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Initial boundary value problem for the mKdV equation on a finite interval

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1. Introduction.

The general method that was announced in [4] for analyzing initial-boundary value problems for two-dimensional linear and integrable non-linear PDEs and was further developed in [5, 6, 7, 10] is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann-Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral functions determining the RH problem are expressed in terms of the initial and boundary values of the solution. The fact that these initial and boundary values are in general related can be expressed in a simple way in terms of an algebraic relation satisfied by the corresponding spectral functions. This relation can be used to characterize a part of boundary values which, on the one hand, are involved in the construction of the solution of the boundary value problem but, on the other hand, are not given as boundary data for a well-posed boundary value problem [2].

The rigorous implementation of the method to the modified Korteweg-de Vries (mKdV) equation

\[ q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1 \]

Keywords: Modified Korteweg-de Vries equation – Initial-boundary value problem – Global relation – Finite interval – Riemann-Hilbert problem.
on the half-line $0 < x < \infty$ is presented in [1]. In the present paper, this methodology is applied to the mKdV equation on a finite interval, $0 < x < L$. The main result was announced in [3]. The similar problem for the nonlinear Schrödinger equation is studied in [9].

The problem we are dealing with is the initial-boundary value problem for the mKdV equation in the domain $\{0 < x < L, 0 < t < T\}$ with $L < \infty$, $T \leq \infty$. We follow the scheme below.

**Step 1.** Assuming that the solution of the mKdV equation, $q(x, t)$, exists, carry out the direct spectral analysis. For this purpose:

- Define appropriate solutions of (2.5) (eigenfunctions) analytic and bounded (in $k$) in domains forming a partition of the Riemann sphere $\overline{C} = C \cup \{\infty\}$.

- Define spectral functions $s(k)$, $S(k)$, and $S_1(k)$ such that:
  1. $s(k)$ is determined by the initial conditions $q(x, 0) = q_0(x)$, $0 < x < L$.
  2. $S(k)$ is determined by the boundary values $q(0, t) = g_0(t)$, $q_x(0, t) = g_1(t)$, $q_{xx}(0, t) = g_2(t)$, $0 < t < T$.
  3. $S_1(k)$ is determined by the boundary values $q(L, t) = f_0(t)$, $q_x(L, t) = f_1(t)$, $q_{xx}(L, t) = f_2(t)$, $0 < t < T$.

- Show that the spectral functions satisfy an algebraic relation, called global relation, expressing the fact that $q_0(x)$, $g_0(t)$, $g_1(t)$, $g_2(t)$, $f_0(t)$, $f_1(t)$, and $f_2(t)$ being the initial and boundary values for the mKdV equation, cannot be chosen arbitrary.

**Step 2.** Given $s(k)$, $S(k)$, and $S_1(k)$, determine a regular Riemann-Hilbert problem, the solution of which gives a solution of the mKdV equation.

**Step 3.** Assuming that $\{g_j(t)\}_{j=0}^2$ and $\{f_j(t)\}_{j=0}^2$ are such that the associated $S(k)$ and $S_1(k)$ together with $s(k)$ satisfy the “global relation”, prove that the function $q(x, t)$ obtained from the solution of the RH problem solves the initial boundary value problem for the mKdV equation with initial data $q(x, 0) = q_0(x)$ and boundary data $q(0, t) = g_0(t)$, $q_x(0, t) = g_1(t)$, $q_{xx}(0, t) = g_2(t)$, $q(L, t) = f_0(t)$, $q_x(L, t) = f_1(t)$, $q_{xx}(L, t) = f_2(t)$.

In Section 2 we define appropriate eigenfunctions of the associated Lax pair and spectral functions and study their properties. In particular, we show that the spectral functions satisfy an algebraic relation, the “global
relation". In Section 3, we express the solution of the initial boundary value problem for the mKdV equation in terms of a matrix-valued Riemann-Hilbert (RH) problem. We show that the solution of this RH problem gives the solution of the mKdV equation with prescribed initial and boundary values provided that the spectral functions satisfy the "global relation".

2. Eigenfunctions and spectral functions.

2.1. Lax pair

The modified Korteweg-de Vries equation

\begin{equation}
q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1
\end{equation}

is the compatibility condition for the Lax pair

\begin{subequations}
\begin{align}
\psi_x - ik\sigma_3 \psi &= Q(x,t)\psi, \\
\psi_t + 4ik^3 \sigma_3 \psi &= \tilde{Q}(x,t,k)\psi
\end{align}
\end{subequations}

where

\begin{equation}
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{equation}

\begin{equation}
Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda q(x,t) & 0 \end{pmatrix},
\end{equation}

\begin{equation}
\tilde{Q}(x,t,k) = \begin{pmatrix} -2i\lambda q^2 k & -4qk^2 + 2iq_x k - 2\lambda q^3 + q_{xx} \\ \lambda (-4qk^2 - 2iq_x k - 2\lambda q^3 + q_{xx}) & 2i\lambda q^2 k \end{pmatrix}.
\end{equation}

We denote \( \hat{\sigma}_3 \) the commutation operation \( \text{ad} \sigma_3 \):

\begin{equation}
\hat{\sigma}_3 A := [\sigma_3, A] = \sigma_3 A - A\sigma_3
\end{equation}

for any \( 2 \times 2 \) matrix \( A \). The Lax pair (2.1) can be rewritten as

\begin{subequations}
\begin{align}
\mu_x - ik\hat{\sigma}_3 \mu &= Q(x,t)\mu, \\
\mu_t + 4ik^3 \hat{\sigma}_3 \mu &= \tilde{Q}(x,t,k)\mu,
\end{align}
\end{subequations}

where

\begin{equation}
\mu := \psi e^{i(-kx + 4k^3 t)\sigma_3}.
\end{equation}
In turn, (2.4) can be written as

\begin{equation}
(2.5) \quad d \left( e^{i(-kx + 4k^3t)\partial_3} \mu \right) = W,
\end{equation}

where

\[
e^{\sigma_3} A = e^{\sigma_3} A e^{-\sigma_3},
\]

and $W$ is the exact 1-form defined by

\begin{equation}
(2.6) \quad W(x, t, k) = e^{i(-kx + 4k^3t)\partial_3} (Q\mu dx + \bar{Q}\mu dt).
\end{equation}

2.2. Eigenfunctions

Assume that there exists a smooth real-valued function $q(x, t)$ satisfying (1.1) in $\{0 < x < L, 0 < t < T\}$ (if $T = \infty$ then it is also assumed a sufficient decay of $q(x, t)$ as $t \to \infty$). Define $\mu_n(x, t, k)$, $n = 1, 2, 3, 4$ as $2 \times 2$-matrix-valued solutions of the integral equations

\begin{equation}
(2.7) \quad \mu_n(x, t, k) = I + \int_{(x, t, \tau)}^{(x, t)} e^{i(kx - 4k^3t)\partial_3} W(y, \tau, k),
\end{equation}

where $(x_1, t_1) = (0, T)$, $(x_2, t_2) = (0, 0)$, $(x_3, t_3) = (L, 0)$, $(x_4, t_4) = (L, T)$, and the paths of integration are chosen to be parallel to the $x$ and $t$ axes:

\begin{align*}
(2.8a) \quad & \mu_1(x, t, k) = I + \int_0^x e^{ik(x-y)\partial_3} (Q\mu_1)(y, t, k) dy \\
& - e^{ikx\partial_3} \int_t^T e^{-4ik^3(t-\tau)\partial_3} (\bar{Q}\mu_1)(0, \tau, k) d\tau,
\end{align*}

\begin{align*}
(2.8b) \quad & \mu_2(x, t, k) = I + \int_0^x e^{ik(x-y)\partial_3} (Q\mu_2)(y, t, k) dy \\
& + e^{ikx\partial_3} \int_0^t e^{-4ik^3(t-\tau)\partial_3} (\bar{Q}\mu_2)(0, \tau, k) d\tau,
\end{align*}

\begin{align*}
(2.8c) \quad & \mu_3(x, t, k) = I - \int_x^L e^{ik(x-y)\partial_3} (Q\mu_3)(y, t, k) dy \\
& + e^{ik(x-L)\partial_3} \int_0^t e^{-4ik^3(t-\tau)\partial_3} (\bar{Q}\mu_3)(L, \tau, k) d\tau,
\end{align*}

\begin{align*}
(2.8d) \quad & \mu_4(x, t, k) = I - \int_x^L e^{ik(x-y)\partial_3} (Q\mu_4)(y, t, k) dy \\
& - e^{ik(x-L)\partial_3} \int_t^T e^{-4ik^3(t-\tau)\partial_3} (\bar{Q}\mu_4)(L, \tau, k) d\tau.
\end{align*}
The domains where the exponentials are bounded, are separated by the three lines $L_0, L_1, L_2$ where $\text{Im} k^3$ vanishes: $L_0 \cup L_1 \cup L_2 = \{ k \in C \mid \text{Im} k^3 = 0 \}$. The relevant domains in the $k$-plane are labelled $I, II, \ldots, VI$, see Fig. 1.

Figure 1: Domains of analyticity and boundedness of eigenfunctions

Let us denote $\mu^{(1)}, \mu^{(2)}$ the columns of a $2 \times 2$ matrix $\mu = (\mu^{(1)} \quad \mu^{(2)})$. Then the columns of $\mu_n$ are analytic and bounded in the following domains in the complex $k$-plane (where the exponentials involved in the corresponding integral equations are bounded):

<table>
<thead>
<tr>
<th>eigenfunctions</th>
<th>domains of analyticity and boundedness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^{(1)}_1$ and $\mu^{(2)}_3$</td>
<td>${ k \mid \text{Im} k \leq 0, \text{Im} k^3 \leq 0 } = IV \cup VI$,</td>
</tr>
<tr>
<td>$\mu^{(2)}_1$ and $\mu^{(1)}_3$</td>
<td>${ k \mid \text{Im} k \geq 0, \text{Im} k^3 \geq 0 } = I \cup III$,</td>
</tr>
<tr>
<td>$\mu^{(1)}_2$ and $\mu^{(2)}_4$</td>
<td>${ k \mid \text{Im} k \leq 0, \text{Im} k^3 \geq 0 } = V$,</td>
</tr>
<tr>
<td>$\mu^{(2)}_2$ and $\mu^{(1)}_4$</td>
<td>${ k \mid \text{Im} k \geq 0, \text{Im} k^3 \leq 0 } = II$.</td>
</tr>
</tbody>
</table>

Thus, in each domain $I, \ldots, VI$, one has an analytic and bounded $2 \times 2$ matrix-valued eigenfunction, consisting of the appropriate vectors $\mu^{(j)}_n, n = 1, 2, 3, 4, j = 1, 2$.

For particular values of $x$ or $t$, the domains of boundedness of the eigenfunctions are larger than indicated above. In particular, for $t = 0$, the domains of boundedness of $\mu_2$ and $\mu_3$ are:

<table>
<thead>
<tr>
<th>eigenfunctions</th>
<th>domains of boundedness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^{(1)}_2(x, 0, k)$ and $\mu^{(2)}_3(x, 0, k)$</td>
<td>${ k \mid \text{Im} k \leq 0 } = IV \cup V \cup VI$,</td>
</tr>
<tr>
<td>$\mu^{(2)}_2(x, 0, k)$ and $\mu^{(1)}_3(x, 0, k)$</td>
<td>${ k \mid \text{Im} k \geq 0 } = I \cup II \cup III$,</td>
</tr>
</tbody>
</table>

for $x = 0$, the domains of boundedness of $\mu_1$ and $\mu_2$ are:
eigenfunctions \( \mu_1^{(1)}(0, t, k) \) and \( \mu_2^{(2)}(0, t, k) \) domains of boundedness 
\( \{ k \mid \text{Im} k^3 \leq 0 \} = \text{II} \cup \text{IV} \cup \text{VI} \),

\( \mu_1^{(2)}(0, t, k) \) and \( \mu_2^{(1)}(0, t, k) \) 
\( \{ k \mid \text{Im} k^3 \geq 0 \} = \text{I} \cup \text{III} \cup \text{V} \),

for \( x = L \), the domains of boundedness of \( \mu_3 \) and \( \mu_4 \) are:

\[ \mu_3^{(2)}(L, t, k) \text{ and } \mu_4^{(1)}(L, t, k) \]
\( \{ k \mid \text{Im} k^3 \leq 0 \} = \text{II} \cup \text{IV} \cup \text{VI} \),

\[ \mu_3^{(1)}(L, t, k) \text{ and } \mu_4^{(2)}(L, t, k) \]
\( \{ k \mid \text{Im} k^3 \geq 0 \} = \text{I} \cup \text{III} \cup \text{V} \).

2.3. Spectral functions

Since the eigenfunctions \( \mu_j \) are solutions of the system of differential equations (2.4), they are simply related (in the domains where they are defined):

\[ \begin{align*}
(2.9a) \quad &\mu_3(x, t, k) = \mu_2(x, t, k)e^{i(kx - 4k^3t)}\hat{\sigma}_3 \mu_3(0, 0, k), \\
(2.9b) \quad &\mu_1(x, t, k) = \mu_2(x, t, k)e^{i(kx - 4k^3t)}\hat{\sigma}_3 \mu_1(0, 0, k) \\
&= \mu_2(x, t, k)e^{i(kx - 4k^3t)}\hat{\sigma}_3 \left( e^{4ik^3T\hat{\sigma}_3} \mu_2(0, T, k) \right)^{-1}, \\
(2.9c) \quad &\mu_4(x, t, k) = \mu_3(x, t, k)e^{i(kx - 4k^3t)}\hat{\sigma}_3 e^{-ikL\hat{\sigma}_3} \mu_4(L, 0, k) \\
&= \mu_3(x, t, k)e^{i(kx - 4k^3t)}\hat{\sigma}_3 e^{-ikL\hat{\sigma}_3} \left( e^{4ik^3T\hat{\sigma}_3} \mu_3(L, T, k) \right)^{-1}.
\end{align*} \]

Relations (2.9) suggest the introduction of the 2 × 2-matrix-valued spectral functions

\[ \begin{align*}
(2.10a) \quad &s(k) := \mu_3(0, 0, k), \\
(2.10b) \quad &S(k) = S(k; T) := \mu_1(0, 0, k) = \left( e^{4ik^3T\hat{\sigma}_3} \mu_2(0, T, k) \right)^{-1}, \\
(2.10c) \quad &S_1(k) = S_1(k; T) := \mu_4(L, 0, k) = \left( e^{4ik^3T\hat{\sigma}_3} \mu_3(L, T, k) \right)^{-1}.
\end{align*} \]

From (2.9) and (2.10) it follows that \( s(k) \) is determined by the initial values of \( q(x, t) \), whereas \( S(k) \) and \( S_1(k) \) are determined by the boundary values at \( x = 0 \) and \( x = L \). Namely,

\[ s(k) = I - \int_0^L e^{-iky\hat{\sigma}_3}(Q\mu_3)(y, 0, k)dy, \]

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where $\mu_3(x, 0, k), 0 < x < L$ is the solution of the integral equation

$$\mu_3(x, 0, k) = I - \int_x^L e^{ik(x-y)\hat{\sigma}_3} (Q\mu_3)(y, 0, k) dy.$$ \hfill (2.12)

Similarly,

$$S(k; T) = \left( I + \int_0^T e^{4ik^3 \tau \hat{\sigma}_3} (\bar{Q}\mu_2)(0, \tau, k) d\tau \right)^{-1},$$ \hfill (2.13a)

$$S_1(k; T) = \left( I + \int_0^T e^{4ik^3 \tau \hat{\sigma}_3} (\bar{Q}\mu_3)(L, \tau, k) d\tau \right)^{-1},$$ \hfill (2.13b)

where $\mu_2(0, t, k)$ and $\mu_3(L, t, k)$, $0 < t < T$ are the solutions of the integral equations

$$\mu_2(0, t, k) = I + \int_0^t e^{-4ik^3(t-\tau)\hat{\sigma}_3} (\bar{Q}\mu_2)(0, \tau, k) d\tau,$$ \hfill (2.14a)

$$\mu_3(L, t, k) = I + \int_0^t e^{-4ik^3(t-\tau)\hat{\sigma}_3} (\bar{Q}\mu_3)(L, \tau, k) d\tau,$$ \hfill (2.14b)

respectively. Note that $Q(x, 0)$ is determined by $q(x, 0)$, whereas $\bar{Q}(0, t, k)$ is determined by $q(0, t)$, $q_x(0, t)$, and $q_{xx}(0, t)$, and $\bar{Q}(L, t, k)$ is determined by $q(L, t)$, $q_x(L, t)$, and $q_{xx}(L, t)$.

In what follows, $s(k)$, $S(k)$, and $S_1(k)$ will be used to construct a Riemann-Hilbert problem (more precisely, a family of Riemann-Hilbert problems parametrized by $x$ and $t$), whose solution gives the eigenfunctions $\mu_n$ and hence $q(x, t)$, the solution of the mKdV problem.

### 2.4. Symmetry properties

Due to the particular symmetries of the equations of the Lax pair (2.1), the spectral matrices $s(k)$, $S(k)$, and $S_1(k)$ can be written as

$$s(k) = \begin{pmatrix} a(k) & b(k) \\ \lambda b(k) & a(k) \end{pmatrix},$$ \hfill (2.15a)

$$S(k) = \begin{pmatrix} \bar{A}(k) & B(k) \\ \lambda B(k) & A(k) \end{pmatrix},$$ \hfill (2.15b)

$$S_1(k) = \begin{pmatrix} \bar{A}_1(k) & B_1(k) \\ \lambda B_1(k) & A_1(k) \end{pmatrix}.$$ \hfill (2.15c)
2.5. Global relation

Evaluating eqs. (2.9c) and (2.9a) at \( x = 0, t = T \), we find

\[
\begin{align*}
\mu_4(0, T, k) &= \mu_3(0, T, k)e^{-ikL\xi_3}e^{-4ik^3T\xi_3}\mu_4(L, 0, k), \\
\mu_3(0, T, k) &= \mu_2(0, T, k)e^{-ik^3T\xi_3}\mu_3(0, 0, k).
\end{align*}
\]

Writing \( \mu_3(0, 0, k), \mu_2(0, T, k), \) and \( \mu_4(L, 0, k) \) in terms of \( s(k), S(k), \) and \( S_1(k) \), respectively (see equations (2.10)), and using eq. (2.8d) to evaluate \( \mu_4(0, T, k) \), eq. (2.16a) becomes

\[
S^{-1}(k)s(k)\left[e^{-ikL\xi_3}S_1(k)\right] = e^{4ik^3T\xi_3}\mu_4(0, T, k)
\]

\[
= I - e^{4ik^3T\xi_3}\int_0^L e^{-iky\xi_3}(Q\mu_4)(y, T, k)dy.
\]

- For \( T < \infty \), the \((1, 2)\) coefficient of (2.17) is, for \( k \in C \),

\[
e^{-2ikL}(a(k)A(k) - \lambda b(k)B(k))B_1(k) - (a(k)B(k) - b(k)A(k))A_1(k) = e^{8ik^3T}c(k),
\]

where

\[
c(k) = c(k; T) = \int_0^L e^{-2iky}(Q\mu_4)_{12}(y, T, k)dy
\]

is an entire function which is \( O \left((1 + e^{-2ikL})/k\right) \) as \( k \to \infty \).

- For \( T = \infty \), the \((1, 2)\) coefficient of (2.17) becomes

\[
e^{-2ikL}(a(k)A(k) - \lambda b(k)B(k))B_1(k)
\]

\[
- (a(k)B(k) - b(k)A(k))A_1(k) = 0,
\]

which is valid for \( k \in I \cup III \cup V \).

Equation (2.18) for \( T < \infty \), resp. (2.19) for \( T = \infty \), is an algebraic relation between the spectral functions. We call it “global relation”, because it express, in spectral terms, the relations between the initial and boundary values of a solution of the mKdV equation.

2.6. Direct and inverse spectral maps

- The spectral maps \( S, Q \). The direct spectral map

\[
S: \{q_0(x)\} \longrightarrow \{a(k), b(k)\}
\]

is defined following (2.10a), (2.15a), by the second column of the solution \( \mu_3(x, 0, k) \) of equation (2.12) where \( Q = Q(x, 0) \) is given by (2.2) with \( q_0(x) \) instead of \( q(x, t) \):

\[
Q = \begin{pmatrix}
0 & q_0(x) \\
\lambda q_0(x) & 0
\end{pmatrix}.
\]
Thus, this direct spectral map is the same as in the inverse scattering theory on the whole line for the Dirac equation

$$\psi_x - ik\sigma_3 \psi = \begin{pmatrix} 0 & q_0(x) \\ \lambda q_0(x) & 0 \end{pmatrix} \psi$$

restricted to the potentials with support on the finite interval $[0, L]$. The analysis of the linear Volterra integral equation (2.12) gives the following properties of $a(k)$ and $b(k)$:

- $a(k)$ and $b(k)$ are entire functions, bounded for $\text{Im } k \leq 0$;
- they have the asymptotic behavior, as $k \to \infty$:

$$a(k) = 1 + O \left( \frac{1 + e^{-2ikL}}{k} \right), \quad b(k) = O \left( \frac{1 + e^{-2ikL}}{k} \right).$$

The fact that $Q$ has zero trace together with the behavior as $k \to \infty$ given above imply that $\det s(k) = 1$ which, in terms of $a(k)$ and $b(k)$, reads

- $a(k)a(\overline{k}) - \lambda b(k)b(\overline{k}) = 1, \quad k \in C$;
- $a(-k) = \overline{a(k)}, \quad b(-k) = \overline{b(k)}$.

Define a sectionally holomorphic, matrix-valued function $M^{(x)}(x, k)$:

$$M^{(x)}(x, k) = \begin{cases} \begin{pmatrix} \mu_3^{(1)}(x, 0, k) & \mu_2^{(2)}(x, 0, k) \\ \mu_2(x, 0, k) & \mu_3^{(1)}(x, 0, k) \end{pmatrix} & |k| > R, \quad \text{Im } k > 0, \\
\begin{pmatrix} \mu_2^{(1)}(x, 0, k) & \mu_3^{(2)}(x, 0, k) \\ \mu_2(x, 0, k) & \mu_3^{(1)}(x, 0, k) \end{pmatrix} & |k| > R, \quad \text{Im } k < 0, \\
\mu_2(x, 0, k) & \mu_3^{(1)}(x, 0, k) \end{cases}$$

where $R$ is chosen to be sufficiently large such that the disk $\{ k \in C \mid |k| < R \}$ contains all the zeros of $a(k)$ with $\text{Im } k \leq 0$.

Let $\Sigma^{(x)}$ be the oriented contour in the complex $k$-plane consisting of the real axis and the circle $\Gamma_R = \{ k \mid |k| = R \}$ of radius $R$, and let $\Omega^{(x)}_{\pm}$ be the sectors bounded by $\Sigma^{(x)}$, see Fig. 2.

![Figure 2: The contour $\Sigma^{(x)}$](image-url)
Similarly to $\mu_3(x,0,k)$, $\mu_2(x,0,k)$ in (2.22) is determined by (2.8b) for $t = 0$, where $q(x,0)$ in $Q(x,0)$ is replaced by $q_0(x)$. Then the limits $M_{\pm}^{(x)}(x,\zeta)$ of $M^{(x)}(x,k)$ as $k$ approaches $\zeta \in \Sigma^{(x)}$ from $\Omega_{\pm}^{(x)}$ are related by the jump matrix $J^{(x)}(x,\zeta)$:

$$M_{-}^{(x)}(x,\zeta) = M_{+}^{(x)}(x,\zeta)J^{(x)}(x,\zeta), \quad \zeta \in \Sigma^{(x)},$$

where

(2.23)

$$J^{(x)}(x,k) = \begin{cases}
\begin{pmatrix}
1 & -\frac{b(k)}{\bar{a}(k)}e^{2ikx} \\
\lambda \frac{\bar{b}(k)}{a(k)} e^{-2ikx} & 1 - \lambda \left| \frac{b(k)}{\bar{a}(k)} \right|^2
\end{pmatrix} & |k| > R, \quad \text{Im} \ k = 0, \\
I & |k| < R, \quad \text{Im} \ k = 0,
\end{cases}$$

(2.24)

The inverse spectral map $Q: \{a(k), b(k)\} \mapsto \{q_0(x)\}$ is defined as

(2.25)

$$q_0(x) = -2i \lim_{k \to \infty} k(M^{(x)}(x,k))_{12},$$

where $M^{(x)}(x,k)$ is the solution of the following RH problem:

- $M^{(x)}(x,k)$ is a sectionally holomorphic function relative to the contour $\Sigma^{(x)}$.
- The limits $M_{\pm}^{(x)}(x,\zeta)$ of $M^{(x)}(x,k)$ as $k$ approaches $\zeta \in \Sigma^{(x)}$ from $\Omega_{\pm}$ are related by (2.23), where the jump matrix $J^{(x)}(x,\zeta)$ is constructed from $a(k)$ and $b(k)$ following (2.24).
- $M^{(x)}(x,k) = I + O \left( \frac{1}{k} \right), \quad k \to \infty$.

Remark 1. — In [1] the corresponding inverse spectral map was defined via the solution of a singular RH problem, where $M^{(x)}(x,k)$ can have poles (at possible zeros of $a$ and $\bar{a}$). In that case, residue relations were also added to the formulation of the RH problem. Here we give a
formulation of the inverse map which is based on the solution of a regular RH problem relative to a contour containing the circle $\Gamma_R$ as additional part (see [13]). In such a formulation, we do not need any assumption on the zeros of $a(k)$. Notice that the cyclic conditions for the jump matrices at $k = R$ and $k = -R$, the intersection points of $\Sigma^{(x)}$, which are necessary for the solvability of the Riemann-Hilbert problem, are automatically satisfied by the construction of $J^{(x)}$.

- **The spectral maps $\tilde{S}$, $\tilde{Q}$ and $\tilde{S}_1$, $\tilde{Q}_1$.** The direct spectral map

$$(2.26) \quad \tilde{S}: \{g_0(t), g_1(t), g_2(t)\} \mapsto \{A(k), B(k)\}$$

is defined following (2.10b), (2.15b), by the second column of the solution $\mu_2(0, t, k)$ of equation (2.14a) where $\tilde{Q} = \tilde{Q}(0, t, k)$ is determined by (2.3) with $q$, $q_x$, and $q_{xx}$ replaced by $g_0(t)$, $g_1(t)$, and $g_2(t)$, respectively:

$$(2.27) \quad \begin{pmatrix} -e^{-ik^3T}B(k) \\ A(k) \end{pmatrix} = \mu_2^{(2)}(0, T, k).$$

Similarly, the direct spectral map

$$(2.28) \quad \tilde{S}_1: \{f_0(t), f_1(t), f_2(t)\} \mapsto \{A_1(k), B_1(k)\}$$

is defined following (2.10c), (2.15c), by the second column of the solution $\mu_3(L, t, k)$ of equation (2.14b) where $\tilde{Q} = \tilde{Q}(L, t, k)$ is determined by (2.3) with $q$, $q_x$, and $q_{xx}$ replaced by $f_0(t)$, $f_1(t)$, and $f_2(t)$, respectively:

$$(2.29) \quad \begin{pmatrix} -e^{-ik^3T}B_1(k) \\ A_1(k) \end{pmatrix} = \mu_3^{(2)}(L, T, k).$$

The properties of $A$ and $B$ are discussed in [1], and $A_1$ and $B_1$ have the same properties. In particular:

- $A(-k) = \overline{A(k)}$, $B(-k) = \overline{B(k)}$.
- If $T < \infty$, $A(k)$ and $B(k)$ are entire functions, bounded in $I \cup III \cup V$.
- If $T = \infty$, $A(k)$ and $B(k)$ are only defined in $I \cup III \cup V$, being analytic and bounded there.

- $A(k)$ and $B(k)$ have the asymptotic behavior, as $k \to \infty$:

$$(2.30) \quad A(k) = 1 + O\left(\frac{1 + e^{i k^3 T}}{k}\right), \quad B(k) = O\left(\frac{1 + e^{i k^3 T}}{k}\right).$$

- $A(k)$ and $B(k)$ are related by the relation

$$(2.31) \quad A(k)A(\overline{k}) - \lambda B(k)B(\overline{k}) = 1, \quad k \in C \quad (\text{with } \Im k^3 = 0 \text{ if } T = \infty).$$
Define a sectionally holomorphic, matrix-valued function

$$M^{(t)}(t, k) = \begin{cases} 
\left( \mu_1^{(1)}(0, t, k) \frac{\mu_2^{(2)}(0, t, k)}{A(k)} \right) & |k| > R, \quad \text{Im} k^3 < 0, \\
\left( \mu_2^{(1)}(0, t, k) \frac{\mu_1^{(2)}(0, t, k)}{A(k)} \right) & |k| > R, \quad \text{Im} k^3 > 0, \\
\mu_2(0, t, k) & |k| < R,
\end{cases}$$

where $R$ is again chosen to be sufficiently large such that the disk $\{k \in \mathbb{C} \mid |k| < R\}$ contains all the zeros of $A(k)$ with $\text{Im} k^3 \geq 0$.

Let $\Sigma$ be the oriented contour in the complex $k$-plane consisting of the lines $L_0$, $L_1$, $L_2$ and the circle $\Gamma_R = \{k \mid |k| = R\}$ of radius $R$, and let $\Omega_{\pm}$ be the sectors bounded by $\Sigma$, see Fig. 3.

Similarly to $\mu_2(0, t, k)$, $\mu_1(0, t, k)$ in (2.32) is determined by (2.8a) for $x = 0$, where $q(0, t)$, $q_x(0, t)$, and $q_{xx}(0, t)$ in $\tilde{Q}(0, t, k)$ are replaced by $g_0(t)$, $g_1(t)$, and $g_2(t)$, respectively. Then the limits $M_{\pm}^{(t)}(t, \zeta)$ of $M^{(t)}(t, k)$ as $k$ approaches $\zeta \in \Sigma$ from the domains $\Omega_{\pm}$ are related by the jump matrix $J^{(t)}(t, \zeta)$:

$$M_{\pm}^{(t)}(t, \zeta) = M_{\pm}^{(t)}(t, \zeta)J^{(t)}(t, \zeta), \quad \zeta \in \Sigma,$$

where
The map inverse to $S$ is defined as follows:

\[
J(t, k) = \begin{cases} 
1 & -\frac{B(k)}{A(k)} e^{-8i\kappa^3 t} \\
\lambda B(k) e^{8i\kappa^3 t} & 1 \\
A(k) & 0 \\
\lambda B(k) e^{8i\kappa^3 t} & \frac{1}{A(k)}
\end{cases}
\]

where $I'$ are given by the large $k$-asymptotic expansion of the solution of the following RH problem:

\[
\text{Im} k^3 = 0, \ |k| > R \\
\text{Im} k^3 = 0, \ |k| < R \\
\text{Im} k^3 > 0, \ |k| = R \\
\text{Im} k^3 < 0, \ |k| = R.
\]

The spectral map

\[\tilde{Q}: \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t), g_2(t)\}\]

inverse to $\tilde{S}$ is defined as follows:

\[2.35a\] \[g_0(t) = -2i(M_{12}^{(t)})_{12}(t),\]

\[2.35b\] \[g_1(t) = 4(M_{22}^{(t)})_{12}(t) - 2i g_0(t)(M_{12}^{(t)})_{22}(t),\]

\[2.35c\] \[g_2(t) = \lambda g_0^3(t) + 8i(M_{32}^{(t)})_{12}(t) + 4g_0(t)(M_{22}^{(t)})_{22}(t) - 2i g_1(t)(M_{12}^{(t)})_{22}(t),\]

where $M_j^{(t)}(t, k), j = 1, 2, 3$ are given by the large $k$-asymptotic expansion

\[M^{(t)}(t, k) = I + \frac{M_1^{(t)}(t)}{k} + \frac{M_2^{(t)}(t)}{k^2} + \frac{M_3^{(t)}(t)}{k^3} + O\left(\frac{1}{k^4}\right), \quad k \to \infty,\]

of the solution $M^{(t)}(t, k)$ of the following RH problem:

- $M^{(t)}(t, k)$ is a sectionally holomorphic function relative to the contour $\Sigma$.

- The limits $M_{\pm}^{(t)}(t, \zeta)$ of $M^{(t)}(t, k)$ as $k$ approaches $\zeta \in \Sigma$ from $\Omega_{\pm}$ are related by (2.33), where the jump matrix $J^{(t)}(t, \zeta)$ is constructed from $A(k)$ and $B(k)$ following (2.34).

- $M^{(t)}(t, k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty.$

The spectral map

\[\tilde{Q}_1: \{A_1(k), B_1(k)\} \mapsto \{f_0(t), f_1(t), f_2(t)\},\]
inverse to \( \tilde{S}_1 \) is defined via the solution of a similar RH problem, with \( g_k, A, B \) replaced by \( f_k, A_1, B_1 \).

Remark 2. — As in the case of the \( x \)-problem, the two inverse spectral \( t \)-problems described above are based on regular RH problems: the residue conditions at zeros of \( A \) in the singular RH problem \([1]\) are replaced by additional jump conditions at the auxiliary part \( \Gamma_R \) of the contour. Here again, the cyclic conditions for the jump matrices at the points of self-intersection of \( \Sigma \) are satisfied by the construction of \( J^{(t)} \).


Relating the vector solutions of (2.4a)–(2.4b) (analytic in domains \( I, \ldots, VI \)) by using (2.9) and the definitions of the spectral functions (2.11)–(2.13), we find

\[
M_-(x, t, k) = M_+(x, t, k) J(x, t, k), \quad k \in \Sigma,
\]

where \( M_{\pm}(x, t, k) \) are the limit values (as \( k \) approaches \( \Sigma \) from \( \Omega_{\pm} \)) of a sectionally holomorphic function \( M(x, t, k) \) defined as follows:

\[
M = \begin{cases}
\left( \begin{array}{c}
\mu_3^{(1)} \\
\frac{\mu_2^{(2)}}{d(k)}
\end{array} \right), & \text{for } |k| > R, \quad k \in I \cup III, \\
\left( \begin{array}{c}
\frac{\mu_3^{(1)} a(k)}{d_1(k)} \\
\frac{\mu_2^{(2)}}{a(k)}
\end{array} \right), & \text{for } |k| > R, \quad k \in II, \\
\left( \begin{array}{c}
\frac{\mu_1^{(1)}}{d(k)} \\
\mu_3^{(2)}
\end{array} \right), & \text{for } |k| > R, \quad k \in IV \cup VI, \\
\left( \begin{array}{c}
\frac{\mu_2^{(1)}}{a(k)} \\
\frac{\mu_4^{(2)} a(k)}{d_1(k)}
\end{array} \right), & \text{for } |k| > R, \quad k \in V, \\
\mu_2, & \text{for } |k| < R,
\end{cases}
\]

where

\[
d(k) = a(k) \overline{A(k)} - \lambda b(k) \overline{B(k)}, \quad k \in II \cup IV \cup VI,
\]

\[
d_1(k) = a(k) A_1(k) + \lambda e^{-2ik} \overline{b(k)} B_1(k), \quad k \in I \cup III \cup V.
\]

The jump matrix \( J(x, t, k) \) has an explicit \((x, t)\)-dependence:

\[
J(x, t, k) = e^{(ikx - 4ik^3t) \overline{\alpha_3}} J_0(k),
\]

where \( J_0(k) \) is constructed from the elements of the spectral functions:
Here
\begin{align*}
(3.6a) \quad \gamma(k) &= \frac{b(k)}{a(k)}, \\
(3.6b) \quad \Gamma(k) &= \frac{\lambda B(k)}{a(k)d(k)}, \\
(3.6c) \quad \Gamma_1(k) &= \frac{e^{-2ikL}a(k)B_1(k)}{d_1(k)}, \\
(3.6d) \quad \Gamma_2(k) &= \frac{a(k)}{d_1(k)}(e^{-2ikL}a(k)B_1(k) + b(k)A_1(k)),
\end{align*}
and \( R \) is such that all zeros of \( a(k), d(k), \) and \( d_1(k) \) with \( \text{Im} \ k \leq 0 \) are in the disk \( |k| < R \).

Remark 3. — Since
The jump data for the RH problem (3.4) are determined by:

\[ \Gamma(k) = \lambda \frac{B(k)/A(k)}{a(k) \left( a(k) - \lambda b(k)(B(k)/A(k)) \right)} , \]

\[ \Gamma_1(k) = \frac{e^{-2ikL} a(k)(B_1(k)/A_1(k))}{a(k) + \lambda e^{-2ikL} b(k)(B_1(k)/A_1(k))} , \]

\[ \Gamma_2(k) = a(k) \frac{e^{-2ikL} a(k)(B_1(k)/A_1(k)) + b(k)}{a(k) + \lambda e^{-2ikL} b(k)(B_1(k)/A_1(k))} , \]

the jump data for the RH problem (3.4) are determined by:

- \( a(k) \) and \( b(k) \) for \( k \in C, \ |k| \geq R, \)
- \( \frac{B(k)}{A(k)} \) and \( \frac{B_1(k)}{A_1(k)} \) for \( k \in I \cup III \cup V, \ |k| \geq R. \)

The main result on the inverse spectral problem is the following:

**Theorem.** — Let \( q_0(x) \in C^\infty([0, L]) \). Let \( \{g_j(t)\}_{j=0}^2 \) and \( \{f_j(t)\}_{j=0}^2 \) be smooth functions such that:

- \( (\partial_x)^j q_0(0) = g_j(0), \quad (\partial_x)^j q_0(L) = f_j(0), \quad j = 0, 1, 2, \)
- the associated spectral functions \( s(k), S(k), \) and \( S_1(k) \) satisfy the global relation (2.18) for \( T < \infty, \) or (2.19) for \( T = \infty, \) where \( c(k) \) is an entire function such that \( c(k) = O \left( (1 + e^{-2ikL})/k \right) \) as \( |k| \to \infty. \)

Let \( M(x,t,k) \) be a solution of the following \( 2 \times 2 \) matrix RH problem:

- \( M \) is sectionally holomorphic in \( k \in C \setminus \Sigma. \)
- At \( k \in \Sigma, \ M \) satisfies the jump conditions (3.1), where the jump matrix \( J \) is defined in terms of the spectral functions \( a, b, A, B, A_1 \) and \( B_1 \) by equations (3.4)-(3.6).
- As \( k \to \infty, \)

\[ M(x,t,k) = I + O\left( \frac{1}{k} \right). \]

Then:

1. \( M(x,t,k) \) exists and is unique;
2. \( q(x,t) \) defined in terms of \( M(x,t,k) \) by

\[ q(x,t) = -2i \lim_{k \to \infty} (kM(x,t,k))_{12} \]

satisfies the mKdV equation (1.1);
3. \( q(x, t) \) satisfies the initial and boundary conditions

\[
\begin{align*}
q(x, 0) &= q_0(x), \\
q(0, t) &= g_0(t), \quad q_x(0, t) = g_1(t), \quad q_{xx}(0, t) = g_2(t), \\
q(L, t) &= f_0(t), \quad q_x(L, t) = f_1(t), \quad q_{xx}(L, t) = f_2(t).
\end{align*}
\]  

\textbf{Proof.} — The proof follows the same lines as in the case of nonlinear integrable equations on the half-line. For the mKdV equation on the half-line, see [1]; for a more detailed presentation, see [10] in the case of the NLS equation. The main steps of the proof include the following.

\textbf{Step 1.} The RH problem in question is regular; its unique solvability is a consequence of a “vanishing lemma” for the associated RH problem with the vanishing condition at infinity \( M = O(1/k), \ k \to \infty \) (see [12]).

\textbf{Step 2.} The proof that the constructed \( q(x, t) \) solves the mKdV equation is straightforward and follows the proof in the case of the whole line problem; see [11].

\textbf{Step 3.} The proof that \( q \) satisfies the initial condition \( q(x, 0) = q_0(x) \) follows from the fact that it is possible to map the RH problem for \( M(x, 0, k) \) to that for \( M^{(x)}(x, k) \) such that

\[
M^{(x)}(x, k) = M(x, 0, k) P^{(x)}(x, k)
\]

where \( P^{(x)}(x, k) \) is piecewise holomorphic relative to \( \Sigma \) and \( P^{(x)}(x, k) = I + O \left( \frac{1}{k} \right) \) as \( k \to \infty \). For instance, for \( \text{Im} \ k > 0 \), one has

\[
P^{(x)}(x, k) = \begin{cases} 
\begin{pmatrix} 1 & -\lambda \Gamma(k) e^{2ikx} \\
0 & 1 \end{pmatrix}, & k \in \text{I} \cup \text{III}, \ |k| > R, \\
\begin{pmatrix} 1 & 0 \\
-\lambda \overline{\Gamma(k)} e^{-2ikx} & 1 \end{pmatrix}, & k \in \text{II}, \ |k| > R.
\end{cases}
\]  

Now (2.21), (2.30), and (3.7) imply that \( \overline{\Gamma} \) is \( O \left( \frac{1}{k} \right) \) in \( \text{I} \cup \text{III} \) and \( \overline{\Gamma}_1 e^{2ikx} = O \left( \frac{1}{k} \right) e^{2ik(L-x)} \) in \( \text{II} \), respectively. Therefore, \( P^{(x)}(x, k) = I + P^{(x)}_{\text{off}}(x, k) \), where \( P^{(x)}_{\text{off}}(x, k) \) is off-diagonal and exponentially decaying as \( k \to \infty \) for \( \text{Im} \ k \neq 0 \), so that the asymptotics (2.25) and (3.9) for \( t = 0 \) yield \( q_0(x) = q(x, 0) \).

\textbf{Step 4.} The proof that \( q \) satisfies the boundary conditions \( q(0, t) = g_0(t), \ q_x(0, t) = g_1(t), \ q_{xx}(0, t) = g_2(t) \) and \( q(L, t) = f_0(t), \ q_x(L, t) = f_1(t), \ q_{xx}(L, t) = f_2(t) \),
\[ q_{xx}(L, t) = f_2(t) \] is, in turn, based on the maps \( M(0, t, k) \rightarrow M(t)^{(t)}(t, k) \) and \( M(L, t, k) \rightarrow M(t)^{(t)}(t, k) \):

(3.13) \[ M(t)^{(t)}(t, k) = M(0, t, k)P(t)^{(t)}(t, k), \quad M_1(t)^{(t)}(t, k) = M(L, t, k)P_1(t)^{(t)}(t, k). \]

In this case, the fact that \( P(t)^{(t)} \) and \( P_1(t)^{(t)} \) in the different sectors \( I, \ldots, VI \) are \( I + O\left(\frac{1}{k}\right) \), is a consequence of the global relation. For instance, for \( \text{Im} \, k > 0 \), one has

\[
\begin{pmatrix}
\frac{1}{d(k)} & 0 \\
-\lambda \frac{b(k)}{A(k)} e^{8ik^3t} & \frac{1}{d(k)}
\end{pmatrix}, \quad |k| > R, \ k \in I \cup II,
\]

(3.14) \[ P(t) = \frac{A(k)}{d(k)} e^{8ik^3t}, \quad |k| > R, \ k \in II, \]

where \( \text{GR}(k) \) is the l.h.s. of the global relation (2.18). Therefore, it is the global relation (2.18) for \( k \in V \) that gives \( \text{GR}(k) e^{8ik^3t} = O\left(\frac{1}{k}\right) \) for \( k \in II \) and, consequently, \( P_{21}^{(t)}(t, k) = O\left(\frac{1}{k}\right) \) for \( k \in II \) and \( t < T \). Similarly,

\[
\begin{pmatrix}
\frac{1}{A_1(k)} & \frac{\text{GR}(k)}{d(k)} e^{2ikL - 8ik^3t} \\
0 & A_1(k)
\end{pmatrix}, \quad |k| > R, \ k \in I \cup III,
\]

(3.15) \[ P(t) = \begin{pmatrix}
\frac{d_1(k)}{a(k)} & \frac{b(k)}{a(k) A_1(k)} e^{2ikL - 8ik^3t} \\
0 & \frac{a(k)}{d_1(k)}
\end{pmatrix}, \quad |k| > R, \ k \in II.
\]

Now (2.18) in \( I \cup III \) gives

\[ \text{GR}(k) e^{2ikL - 8ik^3t} = O\left(\frac{1}{k}\right), \]

hence, \( (P(t))_{12}(t, k) = O\left(\frac{1}{k}\right) \) for \( t < T \).

Finally, one obtains \( P(t)(t, k) = P(t)_{\text{diag}}(t, k) + P(t)_{\text{off}}(t, k) \), where \( P(t)_{\text{diag}}(t, k) \) is a diagonal matrix and \( P(t)_{\text{off}}(t, k) \) is off-diagonal and exponentially decaying as \( k \to \infty \), and \( P(t)_{\text{diag}} = I + O\left(\frac{1}{k}\right) \). Similarly, \( P(t)(t, k) = (P(t))_{\text{diag}}(t, k) + (P(t))_{\text{off}}(t, k) \) with the same properties. Now the fact that the multiplication by a diagonal matrix does not affect the r.h.s. of (2.35) provides the boundary conditions at \( x = 0 \) and \( x = L \). □
BIBLIOGRAPHY


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