ANNALES

DE

L'INSTITUT FOURIER

Taku ISHII

A remark on Whittaker functions on SL$(n, \mathbb{R})$

Tome 55, nº 2 (2005), p. 483-492.

<http://aif.cedram.org/item?id=AIF_2005__55_2_483_0>

Ann. Inst. Fourier, Grenoble
55, 2 (2005), 483–492

A REMARK ON WHITTAKER FUNCTIONS ON $\mathrm{SL}(n,\mathbb{R})$

by Taku ISHII

1. Preliminaries.

We first recall the definition and some basic facts on class one Whittaker functions on $G = \mathrm{SL}(n,\mathbb{R})$. Let N be the subgroup of G consisting of upper triangular unipotent matrices. Fix a nondegenerate unitary character η of N by

$$\eta(n) = \exp\Big(2\pi\sqrt{-1}\sum_{i=1}^{n-1} n_{i,i+1}\Big)$$

for $n = (n_{i,j}) \in N$. Also let $A = \{\mathrm{diag}(a_1,\dots,a_n) \mid a_i > 0,\ \prod_{i=1}^n a_i = 1\}$ and $K = \mathrm{SO}(n)$. Then the Iwasawa decomposition $G = NAK$ holds. We denote by $\mathfrak{g},\mathfrak{n},\mathfrak{a}$ and $\mathfrak{k}$ the Lie algebras of G, N, A and K respectively.

Let $P_0 = MAN$ be the standard minimal parabolic subgroup of G with $M = Z_K(A)$. For a linear form $\nu = (\nu_1,\dots,\nu_n) \in \mathfrak{a}_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a},\mathbb{C}) \cong \mathbb{C}^{n-1}$ ($\sum_{i=1}^n \nu_i = 0$), define a character e^ν of A by $e^\nu(\mathrm{diag}(a_1,\dots,a_n)) = \exp(\nu_1(\log a_1) + \cdots + \nu_n(\log a_n))$. We call the induced representation

$$\pi_\nu = L^2 - \mathrm{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$$

the *class one principal series representation* of G. Here $\rho = (n-1, n-2,\dots,0)$ is the half sum of positive roots.

Let $U(\mathfrak{g}_{\mathbb{C}})$ and $U(\mathfrak{a}_{\mathbb{C}})$ be the universal enveloping algebras of the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. Let p be the projection $U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{a}_{\mathbb{C}})$

Keywords: Whittaker functions, automorphic forms.
Math. classification: 11F55, 33C20.

corresponding to the decomposition $U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{a}_{\mathbb{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k})$. Set

$$U(\mathfrak{g}_{\mathbb{C}})^K = \{X \in U(\mathfrak{g}_{\mathbb{C}}) \mid \mathrm{Ad}(k)X = X \text{ for all } k \in K\}.$$

Define the automorphism γ of $U(\mathfrak{a}_{\mathbb{C}})$ by $\gamma(H) = H + \rho(H)$ for $H \in \mathfrak{a}_{\mathbb{C}}$. Then $\gamma \circ p$ induces an algebra homomorphism from $U(\mathfrak{g}_{\mathbb{C}})^K$ to $U(\mathfrak{a}_{\mathbb{C}})$ with the kernel $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}$. For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, define the algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbb{C}})^K \to \mathbb{C}$ by

$$\chi_\nu(z) = \nu(\gamma \circ p(z))$$

for $z \in U(\mathfrak{g}_{\mathbb{C}})^K$. Note that χ_ν is trivial on $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}$ and the restriction of χ_ν to the center $Z(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$ coincides with the infinitesimal character of the class one principal series representation π_ν.

Definition 1. — *A smooth function f on G is called class one Whittaker function if*

(i) $f(ngk) = \eta(n)f(g)$ *for all* $n \in N$, $g \in G$ *and* $k \in K$,

(ii) $zf = \chi_\nu(z)f$ *for all* $z \in U(\mathfrak{g}_{\mathbb{C}})^K$.

We denote by $\mathrm{Wh}(\nu)$ *the space of class one Whittaker functions.*

It is well known ([5]) that for almost all ν, the dimension of $\mathrm{Wh}(\nu)$ is equal to the order of the Weyl group $W \cong \mathfrak{S}_n$ of G. Hashizume ([3]) constructed the basis function $M_n(\nu; g)$ of $\mathrm{Wh}(\nu)$, which will be explained below. Shalika ([7]) proved the uniqueness of the element in $\mathrm{Wh}(\nu)$ which has the moderate growth property and contributes to the Fourier expansions of automorphic forms. Further, this unique element can be written as Jacquet integral ([4]):

$$W_n(\nu; g) = \int_N a(s_0^{-1}ng)^{\nu+\rho}\eta(n)^{-1}dn.$$

Here s_0 is the longest element in the Weyl group W and $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. In case of $G = \mathrm{SL}(n, \mathbb{R})$, Stade ([9]) found a remarkable recursive integral formula of $W_n(\nu; g)$ involving K-Bessel functions by evaluating the above integral and applied it for the computation of the gamma factors of certain automorphic L-functions.

Since $W_n(\nu; g)$ can be explicitly written by the linear combination of $M_n(\nu; g)$'s ([3, Theorem 7.8]), the basis function is related to the theory of automorphic forms. On the other hand, one of the possible direct applications of the explicit formula of $M_n(\nu; g)$ is the construction of Poincaré series (*cf.* [6], [8]).

Let us review the construction of $M_n(\nu;g)$. For $\mathbf{k}=(k_1,\dots,k_{n-1})\in(\mathbb{Z}_{\geqslant 0})^{n-1}$, define the function $G_{n,\mathbf{k}}(\nu)$ by the following recurrence relation.

$$G_{n,(0,\dots,0)}(\nu)=1,$$
$$\left[\sum_{i=1}^{n-1}k_i^2-\sum_{i=1}^{n-2}k_ik_{i+1}+\frac{1}{2}\sum_{i=1}^{n-1}(\nu_i-\nu_{i+1})k_i\right]G_{n,\mathbf{k}}(\nu)=\sum_{i=1}^{n-1}G_{n,\mathbf{k}-\mathbf{e}_i}(\nu). \tag{1}$$

Here $\mathbf{e}_i=(0,\dots,1,\dots,0)$ (1 is at i-th entry) and $G_{n,\mathbf{k}}(\nu)=0$ if $k_i<0$ for some i. Let

$$'\mathfrak{a}_{\mathbb{C}}^*=\left\{\nu\in\mathfrak{a}_{\mathbb{C}}^*\mid\sum_{i=1}^{n-1}k_i^2-\sum_{i=1}^{n-2}k_ik_{i+1}+\frac{1}{2}\sum_{i=1}^{n-1}(\nu_i-\nu_{i+1})k_i\neq 0\right.$$
$$\left.\text{for all }\mathbf{k}\in(\mathbb{Z}_{\geqslant 0})^{n-1}\backslash\{0\}\right\}.$$

Then if $\nu\in{}'\mathfrak{a}_{\mathbb{C}}^*$, we can determine $G_{n,\mathbf{k}}(\nu)$ inductively and define a series on A by

$$M_n(\nu;a)=a^{\nu+\rho}\sum_{\mathbf{k}\in(\mathbb{Z}_{\geqslant 0})^{n-1}}G_{n,\mathbf{k}}(\nu)\left(\pi\frac{a_1}{a_2}\right)^{2k_1}\cdots\left(\pi\frac{a_{n-1}}{a_n}\right)^{2k_{n-1}}$$

($a=\mathrm{diag}(a_1,\dots,a_n)\in A$). Here $a^{\nu+\rho}=\prod_{i=1}^n a_i^{\nu_i+(n-i)}$. We extend it to a function on G by

$$M_n(\nu;g)=\eta(n(g))M_n(\nu;a(g)).$$

Theorem 2 ([3]). — *Let Ω_n be the set of $\nu\in\mathfrak{a}_{\mathbb{C}}^*$ satisfying the conditions*

(i) *$s\nu\in{}'\mathfrak{a}_{\mathbb{C}}^*$ for all $s\in W$,*

(ii) *$s\nu-t\nu\notin(\mathbb{Z}_{\geqslant 0})^n$ for all distinct pair s,t in W. If $\nu\in\Omega_n$, then the set $\{M_n(s\nu;g)\mid s\in W\}$ spans* $\mathrm{Wh}(\nu)$.

The explicit formulas of $G_{n,\mathbf{k}}(\nu)$ are obtained by Bump ([2]) for $n=3$ and Stade ([10]) for $n=4$. By using them, Stade ([11]) found integral formulas for $M_3(\nu;a)$ and $M_4(\nu;a)$. He also conjectured a recursive integral formula for general n. The proof of Stade's conjecture is the main result of this paper.

Theorem 3 (Stade's Conjecture). — *Let us use the coordinate $y=(y_1,\dots,y_{n-1})$ with $y_i=a_i/a_{i+1}$ for A and put*

$$\mu_i=\nu_{i+1}+\frac{\nu_1+\nu_n}{n-2} \tag{$\natural$}$$

for $1 \leqslant i \leqslant n-2$ *and* $\mu = (\mu_1, \ldots, \mu_{n-2})$. *Formally define* $u_0 = 1/u_{n-1} = 0$. *If* $\nu \in \Omega_n$, $\mu \in \Omega_{n-2}$ *and* $\mathrm{Re}((\nu_1 - \nu_n)/2 + 1) > 0$, *then*

$$M_n(\nu; a) = \pi^{\sum_{i=1}^{n-2}(-n+i)\nu_i} \prod_{i=1}^{n-1} \Gamma\Big(\frac{\nu_1 - \nu_{i+1}}{2} + 1\Big)\Gamma\Big(\frac{\nu_{i+1} - \nu_n}{2} + 1\Big)$$
$$\cdot \prod_{i=1}^{n-1} y_i^{(n/2-i)(\nu_1+\nu_n)/(n-2)+(n-1)/2} \frac{1}{(2\pi\sqrt{-1})^{n-2}} \int_{|u_1|=1} \cdots$$
$$\cdots \int_{|u_{n-2}|=1} \prod_{i=1}^{n-1} I_{(\nu_1-\nu_n)/2}\Big(2\pi y_i \sqrt{(1+u_{i-1})(1+1/u_i)}\Big)$$
$$\cdot M_{n-2}\Big(\mu; \big(y_2\sqrt{u_1/u_2}, \ldots, y_{n-2}\sqrt{u_{n-3}/u_{n-2}}\,\big)\Big)$$
$$\cdot \prod_{i=1}^{n-2} u_i^{(-n+2i+1)/4-n(\nu_1+\nu_n)/4(n-2)} \frac{du_i}{u_i},$$

(each integral is taken counterclockwise in the complex plane), with I_ν *the modified Bessel function.*

2. Relation between $G_{n,\mathbf{k}}(\nu)$ and $G_{n-2,\mathbf{k}}(\nu)$.

Theorem 4. — *Let* $\mathbf{m} = (m_1, \ldots, m_{n-3}) \in (\mathbb{Z}_{\geqslant 0})^{n-3}$. *For* $\mathbf{k} = (k_1, \ldots, k_{n-1}) \in (\mathbb{Z}_{\geqslant 0})^{n-1}$ *set*

$$S(\mathbf{k}) = \{\mathbf{m} \in (\mathbb{Z}_{\geqslant 0})^{n-3} \mid 0 \leqslant m_1 \leqslant k_2, \ldots, 0 \leqslant m_{n-3} \leqslant k_{n-2}\}.$$

Then for $\nu \in \Omega_n$ *and* $\mu \in \Omega_{n-2}$ *satisfying* ($\natural$),

$$G_{n,\mathbf{k}}(\nu) = P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m} \in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu),$$

where

$$P_{n,\mathbf{k}}(\nu) = \prod_{i=1}^{n-1} \frac{(\frac{\nu_1-\nu_n}{2}+1)_{k_i+k_{i+1}}}{(\frac{\nu_1-\nu_n}{2}+1)_{k_i}(\frac{\nu_1-\nu_{i+1}}{2}+1)_{k_i}(\frac{\nu_{i+1}-\nu_n}{2}+1)_{k_{i+1}}}$$

and

$$Q_{n,\mathbf{k},\mathbf{m}}(\nu) = \prod_{i=1}^{n-1} \frac{(-1)^{m_{i-1}}(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1})_{m_{i-1}}(\frac{-\nu_1+\nu_n}{2}-k_i)_{m_{i-1}}(\frac{-\nu_1+\nu_{i+1}}{2}-k_i)_{m_i}}{(k_i-m_{i-1})!(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1})_{m_{i-1}+m_i}}.$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$. *Moreover, we set* $m_j = 0$ *for* $j \neq 1, \ldots, n-3$ *and* $k_j = 0$ *for* $j \neq 1, \ldots, n-1$.

Proof. — We have only to check that

$$F_{\mathbf{k}}(\nu) := P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu)$$

satisfies the recurrence relation (1). Let us compute $F_{\mathbf{k}-\mathbf{e}_i}(\nu)$ for $1 \leqslant i \leqslant n-1$. Since

$$\frac{P_{n,\mathbf{k}-\mathbf{e}_i}(\nu)}{P_{n,\mathbf{k}}(\nu)} = \frac{(\frac{\nu_1-\nu_n}{2}+k_i)(\frac{\nu_1-\nu_{i+1}}{2}+k_i)(\frac{\nu_i-\nu_n}{2}+k_i)}{(\frac{\nu_1-\nu_n}{2}+k_{i-1}+k_i)(\frac{\nu_1-\nu_n}{2}+k_i+k_{i+1})}$$

and

$$\begin{aligned}\frac{Q_{n,\mathbf{k}-\mathbf{e}_i,\mathbf{m}}(\nu)}{Q_{n,\mathbf{k},\mathbf{m}}(\nu)} &\\ = \frac{(k_i-m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i)(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1})}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i)} &\\ \cdot\frac{(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2})(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1})(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i)}{(\frac{-\nu_i+\nu_n}{2}-k_i)(\frac{-\nu_1+\nu_n}{2}-k_i)(\frac{-\nu_1+\nu_{i+1}}{2}-k_i)},\end{aligned}$$

we have

$$\begin{aligned}F_{\mathbf{k}-\mathbf{e}_i}(\nu) =& P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k}-\mathbf{e}_i)} \\ &\cdot G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \frac{(-1)(k_i-m_{i-1})(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2})}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1})} \\ &\cdot \frac{(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1})(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i)}{(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i)}.\end{aligned}$$

Suppose that $2 \leqslant i \leqslant n-2$. If we decompose this summation by means of

$$\begin{aligned}\Big(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2}\Big)\Big(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i\Big) &\\ = \Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big) &\\ + \Big\{\Big(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2}\Big)\Big(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i\Big) &\\ - \Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big)\Big\},\end{aligned}$$

$F_{\mathbf{k}-\mathbf{e}_i}(\nu)$ becomes the sum of following two terms:

$$\begin{aligned}P_{n,\mathbf{k}}(\nu) &\sum_{\mathbf{m}+\tilde{\mathbf{e}}_i\in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\ &\cdot \Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big)\end{aligned}$$

$$
\cdot \frac{(-1)(k_i-m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1})}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i)}
$$

$$
\begin{aligned}
&= P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}-\tilde{\mathbf{e}}_i}(\nu) \\
&\quad \cdot \Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}-1\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}-1\Big) \\
&\quad \cdot \frac{(-1)(k_i-m_{i-1}+1)(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1}-1)}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1}-1)(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i-1)} \\
&= P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu)
\end{aligned}
$$

($\tilde{\mathbf{e}}_i$ is $(n-3)$-dimensional row vector with 1 at $(i-1)$-th entry and 0 otherwise) and

$$
\begin{aligned}
&P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k}-\mathbf{e}_i)} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
&\quad \cdot \Big\{\Big(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2}\Big)\Big(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i\Big) \\
&\qquad -\Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big)\Big\} \\
&\quad \cdot \frac{(-1)(k_i-m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1})}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i)} \\
&= P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
&\quad \cdot \Big\{\Big(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2}\Big)\Big(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i\Big) \\
&\qquad -\Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big)\Big\} \\
&\quad \cdot \frac{(-1)(k_i-m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i+m_{i-1})}{(\frac{-\nu_1+\nu_n}{2}-k_{i-1}-k_i+m_{i-2}+m_{i-1})(\frac{-\nu_1+\nu_n}{2}-k_i-k_{i+1}+m_{i-1}+m_i)}.
\end{aligned}
$$

Thus $\sum_{i=1}^{n-1} F_{\mathbf{k}-\mathbf{e}_i}(\nu)$ is equal to

$$
\begin{aligned}
&P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} \Big[\sum_{i=2}^{n-2} G_{n-2,\mathbf{m}-\tilde{\mathbf{e}}_i}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) + \sum_{i=1}^{n-1} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \\
&\quad \cdot \Big\{\Big(\frac{-\nu_i+\nu_n}{2}-k_i+m_{i-2}\Big)\Big(\frac{-\nu_1+\nu_{i+1}}{2}-k_i+m_i\Big) \\
&\qquad -\Big(\frac{-\nu_1+\nu_i}{2}-k_{i-1}+m_{i-1}\Big)\Big(\frac{-\nu_{i+1}+\nu_n}{2}-k_{i+1}+m_{i-1}\Big)\Big\}
\end{aligned}
$$

$$\cdot \frac{(-1)(k_i - m_{i-1})(\frac{-\nu_1+\nu_n}{2} - k_i + m_{i-1})}{(\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1})(\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i)}\Bigg].$$

By using the recurrence relation of $G_{n-2,\mathbf{m}}(\mu)$ for $\sum_{i=2}^{n-2}$ and after some rearrangement for $\sum_{i=1}^{n-1}$, the sum $\sum_{i=1}^{n-1} F_{\mathbf{k}-\mathbf{e}_i}(\nu)$ becomes

$$\begin{aligned}
&P_{n,\mathbf{k}}(\nu) \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu) \Bigg[\sum_{i=1}^{n-3} m_i^2 - \sum_{i=1}^{n-4} m_i m_{i+1} \\
&\qquad + \frac{1}{2}\sum_{i=1}^{n-3} (\nu_{i+1} - \nu_{i+2}) m_i \\
&\qquad + \sum_{i=1}^{n-1} \Big\{ k_i^2 + \frac{1}{2}(\nu_i - \nu_{i+1}) k_i - m_{i-1}^2 - \frac{1}{2}(\nu_i - \nu_{i+1}) m_{i-1} \\
&\qquad + \frac{(\frac{\nu_i-\nu_n}{2} + k_i - m_{i-1}) k_{i-1} k_i + (\frac{-\nu_1+\nu_i}{2} - k_i + m_{i-1}) m_{i-2} m_{i-1}}{\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}} \\
&\qquad + \frac{\frac{1}{2}(-\nu_i + \nu_n) k_{i-1} m_{i-1} + \frac{1}{2}(\nu_1 - \nu_i) k_i m_{i-2}}{\frac{-\nu_1+\nu_n}{2} - k_{i-1} - k_i + m_{i-2} + m_{i-1}} \\
&\qquad + \frac{(\frac{\nu_1-\nu_{i+1}}{2} + k_i - m_{i-1}) k_i k_{i+1} + (\frac{-\nu_{i+1}+\nu_n}{2} - k_i + m_{i-1}) m_{i-1} m_i}{\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i} \\
&\qquad + \frac{\frac{1}{2}(\nu_{i+1} - \nu_n) k_i m_i + \frac{1}{2}(-\nu_1 + \nu_{i+1}) k_{i+1} m_{i-1}}{\frac{-\nu_1+\nu_n}{2} - k_i - k_{i+1} + m_{i-1} + m_i} \Big\} \Bigg] \\
&= \Big\{ \sum_{i=1}^{n-1} k_i^2 - \sum_{i=1}^{n-2} k_i k_{i+1} + \frac{1}{2}\sum_{i=1}^{n-1} (\nu_i - \nu_{i+1}) k_i \Big\} P_{n,\mathbf{k}}(\nu) \\
&\qquad \cdot \sum_{\mathbf{m}\in S(\mathbf{k})} G_{n-2,\mathbf{m}}(\mu) Q_{n,\mathbf{k},\mathbf{m}}(\nu).
\end{aligned}$$

This completes the proof. □

Remark. —Let $n=4$ in this theorem. Since $G_{2,\mathbf{k}}(\nu) = 1/(\frac{\nu_1-\nu_2}{2}+1)_{k_1}$,

$$\begin{aligned}
G_{4,\mathbf{k}}(\nu) = &\frac{1}{(\frac{\nu_1-\nu_4}{2}+1)_{k_1} (\frac{\nu_1-\nu_4}{2}+1)_{k_2} (\frac{\nu_1-\nu_4}{2}+1)_{k_3}} \\
&\cdot \frac{(\frac{\nu_1-\nu_4}{2}+1)_{k_1+k_2}}{(\frac{\nu_1-\nu_2}{2}+1)_{k_1} (\frac{\nu_2-\nu_4}{2}+1)_{k_2}} \frac{(\frac{\nu_1-\nu_4}{2}+1)_{k_2+k_3}}{(\frac{\nu_1-\nu_3}{2}+1)_{k_2} (\frac{\nu_3-\nu_4}{2}+1)_{k_3}} \\
&\cdot \frac{1}{k_1! k_2! k_3!}\, {}_4F_3\left(\begin{matrix} -k_2,\ \frac{-\nu_1+\nu_2}{2} - k_1,\ \frac{-\nu_1+\nu_4}{2} - k_2,\ \frac{-\nu_3+\nu_4}{2} - k_3 \\ \frac{\nu_2-\nu_3}{2} + 1,\ \frac{-\nu_1+\nu_4}{2} - k_1 - k_2,\ \frac{-\nu_1+\nu_4}{2} - k_2 - k_3 \end{matrix} \middle| 1 \right).
\end{aligned}$$

Here ${}_4F_3$ is the generalized hypergeometric series,

$${}_4F_3\left(\begin{matrix} a_1,\ a_2,\ a_3,\ a_4 \\ b_1,\ b_2,\ b_3 \end{matrix} \middle| z \right) = \sum_{n\geqslant 0} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{z^n}{n!}.$$

Stade ([10, Theorem 3.1]) found the similar formula including ${}_4F_3(1)$. The equivalence of these two formulas can be checked by using Saalschützian formula for ${}_4F_3(1)$ ([1, 7.2(1)]).

3. Proof of Stade's Conjecture.

In the same way as the proof for $n = 3, 4$, we use the following lemma.

LEMMA 5 ([11, Theorem 1]). — *If* $\mathrm{Re}(x+y-1) > 0$, *then*

$$\frac{\Gamma(x+y-1)}{\Gamma(x)\Gamma(y)} = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=1}(1+1/u)^{x-1}(1+u)^{y-1}\frac{du}{u}.$$

Here the path is taken counterclockwise in the complex plane.

If we denote $a^{\nu+\rho} = \prod_{i=1}^{n-1} y_i^{i(n-i)/2+\sum_{k=1}^{i}\nu_k}$ by $y^{\nu+\rho}$, Theorem 3 implies that

$$\begin{aligned}M_n(\nu;a) &= y^{\nu+\rho}\sum_{\mathbf{k}\in(\mathbb{Z}_{\geqslant 0})^{n-1}} G_{n,\mathbf{k}}(\nu)(\pi y_1)^{2k_1}\cdots(\pi y_{n-1})^{2k_{n-1}}\\ &= y^{\nu+\rho}\prod_{i=1}^{n-1}\Gamma\Big(\frac{\nu_1-\nu_{i+1}}{2}+1\Big)\Gamma\Big(\frac{\nu_{i+1}-\nu_n}{2}+1\Big)\\ &\quad\cdot\sum_{\mathbf{k}\in(\mathbb{Z}_{\geqslant 0})^{n-1}}(\pi y_1)^{2k_1}\cdots(\pi y_{n-1})^{2k_{n-1}}\sum_{\mathbf{m}\in S(\mathbf{k})}G_{n-2,\mathbf{m}}(\mu)\\ &\quad\cdot\prod_{i=1}^{n-1}\frac{1}{(k_i-m_{i-1})!\Gamma(\frac{\nu_1-\nu_n}{2}+k_i-m_{i-1}+1)}\\ &\quad\cdot\prod_{i=1}^{n-2}\frac{\Gamma(\frac{\nu_1-\nu_n}{2}+k_i+k_{i+1}-m_{i-1}-m_i+1)}{\Gamma(\frac{\nu_{i+1}-\nu_n}{2}+k_{i+1}-m_{i-1}+1)\Gamma(\frac{\nu_1-\nu_{i+1}}{2}+k_i-m_i+1)}\end{aligned}$$

If we put $k_i = m_{i-1}+l_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathbf{l} = (l_1,\ldots,l_{n-1})$, then $M_n(\nu;a)$ becomes $y^{\nu+\rho}\prod_{i=1}^{n-1}\Gamma(\frac{\nu_1-\nu_{i+1}}{2}+1)\Gamma(\frac{\nu_{i+1}-\nu_n}{2}+1)$ times

$$\sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geqslant 0})^{n-1}\\ \mathbf{m}\in(\mathbb{Z}_{\geqslant 0})^{n-3}}}(\pi y_1)^{2l_1}(\pi y_2)^{2(l_2+m_1)}\cdots(\pi y_{n-2})^{2(l_{n-2}+m_{n-3})}(\pi y_{n-1})^{2l_{n-1}}$$

$$(2)\quad \cdot G_{n-2,\mathbf{m}}(\mu)\prod_{i=1}^{n-1}\frac{1}{l_i!\Gamma(\frac{\nu_1-\nu_n}{2}+l_i+1)}$$
$$\cdot\prod_{i=1}^{n-2}\frac{\Gamma(\frac{\nu_1-\nu_n}{2}+l_i+l_{i+1}+1)}{\Gamma(\frac{\nu_{i+1}-\nu_n}{2}+l_{i+1}-m_{i-1}+m_i+1)\Gamma(\frac{\nu_1-\nu_{i+1}}{2}+l_i+m_{i-1}-m_i+1)}.$$

Suppose $\mathrm{Re}(\frac{\nu_1-\nu_n}{2}+1)>0$. By Lemma 5, the last term $\prod_{i=1}^{n-2}$ in (2) can be written as

$$\prod_{i=1}^{n-2}\frac{1}{2\pi\sqrt{-1}}\int_{|u_i|=1}(1+1/u_i)^{(\nu_1-\nu_{i+1})/2+l_i+m_{i-1}-m_i} \cdot(1+u_i)^{(\nu_{i+1}-\nu_n)/2+l_{i+1}-m_{i-1}+m_i}\frac{du_i}{u_i}$$
$$=\prod_{i=1}^{n-2}\frac{1}{2\pi\sqrt{-1}}\int_{|u_i|=1}(1+1/u_i)^{(\nu_1-\nu_n)/4+l_i}(1+u_i)^{(\nu_1-\nu_n)/4+l_{i+1}} \cdot u_i^{(\nu_{i+1})/2-(\nu_1+\nu_n)/4-m_{i-1}+m_i}\frac{du_i}{u_i}.$$

In view of

$$\left|\int_{|u_i|=1}(1+1/u_i)^{(\nu_1-\nu_n)/4+l_i}(1+u_i)^{(\nu_1-\nu_n)/4+l_{i+1}} \cdot u_i^{(\nu_{i+1})/2-(\nu_1+\nu_n)/4-m_{i-1}+m_i}\frac{du_i}{u_i}\right|\ll 2^{l_i+l_{i+1}}$$

([11, pp. 102-103]) and the estimate of $|G_{n-2,\mathbf{m}}(\mu)|$ in [3, Lemma 4.5], we can verify the change of order of integration and summation. Then (2) is equal to

$$\frac{1}{(2\pi\sqrt{-1})^{n-2}}\int_{|u_1|=1}\cdots\int_{|u_{n-2}|=1} \cdot\prod_{i=1}^{n-1}\left[\sum_{l_i\geqslant 0}\frac{(\pi y_i)^{2l_i}\{(1+u_{i-1})(1+1/u_i)\}^{(\nu_1-\nu_n)/4+l_i}}{l_i!\,\Gamma(\frac{\nu_1-\nu_n}{2}+l_i+1)}\right]$$
$$\cdot\left[\sum_{\mathbf{m}\in(\mathbb{Z}_{\geqslant 0})^{n-3}}G_{n-2,\mathbf{m}}(\mu)\prod_{i=1}^{n-3}\left(\pi y_{i+1}\sqrt{u_i/u_{i+1}}\right)^{2m_i}\right] \cdot\prod_{i=1}^{n-2}u_i^{\nu_{i+1}/2-(\nu_1+\nu_n)/4}\frac{du_i}{u_i}$$
$$=\frac{1}{(2\pi\sqrt{-1})^{n-2}}\int_{|u_1|=1}\cdots\int_{|u_{n-2}|=1}\prod_{i=1}^{n-1}\Big[y_i^{-(\nu_1-\nu_n)/2} \cdot I_{(\nu_1-\nu_n)/2}\Big(2\pi y_i\sqrt{(1+u_{i-1})(1+1/u_i)}\Big)\Big]$$
$$\cdot M_{n-2}\Big(\mu;\big(y_2\sqrt{u_1/u_2},\dots,y_{n-2}\sqrt{u_{n-3}/u_{n-2}}\,\big)\Big) \cdot\prod_{i=1}^{n-3}\left(\pi y_{i+1}\sqrt{u_i/u_{i+1}}\right)^{-i(n-2-i)/2-\sum_{k=1}^{i}\mu_k}\prod_{i=1}^{n-2}u_i^{\nu_{i+1}/2-(\nu_1+\nu_n)/4}\frac{du_i}{u_i}.$$

Collecting the exponents of y_i and u_i, we complete the proof.

Acknowledgment. — The author would like to thank Professor Eric Stade for his comment on this paper and the referee for careful reading and correcting errors.

BIBLIOGRAPHY

[1] W. N. Bailey, Generalized Hypergeometric Series, Cambridge, 1935.

[2] D. Bump, Automorphic Forms on $GL(3,\mathbb{R})$, Lect. Notes in Math., 1083 (1984).

[3] M. Hashizume, Whittaker functions on semisimple Lie groups, Hiroshima Math. J., 12 (1982), 259–293.

[4] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France, 95 (1967), 243–309.

[5] B. Kostant, On Whittaker vectors and representation theory, Invent. Math., 48 (1978), 101–184.

[6] R. Miatello and N. Wallach, Automorphic Forms Constructed from Whittaker Vectors, J. of Funct. Anal., 86 (1989), 411–487.

[7] J. Shalika, The multiplicity one theorem for $GL(n)$, Ann. of Math., 100 (1974), 171–193.

[8] E. Stade, Poincaré Series For $GL(3,\mathbb{R})$-Whittaker Functions, Duke Math. J., 58-3 (1989), 695–729.

[9] E. Stade, On Explicit Integral Formulas For $GL(n,\mathbb{R})$-Whittaker Functions, Duke Math. J., 60-2 (1990), 313–362.

[10] E. Stade, $GL(4,\mathbb{R})$-Whittaker functions and ${}_4F_3(1)$ hypergeometric series, Trans. Amer. Math. Soc., 336-1 (1993), 253–264.

[11] E. Stade, The reciprocal of the beta function and $GL(n,\mathbb{R})$ Whittaker functions, Ann. Inst. Fourier, Grenoble, 44-1 (1994), 93–108.

Manuscrit reçu le 30 mai 2003,
accepté le 11 septembre 2004.

Taku ISHII,
Tokyo Institute of Technology
Department of Mathematics
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8551 (Japan)
taku@math.titech.ac.jp