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The ring of multisymmetric functions

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THE RING OF MULTISYMMETRIC FUNCTIONS

by Francesco VACCARINO

Introduction.

Let \( R \) be a commutative ring and let \( n, m \) be two positive integers. Let \( A_R(n, m) \) be the polynomial ring in the commuting independent variables \( x_i(j) \) with \( i = 1, \ldots, m; j = 1, \ldots, n \) and coefficients in \( R \). The symmetric group on \( n \) letters \( S_n \) acts on \( A_R(n, m) \) by means of \( \sigma(x_i(j)) = x_i(\sigma(j)) \) for all \( \sigma \in S_n \) and \( i = 1, \ldots, m; j = 1, \ldots, n \). Let us denote by \( A_R(n, m)^{S_n} \) the ring of invariants for this action: its elements are usually called multisymmetric functions and they are the usual symmetric functions when \( m = 1 \). In this case, \( A_R(n, 1) \cong R[x_1, x_2, \ldots, x_n] \) and \( R[x_1, x_2, \ldots, x_n]^{S_n} \) is freely generated by the elementary symmetric functions \( e_1, \ldots, e_n \) given by the equality

\[
\sum_{k=0}^{n} t^k e_k := \prod_{i=1}^{n} (1 + tx_i).
\]

Here \( e_0 = 1 \) and \( t \) is a commuting independent variable (see [M]). Furthermore one has

\[
e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \ldots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}
\]

Unless otherwise stated, we now assume that \( m > 1 \). We first obtain generators of the ring \( A_R(n, m)^{S_n} \).

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Let $A_R(m) := R[y_1, \ldots, y_m]$, where $y_1, \ldots, y_m$ are commuting independent variables, let $f = f(y_1, \ldots, y_m) \in A_R(m)$ and define
\[(0.3) \quad f(j) := f(x_1(j), \ldots, x_m(j)) \text{ for } 1 \leq j \leq n.\]
Notice that $f(j) \in A_R(n, m)$ for all $1 \leq j \leq n$ and that $\sigma(f(j)) = f(\sigma(j))$, for all $\sigma \in S_n$ and $j = 1, \ldots, n$.

Define $e_k(f) := e_k(f(1), f(2), \ldots, f(n))$ i.e.
\[(0.4) \quad \sum_{k=0}^{n} t^k e_k(f) := \prod_{i=1}^{n} (1 + tf(i)),\]
where $t$ is a commuting independent variable. Then $e_k(f) \in A_R(n, m)^{S_n}$.

One may think about the $y_i$ as diagonal matrices in the following sense: let $M_n(A_R(n, m))$ be the full ring of $n \times n$ matrices with coefficients in $A_R(n, m)$. Then there is an embedding
\[(0.5) \quad \rho_n : A_R(m) \hookrightarrow M_n(A_R(n, m))\]
given by
\[(0.6) \quad \rho_n(y_i) := \begin{pmatrix} x_i(1) & 0 & \ldots & 0 \\ 0 & x_i(2) & \ldots & 0 \\ 0 & 0 & \ldots & x_i(n) \end{pmatrix} \text{ for } i = 1, \ldots, m.\]
Now (0.4) gives
\[(0.7) \quad \sum_{k=0}^{n} t^k e_k(f) = \prod_{j=1}^{n} (1 + t\rho_n(f)_{jj}) = \det(1 + t\rho_n(f)),\]
where $\det(-)$ is the usual determinant of $n \times n$ matrices.

Let $\mathcal{M}_m$ be the set of monomials in $A_R(m)$. For $\mu \in \mathcal{M}_m$ let $\partial_i(\mu)$ denote the degree of $\mu$ in $y_i$, for all $i = 1, \ldots, m$. We set
\[(0.8) \quad \partial(\mu) := (\partial_1(\mu), \ldots, \partial_m(\mu))\]
for its multidegree. The total degree of $\mu$ is $\sum_i \partial_i(\mu)$. Let $\mathcal{M}_m^+$ be the set of monomials of positive degree. A monomial $\mu \in \mathcal{M}_m^+$ is called primitive if it is not a power of another one. We denote by $\mathfrak{M}_m^+$ the set of primitive monomials. We define an $S_n$ invariant multidegree on $A_R(n, m)$ by setting $\partial(x_i(j)) = \partial(y_i) \in \mathbb{N}^m$ for all $1 \leq j \leq n$ and $1 \leq i \leq m$. If $f \in A_R(m)$ is homogeneous of total degree $l$, then $e_k(f)$ has total degree $kl$ (for all $k$ and $n$).

We are now in a position to state the first part of our result (recall that $m > 1$).

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THEOREM 1 (generators). — The ring of multisymmetric functions $A_R(n, m)^{S_n}$ is generated by the $e_k(\mu)$, where $\mu \in M^+_m$, $k = 1, \ldots n$ and the total degree of $e_k(\mu)$ is less or equal than $m(n-1)$. If $n = p^s$ is a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$, then at least one generator has degree equal to $m(n-1)$.

If $R \supset \mathbb{Q}$ then $A_R(n, m)^{S_n}$ is generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}^+_m$ and the degree of $\mu$ is less or equal than $n$.

To obtain the relations between these generators, we need more notation on (multi)symmetric functions.

The action of $S_n$ on $A_R(n, 1) \cong R[x_1, x_2, \ldots, x_n]$ preserves the usual degree. We denote by $\Lambda_{R,n}^k$ the $R$-submodule of invariants of degree $k$.

Let $q_n : R[x_1, x_2, \ldots, x_n] \rightarrow R[x_1, x_2, \ldots, x_{n-1}]$ be given by $x_n \mapsto 0$ and $x_i \mapsto x_i$, for $i = 1, \ldots, n-1$. This map sends $\Lambda_{R,n}^k$ to $\Lambda_{R,n-1}^k$ and it is easy to see that $\Lambda_{n,R}^k \cong \Lambda_{n-1,R}^k$ for all $n \geq k$. Denote by $\Lambda_R^k$ the limit of the inverse system obtained in this way.

The ring $\Lambda_R := \bigoplus_{k \geq 0} \Lambda_R^k$ is called the ring of symmetric functions (over $R$).

It can be shown [M] that $\Lambda_R$ is a polynomial ring, freely generated by the (limits of the) $e_k$, that are given by

\begin{equation}
(0.9) \quad \sum_{k=0}^{\infty} t^k e_k := \prod_{i=1}^{\infty} (1 + tx_i).
\end{equation}

Furthermore the kernel of the natural projection $\pi_n : \Lambda_R \rightarrow \Lambda_{n,R}$ is generated by the $e_{n+k}$, where $k \geq 1$.

In a similar way we build a limit of multisymmetric functions. For any $a \in \mathbb{N}^m$ we set $A_R(n, m, a)$ for the linear span of the monomials of multidegree $a$. One has

\begin{equation}
(0.10) \quad A_R(n, m) = \bigoplus_{a \in \mathbb{N}^m} A_R(n, m, a).
\end{equation}

Let $\pi_n : A_R(n, m) \rightarrow A_R(n-1, m)$ be given by

\begin{equation}
(0.11) \quad \pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leq n-1 \end{cases} \quad \text{for all } i.
\end{equation}

Then (see (3.5)) we prove that, for all $a \in \mathbb{N}^m$

\begin{equation}
(0.12) \quad \pi_n(A_R(n, m, a)^{S_n}) = A_R(n-1, m, a)^{S_{n-1}}.
\end{equation}
For any $a \in \mathbb{N}^m$ set
\begin{equation}
A_R(\infty, m, a) := \lim_{\leftarrow} A_R(n, m, a)^{S_n},
\end{equation}
where the projective limit is taken with respect to $n$ over the projective system $(A_R(n, m, a)^{S_n}, \pi_n)$.

Set
\begin{equation}
A_R(\infty, m) := \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a).
\end{equation}

We set, by abuse of notation,
\begin{equation}
e_k(f) := \lim_{\leftarrow} e_k(f) \in A_R(\infty, m)
\end{equation}
with $k \in \mathbb{N}$ and $f \in A(m)^+$, the augmentation ideal, i.e.
\begin{equation}
\sum_{k=0}^{\infty} t^k e_k(f) := \prod_{j=1}^{\infty} (1 + tf(j)).
\end{equation}

Then $e_k$ is a homogeneous polynomial of degree $k$. Now, if $f = \sum_{\mu \in \mathcal{M}_m^+} \lambda_{\mu} \mu$, we set
\begin{equation}
e_k(f) := \sum_{\alpha} \lambda^{\alpha} e_\alpha
\end{equation}
where $\alpha := (\alpha_{\mu})_{\mu \in \mathcal{M}_m^+}$ is such that $\alpha_{\mu} \in \mathbb{N}$, $\sum_{\mu \in \mathcal{M}_m^+} \alpha_{\mu} \leq k$ and $\lambda^{\alpha} := \prod_{\mu \in \mathcal{M}_m^+} \lambda^{\alpha_{\mu}}$.

We can now state the second part of our main result.

**Theorem 2** (relations). — (1) The ring $A_R(\infty, m)$ is a polynomial ring, freely generated by the (limits of) the $e_k(\mu)$, where $\mu \in \mathcal{M}_m^+$ and $k \in \mathbb{N}$.

The kernel of the natural projection
\begin{equation}
A_R(\infty, m) \longrightarrow A_R(n, m)^{S_n}
\end{equation}
is generated as $R$-module by the coefficients $e_\alpha$ of the elements
\begin{equation}
en+k(f), \text{ where } k \geq 1 \text{ and } f \in A_R(m)^+.
\end{equation}

(2) If $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$.

The kernel of the natural projection is generated as an ideal by the $e_{n+1}(f)$, where $f \in A_R(m)^+$.
In Dalbec's paper [D] generators and relations are found in the case where $R \supset \mathbb{Q}$. The relations found there are actually the same we find: indeed what Dalbec calls *monomial multisymmetric functions* are exactly those $e_\alpha$ we introduced in (0.17), so that his Proposition 1.9 is a special case of our Proposition 3.1(1) when $R \supset \mathbb{Q}$. Another paper on this theme, giving a minimal presentation when the base ring is a characteristic 2 field, is [A]. Again, its main results on multisymmetric functions are a corollary of ours when $R$ is a characteristic 2 field.

The results of this paper were presented in 1997 at a congress on algebraic groups representations in Ascona (CH) organized by H.P. Kraft. They are published only now for personal reasons.

1. Notations and basic facts.

The monomials of $A_R(n, m)$ form a $R$-basis, permuted by the action of $S_n$. Thus, the sums of monomials over the orbits form a $R$-basis of the ring of multisymmetric functions. We now introduce some notation and preliminary results concerning these functions and orbit sums.

Let $k \in \mathbb{N}$, we denote by $f$ the sequence $(f_1, \ldots, f_k)$ in $A_R(m)$ and by $\alpha$ the element $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, where $\sum \alpha_j \leq n$. Let $t_1, \ldots, t_k$ be commuting independent variables, we set as usual $t^\alpha := \prod_i t_i^{\alpha_i}$. We define elements $e_\alpha(f) \in A_R(n, m)^{S_n}$ by

$$\sum_\alpha t^\alpha e_\alpha(f) := \det \left( 1 + \sum_h t_h \rho_n(f_h) \right) = \prod_{i=1}^n \left( 1 + \sum_h t_h f_i(h) \right).$$

**Example 1.1.** — Let $n = 3$ and $f, g \in A_R(m)$ then

$$e_{(2,1)}(f, g) = f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3).$$

If $n = 4$ then

$$e_{(2,1)}(f, g) = f(1)f(2)g(3) + f(1)g(2)f(3) + g(1)f(2)f(3) + f(1)f(2)g(4) + f(1)g(2)f(4) + g(1)f(2)f(4) + f(1)f(3)g(4) + f(1)g(3)f(4) + g(1)f(3)f(4) + f(2)f(3)g(4) + f(2)g(3)f(4) + g(2)f(3)f(4)$$

Let $k = m$ and $f_j = y_j$ for $j = 1, \ldots, m$, then the $e_\alpha(y) = e_{(\alpha_1, \ldots, \alpha_m)}(y_1, \ldots, y_m)$ where $\sum \alpha_j \leq n$ are the well–known elementary.


multisymmetric functions. These generate $A_R(n, m)S_n$ when $R \supset \mathbb{Q}$ (see [G] or [W]), and satisfy

$$
\sum_{\alpha} t^\alpha e_\alpha(y) = \det \left( 1 + \sum_j t_j \rho_n (y_j) \right) = \prod_{i=1}^{n} \left( 1 + \sum_{j=1}^{m} t_j x_j (i) \right).
$$

Lemma 1.2. — The multisymmetric function $e_{(\alpha_1, \ldots, \alpha_k)}(f_1, \ldots, f_k)$ is the orbit sum (under the considered action of $S_n$) of

$$
f_1(1)f_1(2) \cdots f_1(\alpha_1)f_2(\alpha_1+1) \cdots f_2(\alpha_1+\alpha_2) \cdots f_k(\sum_h \alpha_h).
$$

Proof. — Let $E$ be the set of mappings $\phi : \{1, \ldots, n\} \to \{1, \ldots, k+1\}$. We define a mapping $\phi \mapsto \phi^*$ of $E$ into $\mathbb{N}^{k+1}$ by putting $\phi^*(i)$ equal to the cardinality of $\phi^{-1}(i)$. For two elements $\phi_1, \phi_2$ of $E$, to satisfy $\phi_1^* = \phi_2^*$ it is necessary and sufficient that there should exist $\sigma \in S_n$ such that $\phi_2 = \phi_1 \circ \sigma$. Set $f_{k+1} := 1_R$ and $E(\alpha) := \{ \phi \in E \mid \phi^* = (\alpha_1, \ldots, \alpha_k, n - \sum_i \alpha_i) \}$, then we have

$$
e_\alpha(f) = \sum_{\phi \in E(\alpha)} f_{\phi(1)}(1)f_{\phi(2)}(2) \cdots f_{\phi(n)}(n)
$$

and the lemma is proved. \hfill \Box

It is clear that $e_{(\alpha_1, \ldots, \alpha_k)}(f_1, \ldots, f_k) = e_{(\alpha_1, \ldots, \alpha_k)}(f_{\tau(1)}, \ldots, f_{\tau(k)})$ for all $\tau \in S_k$. If two entries are equal, say $f_1 = f_2$, then, by (1.1)

$$
e_{(\alpha_1, \ldots, \alpha_k)}(f_1, \ldots, f_k) = \frac{(\alpha_1 + \alpha_2)!}{\alpha_1! \alpha_2!} e_{(\alpha_1 + \alpha_2, \ldots, \alpha_k)}(f_1, f_3 \ldots, f_k).
$$

Let $\mathbb{N}(\mathcal{M}_m^+)\to \mathbb{N}$ with finite support. We set

$$
|\alpha| := \sum_{\mu \in \mathcal{M}_m^+} \alpha(\mu).
$$

Let $\alpha \in \mathbb{N}(\mathcal{M}_m^+)$, then there exist $k \in \mathbb{N}$ and $\mu_1, \ldots, \mu_k \in \mathcal{M}_m^+$ such that $\alpha(\mu_i) = \alpha_i \neq 0$ for $i = 1, \ldots, k$ and $\alpha(\mu) = 0$ when $\mu \neq \mu_1, \ldots, \mu_k$. We set

$$
e_\alpha := e_{(\alpha_1, \ldots, \alpha_k)}(\mu_1, \ldots, \mu_k),
$$

i.e. we substitute $(\mu_1, \ldots, \mu_k)$ to variables in the elementary multisymmetric function $e_{(\alpha_1, \ldots, \alpha_k)}(y_1, \ldots, y_k)$. 

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Then

\[(1.7) \quad \sum_{|\alpha| \leq n} t^{\alpha} e_{\alpha} = \prod_{i=1}^{n} \left(1 + \sum_{\mu \in \mathcal{M}_m^+} t_{\mu}(i)\right),\]

where \(t_{\mu}\) are commuting independent variables indexed by monomials and

\[(1.8) \quad t^{\alpha} := \prod_{\mu \in \mathcal{M}_m^+} t_{\mu}^{\alpha(\mu)}\]

for all \(\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}.\)

If \(\alpha \in \mathbb{N}^{(\mathcal{M}_m^+)}\) is such that \(\alpha(\mu) = k\) for some \(\mu \in \mathcal{M}_m^+\) and \(\alpha(\nu) = 0\) for all \(\nu \in \mathcal{M}_m^+\) with \(\nu \neq \mu\), we see that \(e_{\alpha} = e_k(\mu)\), the \(k\)-th elementary symmetric function evaluated at \((\mu(1), \mu(2), \ldots, \mu(n))\).

**Lemma 1.3.** — Given a monomial \(\mu \in A_R(n, m)\), there exist \(\mu_1, \ldots, \mu_n \in A_R(m)\) such that \(\mu = \mu_1(1) \cdots \mu_n(n)\).

**Proof.** — Let \(\mu = \prod_{ij} x_i(j)^{a_{ij}}\) then \(\mu_j = \prod_i y_i^{a_{ij}}\) for \(j = 1, \ldots, n\). \(\square\)

**Proposition 1.4.** — The set

\[\mathcal{B}_{n,m,R} := \{e_{\alpha} : |\alpha| \leq n\}\]

is a \(R\)-basis of \(A_R(n, m)^{S_n}\).

The set

\[\mathcal{B}_{n,m,a,R} := \{e_{\alpha} : |\alpha| \leq n \text{ and } \partial(e_{\alpha}) = a\}\]

is a \(R\)-basis of \(A_R(n, m, a)^{S_n}\), for all \(a \in \mathbb{N}^m\).

**Proof.** — By Lemma 1.2 and (1.6), the \(e_{\alpha}\) are a complete system of representatives (for the action of \(S_n\)) of the orbit sums of the products

\[\{\mu_1(1)\mu_2(2) \cdots \mu_n(n) : \mu_i \in \mathcal{M}_m, i = 1, \ldots, n\}\]

So the first statement follows by Lemma 1.3.

Notice that \(\partial(e_{\alpha}) = \sum_{\mu \in \mathcal{M}_m^+} \alpha_{\mu} \partial(\mu)\) to prove the second statement. \(\square\)
2. Generators.

Let us calculate the product between two elements \( e_\alpha, e_\beta \in B_{n,m,R} \) of the basis \( B_{n,m,R} \).

**Theorem 2.1** (Product Formula). — Let \( k, h \in \mathbb{N}, f_1, \ldots, f_k, g_1, \ldots, g_h \in A_R(m) \) and \( t_1, \ldots, t_k, s_1, \ldots, s_h \) be commuting independent variables. Set as in (1.1)

\[
e_\alpha(f) := e_{(\alpha_1, \ldots, \alpha_k)}(f_1, \ldots, f_k) \quad \text{and} \quad e_\beta(g) := e_{(\beta_1, \ldots, \beta_h)}(g_1, \ldots, g_h).
\]

Then

\[
e_\alpha(f) e_\beta(g) = \sum_\gamma e_\gamma(f, g, fg),
\]

where \( fg := (f_1 g_1, f_1 g_2, \ldots, f_k g_h) \) and \( \gamma := (\gamma_{10}, \ldots, \gamma_{k0}, \gamma_{01}, \ldots, \gamma_{0h}, \gamma_{11}, \ldots, \gamma_{kh}) \) are such that

\[
\begin{cases}
\gamma_{ij} \in \mathbb{N} \\
|\gamma| \leq n \\
\sum_{j=0}^{h} \gamma_{ij} = \alpha_i \quad \text{for} \quad i = 1, \ldots, k \\
\sum_{i=0}^{k} \gamma_{ij} = \beta_j \quad \text{for} \quad j = 1, \ldots, h.
\end{cases}
\]

**Proof.** — The result follows from

\[
\left( \sum_{\alpha_i \leq n} \prod_{j=1}^{k} t_j^{\alpha_j} e_\alpha(f) \right) \left( \sum_{\beta_l \leq n} \prod_{l=1}^{h} s_l^{\beta_l} e_\beta(g) \right) = \left( \sum_{\alpha} t^\alpha e_\alpha(f) \right) \left( \sum_{\beta} s^\beta e_\beta(g) \right)
\]

\[
= \prod_{i=1}^{n} \left( 1 + \sum_{j=1}^{k} t_j f_j(i) \right) \prod_{i=1}^{n} \left( 1 + \sum_{l=1}^{h} s_l g_l(i) \right) = \prod_{i=1}^{n} \left( 1 + \sum_{j=1}^{k} t_j f_j(i) + \sum_{l=1}^{h} s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i) \right).
\]

Introduce the new variables \( u_{jl} \) with \( j = 1, \ldots, k \) and \( l = 1, \ldots, h \), then

\[
\prod_{i=1}^{n} \left( 1 + \sum_{j=1}^{k} t_j f_j(i) + \sum_{l=1}^{h} s_l g_l(i) + \sum_{j,l} t_j s_l f_j(i) g_l(i) \right) = \prod_{i=1}^{n} \left( 1 + \sum_{j=1}^{k} t_j f_j(i) + \sum_{l=1}^{h} s_l g_l(i) + \sum_{j,l} u_{jl}(i) g_l(i) \right) = \sum_\gamma v^\gamma e_\gamma(f, g, fg)
\]
where \( v \) is the cumulative variable \( t, s, u \). Then substitute \( u_{jl} = t_j s_l \) to obtain

\[
\sum_{\gamma} v^\gamma e_\gamma(f, g, fg) = \sum_{\gamma} \left( \prod_{a=1}^{k} t_{a}^{\gamma_{a0}} \prod_{b=1}^{h} s_{b}^{\gamma_{b0}} \prod_{c=1}^{k} \prod_{d=1}^{h} (t_{a} s_{b})^{\gamma_{ad}} e_{\gamma}(f, g, fg) \right),
\]

where \( fg = (f_1 g_1, f_1 g_2, \ldots, f_k g_1, \ldots, f_k g_h) \) and \( \gamma \) satisfy the condition of the theorem.

**Example 2.2.** — Let us calculate in \( A_R(2, 3)^{S_2} \)

\[
e_{(1,1)}(a, b) e_2(c) = \sum_{0 \leq k, h \leq 1} e_{(1-k, 1-h, 2-k-h, h, k)}(a, b, c) = e_{(1,1)}(ac, bc),
\]

since \( 1 - k + 1 - h + 2 - k - h + h + k = 4 - k - h \leq 2 \).

**Corollary 2.3.** — Let \( k \in \mathbb{N}, a_1, \ldots, a_k \in A_R(m), \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) with \( \sum \alpha_j \leq n \). Then \( e_{(\alpha_1, \ldots, \alpha_k)}(a_1, \ldots, a_k) \) belongs to the subring of \( A_R(n, m)^{S_n} \) generated by the \( e_i(\mu) \), where \( i = 1, \ldots, n \) and \( \mu \) is a monomial in the \( a_1, \ldots, a_k \).

**Proof.** — We prove the claim by induction on \( \sum_j \alpha_j \) (notice that \( 1 \leq k \leq \sum_j \alpha_j \)) assuming that \( \alpha_i > 0 \) for all \( i \). If \( \sum_j \alpha_j = 1 \) then \( k = 1 \) and \( e_{(\alpha_1, \ldots, \alpha_k)}(a_1, \ldots, a_k) = e_1(a_1) \). Suppose the claim true for all \( e_{(\beta_1, \ldots, \beta_h)}(b_1, \ldots, b_h) \) with \( b_1, \ldots, b_h \in A_R(m) \) and \( \sum_i \beta_i < \sum_j \alpha_j \). Let \( k, a_1, \ldots, a_k, \alpha \) be as in the statement, then we have by Theorem 2.1

\[
e_{\alpha_1}(a_1) e_{(\alpha_2, \ldots, \alpha_k)}(a_2, \ldots, a_k) = e_{(\alpha_1, \ldots, \alpha_k)}(a_1, \ldots, a_k) + \sum_{\gamma} e_{\gamma}(a_1, \ldots, a_k, a_1 a_2, \ldots, a_1 a_k),
\]

where

\( \gamma = (\gamma_{10}, \gamma_{01}, \ldots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1h}) \)

with \( h = k - 1 \), \( \sum_{j=0}^{h} \gamma_{1j} = \alpha_1 \) with \( \sum_{j=1}^{h} \gamma_{1j} > 0 \), and \( \gamma_{0j} + \gamma_{1j} = \alpha_j \) for \( j = 1, \ldots, h \). Thus

\[
\gamma_{10} + \gamma_{01} + \ldots + \gamma_{0h} + \gamma_{11} + \ldots + \gamma_{1h} = \sum_{j} \alpha_j - \sum_{j=1}^{h} \gamma_{1j} < \sum_{j} \alpha_j.
\]

Hence

\[
e_{(\alpha_1, \ldots, \alpha_k)}(a_1, \ldots, a_k) = e_{\alpha_1}(a_1) e_{(\alpha_2, \ldots, \alpha_k)}(a_2, \ldots, a_k) - \sum_{\gamma} e_{\gamma}(a_1, \ldots, a_k, a_1 a_2, a_1 a_3, \ldots, a_1 a_k),
\]

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where \( \sum_{r,s} \gamma_{rs} < \sum_{j} \alpha_j \). So the claim follows by induction hypothesis.

**Example 2.4.** — Consider \( e_{(2,1)}(a,b) \) in \( A_{R}(3,m) \) as in Example 1.2, then
\[
e_{(2,1)}(a,b) = e_2(a)e_1(b) - e_{(1,1)}(a,ab) = e_2(a)e_1(b) - e_1(a)e_1(ab) + e_1(a^2b).
\]

We now recall some basic facts about classical symmetric functions, for further reading on this topic see [M].

We have another distinguished kind of functions in \( \Lambda_R \) beside the elementary symmetric ones: the power sums.

For any \( r \in \mathbb{N} \) the \( r \)-th power sum is
\[
p_r := \sum_{i \geq 1} x_i^r.
\]

Let \( g \in \Lambda_R \), set \( g \cdot p_r = g(x_1^r, x_2^r, \ldots, x_k^r, \ldots) \), this is again a symmetric function. Since the \( e_i \) generate \( \Lambda_R \) we have that \( g \cdot p_r \) can be expressed as a polynomial in the \( e_i \). In particular,
\[
P_{h,k} := e_h \cdot p_k
\]
is a polynomial in the \( e_i \).

**Proposition 2.5.** — For all \( f \in A_R(m) \), and \( k, h \in \mathbb{N} \), \( e_h(f^k) \) belongs to the subring of \( A_R(n,m)^{S_n} \) generated by the \( e_j(f) \).

**Proof.** — Let \( f \in A_R(m) \) and consider \( e_h(f^k) \in A_R(n,m)^{S_n} \), we have (see Introduction)
\[
e_h(f^k) = e_h(f(1)^k, \ldots, f(n)^k) = P_{h,k}(e_1(f(1)), \ldots, f(n)), \ldots, e_n(f(1), \ldots, f(n)))
\]
and the result is proved.

We are now ready to prove Theorem 1 stated in the introduction.

**Proof of Theorem 1.** — Recall that a monomial \( \mu \in \mathcal{M}_m^+ \) is called **primitive** if it is not a power of another one and we denote by \( \mathfrak{M}_m^+ \) the set of primitive monomials. The elements \( e_\alpha \in \mathcal{B}_{n,m,R} \), that form a \( R \)-basis by Proposition 1.4, can be expressed as polynomials in \( e_i(\mu) \) with \( i = 1, \ldots, n \) and \( \mu \in \mathcal{M}_m^+ \), by Corollary 2.3. If \( \mu = \nu^k \) with \( \nu \in \mathfrak{M}_m^+ \), then \( e_i(\mu) \) can be expressed as a polynomial in the \( e_j(\nu) \), by Proposition 2.5. Since for all \( \mu \in \mathcal{M}_m^+ \) there exist \( k \in \mathbb{N} \) and \( \nu \in \mathfrak{M}_m^+ \) such that \( \mu = \nu^k \), we have that
$A(n, m)^{S_n}$ is generated as a commutative ring by the $e_j(\nu)$, where $\nu \in M^+_m$ and $j = 1, \ldots, n$.

The theorem then follows by the following result due to Fleischmann [F]: the ring $A_R(n, m)^{S_n}$ is generated by elements of total degree $\ell \leq m(n - 1)$, for any commutative ring $R$, with sharp bound if $n = p^s$ a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$. If $R \supset \mathbb{Q}$ then the result follows from Newton’s Formulas and a well-known result of H. Weyl (see [G], [W]).

3. Relations.

We write a generating series for the orbits of monomials

$$G_n(t) := \prod_{i=1}^{n} \left(1 + \sum_{\mu(i) \in M^+_m} t^\mu(i)\right) = \sum_{\alpha, |\alpha| \leq n} t^\alpha e_\alpha(n),$$

where $\alpha \in \mathbb{N}^{(M^+_m)}$ and $t^\alpha e_\alpha(n) = 0$ when $\alpha = 0$.

Recall the map $\pi_n : A_R(n, m) \rightarrow A_R(n - 1, m)$ defined by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leq n - 1 \end{cases} \text{ for all } i.$$ 

Then we have of course that $\pi_n(G_n(t)) = G_{n-1}(t)$, so that

$$\pi_n((e_\alpha)) = \begin{cases} e_\alpha & \text{if } |\alpha| < n \\ 0 & \text{otherwise}. \end{cases}$$

Thus, by Proposition 1.4, for all $a \in \mathbb{N}^m$ the restriction

$$\pi_{n,a} : A_R(n, m, a) \rightarrow A_R(n - 1, m, a)$$

is such that

$$\pi_{n,a}(A_R(n, m, a)^{S_n}) = A_R(n - 1, m, a)^{S_{n-1}}$$

and then $(A_R(n, m, a)^{S_n}, \pi_{n,a})$ is a projective system.

For any $a \in \mathbb{N}^m$ set

$$A_R(\infty, m, a) := \lim_{\leftarrow} A_R(n, m, a)^{S_n},$$

where the projective limit is taken with respect to $n$ over the above projective system and set

$$\tilde{\pi}_{n,a} : A_R(\infty, m, a) \rightarrow A_R(n, m, a)^{S_n}$$
for the natural projection.

Set
\[
A_R(\infty, m) := \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a)
\]
and
\[
\tilde{\pi}_n := \bigoplus_{a \in \mathbb{N}^m} \tilde{\pi}_{n,a}.
\]

Similarly to the classical case \((m = 1)\) and recalling (3.1), (3.3) we make an abuse of notation and set
\[
e_\alpha := \lim_{\leftarrow} e_\alpha(n),
\]
for any \(\alpha \in \mathbb{N}^{(M^+_m)}\). In the same way we set \(e_j(f) := \lim_{\leftarrow} e_j(f)\) with \(j \in \mathbb{N}\), where \(f \in A_R(m)^+\) is homogeneous of positive multidegree, so that \(j \partial(f) = a\).

**Proposition 3.1.** — Let \(a \in \mathbb{N}^m\).

1. The \(R\)-module \(\ker \tilde{\pi}_{n,a}\) is the linear span of
   \[
   \{e_\alpha \in A_R(\infty, m, a) : |\alpha| > n\}.
   \]
2. The \(R\)-module homomorphisms \(\tilde{\pi}_{n,a} : A_R(\infty, m, a) \to A_R(n, m, a)^{S_n}\) are onto for all \(n \in \mathbb{N}\) and \(A_R(\infty, m, a) \cong A_R(n, m, a)^{S_n}\) for all \(n \geq |a|\).
3. The \(R\)-module \(A_R(\infty, m, a)\) is free with basis
   \[
   \{e_\alpha : \partial(e_\alpha) = a\},
   \]
4. The \(R\)-module \(A_R(\infty, m)\) is free with basis
   \[
   \{e_\alpha : \alpha \in \mathbb{N}^{(M^+_m)}\}.
   \]

**Proof.** — (1) By (3.3) and (3.5), for all \(a \in \mathbb{N}^m\), the following is a split exact sequence of \(R\)-modules
\[
0 \to \ker \pi_{n,a} \to A(n, m, a)^{S_n} \xrightarrow{\pi_{n,a}} A(n - 1, m, a)^{S_{n-1}} \to 0,
\]
and the claim follows.

(2) If \(\sum_{j=1}^m a_j < n\), then \(\ker \tilde{\pi}_{n,a} = 0\), indeed
\[
\partial(e_\alpha) = \sum_{\mu \in M^+_m} a_\mu \partial(\mu) = a \implies |\alpha| \leq \sum_{j=1}^m a_j < n.
\]
Hence \( A(h, m, a)^{S_h} \cong A(b, m, a)^{S_b} \) where \( b := \sum_{j=1}^{m} a_j \), for all \( h \geq \sum_{j=1}^{m} a_j \) and the claim follows by (3.5).

(3) follows from (1) and (2).

(4) follows from (3) and (3.8) \( \Box \)

**Remark 3.2.** — Notice that \( A_R(m)^{\otimes n} \cong A_R(n, m) \) as multigraded \( S_n \)-algebras by means of

\[
(3.10) \quad f_1 \otimes \cdots \otimes f_n \leftrightarrow f_1(1)f_2(2)\cdots f_n(n)
\]

for all \( f_1, \ldots, f_n \in A_R(m) \). Hence \( A_R(n, m)^{S_n} \cong TS^n(A_R(m)) \), where \( TS^n(-) \) denotes the symmetric tensors functor. Since \( TS^n(A_R(m)) \cong R \otimes TS^n(A_Z(m)) \) (see [B]), we have

\[
(3.11) \quad A_R(n, m)^{S_n} \cong R \otimes A_Z(n, m)^{S_n}
\]

for any commutative ring \( R \).

We then work with \( R = Z \) and we suppress the \( Z \) subscript for the sake of simplicity.

**Remark 3.3.** — The \( Z \)-module \( A(\infty, m) \) can be endowed with a structure of \( \mathbb{N}^m \)-graded ring such that the \( \pi_n \) are \( \mathbb{N}^m \)-graded ring homomorphisms: the product \( e_\alpha e_\beta \), where \( \alpha, \beta \in \mathbb{N}(M_m^+) \), is defined by using the product formula of Theorem 2.1 with no upper bound on \( |\gamma| \), where \( \gamma \) appears in the summation.

**Proposition 3.4.** — Consider the free polynomial ring

\[
C(m) := \bigoplus_{a \in \mathbb{N}^m} C(m, a) := \mathbb{Z}[e_{i,\mu}]_{i \in \mathbb{N}, \mu \in M_m^+}
\]

with multidegree given by \( \partial(e_{i,\mu}) = \partial(\mu)i \).

Then the multigraded ring homomorphism

\[
\sigma_m : \mathbb{Z}[e_{i,\mu}]_{i \in \mathbb{N}, \mu \in M_m^+} \rightarrow A(\infty, m)
\]

given by

\[
\sigma_m : e_{i,\mu} \mapsto e_i(\mu), \text{ for all } i \in \mathbb{N}, \mu \in M_m^+
\]

is an isomorphism, i.e. \( A(\infty, m) \) is freely generated as a commutative ring by the \( e_i(\mu) \), where \( i \in \mathbb{N} \) and \( \mu \in M_m^+ \).

**Proof.** — Since we defined the product in \( A(\infty, m) \) as in Theorem 2.1, it is easy to verify, repeating the reasoning of the previous section,
that $A(\infty, m)$ is generated as a commutative ring by the $e_i(\mu)$, where $i \in \mathbb{N}$, $\mu \in \mathbb{M}_m^+$. Hence $\sigma_m$ is onto for all $m \in \mathbb{N}$.

Let $a \in \mathbb{N}^m$ and consider the restriction $\sigma_{m,a} : C(m, a) \rightarrow A(\infty, m, a)$. It is onto as we have just seen. A $\mathbb{Z}$-basis of $C(m, a)$ is
\[
\left\{ \prod_{i \in \mathbb{N}, \mu \in \mathbb{M}_m^+} e_{i,\mu} : \sum_{i \in \mathbb{N}, \mu \in \mathbb{M}_m^+} i k \partial(\mu) = a \right\}.
\]

On the other hand, a $\mathbb{Z}$-basis of $A(\infty, m, a)$ is
\[
\left\{ e_{\alpha} : \sum_{\alpha \in \mathbb{N}, \mu \in \mathbb{M}_m^+} \alpha \mu \partial(\mu) = a \right\}.
\]

Let $\mu \in \mathbb{M}_m^+$, then there are an unique $k \in \mathbb{N}$ and an unique $\nu \in \mathbb{M}_m^+$ such that $\mu = \nu^k$. Hence
\[
\sum_{\alpha \in \mathbb{N}, \mu \in \mathbb{M}_m^+} \alpha \mu \partial(\mu) = \sum_{k \in \mathbb{N}, \alpha \in \mathbb{N}, \nu \in \mathbb{M}_m^+} \alpha_k \partial(\nu),
\]
so that $C(m, a)$ and $A(\infty, m, a)$ have the same (finite) $\mathbb{Z}$-rank and thus are isomorphic via $\sigma_{m,a}$. \qed

**Corollary 3.5.** — Let $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is a polynomial ring freely generated by the $e_1(\mu)$, where $\mu \in \mathbb{M}_m^+$.

**Proof.** — By Proposition 3.4 and Theorem 1. \qed

**Proof of Theorem 2.** — (1) As before we set $R = \mathbb{Z}$ and the result follows by Remark 3.2, Proposition 3.4. and Proposition 3.1.

(2) By Proposition 3.1 the kernel of
\[
A(\infty, m) \xrightarrow{\pi_n} A(n, m)^{S_n}
\]
has basis $\{ e_\alpha : | \alpha | > n \}$. Let $V_k$ be the submodule of $A(\infty, m)$ with basis $\{ e_\alpha : | \alpha | = k \}$. Let $A_k$ be the sub-$\mathbb{Z}$-module of $\mathbb{Q} \otimes V_k$ generated by the $e_k(f)$ with $f \in A(m)^+$. Let $g : \mathbb{Q} \otimes V_k \rightarrow \mathbb{Q}$ be a linear form identically zero on $A_k$. Then
\[
0 = g(e_k(f)) = g\left( e_k \left( \sum_{\mu \in \mathbb{M}_m^+} \lambda_{\mu} \mu \right) \right) = \left( \sum_{|\alpha| = k} \left( \prod_{\mu \in \mathbb{M}_m^+} \lambda_{\mu}^{\alpha_{\mu}} \right) g(e_\alpha) \right),
\]
for all $\sum_{\mu \in \mathbb{M}_m^+} \lambda_{\mu} \mu \in A(m)^+$. Hence $g(e_\alpha) = 0$ for all $e_\alpha$ with $| \alpha | = k$; thus $g = 0$. If $R \supset \mathbb{Q}$ the result then follows from Newton’s formulas and Corollary 3.5. \qed
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