)$$

are the maps induced by α .

The above theorem allows us to parametrize the set of homogeneous bundles on \mathbb{P}^2 by a set of sequences of injective matrices with a certain shape up to the action of invertible matrices with a certain shape. An interesting problem is to use this parametrization to study the related moduli spaces.

Then we begin to study which minimal free resolutions give simple or stable homogeneous bundles. We will consider the case A is irreducible; we call a homogeneous bundle *elementary* if it has minimal free resolution $0 \rightarrow A \rightarrow B \rightarrow E \rightarrow 0$ with A irreducible; besides we say that a bundle E on \mathbb{P}^2 is *regular* if the minimal free resolution is $0 \rightarrow A \rightarrow B \rightarrow E \rightarrow 0$ with all the components of A with the same twist and all the components of B with the same twist.

First we study simplicity and stability of regular elementary homogeneous bundles. Fundamental tool are quivers and representations of quivers associated to homogeneous bundles introduced by Bondal and Kapranov [BK]. The quivers allow us to handle well and “to make explicit” the homogeneous subbundles of a homogeneous bundle E and Rohnfeld’s criterion (see [Ro]) in this context is equivalent to saying that E is semistable if and only if the slope of every subbundle associated to a subrepresentation of the quiver representation of E is less or equal than the slope of E .

The simplest regular elementary homogeneous bundles are the bundles E defined by the exact sequence

$$0 \rightarrow S^{p,q}V \otimes \mathcal{O}(-s) \xrightarrow{\varphi} S^{p+s,q}V \otimes \mathcal{O} \longrightarrow E \rightarrow 0$$

for some $p,q,s \in \mathbb{N}$, $p \geq q$, φ an $SL(V)$ -invariant nonzero map; we prove that such bundles are stable, see Theorem 36 (observe that Ramanan’s theorem does not apply here). In the particular case $p = q = 0$ the stability of E was already proved in [Ba].

Besides we prove:

THEOREM 3. — *A regular elementary homogeneous bundle E on \mathbb{P}^2 is simple if and only if its minimal free resolution is of the following kind*

$$0 \rightarrow S^{p,q}V \otimes \mathcal{O}(-s) \xrightarrow{\varphi} W \otimes \mathcal{O} \rightarrow E \rightarrow 0$$

where $p, q, s \in \mathbb{N}$, $p \geq q$, W is a nonzero $\mathrm{SL}(V)$ -submodule of $S^{p,q}V \otimes S^sV$, all the components of φ are nonzero $\mathrm{SL}(V)$ -invariant maps and we are in one of the following cases:

- (i) $p = 0$,
- (i) $p > 0$ and $W \neq S^{p,q}V \otimes S^sV$.

By using the above theorem and Theorem 47, which characterizes stability of the minimal free resolution of regular elementary homogeneous bundles when (with the above notation) $s = 1$, we find infinite examples of unstable simple homogeneous bundles.

Finally we state a criterion, generalizing Theorem 3, to say when an elementary (not necessarily regular) homogeneous bundle is simple by means of its minimal free resolution, see Theorem 48.

The sketch of the paper is the following: in §2 we recall some basic facts on representation theory; in §3 we characterize the resolutions of homogeneous bundles on \mathbb{P}^2 : in this section we prove Theorem 9, which contains Theorem 2; in §4 we recall the theory of quivers; in §5 we prove some lemmas by using quivers and we fix some notation; in §6 we study stability and simplicity of elementary homogeneous bundles.

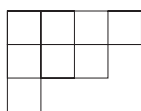
2. Notation and preliminaries.

We recall some facts on representation theory (see for instance [FH]).

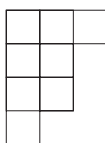
Let d be a natural number and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d with $\lambda_1 \geq \dots \geq \lambda_k$.

We can associate to λ a *Young diagram* with λ_i boxes in the i -th row, the rows lined up on the left. The conjugate partition λ' is the partition of d whose Young diagram is obtained from the Young diagram of λ interchanging rows and columns.

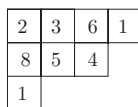
A *tableau* with entries in $\{1, \dots, n\}$ on the Young diagram of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of d is a numbering of the boxes by the integers $1, \dots, n$, allowing repetitions (we say also that it is a tableau on λ).



Young diagram of (4,3,1)



Young diagram of (3,2,2,1), conjugate of (4,3,1)



A tableau on (4,3,1)

Let V be a complex vector space of dimension n . Let $d \in \mathbb{N}$ and let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d , with $\lambda_1 \geq \dots \geq \lambda_k$. Number the boxes of the Young diagram of λ with the numbers $1, \dots, d$ from left to right beginning from the top row. Let Σ_d be the group of permutations on d elements; let R be the subgroup of Σ_d given by the permutations preserving the rows and let C be the subgroup of Σ_d given by the permutations preserving the columns.

DEFINITION 4. — We define

$$S^\lambda V := \text{Im} \left(\sum_{\substack{a \in C \\ s \in R}} \text{sign}(a) s \circ a : \otimes^d V \rightarrow \otimes^d V \right).$$

The $S^\lambda V$ are called *Schur representations*

The $S^\lambda V$ are irreducible $\text{SL}(V)$ -representations and it is well-known that all the irreducible $\text{SL}(V)$ -representations are of this form.

NOTATION 5. — Let V be a complex vector space and let $\{v_j\}_{1, \dots, n}$ be a basis of V . Let $d \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d with $\lambda_1 \geq \dots \geq \lambda_k$. Let μ be the conjugate partition. Let \mathcal{G}_λ be the free abelian group generated by the tableaux on λ with entries in $\{1, \dots, n\}$ and let $\mathcal{T}_\lambda = \mathcal{G}_\lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

Let t be the map associating to a tableau T on λ with entries in $\{1, \dots, n\}$ the following element of $V^{\otimes d}$:

$$v_{T_1^1} \otimes \dots \otimes v_{T_{\lambda_1}^1} \otimes \dots \otimes v_{T_1^k} \otimes \dots \otimes v_{T_{\lambda_k}^k}$$

where $(T_1^j, \dots, T_{\lambda_j}^j)$ is the j -th row of T .

• We define $\text{ant} : \mathcal{T}_\lambda \rightarrow \mathcal{T}_\lambda$ to be the linear map such that for every tableau T on λ

$$\text{ant}(T) = \sum_{(\sigma_1, \dots, \sigma_{\lambda_1}) \in \Sigma_{\mu_1} \times \dots \times \Sigma_{\mu_{\lambda_1}}} \text{sign}(\sigma_1) \dots \text{sign}(\sigma_{\lambda_1}) T^{\sigma_1, \dots, \sigma_{\lambda_1}},$$

where $T^{\sigma_1, \dots, \sigma_{\lambda_1}}$ is the tableau obtained from T permuting the elements of the j -th column with σ_j for all j . For instance

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \xrightarrow{\text{ant}} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 4 & 2 & 3 \\ \hline 1 & 5 & \\ \hline \end{array} \\ - \begin{array}{|c|c|c|} \hline 1 & 5 & 3 \\ \hline 4 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 4 & 5 & 3 \\ \hline 1 & 2 & \\ \hline \end{array}$$

• Analogously we define $\text{sim} : \mathcal{T}_\lambda \rightarrow \mathcal{T}_\lambda$ to be the linear map such that for every tableau T on λ

$$\text{sim}(T) = \sum_{(\sigma_1, \dots, \sigma_{\mu_1}) \in \Sigma_{\lambda_1} \times \dots \times \Sigma_{\lambda_{\mu_1}}} T_{\sigma_1, \dots, \sigma_{\mu_1}},$$

where $T_{\sigma_1, \dots, \sigma_{\mu_1}}$ is the tableau obtained from T permuting the elements of the j -th row with σ_j for all j . For instance

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \xrightarrow{\text{sim}} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 4 \\ \hline \end{array}$$

• We call ord the linear map associating to a tableau T the tableau obtained from T ordering the entries of every row in nondecreasing way.

Observe that

$$\text{ord} \circ \text{sim} = \lambda_1! \dots \lambda_k! \text{ord} \quad \text{and} \quad \text{sim} \circ \text{ord} = \text{sim}.$$

• Let $\mathcal{S} = t \circ \text{sim} \circ \text{ant}$.

Obviously the space $S^\lambda V$ can be described as the image of $\mathcal{S} : \mathcal{T}_\lambda \rightarrow V^{\otimes d}$.

We recall that *Pieri's formula* says that, if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of a natural number d with $\lambda_1 \geq \lambda_2 \geq \dots$ and t is a natural number, then

$$S^\lambda V \otimes S^t V = \bigoplus_{\nu} S^\nu V$$

as $\text{SL}(V)$ -representations, where the sum is performed on all the partitions $\nu = (\nu_1, \dots)$ with $\nu_1 \geq \nu_2 \geq \dots$ of $d + t$ whose Young diagrams are obtained from the Young diagram of λ adding t boxes not two in the same column.

Finally we observe that if V is a complex vector space of dimension n then $S^{(\lambda_1, \dots, \lambda_{n-1})} V$ is isomorphic to $S^{(\lambda_1+r, \dots, \lambda_{n-1}+r, r)} V$ for all r as $\text{SL}(V)$ -representation. Besides $(S^{(\lambda_1, \dots, \lambda_n)} V)^\vee$ is isomorphic as $\text{SL}(V)$ -representation to $S^{(\lambda_1-\lambda_n, \dots, \lambda_1-\lambda_2)} V$. Moreover $(S^\lambda V)^\vee \simeq S^\lambda V^\vee$.

NOTATION 6. — In all the paper V will be a complex vector space of dimension 3 if not otherwise specified.

If E is a vector bundle on $\mathbb{P}(V)$ then $\mu(E)$ will denote the slope of E , i.e. the first Chern class divided by the rank.

3. Resolutions of homogeneous vector bundles.

The aim of this section is to characterize the minimal free resolutions of the homogeneous bundles on \mathbb{P}^2 .

LEMMA 7. — *Let E be a homogeneous vector bundle on $\mathbb{P}^2 = \mathbb{P}(V)$. By Horrocks' theorem [Hor] the bundle E has a minimal free resolution*

$$0 \rightarrow \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} A_q \longrightarrow \bigoplus_q \mathcal{O}(-q) \otimes_{\mathbb{C}} B_q \xrightarrow{\psi} E \rightarrow 0$$

(A_q, B_q \mathbb{C} -vector spaces). Since E is homogeneous we can suppose the maps are $\mathrm{SL}(V)$ -invariant maps (A_q and B_q are $\mathrm{SL}(V)$ -representations).

Proof (see also [Ka]). — Let

$$M := \bigoplus_k H^0(E(k)) \quad \text{and} \quad S = \bigoplus_k \mathrm{Sym}^k V^\vee.$$

Let $m_i \in M$, $i = 1, \dots, k$ such that no one of them can be written as linear combination with coefficient in S of elements of the $\mathrm{SL}(V)$ -orbits of the others and such that $\bigcup_{g \in \mathrm{SL}(V), i=1, \dots, k} g(m_i)$ generates M on S . Let $q_i = \deg(m_i)$. Let

$$B_i := \left\langle \bigcup_{g \in \mathrm{SL}(V)} g(m_i) \right\rangle_{\mathbb{C}}$$

(finite dimensional $\mathrm{SL}(V)$ -representations). Let

$$P = \bigoplus_i B_i \otimes_{\mathbb{C}} S(-q_i)$$

and $\varphi: P \rightarrow M$ be the $\mathrm{SL}(V)$ -invariant map given by multiplication. Let $\psi: B \rightarrow E$ be the sheafification of $\varphi: P \rightarrow M$. Thus $B = \bigoplus_i B_i \otimes \mathcal{O}(-q_i)$. Let $A = \mathrm{Ker}(\psi)$; it is a homogeneous vector bundle; we have $H^1(A(t)) = 0$ for all $t \in \mathbb{Z}$, because $H^0(B(t)) \rightarrow H^0(E(t))$ is surjective for all t and $H^1(B(t)) = 0$ for all t , hence by Horrocks' criterion A splits. \square

Remark 8.

a) If U, W, V are three vector spaces then on $P(V)$ we have $\text{Hom}(U \otimes \mathcal{O}(-s), W) = \text{Hom}(U \otimes S^s V, W)$ (the isomorphism can be given by $H^0(\bullet^\vee)^\vee$).

b) Let V be a vector space. For any λ, μ partitions, $s \in \mathbb{N}$, up to multiples there is a unique $\text{SL}(V)$ -invariant map

$$S^\lambda V \otimes \mathcal{O}(-s) \longrightarrow S^\mu V \otimes \mathcal{O}$$

by Pieri's formula, Schur's lemma and part a) of the remark.

Theorem 9, which implies Theorem 2, is the aim of this section. It allows us to classify all homogeneous vector bundles on \mathbb{P}^2 ; in fact it characterizes their minimal free resolutions.

Precisely part (i) allows us to say which A and B can occur in a minimal free resolution $0 \rightarrow A \rightarrow B \rightarrow E \rightarrow 0$ of a homogeneous vector bundle E on \mathbb{P}^2 ; first we investigate when, given A and B direct sums of bundles of the kind $S^{p,q}V(-i)$, there exists an injective $\text{SL}(V)$ -invariant map $A \rightarrow B$; roughly speaking this is true if and only if for every $\text{SL}(V)$ -irreducible subbundle $S = S^{p,q}V(i)$ of A there exists a subbundle $M(S)$ of B of the kind $S^{p+s,q}V(i+s)$ for some $s \in \mathbb{N}$ and we can choose $M(S)$ in such way that the map $S \mapsto M(S)$ is injective.

A crucial point of the proof is the fact that an $\text{SL}(V)$ -invariant map $S^{p,q}V \rightarrow S^{p+s_1,q+s_2,s_3}V(s_1+s_2+s_3)$ is injective if and only if $s_2 = s_3 = 0$ and the intersection of the kernels of the ones of such maps with $s_2 + s_3 > 0$ is nonzero.

Part (ii) allows us to say which maps $A \rightarrow B$ can occur in a minimal free resolution $0 \rightarrow A \rightarrow B \rightarrow E \rightarrow 0$ of a homogeneous bundle E ; first we study when an $\text{SL}(V)$ -invariant map $\alpha: A \rightarrow B$ is injective (A and B direct sums of bundles of the kind $S^{p,q,r}V(-i)$). We remark that, if $\alpha: A \rightarrow B$ is an $\text{SL}(V)$ -invariant map, we can suppose that the sum $p + q + r - i$ is constant for $S^{p,q,r}V(i)$ varying among all $\text{SL}(V)$ -irreducible subbundles of A or B (by using the isomorphism $S^{p,q}V \simeq S^{p+u,q+u,u}V$ for all $u \in \mathbb{N}$).

In the sequel we will use Notation 1.

THEOREM 9. — (i) *On $\mathbb{P}^2 = \mathbb{P}(V)$ let*

$$A = \bigoplus_{p,q,i} A_i^{p,q} \otimes S^{p,q}V(i), \quad B = \bigoplus_{p,q,i} B_i^{p,q} \otimes S^{p,q}V(i)$$

with p, q, i varying in a finite subset of \mathbb{N} , $A_i^{p,q}$ and $B_i^{p,q}$ finite dimensional

vector spaces. There exists an injective $SL(V)$ -invariant map $A \rightarrow B$ if and only if for all $c \in \mathbb{Z}$, $q, \tilde{p} \in \mathbb{N}$,

$$(1) \quad \dim \left(\bigoplus_{p \geq \tilde{p}} A_{c+p}^{p,q} \right) \leq \dim \left(\bigoplus_{\rho \geq \tilde{p}} B_{c+\rho}^{\rho,q} \right).$$

(i') Besides A and B are the first two terms of a minimal free resolution of a homogeneous bundle on \mathbb{P}^2 if and only if for all $c \in \mathbb{Z}$, $q, \tilde{p} \in \mathbb{N}$,

$$(2) \quad \dim \left(\bigoplus_{p \geq \tilde{p}} A_{c+p}^{p,q} \right) \leq \dim \left(\bigoplus_{\rho > \tilde{p}} B_{c+\rho}^{\rho,q} \right).$$

(ii) Let

$$A = \bigoplus_{p,q,r} A^{p,q,r} \otimes S^{p,q,r}V(p+q+r), \quad B = \bigoplus_{p,q,r} B^{p,q,r} \otimes S^{p,q,r}V(p+q+r)$$

p, q, r varying in a finite subset of \mathbb{N} , $A^{p,q,r}$ and $B^{p,q,r}$ finite dimensional vector spaces; let α be an $SL(V)$ -invariant map $A \rightarrow B$. Then α is injective if and only if for all q, r the induced map

$$\alpha_{q,r}: \bigoplus_p A^{p,q,r} \otimes S^{p,q,r}V(p+q+r) \longrightarrow \bigoplus_\rho B^{\rho,q,r} \otimes S^{\rho,q,r}V(\rho+q+r)$$

is injective and (by Lemma 13) this is true if and only if

$$M(\alpha_{q,r}): \bigoplus_p A^{p,q,r} \longrightarrow \bigoplus_\rho B^{\rho,q,r}$$

is injective. Besides obviously there exists a homogeneous bundle E on \mathbb{P}^2 such that

$$0 \rightarrow A \xrightarrow{\alpha} B \rightarrow E \rightarrow 0$$

is the minimal free resolution of E if and only if

- $M(\alpha_{q,r}): \bigoplus_p A^{p,q,r} \rightarrow \bigoplus_\rho B^{\rho,q,r}$ is injective for all q, r

and

- $M(\alpha_{p,q,r}): A^{p,q,r} \rightarrow B^{p,q,r}$ is zero for all p, q, r .

To prove Theorem 9 we need some lemmas.

LEMMA 10. — Let V be a complex vector space of dimension n . Let $\lambda_1, \dots, \lambda_{n-1}, s \in \mathbb{N}$ with $\lambda_1 \geq \dots \geq \lambda_{n-1}$ and

$$\pi: S^{\lambda_1, \dots, \lambda_{n-1}}V \otimes S^sV \longrightarrow S^{\lambda_1+s, \lambda_2, \dots, \lambda_{n-1}}V$$

be an $SL(V)$ -invariant nonzero map. Then (up to multiples) π can be

described in the following way: let T be a tableau on $(\lambda_1, \dots, \lambda_{n-1})$ and let R be a tableau on (s) ; then

$$\pi(\mathcal{S}(T) \otimes \mathcal{S}(R)) = \mathcal{S}(TR),$$

where TR is the tableau obtained from T adding R at the end of its first row and \mathcal{S} is defined in Notation 5. In particular on $\mathbb{P}^{n-1} = \mathbb{P}(V)$ any $\text{SL}(V)$ -invariant nonzero map

$$S^{\lambda_1, \dots, \lambda_{n-1}} V(-s) \longrightarrow S^{\lambda_1+s, \lambda_2, \dots, \lambda_{n-1}} V$$

is injective.

Proof. — Let $\varphi : S^{\lambda_1, \dots, \lambda_{n-1}} V \otimes S^s V \rightarrow S^{\lambda_1+s, \lambda_2, \dots, \lambda_{n-1}} V$ be the linear map such that $\varphi(\mathcal{S}(T) \otimes \mathcal{S}(R)) = \mathcal{S}(TR)$ for all T tableau on $(\lambda_1, \dots, \lambda_{n-1})$ and for all R tableau on (s) ; it is sufficient to prove that φ is well defined, $\text{SL}(V)$ -invariant and nonzero.

To show that it is well defined it is sufficient to show that, if $T, T' \in \mathcal{T}_{(\lambda_1, \dots, \lambda_{n-1})}$ are such that $\mathcal{S}(T) = \mathcal{S}(T')$ and $R, R' \in \mathcal{T}_{(s)}$ such that $\mathcal{S}(R) = \mathcal{S}(R')$, then $\mathcal{S}(TR) = \mathcal{S}(T'R')$ (with the obvious definition of TR). Observe that

$$\begin{aligned} \mathcal{S}(TR) = \mathcal{S}(T'R') &\iff \text{ord}(\text{ant}(TR)) = \text{ord}(\text{ant}(T'R')) \\ &\iff \text{ord}(\text{ant}(T)R) = \text{ord}(\text{ant}(T')R') \end{aligned}$$

and the last equality follows from the fact that $\text{ord}(\text{ant}(T)) = \text{ord}(\text{ant}(T'))$ because $\text{sim}(\text{ant}(T)) = \text{sim}(\text{ant}(T'))$ and $\text{ord}(R) = \text{ord}(R')$ because $\text{sim}(R) = \text{sim}(R')$.

Besides obviously the map $\mathcal{S}(T) \otimes \mathcal{S}(R) \mapsto \mathcal{S}(TR)$ is $\text{SL}(V)$ -invariant and nonzero. Thus, up to multiples, it is the map π .

This implies the injectivity of any $\text{SL}(V)$ -invariant nonzero map $S^{\lambda_1, \dots, \lambda_{n-1}} V(-s) \rightarrow S^{\lambda_1+s, \lambda_2, \dots, \lambda_{n-1}} V$; in fact the induced map on the fibre on $[0 : \dots : 0 : 1]$ is

$$\mathcal{S}(T) \longmapsto \mathcal{S}(T[n \dots n])$$

for all $T \in \mathcal{T}_{(\lambda_1, \dots, \lambda_{n-1})}$ and if $\mathcal{S}(T[n \dots n]) = 0$, then

$$\text{ord} \circ \text{sim} \circ \text{ant}(T[n \dots n]) = 0.$$

Thus $\text{ord} \circ \text{ant}(T[n \dots n]) = 0$, but

$$\text{ord} \circ \text{ant}(T[n \dots n]) = (\text{ord} \circ \text{ant}(T))[n \dots n],$$

hence $\text{ord} \circ \text{ant}(T) = 0$, i.e. $\mathcal{S}(T) = 0$. □

The injectivity statement of Lemma 10 (probably well known) will be obvious by the theory of quivers, precisely it will follow from Lemma 24, but we wanted to show the above proof because it is more elementary and intuitive.

LEMMA 11. — Let $\mathbb{P}^{n-1} = \mathbb{P}(V)$. For every $\lambda_1, \dots, \lambda_n \in \mathbb{N}$, $t \in \mathbb{Z}$ with $\lambda_1 \geq \dots \geq \lambda_n$, there exist $y = y_{\lambda_1, \dots, \lambda_n}^t \in S^{\lambda_1, \dots, \lambda_n} V(t)$, $y \neq 0$, such that $\varphi(y) = 0$ for every $SL(V)$ -invariant map

$$\varphi : S^{\lambda_1, \dots, \lambda_n} V(t) \longrightarrow S^{\lambda_1+s_1, \dots, \lambda_n+s_n} V(t + s_1 + \dots + s_n)$$

for all $s_1, \dots, s_n \in \mathbb{N}$ such that $s_2 + \dots + s_n > 0$.

Proof. — It is sufficient to prove the statement when $t = 0$. It is sufficient to take as y a nonzero element of the image of an $SL(V)$ -invariant nonzero map

$$\psi : S^{\lambda_2, \dots, \lambda_n} V(-\lambda_1) \longrightarrow S^{\lambda_1, \dots, \lambda_n} V$$

(such a map exists because, by Pieri’s formula, $S^{\lambda_1, \dots, \lambda_n} V$ is a summand of $S^{\lambda_2, \dots, \lambda_n} V \otimes S^{\lambda_1} V$, thus we can take the map induced by the projection; besides an $SL(V)$ -invariant map

$$S^{\lambda_2, \dots, \lambda_n} V(-\lambda_1 - s_1 \dots - s_n) \longrightarrow S^{\lambda_1+s_1, \dots, \lambda_n+s_n} V$$

is zero for all s_1, \dots, s_n with $s_2 + \dots + s_n > 0$, because by Pieri’s formula the induced map (i.e. $H^0(\bullet^\vee)^\vee$) $S^{\lambda_2, \dots, \lambda_n} V \otimes S^{\lambda_1+s_1+\dots+s_n} V \rightarrow S^{\lambda_1+s_1, \dots, \lambda_n+s_n} V$ is zero, thus $\varphi \circ \psi = 0$ for all φ as in the statement and then $\varphi(y) = 0$). □

Remark 12. — Let W be a finite dimensional \mathbb{C} -vector space and $v_1, \dots, v_k \in W$ not all zero. Let B be a matrix with k columns such that $\sum_{j=1, \dots, k} B_{i,j} v_j = 0$ for all i (i.e. the coefficients of every row of B are the coefficients of a linear relation among the v_j). Then $B : \mathbb{C}^k \rightarrow \mathbb{C}^s$ (where s is the number of the rows of B) is not injective.

LEMMA 13. — Fix $q, r \in \mathbb{N}$ with $q \geq r$ and let \mathcal{P}, \mathcal{R} be finite subsets of $\{p \in \mathbb{N} \mid p \geq q\}$. A map

$$\alpha : \bigoplus_{p \in \mathcal{P}} A^p \otimes S^{p, q, r} V(p) \longrightarrow \bigoplus_{\rho \in \mathcal{R}} B^\rho \otimes S^{\rho, q, r} V(\rho)$$

(A^p, B^ρ nonzero finite-dimensional vector spaces) is injective if and only if $M(\alpha)$ is injective.

Proof. — Let $\underline{p} = \min \mathcal{P}$ and $\bar{p} = \max \mathcal{P}$. Let

$$\psi = \bigoplus_{p \in \mathcal{P}} \text{Id}_{A^p} \otimes \varphi_{p, \underline{p}} : \left(\bigoplus_{\substack{p \in \mathcal{P} \\ p \geq \underline{p}}} A^p \right) \otimes S^{\underline{p}, q, r} V(\underline{p}) \longrightarrow \bigoplus_{p \in \mathcal{P}} A^p \otimes S^{\underline{p}, q, r} V(\underline{p}).$$

The map ψ is injective by Lemma 10. We have

$$(3) \quad \alpha \circ \psi = \left(\bigoplus_{\substack{\rho \in \mathcal{R} \\ \rho \geq \underline{p}}} \text{Id}_{B^\rho} \otimes \varphi_{\rho, \underline{p}} \right) \circ \left(M(\alpha) \otimes \text{Id}_{S^{\underline{p}, q, r} V(\underline{p})} \right).$$

Suppose α is injective. Since ψ is injective, $\alpha \circ \psi$ is injective. Thus, by (3), $M(\alpha)$ is injective.

Suppose now $M(\alpha)$ is injective. Let $x \in \mathbb{P}^2$. Let α^x be the map induced on the fibres on x by α and for any bundle E , E^x will denote the fibre on x .

Let $\mathcal{P} = \{p_1, \dots, p_n\}$ and $v = (v_1^1, \dots, v_{a_1}^1, \dots, v_1^n, \dots, v_{a_n}^n) \in \text{Ker}(\alpha^x)$ where $a_i = \dim A^{p_i}$ and $v_j^i \in S^{p_i, q, r} V(p_i)^x$. We want to show $v = 0$.

We can see every v_j^i in $S^{\bar{p}, q, r}(\bar{p})^x$ (by $\varphi_{\bar{p}, p_i}$, which is an injection); (after fixing bases and seeing $M(\alpha)$ as a matrix) the coefficients of the rows of $M(\alpha)$ are the coefficients of linear relations among the v_j^i seen in $S^{\bar{p}, q, r}(\bar{p})^x$, since $\alpha^x(v) = 0$. Thus, since $M(\alpha)$ is injective, by Remark 12, the v_j^i must be all zero, i.e. $v = 0$. □

Proof of Theorem 9. — (i) Let $\alpha: A \rightarrow B$ be an $\text{SL}(V)$ -invariant injective map. For any c, q, \tilde{p} , let

$$\alpha_{c, q, \geq \tilde{p}} : \bigoplus_{p \geq \tilde{p}} A_{c+p}^{p, q} \otimes S^{p, q} V(c+p) \longrightarrow \bigoplus_{\rho \geq \tilde{p}} B_{c+\rho}^{\rho, q} \otimes S^{\rho, q} V(c+\rho)$$

the map induced by α , and let

$$\psi_{\tilde{p}} = \bigoplus_{p \geq \tilde{p}} \text{Id}_{A_{c+p}^{p, q}} \otimes \varphi_{p, \tilde{p}} : \left(\bigoplus_{p \geq \tilde{p}} A_{c+p}^{p, q} \right) \otimes S^{\tilde{p}, q} V(c+\tilde{p}) \longrightarrow \bigoplus_{p \geq \tilde{p}} A_{c+p}^{p, q} \otimes S^{p, q} V(c+p).$$

The map $\psi_{\tilde{p}}$ is injective by Lemma 10. We can write

$$(4) \quad \alpha_{c, q, \geq \tilde{p}} \circ \psi_{\tilde{p}} = \left(\bigoplus_{\rho \geq \tilde{p}} \text{Id}_{B_{c+\rho}^{\rho, q}} \otimes \varphi_{\rho, \tilde{p}} \right) \circ \left(M(\alpha_{c, q, \geq \tilde{p}}) \otimes \text{Id}_{S^{\tilde{p}, q} V(c+\tilde{p})} \right).$$

Let $v \in \text{Ker} M(\alpha_{c, q, \geq \tilde{p}})$. If $v \neq 0$ then $\psi_{\tilde{p}}(v \otimes y_{\tilde{p}, q}^{c+\tilde{p}})$ (see Lemma 11 for the definition of $y_{\tilde{p}, q}^{c+\tilde{p}}$) is nonzero (since $\psi_{\tilde{p}}$ is injective) and it is in $\text{Ker}(\alpha)$ (in fact

it is in $\text{Ker}(\alpha_{c,q,\geq \tilde{p}})$ by (4) and thus in $\text{Ker}(\alpha)$ by the definition of $y_{\tilde{p},q}^{c+\tilde{p}}$. Since α is injective we get a contradiction, thus $v = 0$. Thus $M(\alpha_{c,q,\geq \tilde{p}})$ is injective. Then (1) holds.

Suppose now (1) holds. Since

$$A = \bigoplus_{c,q} \left(\bigoplus_p A_{c+p}^{p,q} \otimes S^{p,q}V(c+p) \right), \quad B = \bigoplus_{c,q} \left(\bigoplus_\rho B_{c+\rho}^{\rho,q} \otimes S^{\rho,q}V(c+\rho) \right),$$

to find an injective map $\alpha : A \rightarrow B$ it is sufficient to find for all c, q an injective map

$$\alpha_{c,q} : \bigoplus_p A_{c+p}^{p,q} \otimes S^{p,q}V(c+p) \longrightarrow \bigoplus_\rho B_{c+\rho}^{\rho,q} \otimes S^{\rho,q}V(c+\rho).$$

Order p and ρ in decreasing way, fix a basis of $A_{c+p}^{p,q}$ for all p and let $\alpha_{c,q}$ be the map such that $M(\alpha_{c,q}) = \begin{pmatrix} I \\ 0 \end{pmatrix}$. Observe that, since p and ρ are ordered in decreasing way and $\dim(\bigoplus_{p \geq \tilde{p}} A_{c+p}^{p,q}) \leq \dim(\bigoplus_{\rho \geq \tilde{p}} B_{c+\rho}^{\rho,q})$, then the entries equal to 1 are “where $\rho \geq p$ ”.

(i') First observe that an $\text{SL}(V)$ -invariant injective map $\alpha : A \rightarrow B$ is such that $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow \text{Coker } \alpha \rightarrow 0$ is a minimal free resolution if and only if the maps induced by α

$$\alpha_{p,q,i} : A_i^{p,q} \otimes S^{p,q}V(i) \longrightarrow B_i^{p,q} \otimes S^{p,q}V(i)$$

are zero.

Let $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow \text{Coker } \alpha \rightarrow 0$ be a minimal free resolution with α $\text{SL}(V)$ -invariant injective map. For any c, q, \tilde{p} , let

$$\alpha_{c,q,>\tilde{p},\geq\tilde{p}} : \bigoplus_{p \geq \tilde{p}} A_{c+p}^{p,q} \otimes S^{p,q}V(c+p) \longrightarrow \bigoplus_{\rho > \tilde{p}} B_{c+\rho}^{\rho,q} \otimes S^{\rho,q}V(c+\rho)$$

the map induced by α .

We can prove that $M(\alpha_{c,q,>\tilde{p},\geq\tilde{p}})$ is injective as in the implication ‘ \Rightarrow ’ of (i), i.e. by considering $\alpha_{c,q,>\tilde{p},\geq\tilde{p}} \circ \psi_{\tilde{p}}$ with $\psi_{\tilde{p}}$ as above (to see that $\psi_{\tilde{p}}(v \otimes y_{\tilde{p},q}^{c+\tilde{p}})$ is in $\text{Ker}(\alpha)$, use also that the maps $\alpha_{p,q,i} : A_i^{p,q} \otimes S^{p,q}V(i) \rightarrow B_i^{p,q} \otimes S^{p,q}V(i)$ are zero). Then (2) holds.

The other implication is completely analogous to the implication “if” of (i).

(ii) Suppose α is injective. Fix q and r . Let $\underline{p} = \min\{p \mid A^{p,q,r} \neq 0\}$ and let

$$\psi_{\underline{p}} = \bigoplus_{p \geq \underline{p}} \text{Id}_{A^{p,q,r}} \otimes \varphi_{p,\underline{p}} : \left(\bigoplus_{p \geq \underline{p}} A^{p,q,r} \right) \otimes S^{\underline{p},q,r} V(\underline{p} + q + r) \longrightarrow \bigoplus_{p \geq \underline{p}} A^{p,q,r} \otimes S^{p,q,r} V(p + q + r).$$

It holds

$$\alpha_{q,r} \circ \psi_{\underline{p}} = \left(\bigoplus_{\rho} \text{Id}_{B^{\rho,q,r}} \otimes \varphi_{\rho,\underline{p}} \right) \circ (M(\alpha_{q,r}) \otimes \text{Id}_{S^{\underline{p},q,r} V(\underline{p}+q+r)}).$$

Let $v \in \text{Ker } M(\alpha_{q,r})$. If $v \neq 0$ then $\psi_{\underline{p}}(v \otimes y_{\underline{p},q,r}^{p+q+r})$ (see Lemma 11 for the definition of $y_{\underline{p},q,r}^{p+q+r}$) is nonzero (since $\psi_{\underline{p}}$ is injective) and it belongs to $\text{Ker}(\alpha)$ (in fact it is in $\text{Ker}(\alpha_{q,r})$ by the above formula and thus in $\text{Ker}(\alpha)$ by the definition of $y_{\underline{p},q,r}^{p+q+r}$). Hence we get a contradiction since α is injective, therefore $v = 0$. Thus $M(\alpha_{q,r})$ is injective; hence $\alpha_{q,r}$ is injective by Lemma 13.

Now suppose $\alpha_{q,r}$ is injective for all q, r . Observe that α is “triangular” with respect to q and r , thus, if for all q the induced map

$$\alpha_q : \bigoplus_{p,r} A^{p,q,r} \otimes S^{p,q,r} V(p + q + r) \longrightarrow \bigoplus_{\rho,r} B^{\rho,q,r} \otimes S^{\rho,q,r} V(\rho + q + r)$$

is injective then α is injective; besides α_q is injective for all q if for all q, r the induced map $\alpha_{q,r}$ is injective. □

Theorem 9 is easily generalizable to \mathbb{P}^n . Obviously the statement on minimal resolutions is generalizable to minimal free resolutions with two terms of bundles on \mathbb{P}^n , but for a generic homogeneous bundle on \mathbb{P}^n with $n \geq 3$ the minimal free resolution has more than two terms.

Finally we observe that Theorem 2 and the following lemma (which will be useful also later to study simplicity) allow us to parametrize the set of homogeneous bundles on \mathbb{P}^2 by a set of sequences of injective matrices with a certain shape up to the action of invertible matrices with a certain shape.

LEMMA 14. — (i) *Let E and E' be two homogeneous vector bundles on \mathbb{P}^2 and*

$$(5) \quad \begin{aligned} \text{a) } & 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0 \\ \text{b) } & 0 \rightarrow R' \xrightarrow{f'} S' \xrightarrow{g'} E' \rightarrow 0 \end{aligned}$$

be two minimal free resolutions. Any map $\eta: E \rightarrow E'$ induces maps A and B such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \rightarrow & R & \xrightarrow{f} & S & \xrightarrow{g} & E & \rightarrow & 0 \\ & & \downarrow A & & \downarrow B & & \downarrow \eta & & \\ 0 & \rightarrow & R' & \xrightarrow{f'} & S' & \xrightarrow{g'} & E' & \rightarrow & 0, \end{array}$$

and given A and B such that the above diagram commutes we have a map $E \rightarrow E'$. Given η , the maps A and B are unique if and only if $\text{Hom}(S, R') = 0$. (In particular if $\text{Hom}(S, R') = 0$ and we find A and B such that the diagram commutes and B is not a multiple of the identity, we can conclude that E is not simple.)

(ii) Let

$$(6) \quad \begin{array}{l} \text{a) } 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0, \\ \text{b) } 0 \rightarrow R \xrightarrow{f'} S \xrightarrow{g'} E \rightarrow 0, \end{array}$$

be two minimal free resolutions with $\text{SL}(V)$ -invariant maps of a homogeneous bundle E on $\mathbb{P}^2 = \mathbb{P}(V)$; then there exist $\text{SL}(V)$ -invariant automorphisms $A: R \rightarrow R$, $B: S \rightarrow S$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \rightarrow & R & \xrightarrow{f} & S & \xrightarrow{g} & E & \rightarrow & 0 \\ & & \downarrow A & & \downarrow B & & \downarrow \text{Id} & & \\ 0 & \rightarrow & R & \xrightarrow{f'} & S & \xrightarrow{g'} & E & \rightarrow & 0. \end{array}$$

Proof. — (i) The composition $S \xrightarrow{g} E \xrightarrow{\eta} E'$ can be lifted to a map $B: S \rightarrow S'$ by the exact sequence

$$0 \rightarrow \text{Hom}(S, R') \rightarrow \text{Hom}(S, S') \rightarrow \text{Hom}(S, E') \rightarrow 0$$

(obtained by applying $\text{Hom}(S, \bullet)$ to (5) b). By the exact sequence

$$\text{Hom}(R, R') \rightarrow \text{Hom}(R, S') \rightarrow \text{Hom}(R, E')$$

(obtained by applying $\text{Hom}(R, \bullet)$ to (5) b) the composition $R \xrightarrow{f} S \xrightarrow{B} S'$, which goes to zero in $\text{Hom}(R, E')$, can be lifted to a map $A: R \rightarrow R'$.

(ii) As in (i) we can prove there exist $SL(V)$ -invariant maps $B : S \rightarrow S$ and $B' : S \rightarrow S$ such that the following diagrams commute

$$\begin{array}{ccc}
 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0 & & 0 \rightarrow R \xrightarrow{f'} S \xrightarrow{g'} E \rightarrow 0 \\
 & \downarrow B & \downarrow \text{Id} \\
 0 \rightarrow R \xrightarrow{f'} S \xrightarrow{g'} E \rightarrow 0 & & 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0,
 \end{array}$$

then we get the commutative diagram

$$\begin{array}{ccc}
 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0 & & \\
 & \downarrow B' \circ B & \downarrow \text{Id} \\
 0 \rightarrow R \xrightarrow{f} S \xrightarrow{g} E \rightarrow 0 & &
 \end{array}$$

Since also $\text{Id} : S \rightarrow S$ lifts $\text{Id} : E \rightarrow E$ in the above diagram (i.e. $\text{Id} \circ g = g \circ \text{Id}$), there exists a map $a : S \rightarrow R$ such that $B' \circ B - \text{Id} = f \circ a$; hence, by the minimality of (6) a), $\det(B' \circ B) = 1 + P$ where P is a polynomial without terms of degree 0; since $\det(B' \circ B) : \det(S) \rightarrow \det(S)$ must be homogeneous, $P = 0$; hence $B' \circ B$ is invertible and thus B is invertible. □

4. Quivers.

We recall now the main definitions and results on quivers and representations of quivers associated to homogeneous bundles introduced by Bondal and Kapranov [BK]. The quivers will allow us to handle well and “to make explicit” the homogeneous subbundles of a homogeneous bundle.

DEFINITION 15 (see [Si], [Ki], [Hi1], [GR]). — A *quiver* is an oriented graph \mathcal{Q} with the set \mathcal{Q}_0 of vertices (or points) and the set \mathcal{Q}_1 of arrows.

A *path* in \mathcal{Q} is a formal composition of arrows $\beta_m \cdots \beta_1$ where the source of an arrow β_i is the sink of the previous arrow β_{i-1} .

A *relation* in \mathcal{Q} is a linear form $\lambda_1 c_1 + \cdots + \lambda_r c_r$ where c_i are paths in \mathcal{Q} with a common source and a common sink and $\lambda_i \in \mathbb{C}$.

A *representation of a quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$, or *\mathcal{Q} -representation*, is the couple of a set of vector spaces $\{X_i\}_{i \in \mathcal{Q}_0}$ and of a set of linear maps $\{\varphi_\beta\}_{\beta \in \mathcal{Q}_1}$ where $\varphi_\beta : X_i \rightarrow X_j$ if β is an arrow from i to j .

A representation of a quiver \mathcal{Q} with relations \mathcal{R} is a \mathcal{Q} -representation such that

$$\sum_j \lambda_j \varphi_{\beta_{m_j}^j} \cdots \varphi_{\beta_1^j} = 0$$

for every $\sum_j \lambda_j \beta_{m_j}^j \cdots \beta_1^j \in \mathcal{R}$.

Let $(X_i, \varphi_\beta)_{i \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ and $(Y_i, \psi_\beta)_{i \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ be two representations of a quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$. A morphism f from $(X_i, \varphi_\beta)_{i \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ to $(Y_i, \psi_\beta)_{i \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ is a set of linear maps $f_i: X_i \rightarrow Y_i, i \in \mathcal{Q}_0$ such that, for every $\beta \in \mathcal{Q}_1, \beta$ arrow from i to j , the following diagram is commutative:

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \varphi_\beta \downarrow & & \downarrow \psi_\beta \\ X_j & \xrightarrow{f_j} & Y_j. \end{array}$$

A morphism f is injective if the f_i are injective.

NOTATION 16. — We will say that a representation $(X_i, \varphi_\beta)_{i \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ of a quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ has multiplicity m in a point i of \mathcal{Q} if $\dim X_i = m$. The support (with multiplicities) of a representation of a quiver \mathcal{Q} is the subgraph of \mathcal{Q} constituted by the points of multiplicity ≥ 1 and the nonzero arrows (with the multiplicities associated to every point of the subgraph).

We recall now from [BK], [Hi1], [Hi2] the definition of a quiver \mathcal{Q} such that the category of the homogeneous bundles on \mathbb{P}^2 is equivalent to the category of finite dimensional representations of \mathcal{Q} with some relations \mathcal{R} . Bondal and Kapranov defined such a quiver in a more general setting but we recall such a construction only for \mathbb{P}^2 . See also [OR].

First some notation. Let P and R be the following subgroup of $SL(3)$:

$$P = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in SL(3) \right\}, \quad R = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in SL(3) \right\}.$$

Observe that R is reductive. We can see \mathbb{P}^2 as $SL(3)/P$, where P is the stabilizer of $[1 : 0 : 0]$. Let p and r be the Lie algebras associated respectively to P and R . Let n be the Lie algebra

$$n = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{C} \right\}.$$

Thus $p = r \oplus n$ (Levi decomposition).

We recall that the homogeneous bundles on $\mathbb{P}^2 = \mathrm{SL}(3)/P$ are given by the representations of P (this bijection is given by taking the fibre over $[1 : 0 : 0]$ of a homogeneous vector bundle); by composing the projection from P to R with a representation of R we get a representation of P and the set of the homogenous bundles obtained in this way from the irreducible representations of R are

$$\{S^\ell Q(t) \mid \ell \in \mathbb{N}, t \in \mathbb{Z}\},$$

where $Q = T_{\mathbb{P}^2}(-1)$.

DEFINITION 17. — From now on \mathcal{Q} will be the following quiver:

- let

$$\begin{aligned} \mathcal{Q}_0 &= \{\text{irreducible representations of } R\} = \{S^\ell Q(t) \mid \ell \in \mathbb{N}, t \in \mathbb{Z}\} \\ &= \{\text{dominant weights of } r\}; \end{aligned}$$

- let \mathcal{Q}_1 be defined in the following way: there is an arrow from λ to μ , $\lambda, \mu \in \mathcal{Q}_0$, if and only if $n \otimes \Sigma_\lambda \supset \Sigma_\mu$, where Σ_λ denotes the representation of r with dominant weight λ .

LEMMA 18. — *The adjoint representation of p on n corresponds to $Q(-2) = \Omega^1$, more precisely: let $\rho: p \rightarrow \mathfrak{gl}(n)$ be the following representation:*

$$\rho(B) h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = Bh\begin{pmatrix} x \\ y \end{pmatrix} - h\begin{pmatrix} x \\ y \end{pmatrix}B,$$

where $h: \mathbb{C}^2 \rightarrow n$ is the isomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$Q(-2)$ is the homogeneous bundle whose fibre as p -representation is n .

Proof. — Observe that if

$$B = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in p \quad \text{and} \quad A = \begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

then

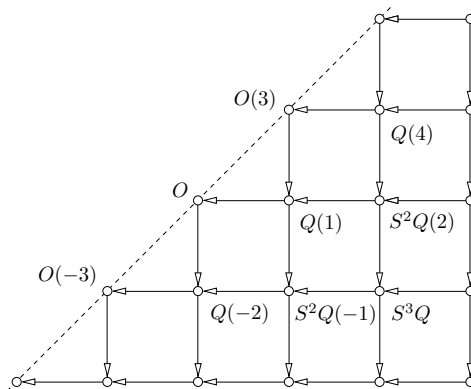
$$h^{-1}\left(Bh\begin{pmatrix} x \\ y \end{pmatrix} - h\begin{pmatrix} x \\ y \end{pmatrix}B\right) = (a\mathrm{Id} - {}^tA)\begin{pmatrix} x \\ y \end{pmatrix};$$

since the representation $p \rightarrow \mathfrak{gl}(1)$, $B \mapsto a\mathrm{Id}$ corresponds to the

bundle $\mathcal{O}(-1)$ and the representation $p \rightarrow \mathfrak{gl}(2)$, $B \mapsto -^tA$ corresponds to the bundle $Q^\vee = Q(-1)$, we conclude. \square

By the previous remark, if Σ_λ is the representation corresponding to $S^\ell Q(t)$ and Σ_μ is the representation corresponding to $S^{\ell'} Q(t')$, the condition $n \otimes \Sigma_\lambda \supset \Sigma_\mu$ is equivalent to the fact $S^{\ell'} Q(t')$ is a direct summand of $Q(-2) \otimes S^\ell Q(t)$ and this is true if and only if $(\ell', t') = (\ell - 1, t - 1)$ or $(\ell', t') = (\ell + 1, t - 2)$ (we recall that, by the Euler sequence, $\wedge^2 Q = \mathcal{O}(1)$).

Thus our quiver has three connected components $\mathcal{Q}^{(1)}$, $\mathcal{Q}^{(2)}$, $\mathcal{Q}^{(3)}$ (given by the congruence class modulo $\frac{3}{2}$ of the slope of the homogeneous vector bundles corresponding to the points of the connected component); the figure shows one of them (the one whose points correspond to the bundles with $\mu \equiv 0 \pmod{\frac{3}{2}}$): we identify the points of every connected component $\mathcal{Q}^{(j)}$ of \mathcal{Q} with a subset of \mathbb{Z}^2 for convenience.



DEFINITION 19. — Let \mathcal{R} be the set of relations on \mathcal{Q} given by the commutativity of the squares, i.e. (denoting by $\beta_{w,v}$ the arrow from v to w):

$$\beta_{(x-1,y-1),(x-1,y)}\beta_{(x-1,y),(x,y)} - \beta_{(x-1,y-1),(x,y-1)}\beta_{(x,y-1),(x,y)}$$

for all $(x,y) \in \mathcal{Q}^{(j)} \subset \mathbb{Z}^2$ for some j , such that $(x-1,y) \in \mathcal{Q}$ and

$$\beta_{(x-1,y-1),(x,y-1)}\beta_{(x,y-1),(x,y)}$$

for all $(x,y) \in \mathcal{Q}^{(j)} \subset \mathbb{Z}^2$ for some j , such that $(x-1,y) \notin \mathcal{Q}$.



DEFINITION 20. — Let E be a homogeneous vector bundle on \mathbb{P}^2 . The \mathcal{Q} -representation associated to E is the following (see [BK], [Hi1], [Hi2], [OR]). Consider the fiber $E_{[1:0:0]}$ of E on $[1:0:0]$; it is a representation of p ; as r -representation we have

$$E_{[1:0:0]} = \bigoplus_{\lambda \in \mathcal{Q}_0} X_\lambda \otimes \Sigma_\lambda$$

for some vector spaces X_λ ; we associate to $\lambda \in \mathcal{Q}_0$ the vector space X_λ ; we fix for all λ a dominant weight vector $v_\lambda \in \Sigma_\lambda$ and η_1, η_2 eigenvectors of the p -representation n ; let ψ_1, ψ_2 be their weights respectively; let i be such that $\lambda + \psi_i = \mu$; we associate to an arrow $\lambda \rightarrow \mu$ a map $f: X_\lambda \rightarrow X_\mu$ defined in the following way: consider the composition

$$\Sigma_\lambda \otimes n \otimes X_\lambda \longrightarrow \Sigma_\mu \otimes X_\mu$$

given by the action of n over $E_{[1:0:0]}$ followed by projection; it maps $v_\lambda \otimes \eta_i \otimes v$ to $v_\mu \otimes w$; we define $f(v) = w$ (it does not depend on the choice of the dominant weight vector).

THEOREM 21 (Bondal-Kapranov, Hille; see [BK], [Hi1], [Hi2], [OR]). — *The category of the homogeneous bundles on \mathbb{P}^2 is equivalent to the category of finite dimensional representation of the quiver \mathcal{Q} with the relations \mathcal{R} .*

Observe that in Definition 20 with respect to Bondal-Kapranov-Hille’s convention in [BK], [Hi1], [Hi2], we preferred to invert the arrows in order that an injective $SL(V)$ -equivariant map of bundles corresponds to an injective morphism of \mathcal{Q} -representations. For example \mathcal{O} injects in $V(1)$ whose support is the arrow from $Q(1)$ to \mathcal{O} :

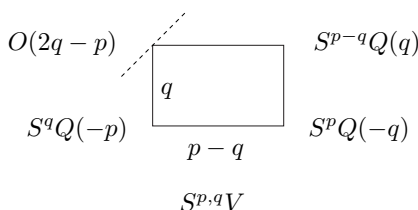
$$\mathcal{O} \xleftarrow{\quad} \circ \quad Q(1)$$

NOTATION 22. — • We will often speak of the \mathcal{Q} -support of a homogeneous bundle E instead of the support with multiplicities of the \mathcal{Q} -representation of E and we will denote it by $\mathcal{Q}\text{-supp}(E)$.

- The word *rectangle* will denote the subgraph with multiplicities of \mathcal{Q} given by the subgraph of \mathcal{Q} included in a rectangle whose sides are unions of arrows of \mathcal{Q} , with the multiplicities of all its points equal to 1.
- The word *segment* will denote a rectangle with base or height equal to 0.

- If A and B are two subgraphs of \mathcal{Q} , $A \cap B$ is the subgraph of \mathcal{Q} whose vertices and arrows are the vertices and arrows both of A and of B ; $A - B$ is the subgraph of \mathcal{Q} whose vertices are the vertices of A not in B and the arrows are the arrows of A joining two vertices of $A - B$.

Remark 23 (see [BK]). — The \mathcal{Q} -support of $S^{p,q}V$ is a rectangle as in the figure



In fact, by the Euler sequence $S^{p,q}V = S^{p,q}(\mathcal{O}(-1) \oplus Q)$ as R -representation; by the formula of a Schur functor applied to a direct sum (see [FH, Exercise 6.11]) we get

$$S^{p,q}V = \bigoplus S^\lambda Q \otimes S^m \mathcal{O}(-1)$$

as R -representations, where the sum is performed on $m \in \mathbb{N}$ and on λ Young diagram obtained from the Young diagram of (p, q) by taking off m boxes not two in the same column; thus

$$S^{p,q}V = \bigoplus_{\substack{0 \leq m_1 \leq p-q \\ 0 \leq m_2 \leq q}} S^{p-m_1, q-m_2} Q(-m_1 - m_2)$$

Finally to show the maps associated to the arrows in the rectangle are nonzero we can consider on the set of the vertices of the rectangle the following equivalence relation: $P \sim Q$ if and only if there exist two paths with P and Q respectively as sources and common sink such that the map associated to any arrow of the two paths is nonzero; if a map associated to an arrow (say from P_1 to P_2) of the rectangle is zero then, by the “commutativity of the squares”, precisely by the relations in Definition 19, there would be at least two equivalence classes (the class of P_1 and the class of P_2), but this is impossible by the irreducibility of $S^{p,q}V$.

5. Some lemmas and notation.

In this section we study equivariant maps between some homogeneous bundles on \mathbb{P}^2 by using the language of the quivers and we collect some

technical notation and lemmas, which will be useful in the next section to study homogeneous subbundles (in particular their slope) of homogeneous bundles and then to study stability.

5.1. \mathcal{Q} -representation of kernels and images.

LEMMA 24. — *Let $\varphi: S^{p,q}V \rightarrow S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)$ be an $SL(V)$ -invariant nonzero map; then*

- (i) $\mathcal{Q}\text{-supp}(\text{Ker}(\varphi)) = \mathcal{Q}\text{-supp}(S^{p,q}V) - \mathcal{Q}\text{-supp}(S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)),$
- (ii) $\mathcal{Q}\text{-supp}(\text{Im}(\varphi)) = \mathcal{Q}\text{-supp}(S^{p,q}V) \cap \mathcal{Q}\text{-supp}(S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)),$
- (iii) $\mathcal{Q}\text{-supp}(\text{Coker}(\varphi)) = \mathcal{Q}\text{-supp}(S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)) - \mathcal{Q}\text{-supp}(S^{p,q}V).$

Proof. — Consider the P -invariant (and thus R -invariant) map $\varphi_{[1:0:0]}$ induced by φ on the fibers on $[1:0:0]$; it is the morphism from the \mathcal{Q} -representation of $S^{p,q}V$ to the \mathcal{Q} -representation of $S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)$.

Obviously the R -representations corresponding to the vertices in the \mathcal{Q} -support of $S^{p,q}V$ and not in the \mathcal{Q} -support of $S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)$ are in $\text{Ker } \varphi_{[1:0:0]}$.

We have to show that the R -representations corresponding to the vertices both in the \mathcal{Q} -support of $S^{p,q}V$ and in the \mathcal{Q} -support of $S^{p+s_1,q+s_2,s_3}V(s_1 + s_2 + s_3)$ are not in $\text{Ker } \varphi_{[1:0:0]}$.

We call I the set of such vertices.

If the R -representation corresponding to an element of I is in $\text{Ker } \varphi_{[1:0:0]}$, then also the R -representation corresponding to another element of I is in $\text{Ker } \varphi_{[1:0:0]}$: in fact by the commutativity of the diagram in the definition of morphism of representations of a quiver, if the R -representation corresponding to $\lambda \in I$ is in $\text{Ker } \varphi_{[1:0:0]}$, then also the R -representation corresponding to any element of I linked to λ by an arrow is in $\text{Ker } \varphi_{[1:0:0]}$ and we conclude since for all $\lambda_1, \lambda_2 \in I$ there is a path of the quiver joining λ_1 and λ_2 . Thus either any R -representation corresponding to an element of I is in $\text{Ker } \varphi_{[1:0:0]}$ or any R -representation corresponding to an element of I is not in $\text{Ker } \varphi_{[1:0:0]}$. But the last case is impossible because φ is nonzero. □

COROLLARY 25. — *Let*

$$\varphi: S^{p,q}V \longrightarrow \bigoplus_{s_1, s_2, s_3} S^{p+s_1, q+s_2, s_3}V(s_1 + s_2 + s_3)$$

(where the sum is on a finite subset of \mathbb{N}^3) be an $SL(V)$ -invariant nonzero map. Then

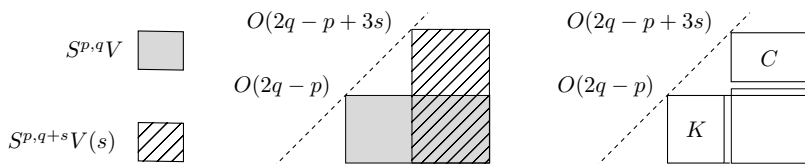
$$\begin{aligned} \mathcal{Q}\text{-supp}(\text{Ker}(\varphi)) \\ = \mathcal{Q}\text{-supp}(S^{p,q}V) - \mathcal{Q}\text{-supp}\left(\bigoplus_{s_1, s_2, s_3} S^{p+s_1, q+s_2, s_3}V(s_1 + s_2 + s_3)\right). \end{aligned}$$

LEMMA 26 (Four Terms Lemma). — *On $\mathbb{P}^2 = \mathbb{P}(V)$ we have the following exact sequence:*

$$\begin{aligned} 0 \rightarrow S^{q+s-1, q}V(-p + q - 1 + s) \rightarrow S^{p,q}V \rightarrow S^{p, q+s}V(s) \\ \rightarrow S^{p-q-1, s-1}V(q + 1 + s) \rightarrow 0, \end{aligned}$$

where the maps are $SL(V)$ -invariant nonzero maps (they are unique up to multiples). (Observe that $S^{p-q-1, s-1}V(q + 1 + s) \simeq S^{p, q+s, q+1}V(q + 1 + s)$.)

Proof. — In the figure below we show the sides of the \mathcal{Q} -supports of $S^{p,q}V$ and $S^{p, q+s}V(s)$:



These supports are rectangles (see Remark 23); thus by Lemma 24 the \mathcal{Q} -support of the kernel of $S^{p,q}V \rightarrow S^{p, q+s}V(s)$ is the rectangle K , which is the \mathcal{Q} -support of $S^{q+s-1, q}V(-p + q - 1 + s)$ and the \mathcal{Q} -support of the cokernel of $S^{p,q}V \rightarrow S^{p, q+s}V(s)$ is the rectangle C , which is the support of $S^{p-q-1, s-1}V(q + 1 + s)$. \square

5.2. Some calculations on the slope.

Remark 27. — (i) The first Chern class of a homogeneous bundle E can be calculated as the sum of the first Chern classes of the irreducible bundles corresponding to the vertices of the \mathcal{Q} -support of E multiplied by the multiplicities. The rank of E is the sum of the ranks of the irreducible bundles corresponding to such vertices multiplied by the multiplicities.

We will often speak of the slope (resp. first Chern class, rank) of a graph with multiplicities instead of the slope (resp. first Chern class, rank) of the vector bundle whose \mathcal{Q} -support is that graph with multiplicities.

(ii) Suppose the set of the vertices of the \mathcal{Q} -support of E is the disjoint union of the vertices of the supports of two \mathcal{Q} -representations A and B ; if $\mu(A) = \mu(B)$ then $\mu(E) = \mu(A) = \mu(B)$, if $\mu(A) < \mu(B)$ then $\mu(A) < \mu(E) < \mu(B)$.

(iii) We recall that the rank of $S^\ell Q(t)$ is $\ell + 1$ and its first Chern class is $(\ell + 1)(\frac{1}{2}\ell + t)$.

LEMMA 28. — *Let R be a rectangle of base h , height k and $S^\ell Q(t)$ as the highest vertex of the left side. Then*

$$\mu(R) = \frac{(h + 1)(k + 1) \left\{ \begin{array}{l} \frac{1}{2}(h^2 - k^2) + h(\ell + \frac{1}{2}t + 1) \\ + k(\frac{1}{2}t - \frac{1}{2}\ell - 1) + (\ell + 1)(\frac{1}{2}\ell + t) \end{array} \right\}}{(h + 1)(k + 1)(\ell + \frac{1}{2}(h + k) + 1)},$$

(where the numerator is the first Chern class and the denominator is the rank).

Proof. — Left to the reader. □

LEMMA 29. — *Let S be a horizontal (resp. vertical) segment in $\mathcal{Q}^{(j)}$ for some j and let S' be obtained translating S by $(0,1)$ (resp $(1,0)$) in $\mathcal{Q}^{(j)} \subset \mathbb{Z}^2$. Then $\mu(S) < \mu(S')$.*



Proof. — Suppose S is horizontal, of length h and its first vertex from left is $S^\ell Q(t)$. By Lemma 28,

$$\begin{aligned} \mu(S') &= \frac{h^2 + 2h(\ell + 1) + \ell(\ell + 1)}{2\ell + 2 + h} + t, \\ \mu(S) &= \frac{h^2 + 2h(\ell + 2) + (\ell + 2)(\ell + 1)}{2(\ell + 1) + 2 + h} + t - 2 \end{aligned}$$

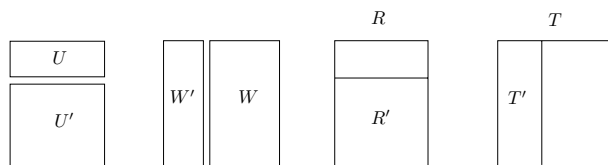
and $\mu(S') > \mu(S)$ is easy to check. The case of vertical segments is similiar. □

LEMMA 30. — (i) Let U and U' be two rectangles with the same base with the vertical sides lined up and U above U' . Then $\mu(U') < \mu(U)$.

(ii) Let W and W' be two rectangles with the same height and the horizontal sides lined up and W at the right side of W' . Then $\mu(W') < \mu(W)$.

(iii) Let R be a rectangle and R' a subrectangle of R with the same base and with the lower side equal to the lower side of R . Then $\mu(R') < \mu(R)$.

(iv) Let T be a rectangle and T' a subrectangle of T with the same height and with the left side equal to the left side of T . Then $\mu(T') < \mu(T)$.



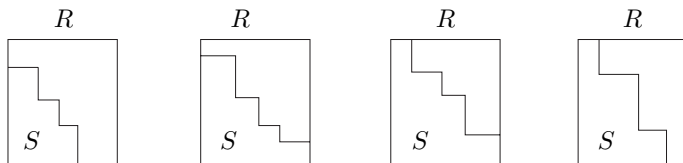
Proof. — (i) By Lemma 29 and Remark 27 the slope of a rectangle is between the slope of the lower side and the slope of the higher side, thus again by Lemma 29 we conclude. Assertion (ii) is analogous to (i); (iii) and (iv) follow from (i) and (ii). □

5.3 Staircases.

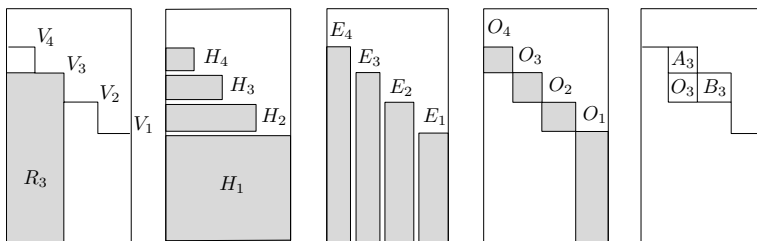
In this subsection we introduce particular \mathcal{Q} -representations, called “staircases”. Their importance is due to the fact that they are the \mathcal{Q} -supports of the homogeneous subbundles of the homogeneous bundles whose \mathcal{Q} -supports are rectangles (in particular of the trivial homogeneous bundles).

Remark 31. — Let E be a homogeneous bundle on \mathbb{P}^2 and F be a homogeneous subbundle. Let S and S' be the \mathcal{Q} -supports of E and F respectively. By Theorem 21 the \mathcal{Q} -representation of F injects into the \mathcal{Q} -representation of E . If the multiplicities of S are all 1 and S' contains the source of an arrow β in S then S' contains β .

DEFINITION 32. — We say that a subgraph with multiplicities of \mathcal{Q} is a *staircase* S in a rectangle R if all its multiplicities are 1 and the graph of S is a subgraph of R satisfying the following property: if V is a vertex of S then the arrows of R having V as source must be arrows of S (and then also their sinks must be vertices of S). We say that a subgraph with multiplicities of \mathcal{Q} is a staircase if it is a staircase in some rectangle.



Observe that a staircase S in a rectangle R has as matter of fact the form of a staircase with base and left side included respectively in the base and the left side of R as in the figure above. By Remark 31 the \mathcal{Q} -support of a homogeneous subbundle of a homogeneous bundle whose \mathcal{Q} -support is a rectangle is a staircase in the rectangle.



NOTATION 33 (see the figure above). — Given a staircase S in a rectangle we define \mathcal{V}_S to be the set of the vertices of S that are not sinks of any arrow of S . We call the elements of \mathcal{V}_S the *vertices of the steps*. Let $\mathcal{V}_S = \{V_1, \dots, V_k\}$ ordered in such a way the projection of V_{i+1} on the base of R is on the left of the projection of V_i for all $i = 1, \dots, k - 1$.

We define R_i as the rectangles with the right higher vertex equal to V_i and left lower vertex equal to the left lower vertex of R .

For any $i = 1, \dots, k$, we define:

- the i -th horizontal step $H_i = R_i - R_{i-1}$ ($R_0 = \emptyset$),
- the i -th vertical step $E_i = R_i - R_{i+1}$ ($R_{k+1} = \emptyset$),
- the i -th sticking out part $O_i = H_i \cap E_i$.

Let s_i and r_i be the lines containing respectively the higher side of O_i and the right side of O_i .

For $i = 1, \dots, k - 1$, let S_i be the rectangle whose sides are on r_i , s_{i+1} , the line of the base of the staircase and the line on the left side of the staircase.

Let A_i , for $i = 1, \dots, k - 1$, and B_i , for $i = 2, \dots, k$, be the rectangles

$$A_i = S_i - R_i - R_{i+1}, \quad B_i = S_{i-1} - R_i - R_{i-1}.$$

6. Results on stability and simplicity of elementary homogeneous bundles.

DEFINITION 34. — We say that a G -homogeneous bundle is multistable if it is the tensor product of a stable G -homogeneous bundle and an irreducible G -representation.

THEOREM 35 (Rohmfeld, Faini). — (i) (see [Ro]) A homogeneous bundle E is semistable if and only if $\mu(F) \leq \mu(E)$ for any subbundle F of E induced by a subrepresentation of the P -representation inducing E .

(ii) (see [Fa]) A homogeneous bundle E is multistable if and only if $\mu(F) < \mu(E)$ for any subbundle F of E induced by a subrepresentation of the P -representation inducing E .

THEOREM 36. — Let $p, q \in \mathbb{N}$ with $p \geq q$ and $s > 0$. Let E be the homogeneous vector bundle on $\mathbb{P}^2 = \mathbb{P}(V)$ defined by the following exact sequence:

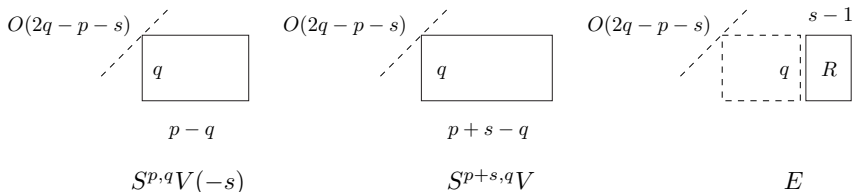
$$(7) \quad 0 \rightarrow S^{p,q}V(-s) \xrightarrow{\varphi} S^{p+s,q}V \rightarrow E \rightarrow 0,$$

where φ is a nonzero $\mathrm{SL}(V)$ -invariant map. Then E is stable (in particular it is simple).

Proof. — To show that E is stable it is sufficient to show that it is multistable; in fact if E is the tensor product of a stable homogeneous vector bundle E' with an $\mathrm{SL}(V)$ -representation W , then the minimal resolution of E must be the tensor product of the minimal resolution of E' with W and from (7) we must have $W = \mathbb{C}$.

To show that E is multistable we consider the \mathcal{Q} -representation associated to E . In the figure we show the sides of the \mathcal{Q} -supports of $S^{p,q}V(-s)$ and $S^{p+s,q}V$; these supports are rectangles (see Remark 23); thus the \mathcal{Q} -support of E is the rectangle R (see figure next page).

By Theorem 35, E is multistable if $\mu(F) < \mu(E)$ for any subbundle F of E induced by a subrepresentation of the P -representation inducing E . Observe that, by Remark 31, the support of the \mathcal{Q} -representation of any such subbundle F must be a staircase C in R and vice versa any \mathcal{Q} -representation whose support is a staircase C in R is the \mathcal{Q} -representation of a subbundle F of E induced by a subrepresentation of the P -representation inducing E .

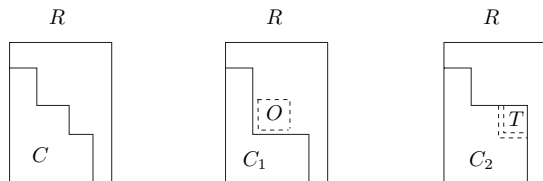


We will show by induction on the number k of steps of C that $\mu(C) < \mu(R)$ for any C staircase in R .

- $k = 1$. — In this case C is a subrectangle in the rectangle R . Thus this case follows from Lemma 30.

- $k - 1$ implies k . — We will show that, given a staircase C in R with k steps, there exists a staircase C' in R with $k - 1$ steps such that $\mu(C) \leq \mu(C')$. If we prove this, we conclude because $\mu(C) \leq \mu(C') < \mu(R)$, where the last inequality holds by induction hypothesis.

Let C_1 and C_2 be two staircases as in the figure



that is, C_1 and C_2 are staircases with $k - 1$ steps obtained from C respectively “removing and adding” two rectangles O and T . Precisely O is a sticking out part O_i of C for some i and T is a nonempty rectangle among the two rectangles A_i, B_i (see Notation 33).

If $\mu(C_1) \geq \mu(C)$ we conclude at once.

Thus we can suppose that $\mu(C_1) < \mu(C)$. We state that in this case $\mu(C_2) \geq \mu(C)$. In fact: let $\mu(C_1) = a/b$, $\mu(O) = c/d$ and $\mu(T) = e/f$, where the numerators are the first Chern classes and the denominators the ranks; since $\mu(C_1) < \mu(C)$, we have $a/b < (a + c)/(b + d)$, thus $a/b < c/d$; besides $\mu(O) < \mu(T)$ by Lemma 30, i.e. $c/d < e/f$; thus $(a + c + e)/(b + d + f) \geq (a + c)/(b + d)$ i.e. $\mu(C_2) \geq \mu(C)$. \square

Observe that, by Lemma 14 (i), the simplicity statement of Theorem 36 is equivalent to:

COROLLARY 37. — *Let $p, q, s \in \mathbb{N}$ with $p \geq q$, $s > 0$ and let A and B*

be two linear maps such that the following diagram commutes:

$$\begin{CD} S^{p,q}V \otimes S^sV @>\pi>> S^{p+s,q}V \\ @V A \otimes \text{Id} VV @VV B V \\ S^{p,q}V \otimes S^sV @>\pi>> S^{p+s,q}V, \end{CD}$$

where π is a nonzero $\text{SL}(V)$ -invariant projection (it is unique up to multiple); then $A = \lambda \text{Id}$ and $B = \lambda \text{Id}$ for some $\lambda \in \mathbb{C}$.

Now we want to prove Theorem 3; first it is necessary to prove several lemmas.

LEMMA 38. — Let $p, q, s \in \mathbb{N}$ with $p \geq q$. For every

$$M \subset \{(s_1, s_2, s_3) \in \mathbb{N}^3 \mid s_1 + s_2 + s_3 = s, s_2 \leq p - q, s_3 \leq q\},$$

let \mathcal{P}_M be the following statement: for every V complex vector space of dimension 3, the commutativity of the diagram of bundles on $\mathbb{P}(V)$:

$$\begin{CD} S^{p,q}V(-s) @>\varphi>> \bigoplus_{(s_1, s_2, s_3) \in M} S^{p+s_1, q+s_2, s_3}V \\ @V A VV @VV B V \\ S^{p,q}V(-s) @>\varphi>> \bigoplus_{(s_1, s_2, s_3) \in M} S^{p+s_1, q+s_2, s_3}V \end{CD}$$

(where A and B are linear maps and the components of φ are nonzero $\text{SL}(V)$ -invariant maps) implies $A = \lambda \text{Id}$ and $B = \lambda \text{Id}$ for some $\lambda \in \mathbb{C}$. Let

$$\{(s_1, s_2, s_3) \in \mathbb{N}^3 \mid s_1 + s_2 + s_3 = s, s_2 \leq p - q, s_3 \leq q\} = R \cup T$$

with $R \cap T = \emptyset, R \neq \emptyset, T \neq \emptyset$. Then \mathcal{P}_R is true if and only if \mathcal{P}_T is true.

Proof. — Suppose \mathcal{P}_T is true. We want to show \mathcal{P}_R is true. Let A and B such that the diagram

$$\begin{CD} S^{p,q}V(-s) @>\varphi>> \bigoplus_{(s_1, s_2, s_3) \in R} S^{p+s_1, q+s_2, s_3}V \\ @V A VV @VV B V \\ S^{p,q}V(-s) @>\varphi>> \bigoplus_{(s_1, s_2, s_3) \in R} S^{p+s_1, q+s_2, s_3}V \end{CD}$$

commutes. It is equivalent to the diagram

$$\begin{array}{ccc}
 S^{p,q}V \otimes S^sV & \xrightarrow{\pi} & \bigoplus_{(s_1,s_2,s_3) \in R} S^{p+s_1,q+s_2,s_3}V \\
 A \otimes \text{Id} \downarrow & & \downarrow B \\
 S^{p,q}V \otimes S^sV & \xrightarrow{\pi} & \bigoplus_{(s_1,s_2,s_3) \in R} S^{p+s_1,q+s_2,s_3}V.
 \end{array}$$

Thus $(A \otimes \text{Id})(\text{Ker } \pi) \subset \text{Ker } \pi$. Observe that

$$\text{Ker } \pi = \bigoplus_{(s_1,s_2,s_3) \in T} S^{p+s_1,q+s_2,s_3}V.$$

Then we get the following commutative diagram:

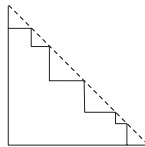
$$\begin{array}{ccc}
 \bigoplus_{(s_1,s_2,s_3) \in T} S^{p+s_1,q+s_2,s_3}V & \longrightarrow & S^{p,q}V \otimes S^sV \\
 \downarrow & & \downarrow A \otimes \text{Id} \\
 \bigoplus_{(s_1,s_2,s_3) \in T} S^{p+s_1,q+s_2,s_3}V & \longrightarrow & S^{p,q}V \otimes S^sV.
 \end{array}$$

Let $W = V^\vee$. Substitute W^\vee for V in the above diagram and dualize; the diagram we obtain is equivalent to the following commutative diagram of bundles on $\mathbb{P}(W)$

$$\begin{array}{ccc}
 S^{p,q}W(-s) & \longrightarrow & \bigoplus_{(s_1,s_2,s_3) \in T} S^{p+s_1,q+s_2,s_3}W \\
 \downarrow A^\vee & & \downarrow \\
 S^{p,q}W(-s) & \longrightarrow & \bigoplus_{(s_1,s_2,s_3) \in T} S^{p+s_1,q+s_2,s_3}W.
 \end{array}$$

By \mathcal{P}_T we conclude that $A^\vee = \lambda \text{Id}$ and then $A = \lambda \text{Id}$. □

DEFINITION 39. — We say (see figure below) that a staircase is *regular* if all the vertices of the steps (see Notation 33) are on a line with angular coefficient equal to -1 .



LEMMA 40. — *The bundles whose support is a regular staircase are multistable.*

Proof of Lemma 40.

Fact 1. — For any regular staircase we have

$$\mu(H_i) > \mu(H_{i-1}) \quad \text{and} \quad \mu(E_i) > \mu(E_{i+1})$$

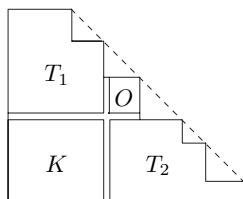
for any i , where H_i are the horizontal steps and E_i are the vertical steps (see Notation 33); recall that H_{i+1} is below H_i and E_{i+1} is on left of E_i .

Proof. — Obviously it is sufficient to prove the statement for a regular staircase with two steps. It is a freshman calculation (even if a bit long) (use Lemma 28). □

Fact 2. — Let S be a regular staircase. Then for every sticking out part O of S we have $\mu(O) > \mu(S - O)$. Therefore $\mu(S) > \mu(S - O)$.

Proof. — Let b be the line on which the base of O is and let ℓ be the line on which the left side of O is. Let T_1 be the staircase whose vertices are the vertices of S that are either above b or on b and on the left of ℓ (see the figure below). Let T_2 be the staircase whose vertices are the vertices of S that are below b and either on the right of ℓ or on ℓ .

Let K be the rectangle $K = S - T_1 - T_2 - O$.



By Lemma 30 $\mu(O) > \mu(K)$. Besides, by applying Fact 1 to the staircases $T_1 + O$ and $T_2 + O$ (where $T_i + O$ is the smallest staircase containing T_i and O), we get $\mu(O) > \mu(T_1)$ and $\mu(O) > \mu(T_2)$. Hence $\mu(O) > \max\{\mu(K), \mu(T_1), \mu(T_2)\} \geq \mu(S - O)$ (see Remark 27). □

Now we are ready to prove that every bundle such that its \mathcal{Q} -support is a regular staircase S is multistable. Let C be the support of a \mathcal{Q} -representation subrepresentation of S (thus again a staircase by Remark 31). We want to prove $\mu(C) < \mu(S)$ by induction on the number k of steps of C .

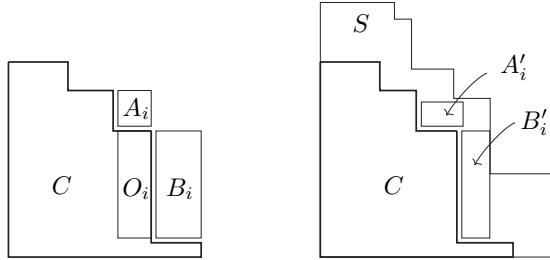
- $k = 1$. — The statement follows from Lemma 30 and Fact 1.

• $k - 1$ implies k . — We do induction on $-\text{length}(bd(C) \cap bd(S))$, where bd denotes the border and the border of a staircase is the border of the part of the plane inside the staircase.

Let C be a staircase with k steps support of a subrepresentation of S . Let O_i be the i -th sticking out part of C .

- ▷ If $\mu(C - O_i) \geq \mu(C)$ for some i we conclude at once because $C - O_i$ has $k - 1$ steps; thus by induction assumption $\mu(S) > \mu(C - O_i)$ and then $\mu(S) > \mu(C)$.
- ▷ Thus we can suppose $\mu(C) > \mu(C - O_i)$ for all i i.e. $\mu(O_i) > \mu(C)$ for all i .

Let A'_i be the biggest rectangle in $A_i \cap S$ with the lower side equal to the lower side of A_i and let B'_i be the biggest rectangle in $B_i \cap S$ with the left side equal to the left side of B_i (see Notation 33 for the definition of A_i and B_i).



Suppose that there exists i such that either A'_i or B'_i is not empty; for instance suppose $B'_i \neq \emptyset$. Since $\mu(B'_i) > \mu(O_i)$ by Lemma 30 and $\mu(O_i) > \mu(C)$ by assumption, we have $\mu(B'_i) > \mu(C)$ and thus

$$(8) \quad \mu(C + B'_i) > \mu(C)$$

where $C + B'_i$ is the smallest staircase containing C and B'_i . If B_i is a subgraph of S i.e. $B_i = B'_i$, then $C + B'_i$ is a staircase with $k - 1$ steps thus, by induction assumption, $\mu(S) > \mu(C + B'_i)$; hence $\mu(S) > \mu(C)$ by (8).

If B_i is not a subgraph of S i.e. $B_i \neq B'_i$, then

$$\text{length}(bd(C + B'_i) \cap bd(S)) > \text{length}(bd(C) \cap bd(S))$$

and by induction assumption $\mu(S) > \mu(C + B'_i)$; hence we conclude again $\mu(S) > \mu(C)$ by (8).

If A'_i and B'_i are empty for all i then there exists a chain of staircases $C = S_0 \subset S_1 \subset \dots \subset S_r = S$ such that S_i is obtained from S_{i+1} taking off one of its sticking out parts, thus, by Fact 2, we conclude. □

LEMMA 41. — a) The \mathcal{Q} -support of $S^\ell Q(t) \otimes S^q V$ is (see also the figure below):

(i) if $\ell \geq q$, the subgraph of \mathcal{Q} , with all the multiplicities equal to 1, included in an isosceles right-angled triangle with horizontal and vertical catheti of length q , the direction of the hypotenuse equal to NW-SE, the vertex opposite to the hypotenuse equal to the lowest left vertex and equal to $S^\ell Q(t - q)$;

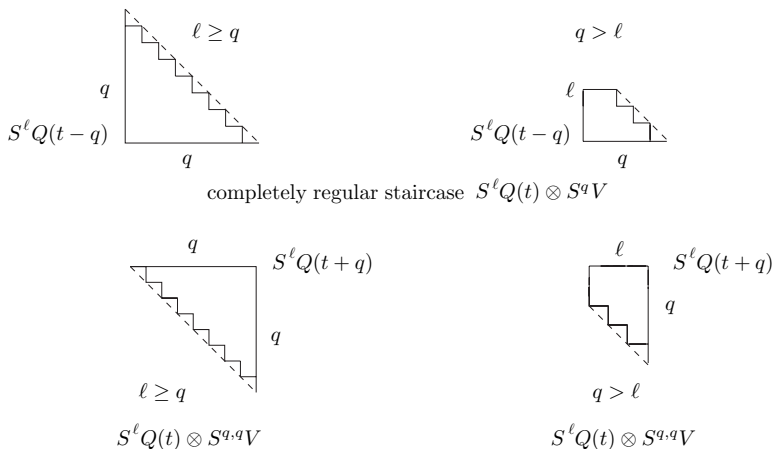
(ii) if $\ell < q$ the subgraph of \mathcal{Q} , with all the multiplicities equal to 1, included in a right-angled trapezium with horizontal bases, left side orthogonal to the bases, right side with angular coefficient -1 , length of the inferior base equal to q , the lowest left vertex equal to $S^\ell Q(t - q)$.

b) By duality (or directly) we can get an analogous statement for the \mathcal{Q} -support of $S^\ell Q(t) \otimes S^{q,q} V$ (see the figure).

Proof. — Use that $S^q V = \bigoplus_{i=0, \dots, q} S^{q-i} Q(-i)$ as R -representation and Clebsch-Gordan's formula, see [FH]: if $\ell \geq m$, then

$$S^\ell Q(t) \otimes S^m Q(r) = S^{\ell+m} Q(t+r) \oplus S^{\ell+m-2} Q(t+r+1) \oplus \dots \oplus S^{\ell-m} Q(t+r+m) \quad \square$$

DEFINITION 42. — We say that a staircase is *completely regular* if it is equal to one of the subgraphs of \mathcal{Q} described in a of Lemma 41.



LEMMA 43. — A bundle whose support is a regular staircase is stable if and only if the staircase is not completely regular.

Proof. — Observe that the \mathcal{Q} -support of $S^\ell Q(t) \otimes S^{p,q}V$ has some multiplicity ≥ 2 if $p \neq q$ and $q \neq 0$; in fact among the vertices of the \mathcal{Q} -support of $S^{p,q}V$ there are $\mathcal{O}(2q - p)$ and $S^2Q(-1 + 2q - p)$, thus, by Clebsch-Gordan’s formula, $S^\ell Q(t + 2q - p)$ occurs at least twice in $S^\ell Q(t) \otimes S^{p,q}V$.

By Lemma 40 if a bundle E has a regular staircase as \mathcal{Q} -support, then $E = E' \otimes T$ where E' is a stable vector bundle and T is a vector space $\mathrm{SL}(V)$ -representation; if $T = S^{p,q}V$ with $p \neq q$ and $q \neq 0$ then by the previous remark the \mathcal{Q} -support of $E' \otimes T$ has some multiplicity ≥ 2 . Thus we must have $T = S^{q,q}V$ or $T = S^qV$ and since the \mathcal{Q} -support of $E' \otimes T$ is a staircase only the last case is possible by Lemma 41. Thus we conclude by Lemma 41, in fact a regular staircase can be the disjoint union of k completely regular staircases with the same length of the base if and only if $k = 1$ i.e. it is a completely regular stairacase (we can see this arguing on the upper part of the border). \square

Remark 44. — If $p + q = p' + q'$, then the rectangles that are the \mathcal{Q} -supports of $S^{p,q}V(t)$ and $S^{p',q'}V(t)$ have the highest right vertices on a line with angular coefficient -1 .

If $p'' + q'' - t'' = p + q - t$ and $t'' < t$, then the highest right vertex of the \mathcal{Q} -support of $S^{p'',q''}V(t'')$ is below this line.

LEMMA 45. — *Let $p, q, s \in \mathbb{N}$ with $p \geq q$ and $s > 0$. Let*

$$T \subset \{(s_1, s_2, s_3) \mid s_1 + s_2 + s_3 = s, s_2 \leq p - q, s_3 \leq q\}, \quad T \not\ni (s, 0, 0).$$

Let A and B be two linear maps such that the following diagram of bundles on $\mathbb{P}(V)$ commutes:

$$\begin{array}{ccc} S^{p,q}V(-s) & \xrightarrow{\varphi} & \bigoplus_{(s_1, s_2, s_3) \in T} S^{p+s_1, q+s_2, s_3}V \\ \downarrow A & & \downarrow B \\ S^{p,q}V(-s) & \xrightarrow{\varphi} & \bigoplus_{(s_1, s_2, s_3) \in T} S^{p+s_1, q+s_2, s_3}V, \end{array}$$

where φ is an $\mathrm{SL}(V)$ -invariant map with all its components nonzero. Then $A = \lambda \mathrm{Id}$ and $B = \lambda \mathrm{Id}$ for some $\lambda \in \mathbb{C}$.

Proof. — Observe that $A(\text{Ker}(\varphi)) \subset \text{Ker}(\varphi)$. Thus we have a commutative diagram

$$(9) \quad \begin{array}{ccc} \text{Ker}(\varphi) & \longrightarrow & S^{p,q}V(-s) \\ A|_{\text{Ker}(\varphi)} \downarrow & & \downarrow A \\ \text{Ker}(\varphi) & \longrightarrow & S^{p,q}V(-s). \end{array}$$

Let $0 \rightarrow R \rightarrow S \rightarrow \text{Ker}(\varphi) \rightarrow 0$ be a minimal free resolution of $\text{Ker}(\varphi)$. By Lemma 14 the map $A|_{\text{Ker}(\varphi)} : \text{Ker}(\varphi) \rightarrow \text{Ker}(\varphi)$ induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & R & \rightarrow & S & \rightarrow & \text{Ker}(\varphi) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow A|_{\text{Ker}(\varphi)} \\ 0 & \rightarrow & R & \rightarrow & S & \rightarrow & \text{Ker}(\varphi) \rightarrow 0. \end{array}$$

Let S_{\max} be the direct sum of the summands of S with maximum twist and let $S = S_{\max} \oplus S'$; thus the previous diagram is

$$\begin{array}{ccccccc} 0 & \rightarrow & R & \rightarrow & S_{\max} \oplus S' & \rightarrow & \text{Ker}(\varphi) \rightarrow 0 \\ & & \downarrow & & \downarrow \swarrow \downarrow & & \downarrow A|_{\text{Ker}(\varphi)} \\ 0 & \rightarrow & R & \rightarrow & S_{\max} \oplus S' & \rightarrow & \text{Ker}(\varphi) \rightarrow 0. \end{array}$$

Then we get a commutative diagram

$$\begin{array}{ccc} S_{\max} & \longrightarrow & \text{Ker}(\varphi) \\ \downarrow \alpha & & \downarrow A|_{\text{Ker}(\varphi)} \\ S_{\max} & \longrightarrow & \text{Ker}(\varphi) \end{array}$$

and, if f is the composition of the map $S_{\max} \rightarrow \text{Ker}(\varphi)$ with the inclusion $\text{Ker}(\varphi) \rightarrow S^{p,q}V(-s)$, by (9) we get the commutative diagram

$$(10) \quad \begin{array}{ccc} S_{\max} & \xrightarrow{f} & S^{p,q}V(-s) \\ \downarrow \alpha & & \downarrow A \\ S_{\max} & \xrightarrow{f} & S^{p,q}V(-s) \end{array}$$

and thus a commutative diagram

$$(11) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{Ker}(f) & \rightarrow & S_{\max} & \xrightarrow{f} & \text{Im}(f) \rightarrow 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \gamma \\ 0 & \rightarrow & \text{Ker}(f) & \rightarrow & S_{\max} & \xrightarrow{f} & \text{Im}(f) \rightarrow 0, \end{array}$$

where $\gamma = A|_{\text{Im}(f)}$.

Now we will prove that if $\text{Im}(f)$ is simple then A is a multiple of the identity. Let $0 \rightarrow K \rightarrow M \rightarrow (\text{Im } f)^\vee \rightarrow 0$ be a minimal free resolution of $(\text{Im } f)^\vee$; for any $\beta: M \rightarrow M$ induced by γ^\vee we have the following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & (\text{Im } f)^\vee \\ \beta \downarrow & & \downarrow \gamma^\vee \\ M & \longrightarrow & (\text{Im } f)^\vee \end{array}$$

and then by (11)

$$\begin{array}{ccc} M & \xrightarrow{r} & S_{\max}^\vee \\ \beta \downarrow & & \downarrow \alpha^\vee \\ M & \xrightarrow{r} & S_{\max}^\vee. \end{array}$$

In particular, since $(\text{Im } f)^\vee$ is simple, γ^\vee is a multiple of the identity, thus β can be taken equal to a multiple of the identity.

Observe that all the components of r are nonzero (because the map $M \rightarrow (\text{Im } f)^\vee$ is surjective and all the components of $f^\vee: (\text{Im } f)^\vee \rightarrow S_{\max}^\vee$ are nonzero, since no component of S_{\max} is sent to 0 by f); besides, up to twisting, we can suppose S_{\max} is a trivial bundle and then $H^0(r^\vee)^\vee$ is a projection. Hence, since β is a multiple of the identity, α^\vee (and then α) is a multiple of the identity.

Thus, by (10) also A is a multiple of the identity as we wanted (twist (10) by s and consider $H^0(\bullet)^\vee$ of every map of the obtained diagram; in this way we get a diagram of $\text{SL}(V)$ -representations whose horizontal maps are surjective, since they are $\text{SL}(V)$ -invariant and nonzero and $S^{p,q}V$ is irreducible).

Observe that, by Remark 44, the \mathcal{Q} -support of $\text{Im } f$ is a regular staircase in the \mathcal{Q} -support of $S^{p,q}V(-s)$, since all the summands of S_{\max} have the same twist.

Thus by Lemma 43 we can conclude at once if the \mathcal{Q} -support of $\text{Im } f$ is not a completely regular staircase.

Therefore we can suppose the \mathcal{Q} -support of $\text{Im } f$ is a completely regular staircase.

Observe that, by Remark 44, if the \mathcal{Q} -support $\text{Im } f$ is a completely regular staircase then $S = S_{\max}$ and thus $\text{Im } f = \text{Ker } \varphi$.

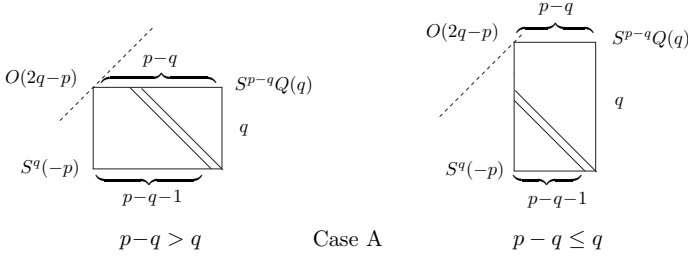
Consider the following exact sequence

$$(12) \quad 0 \rightarrow \text{Ker } \varphi \longrightarrow S^{p,q}V(-s) \longrightarrow \text{Im } \varphi \rightarrow 0.$$

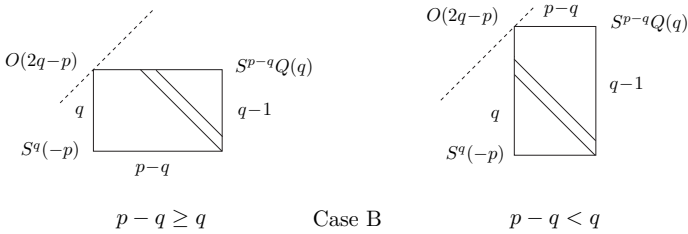
Up to dualizing we can suppose that also the \mathcal{Q} -support of $\text{Im } \varphi$ is of the kind b of Lemma 41.

Thus the unique remaining cases are the cases in which the sequence (12) twisted by s is one of the following:

- Case A: $0 \rightarrow S^q \mathcal{Q}(-q-1) \otimes S^{p-q-1} V \rightarrow S^{p,q} V \rightarrow S^{p-q} \mathcal{Q} \otimes S^{q,q} V \rightarrow 0.$



- Case B: $0 \rightarrow S^q \mathcal{Q}(-q) \otimes S^{p-q} V \rightarrow S^{p,q} V \rightarrow S^{p-q} \mathcal{Q}(1) \otimes S^{q-1,q-1} V \rightarrow 0.$



Observe that Case B is equivalent to Case A (by dualizing and considering $q' = p - q$). Thus it is sufficient to consider Case A. By (9) we get a commutative diagram

$$\begin{array}{ccccccc}
 0 \rightarrow S^q \mathcal{Q}(-q-1) \otimes S^{p-q-1} V & \rightarrow & S^{p,q} V & \rightarrow & S^{p-q} \mathcal{Q} \otimes S^{q,q} V & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow S^q \mathcal{Q}(-q-1) \otimes S^{p-q-1} V & \rightarrow & S^{p,q} V & \rightarrow & S^{p-q} \mathcal{Q} \otimes S^{q,q} V & \rightarrow & 0.
 \end{array}$$

By taking the cohomology (in particular H^0) we get

$$\begin{array}{ccccccc}
 0 & \rightarrow & 0 & \rightarrow & S^{p,q} V & \rightarrow & S^{q,q} V \otimes S^{p-q} V \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & 0 & \rightarrow & S^{p,q} V & \rightarrow & S^{q,q} V \otimes S^{p-q} V
 \end{array}$$

since $H^0(S^q Q(-q-1)) = 0$ and $H^0(S^{p-q} Q) = S^{p-q} V$ (to calculate the cohomology of $S^\ell Q(t)$ use for instance its minimal resolution $0 \rightarrow S^{\ell-1} V(t-1) \rightarrow S^\ell V(t) \rightarrow S^\ell Q(t) \rightarrow 0$). We conclude by applying Lemma 37 to the dual of the right part of the diagram. \square

From Lemmas 45 and 38 we deduce at once:

COROLLARY 46. — *Let $p, q, s \in \mathbb{N}$ with $p \geq q$. Let A and B be two linear maps such that the following diagram of bundles on $\mathbb{P}(V)$ commutes*

$$\begin{array}{ccc} S^{p,q}V(-s) & \xrightarrow{\varphi} & W \\ A \otimes \text{Id} \downarrow & & \downarrow B \\ S^{p,q}V(-s) & \xrightarrow{\varphi} & W, \end{array}$$

where W is a non trivial $\text{SL}(V)$ -submodule of $S^{p,q}V \otimes S^s V$ and all the components of φ are nonzero $\text{SL}(V)$ -invariant maps. Then $A = \lambda \text{Id}$ and $B = \lambda \text{Id}$ for some $\lambda \in \mathbb{C}$.

Proof of Theorem 3. — The case $p = 0$ is trivial. Thus we can suppose $p > 0$. By Corollary 46, the only thing we have to prove is that if in $W \otimes \mathcal{O}$ there are two copies of an irreducible bundle F , i.e. $W = F \oplus F \oplus W'$, then E is not simple: in fact the following diagram induces an automorphism on E not multiple of the identity:

$$\begin{array}{ccccccc} 0 \rightarrow S^{p,q}V(-s) & \longrightarrow & F \oplus F \oplus W' & \longrightarrow & E & \rightarrow & 0 \\ & & \text{Id} \downarrow & & -\text{Id} \downarrow & \swarrow^{2\text{Id}} & \downarrow \text{Id} & \downarrow \text{Id} & \downarrow \text{Id} \\ 0 \rightarrow S^{p,q}V(-s) & \longrightarrow & F \oplus F \oplus W' & \longrightarrow & E & \rightarrow & 0 \end{array}$$

(see Lemma 14). \square

The following theorem gives a precise criterion to see when a regular elementary homogeneous bundle E on \mathbb{P}^2 is stable or simple in the case the difference of the twists of the first bundle and of the middle bundle of the minimal free resolution of E is 1.

THEOREM 47. — *Let E be a homogeneous bundle on \mathbb{P}^2 with minimal free resolution*

$$0 \rightarrow S^{p,q}V \xrightarrow{\varphi} \bigoplus_{\alpha \in \mathcal{A}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V(1) \rightarrow E \rightarrow 0$$

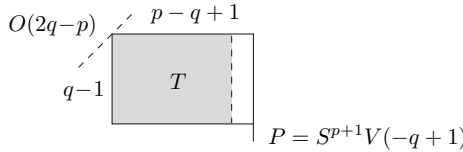
with $p \geq q \geq 0$, $s_i^\alpha \in \mathbb{N}$, $1 = s_1^\alpha + s_2^\alpha + s_3^\alpha$, \mathcal{A} finite subset of indices and all the components of φ nonzero $\text{SL}(V)$ -invariant maps. Then:

(i) E is simple if and only if its minimal free resolution has one of the following forms:

$$\begin{aligned}
 &0 \rightarrow S^{p,q}V \rightarrow S^{p+1,q}V(1) \rightarrow E \rightarrow 0, \\
 &0 \rightarrow S^{p,q}V \rightarrow S^{p+1,q}V(1) \oplus S^{p,q+1}V(1) \rightarrow E \rightarrow 0 \text{ with } q \neq 0, \\
 &0 \rightarrow S^{p,q}V \rightarrow S^{p+1,q}V(1) \oplus S^{p,q,1}V(1) \rightarrow E \rightarrow 0 \text{ with } q \neq 0 \text{ and } p \neq q.
 \end{aligned}$$

(ii) E is stable if and only if E is simple and, moreover, in the case its minimal resolution is of the third type $2q \geq p > q$.

Proof. — Part (i) follows from Theorem 3. So it is enough to check when the bundles described in (i) are stable. The first case follows from Lemma 37. For the second case note that the \mathcal{Q} -support of E is a not completely regular staircase, thus we conclude by Lemma 43. In the third case the \mathcal{Q} -support of E is the following:



The bundle E is multistable if and only if $\mu(T) < \mu(E)$ and $\mu(P) < \mu(E)$ and, by using Lemma 28, one can show that this is true if and only if

$$\begin{aligned}
 &-5p + 2q - 2 - 4p^2 + 3pq + p^2q - p^3 < 0 \quad \text{and} \\
 &3p^2 + p^3 - 8pq - 2p^2q - 4 - 8q < 0
 \end{aligned}$$

respectively. The first inequality always holds since $p \geq q$. The last inequality is equivalent to $p \leq 2q$ since its first member is equal to $(p + 2)^2(p - 1 - 2q)$. Thus E is multistable if and only if $p \leq 2q$. In this case, by Lemma 41, it is stable if and only if $p \neq q$. □

In [Fa] an example of a simple unstable homogeneous bundle on \mathbb{P}^2 is exhibited. Theorem 47 shows infinite examples of such bundles.

We end with a theorem which studies the simplicity of elementary homogeneous bundles.

THEOREM 48. — *Let $p \geq q$. Let E be the homogeneous vector bundle on $\mathbb{P}^2 = \mathbb{P}(V)$ defined by the following exact sequence (\mathcal{A} a finite set of indices)*

$$0 \rightarrow S^{p,q}V \xrightarrow{\varphi} \bigoplus_{\alpha \in \mathcal{A}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V(s^\alpha) \longrightarrow E \rightarrow 0,$$

where $s_1^\alpha + s_2^\alpha + s_3^\alpha = s^\alpha$ for all $\alpha \in \mathcal{A}$ and the components φ_α of φ are $SL(V)$ -invariant maps. Then E is simple if and only if the following five conditions hold:

- a) there do not exist $\alpha, \beta \in \mathcal{A}$ such that $s_i^\alpha \leq s_i^\beta$ $i = 1, 2, 3$;
- b) all the components φ_α of φ are nonzero, in particular $p \geq q + s_2^\alpha$ and $q \geq s_3^\alpha$ for all $\alpha \in \mathcal{A}$;
- c) for all $\alpha, \beta \in \mathcal{A}$ such that $s^\alpha > s^\beta$, we have that if $s_2^\beta > 0$ then $s_3^\beta + s^\alpha - s^\beta < q + 1$ and if $s_1^\beta > 0$ then $q + s_2^\beta + s^\alpha - s^\beta < p + 1$;
- d) for all $\alpha, \beta, \gamma \in \mathcal{A}$ such that $s^\alpha = s^\beta < s^\gamma$, we have $s^\gamma - s^\alpha \geq \max\{|s_i^\alpha - s_i^\beta| \mid i = 1, 2, 3\}$;
- e) if $p > 0$ and if there exists an s such that $s^\alpha = s$ for all $\alpha \in \mathcal{A}$ then $\bigoplus_{\alpha \in \mathcal{A}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V \neq S^{p, q} V \otimes S^s V$.

Sketch of the proof. — Any automorphism η of E induces maps A and B as in Lemma 14 and given A and B we have an automorphism of E . We call

$$B_{\mathcal{K}, \mathcal{J}} : \bigoplus_{\alpha \in \mathcal{J}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V(s^\alpha) \longrightarrow \bigoplus_{\alpha \in \mathcal{K}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V(s^\alpha)$$

the map induced by B , for all \mathcal{J}, \mathcal{K} of \mathcal{A} . We denote $B_J = B_{J, J}$, $B_{\alpha, \beta} = B_{\{\alpha\}, \{\beta\}}$ and $B_\alpha = B_{\{\alpha\}}$ for short. Besides

$$\varphi_{\mathcal{J}} : S^{p, q} V \longrightarrow \bigoplus_{\alpha \in \mathcal{J}} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V(s^\alpha)$$

denotes the map induced by φ , for all $\mathcal{J} \subset \mathcal{A}$.

We show now that the simplicity of E implies a), b), c), d), e); conditions a), b) and e) are left to the reader.

c) Observe that if for some $\bar{\alpha}, \bar{\beta} \in \mathcal{A}$ there exists a nonzero map

$$\gamma : S^{p+s_1^{\bar{\alpha}}, q+s_2^{\bar{\alpha}}, s_3^{\bar{\alpha}}} V(s^{\bar{\alpha}}) \longrightarrow S^{p+s_1^{\bar{\beta}}, q+s_2^{\bar{\beta}}, s_3^{\bar{\beta}}} V(s^{\bar{\beta}})$$

such that $\gamma \circ \varphi_{\bar{\beta}} = 0$ then E is not simple (take $B_{\bar{\alpha}, \bar{\beta}} = \gamma$, $A = \text{Id}$, $B_\alpha = \text{Id}$ for all $\alpha \in \mathcal{A}$, $B_{\alpha, \beta} = 0$ for all $(\alpha, \beta) \neq (\bar{\alpha}, \bar{\beta})$ and use Lemma 14). Such a γ exists if and only if there exists a nonzero map

$$\Gamma : S^{p+s_1^{\bar{\beta}}, q+s_2^{\bar{\beta}}, s_3^{\bar{\beta}}} V \otimes S^{s^{\bar{\alpha}}-s^{\bar{\beta}}} V \longrightarrow S^{p+s_1^{\bar{\alpha}}, q+s_2^{\bar{\alpha}}, s_3^{\bar{\alpha}}} V$$

such that $\Gamma \circ H^0((\varphi_{\bar{\beta}}(-s^{\bar{\alpha}}))^\vee)^\vee = 0$. This is equivalent to the non surjectivity

of $H^0((\varphi_\beta(-s^{\bar{\alpha}})^\vee)^\vee)$, which is the injection followed by the projection

$$S^{p,q}V \otimes S^{s^{\bar{\alpha}}}V \rightarrow S^{p,q}V \otimes S^{s^\beta}V \otimes S^{s^{\bar{\alpha}}-s^\beta}V \rightarrow S^{p+s_1^\beta, q+s_2^\beta, s_3^\beta}V \otimes S^{s^{\bar{\alpha}}-s^\beta}V,$$

and we conclude by Pieri's formula.

d) Obviously if for some $\alpha, \beta, \gamma \in \mathcal{A}$ such that $s^\alpha = s^\beta < s^\gamma$ there exist nonzero maps

$$\delta : S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V(s^\alpha) \longrightarrow S^{p+s_1^\gamma, q+s_2^\gamma, s_3^\gamma}V(s^\gamma)$$

$$\delta' : S^{p+s_1^\beta, q+s_2^\beta, s_3^\beta}V(s^\beta) \longrightarrow S^{p+s_1^\gamma, q+s_2^\gamma, s_3^\gamma}V(s^\gamma),$$

such that $\delta \circ \varphi_\alpha + \delta' \circ \varphi_\beta = 0$ then E is not simple (take $A = \text{Id}$, $B_\alpha = \text{Id}$ for all $\alpha \in \mathcal{A}$, $B_{\gamma, \alpha} = \delta$, $B_{\beta, \alpha} = \delta'$ and $B_{\epsilon, \lambda} = 0$ for all $(\epsilon, \lambda) \neq (\gamma, \alpha), (\gamma, \beta)$). Such δ and δ' exist if and only if

$$S^{p,q}V \otimes S^{s^\gamma}V \xrightarrow{\Upsilon} (S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V \otimes S^{s^\gamma-s^\alpha}V) \oplus (S^{p+s_1^\beta, q+s_2^\beta, s_3^\beta}V \otimes S^{s^\gamma-s^\alpha}V)$$

is not surjective, where $\Upsilon = H^0(\varphi_\alpha(-s^\gamma)^\vee)^\vee \times H^0(\varphi_\beta(-s^\gamma)^\vee)^\vee$; since $H^0(\varphi_\alpha(-s^\gamma)^\vee)^\vee$ and $H^0(\varphi_\beta(-s^\gamma)^\vee)^\vee$ are surjective by c), this is true if and only if $S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V \otimes S^{s^\gamma-s^\alpha}V$ and $S^{p+s_1^\beta, q+s_2^\beta, s_3^\beta}V \otimes S^{s^\gamma-s^\alpha}V$ have a nonzero subrepresentation in common and we conclude.

Suppose now a), b), c), d), e) hold. We may assume $p > 0$. Let η be an automorphism of E and let B and A be the induced maps as above.

Let $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}'' \cup \dots$ be disjoint union such that \mathcal{A}' is the set of indices α in \mathcal{A} such that s_α is the minimum of $\{s_\alpha \mid \alpha \in \mathcal{A}\}$, \mathcal{A}'' is the set of indices α in $\mathcal{A} - \mathcal{A}'$ such that s_α is the minimum of $\{s_\alpha \mid \alpha \in \mathcal{A} - \mathcal{A}'\}$ and so on. Let $s' = s^\alpha$ for $\alpha \in \mathcal{A}'$ and $s'' = s^\alpha$ for $\alpha \in \mathcal{A}''$ and so on.

We have $\varphi_{\mathcal{A}'} \circ A = B_{\mathcal{A}'} \circ \varphi_{\mathcal{A}'}$; thus, by Corollary 46, $A = \lambda \text{Id}$ and $B_{\mathcal{A}'} = \lambda \text{Id}$.

Besides we have $\varphi_{\mathcal{A}''} \circ A = B_{\mathcal{A}''} \circ \varphi_{\mathcal{A}''} + B_{\mathcal{A}'', \mathcal{A}'} \circ \varphi_{\mathcal{A}'}$ thus we get $(\lambda \text{Id} - B_{\mathcal{A}''}) \circ \varphi_{\mathcal{A}''} = B_{\mathcal{A}'', \mathcal{A}'} \circ \varphi_{\mathcal{A}'}$. By applying $H^0(\bullet)^\vee$, we obtain

$$\begin{array}{ccc} S^{p,q}V \otimes S^{s''}V & \xrightarrow{H^0(\varphi_{\mathcal{A}''})^\vee} & \bigoplus_{\alpha \in \mathcal{A}''} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V \\ \downarrow H^0(\varphi_{\mathcal{A}'}^\vee)^\vee & & \downarrow H^0((\lambda \text{Id} - B_{\mathcal{A}''})^\vee)^\vee \\ \bigoplus_{\alpha \in \mathcal{A}'} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V \otimes S^{s''-s'}V & \xrightarrow{H^0(B_{\mathcal{A}'', \mathcal{A}'}^\vee)^\vee} & \bigoplus_{\alpha \in \mathcal{A}''} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha}V. \end{array}$$

By a we have that $\bigoplus_{\alpha \in \mathcal{A}''} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V$ is in the kernel of the map

$$H^0(\varphi_{\mathcal{A}'}^\vee)^\vee : S^{p,q} V \otimes S^{s''} V \longrightarrow \bigoplus_{\alpha \in \mathcal{A}'} S^{p+s_1^\alpha, q+s_2^\alpha, s_3^\alpha} V \otimes S^{s''-s'} V$$

thus $\lambda \text{Id} - B_{\mathcal{A}''} = 0$, then $B_{\mathcal{A}''} = \lambda \text{Id}$ and $B_{\mathcal{A}'', \mathcal{A}'} \circ \varphi_{\mathcal{A}'} = 0$. Hence by c) and d) we get $B_{\mathcal{A}'', \mathcal{A}'} = 0$ (arguing as in the proof of the other implication).

By induction on the number of the subsets $\mathcal{A}', \mathcal{A}'', \dots$ we conclude. \square

BIBLIOGRAPHY

- [Ba] E. BALLICO, On the stability of certain higher rank bundles on \mathbb{P}^n , *Rend. Circ. Mat. Palermo* (2), 41 (1992), 309–314.
- [BK] A.I. BONDAL, M.M. KAPRANOV, Homogeneous Bundles in ‘Seminar Rudakov, Helices and Vector bundles’, *Lecture Notes Series of London Math. Soc.* 148, p. 45–55, Cambridge University Press, 1990.
- [DL] J.-M. DRÉZET, J. LE POTIER, Fibrés stables et fibrés exceptionnels sur le plan projectif, *Ann. Sci. École Norm. Sup.* 4^e série, 18 (1985), 193–244.
- [Fa] S. FAINI, On the stability and simplicity of homogeneous bundles, to appear in *Boll. U.M.I.*
- [FH] W. FULTON, J. HARRIS, *Representation Theory, A First Course*, Graduate Texts in Math., Springer Verlag, 1991.
- [GR] P. GABRIEL, A.V. ROITER, Algebra VIII: Representations of finite dimensional algebras, *Encyclopaedia of Mathematical Sciences* 73, Springer Verlag, 1992.
- [Hi1] L. HILLE, Homogeneous vector bundles and Koszul algebras, *Math. Nach.*, 191 (1998), 189–195.
- [Hi2] L. HILLE, Small homogeneous vector bundles, *Phd Thesis*, Bielefeld, 1994.
- [Ho] G. HORROCKS, Vector bundles on the punctured spectrum of a local ring, *Proc. London Math. Soc.*, 14 (1964), 689–713.
- [Ka] M.M. KAPRANOV, On the derived categories of coherent sheaves on some homogeneous spaces, *Invent. Math.*, 92 (1988), 479–508.
- [Ki] A. KING, Moduli of representations of finite-dimensional algebras, *Quart. J. Math. Oxford Ser. (2)*, 45, no. 180, (1994), 515–530.
- [Mi] L. MIGLIORINI, Stability of homogeneous bundles, *Boll. Unione Mat. Ital. Sez. B*, 10 (1996), 963–990.
- [OR] G. OTTAVIANI, E. RUBEI, Quivers and the cohomology of homogeneous vector bundles, [math.AG/0403307](https://arxiv.org/abs/math/0403307).
- [Ra] S. RAMANAN, Holomorphic vector bundles on homogeneous spaces, *Topology*, 5 (1966), 159–177.

- [Ro] R. ROHMFELD, Stability of homogeneous vector bundles on $\mathbb{C}P^n$, *Geom. Dedicata*, 38 (1991), 159–166.
- [Si] D. SIMPSON, Lectures given on the workshop in Waplewo 8–13 September 1997.

Manuscrit reçu le 17 mai 2004,
révisé le 25 novembre 2004,
accepté le 4 janvier 2005.

Giorgio OTTAVIANI & Elena RUBEI,
Dipartimento di Matematica ‘U. Dini’
Viale Morgagni 67/A
50134 Firenze (Italia)
ottavian@math.unifi.it
rubei@math.unifi.it

