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Partially defined cocycles and the Maslov index for a local ring

<http://aif.cedram.org/item?id=AIF_2004__54_4_875_0>
PARTIALLY DEFINED COCYCLES
AND THE MASLOV INDEX FOR A LOCAL RING

by Amedeo MAZZOLENI

1. Cocycles in general position.

DEFINITION 1. — Let $G$ be a group. Let $Y$ be a subset of $G$. We say that $Y$ is 0-dense if $Y \neq \emptyset$. Let $m \geq 1$. We say that $Y$ is $m$-dense if
\[(g_1 \cdot Y) \cap \ldots \cap (g_m \cdot Y) \neq \emptyset\]
for all $g_1, \ldots, g_m \in G$.

EXAMPLE 2. — Let $G$ be a topological group. If $U$ is an open dense subset of $G$, then $U$ is $m$-dense for all $m \geq 0$.

Proof. — This follows from
1. the set $g \cdot U$ is an open dense set, for $g \in G$;
2. the intersection of two open dense sets is an open dense set. \(\square\)

LEMMA 3. — Let $Y$ be an $m$-dense subset of $G$. Then there exists $(g_1, \ldots, g_m) \in Y^m$ such that $g_1g_{i+1}\ldots g_{i+j} \in Y$, for $1 \leq i \leq m$ and $0 \leq j \leq m - i$.

Keywords: Cocycle – $m$-dense – Simplicial set – Lagrangian – Transversal – Sympletic group.
Math. classification: 20J06 – 11E08.
Proof. — We prove the lemma by induction on $m$. The lemma is true if $m = 0$ or $m = 1$.

We suppose that $m > 1$. By the induction hypothesis there is $(g_1, \ldots, g_{m-1})$ in $Y^{m-1}$ such that the product $g_i g_{i+1} \cdots g_{i+j} \in Y$, for $1 \leq i \leq m - 1$ and $0 \leq j \leq m - 1 - i$. We choose $\bar{g}_m \in Y \cap (g_1 \cdot Y) \cap \cdots \cap (g_1 g_2 \cdots g_{m-1} \cdot Y)$. We let $g_m = (g_1 g_2 \cdots g_{m-1})^{-1} \bar{g}_m$. We have that $g_m \in (g_1 g_2 \cdots g_{i-1} \cdot Y) \cap (g_1 g_2 \cdots g_{m-1} \cdot Y)$, for $2 \leq i \leq m - 1$. Hence $g_i g_{i+1} \cdots g_m \in Y$, for $1 \leq i \leq m$. This proves the lemma.

Let $m \geq 1$. We assume that $Y$ is an $m$-dense subset of $G$. Let $1 \leq n \leq m$. We let $Y^n_{\text{gen}} = \{(g_1, \ldots, g_n) \in Y^n \mid g_i \cdots g_{i+j} \in Y \text{ for } 1 \leq i \leq n \text{ and } 0 \leq j \leq n - i\}$.

Let $B$ be an abelian group with trivial $G$-action. We consider the complex (of groups)

$$
0 \longrightarrow B \xrightarrow{0} C^1_Y \xrightarrow{d^1} C^2_Y \xrightarrow{d^2} \cdots \xrightarrow{d^{m-1}} C^m_Y
$$

where $C^m_Y = \text{Map}(Y_{\text{gen}}^m, B)$ and

$$
d^{m-1}(f)(g_1, g_2, \ldots, g_{n-1}) = f(g_2, \ldots, g_{n-1}) - f(g_1 g_2, \ldots, g_{n-1}) + \cdots + (-1)^{n-1} f(g_1, g_2, \ldots, g_{n-2}).
$$

DEFINITION 4. — Let $0 \leq n \leq m - 1$. An element of $\ker d^n$ is called $n$-cocycle for $Y$. We denote by $H^n_Y(G, B)$ the group $\ker d^n / \text{im } d^{n-1}$.

THEOREM 5. — Let $m \geq 1$. We assume that $Y$ is a $2m$-dense subset of $G$. Let $0 \leq n \leq m - 1$. Then the natural embedding $Y^n_{\text{gen}} \to G^n$ induces an isomorphism between $H^n(G, B)$ and $H^n_Y(G, B)$. Moreover, if $c$ is an $n$-cocycle for $Y$, then there is an $n$-cocycle $\tilde{c}$ such that its restriction to $Y^n_{\text{gen}}$ is $c$.

This result will be proved in Section 3. A consequence of this theorem is the following corollary:

COROLLARY 6. — Let $G$ be a topological group. Let $U$ be an open dense subset of $G$. Then the natural embedding $U^n_{\text{gen}} \to G^n$ induces an isomorphism between $H^n(G, B)$ and $H^n_U(G, B)$. Moreover, if $c$ is an $n$-cocycle for $U$, then there is an $n$-cocycle $\tilde{c}$ such that its restriction to $U^n_{\text{gen}}$ is $c$. 

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2. The generalized Mayer-Vietoris sequence.

**Definition 7.** Let $X$ be a CW-complex. We say that $X$ is $-1$-acyclic if $X \neq \emptyset$. Let $k \geq 0$. We say that $X$ is $k$-acyclic if $X$ is $-1$-acyclic and $\tilde{H}_n(X) = 0$, for all $0 \leq n \leq k$. We say that $X$ is acyclic if it is $k$-acyclic for all $k \in \mathbb{N}$.

Let $X$ be a CW-complex which is the union of a family of non-empty subcomplexes $X_\alpha$, where $\alpha$ ranges over some totally ordered index set $I$. Let $K$ be the abstract simplicial complex whose vertex set is $I$ and whose simplices are the non-empty finite subsets $J$ of $I$ such that the intersection $\cap_{\alpha \in J} X_\alpha$ is non-empty. We denote by $K^{(p)}$ the set of the $p$-simplices of $K$. Then (cf. [1] 166–167).

**Proposition 8.** We have a spectral sequence $E$ such that

$$E^{1}_{p,q} = \bigoplus_{J \in K^{(q)}} H_p\left(\bigcap_{\alpha \in J} X_\alpha\right) \Rightarrow H_{p+q}(X).$$

Let $K$ be a simplicial set. Recall that $\overline{K}$, the geometric realization of $K$, is a CW-complex. Moreover $H_*(K) = H_*(\overline{K})$. We say that $K$ is $k$-acyclic if $\overline{K}$ is $k$-acyclic. The following corollary is a consequence of the Proposition 8.

**Corollary 9.** Let $K$ be a simplicial set which is the union of a family of non-empty simplicial subsets $K_\alpha$, where $\alpha$ ranges over some index set $I$. Let $k \geq -1$. We suppose that $K_{\alpha_1} \cap K_{\alpha_2} \cap \ldots \cap K_{\alpha_n}$ is $k-n+1$-acyclic for all $1 \leq n \leq k+2$ and for all $\{\alpha_1, \ldots, \alpha_n\} \subset I$. Then $K$ is $k$-acyclic.

3. Proof of Theorem 5.

Let $X$ be a subset of the group $G$. We first assume that $1 \in Y$. We let $X_0^0 = X$. Let $n \geq 1$. We let $X_n^0 = \{(g_0, \ldots, g_n) \in X^{n+1} | g_i^{-1} g_j \in Y \text{ for all } i < j\}$. The two following assertions are straightforward.

1. $\partial_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) \in X^{n-1}_Y$, for all $(g_0, \ldots, g_n) \in X^n_\mathbb{Z}$ and for $0 \leq i \leq n$.

2. $s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n) \in X^{n+1}_Y$, for $0 \leq i \leq n$ and for all $(g_0, \ldots, g_n) \in X^n_\mathbb{Z}$. 

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We consider the simplicial set $K_Y(X)$ whose $n$-simplices are the $(g_0, \ldots, g_n) \in X^n_Y$, the face operators are the $\partial_i$’s and the degeneracy operators are the $s_i$’s. (*)

**Lemma 10.** Let $k \geq 0$. Let $X, Y \subset G$ such that $1 \in Y$. Assume that

$$X \cap (g_1 \cdot Y) \cap \ldots \cap (g_{2k} \cdot Y) \neq \emptyset$$

for all $g_1, \ldots, g_{2k} \in X$. Then $K_Y(X)$ is $(k-1)$-acyclic.

**Proof.** We prove the lemma by induction on $k$.

If $k = 0$ then $X \neq \emptyset$. Hence $K_Y(X)$ is $-1$-acyclic and the lemma is true.

We assume that $k > 0$. Let $g \in X$ and denote by $K_g$ the simplicial subset of $K_Y(X)$ whose the $n$-simplices are the $(g_0, \ldots, g_n) \in X^n_Y$ such that $g = g_0$ or $(g, g_0, \ldots, g_n) \in X^{n+1}_Y$. Clearly $K_Y(X) = \bigcup_{g \in X} K_g$.

Let $g_1, \ldots, g_m \in X$ such that $g_i \neq g_j$ for $i \neq j$. We let $K_{g_1, \ldots, g_m} = K_{g_1} \cap \ldots \cap K_{g_m}$. We will prove that $K_{g_1, \ldots, g_m}$ is $(k-m)$-acyclic, for $1 \leq m \leq k+1$ and for $(g_1, \ldots, g_m) \in X^m$.

The geometric realization of $K_g$ is a cone, hence $K_g$ is acyclic. Let $2 \leq m \leq k+1$. Let $g_1, \ldots, g_m \in X$ such that $g_i \neq g_j$ for $i \neq j$. We put $\overline{X} = X \cap (g_1 \cdot Y) \cap \ldots \cap (g_m \cdot Y)$ and $\overline{X}^n_Y = \{(g_0, \ldots, g_n) \in \overline{X}^{n+1} \mid g_i^{-1} g_j \in Y \text{ for all } i < j\}$. Then $K_{g_1, \ldots, g_m} = K_Y(\overline{X})$, the simplicial set whose the $n$-simplices are the $(g_0, \ldots, g_n) \in \overline{X}^n_Y$. Let $h_1, \ldots, h_{2(k-m+1)} \in \overline{X}$. Then

$$\overline{X} \cap (h_1 \cdot Y) \cap \ldots \cap (h_{2(k-m+1)} \cdot Y) \neq \emptyset,$$

since $m + 2(k-m+1) \leq 2k$.

Hence, by induction hypothesis, $K_{g_1, \ldots, g_m}$ is $(k-m)$-acyclic. From Corollary 9 follows that $K_Y(X)$ is $(k-1)$-acyclic. This proves the lemma. \qed

Now we assume that $1 \notin Y$. We let $X^n_Y = X$. Let $n \geq 1$. We let $X^n_Y = \{(g_0, \ldots, g_n) \in X^{n+1} \mid g_i^{-1} g_j \in Y \text{ for all } i < j\}$. Note that

1. If $i \neq j$, then $g_i \neq g_j$, for all $(g_0, \ldots, g_n) \in X^n_Y$.

2. $\partial_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \in X^{n-1}_Y$, for all $(g_0, \ldots, g_n) \in X^n_Y$ and for $0 \leq i \leq n$.

It follows from (1) and (2) that there is a simplicial set $\overline{K}_Y(X)$ whose the non degenerate $n$-simplices are the $(g_0, \ldots, g_n) \in X^n_Y$ and the face operators are the $\partial_i$’s defined above.
Note that $\overline{K}(X) = K_{Y'}(X)$, where $Y' = Y \cup \{1\}$ (see (*)).

**Lemma 11.** — Let $k \geq 0$. Let $X, Y \subset G$ such that $1 \notin Y$. We assume that
\[
X \cap (g_1 \cdot Y) \cap \ldots \cap (g_{2k} \cdot Y) \neq \emptyset
\]
for all $g_1, \ldots, g_{2k} \in G$. Then $\overline{K}(X)$ is $(k-1)$-acyclic.

**Proof.** — We have that $\overline{K}(X) = K_{Y'}(X)$, where $Y' = Y \cup \{1\}$. Clearly
\[
X \cap (g_1 \cdot Y') \cap \ldots \cap (g_{2k} \cdot Y') \neq \emptyset
\]
for all $g_1, \ldots, g_{2k} \in G$. Hence this lemma is a consequence of Lemma 10.$\square$

We consider the complex $C = (C_n, \delta_n)_{n \geq -1}$, where
1. $C_{-1} = \mathbb{Z}$,
2. $C_0 = \mathbb{Z}G$,
3. for $n \geq 1$, $C_n$ is the free abelian group generated by the elements of $G^n_Y = \{(g_0, \ldots, g_n) \in G^{n+1} \mid g_i^{-1}g_j \in Y \text{ for all } i < j\}$,
4. $\delta_0 : C_0 \to C_{-1}$ is the augmentation map,
5. for $n \geq 1$, $\delta_n : C_n \to C_{n-1}$ is defined by
\[
\delta_n(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \partial_i(g_0, \ldots, g_n).
\]

**Corollary 12.** — Let $m \geq 1$. Let $Y$ be a $2m$-dense subset of $G$. Then $H_n(C) = 0$ for all $n \leq m - 1$.

**Proof.** — This corollary is a consequence of Lemma 10 and Lemma 11.$\square$

**Proof of Theorem 5.** — Let $0 \leq n \leq m - 1$. The complex $C$ defined above is a complex of $G$-modules, where the $G$-action is defined by $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$. Then $C_k$ is free with basis $\{(1, g_1, \ldots, g_1 \ldots g_k) \mid (g_1, \ldots, g_k) \in Y^n_{\text{gen}}\}$, for $k \leq 2m$. This means that there is $(\overline{C})_{k \geq 0}$ a free $ZG$-resolution of $Z$ such that $\overline{C}_{n+1} = C_{n+1}$. Hence $H^n_Y(G, B)$ is isomorphic to $H^n(G, B)$. Clearly the isomorphism is induced by the natural embedding $Y^n_{\text{gen}} \to G^n$. This proves part one.

We now prove the second part of the theorem. We consider an $n$-cocycle $\tilde{c}$ and an $n$-cocycle for $Y$ $c$ such that the class of the restriction of
\( \tilde{c} \) to \( Y^n_{\text{gen}} \) in \( H^n_Y(G, B) \) is the same of the class of \( c \). There exists \( f \in C_Y^{n-1} \) such that \( \tilde{c} = c + d^{n-1}(f) \). But \( \text{Hom}(G^{n-1}, B) \) maps onto \( C_Y^{n-1} \). This means that there exists \( \tilde{f} \) in \( \text{Hom}(G^{n-1}, B) \) which maps to \( f \). It then follows that the \( n \)-cocycle \( \tilde{c}' \), defined by \( \tilde{c}'(g_1, g_2) = c(g_1, g_2) - \tilde{f}(g_1) - \tilde{f}(g_1) + \tilde{f}(g_1 g_2) \), maps to \( c \).

**Corollary 13.** — Let \( Y \) be a \( 2m \)-dense subset of \( G \). Let \( 0 \leq n \leq m-1 \). We consider two \( n \)-cocycles \( c, c' \). We suppose that there exists \( g \in G \) such that

\[
c(g_1, \ldots, g_n) = c'(gg_1 g^{-1}, \ldots, gg_n g^{-1}),
\]

for all \( (g_1, \ldots, g_n) \in Y^n_{\text{gen}} \). Then \( c \) and \( c' \) are cohomological equivalent.

**Proof.** — Let \( n \leq m-1 \). The set \( gYg^{-1} \) is a \( 2m \)-dense subset of \( G \). The map \( r_g : G \to G \) defined by \( r_g(h) = ghg^{-1} \) induces two homomorphisms \( i_g : H^n(G, B) \to H^n(G, B) \), \( i_g' : H^n_Y(G, B) \to H^n_{gYg^{-1}}(G, B) \) and the following commutative diagramm

\[
\begin{array}{ccc}
H^n(G, B) & \xrightarrow{i_Y} & H^n_Y(G, B) \\
\downarrow i_g & & \downarrow i_g' \\
H^n(G, B) & \xrightarrow{i_gYg^{-1}} & H^n_{gYg^{-1}}(G, B),
\end{array}
\]

where \( i_Y \) and \( i_gYg^{-1} \) denote the isomorphisms induced by the natural embeddings \( Y^n_{\text{gen}} \to G^n \) and \( (gYg^{-1})^n_{\text{gen}} \to G^n \). Note that \( i_g : H^n(G, B) \to H^n(G, B) \) is the identity map. This proves the corollary.

\[ \square \]

### 4. An application.

In the second part of this paper we give an application of Theorem 5.

Let \( A \) be a local commutative ring such that \( 2 \in A^* \). Let \( \mathfrak{m} \) denote the maximal ideal of \( A \) and \( K = A/\mathfrak{m} \). Let \( V \) be a free \( A \)-module of dimension \( 2n \) with a non-degenerate alternating form \( \varphi \). For a subset \( W \) of \( V \), we set

\[ W^\perp = \{ v \in V \mid \varphi(v, w) = 0 \text{ for all } w \in W \}. \]

A direct summand of \( V \) is called **subspace** and a **Lagrangian** for \( V \) is a subspace \( W \) of dimension \( n \) such that \( W = W^\perp \). Let \( X \) denote the set of the Lagrangians in \( V \). Let \( L_1, L_2 \in X \). We say that \( L_1 \) is **transversal** to \( L_2 \), denoted \( L_1 \pitchfork L_2 \), if \( L_1 + L_2 = V \).

We let \( \text{Sp}(V) \) the symplectic group of \((V, \varphi)\), that is

\[ \text{Sp}(V) = \{ \alpha \in \text{GL}(V) \mid \varphi(\alpha(x), \alpha(y)) = \varphi(x, y) \text{ for all } x, y \in V \}. \]
Let $W$ be a submodule of $V$. We let $\overline{W} = W \otimes_A K$ and $\overline{\varphi} : \overline{V} \times \overline{V} \to K$ denote the non-degenerate alternating form induced by $\varphi$. Finally $\overline{X}$ denotes the set of the Lagrangians in $\overline{V}$. We have

**Lemma 14.** Let $\{v_1, \ldots, v_{2n}\}$ be a basis of $V$. Then there exists a basis $\{u_1, \ldots, u_{2n}\}$ of $V$ such that $\varphi(v_i, u_j) = \delta_{ij}$.

**Proof.** The space $V'$ denotes the dual of $V$. Then $d_\varphi : V \to V'$ defined by $d_\varphi(x) = \varphi(-, x)$ is an isomorphism because $\varphi$ is non-degenerate. We consider the dual basis $\{\tilde{z}_1, \ldots, \tilde{z}_{2n} \in V'\}$ of $\{v_1, \ldots, v_{2n}\}$ and we let $u_i = d_\varphi^{-1}(z_i)$. Then $\delta_{ij} = z_i(v_j) = d_\varphi d_\varphi^{-1}(z_i)(v_j) = \varphi(v_j, u_i)$. \hfill $\square$

**Corollary 15.** Let $v_1, v, \in V$ such that $v_1, \ldots, v_n$ are linear independents in $V$. Then there exists $\{u_1, \ldots, u_n\}$ a subset of $V$ such that $\varphi(v_i, u_j) = \delta_{ij}$. Moreover, if $L_2$ is a Lagrangian of $V$ transversal to $L_1 \in X$ and $\{v_1, \ldots, v_n\}$ is a basis of $L_1$, then there exists a basis $\{w_1, \ldots, w_n\}$ of $L_2$ such that $\varphi(v_i, w_j) = \delta_{ij}$.

**Proof.** We prove only the second part of the corollary. We consider $\{v_1, \ldots, v_n\}$ a basis of $L_1$ and $\{v_{n+1}, \ldots, v_{2n}\}$ a basis of $L_2$. There is a basis $\{w_1, \ldots, w_{2n}\}$ of $V$ such that $\varphi(v_i, w_j) = \delta_{ij}$. This means that $w_1, \ldots, w_n \in L_2$. But $L_2 = L_2^\perp$, hence $\{w_1, \ldots, w_n\}$ is a basis of $L_2$. \hfill $\square$

**Corollary 16.** $X$ maps onto $\overline{X}$.

**Proof.** Let $\{\tilde{v}_1, \ldots, \tilde{v}_n \in \overline{V}\}$ be a basis of $\overline{L}$, a Lagrangian for $\overline{V}$. We consider $\{v_1, \ldots, v_n\}$ a lift of $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ in $V$ and $m = \max \{k \mid \varphi(v_i, v_j) = 0 \text{ for all } 1 \leq i, j \leq k\}$. We prove the corollary by induction on $n - m$.

If $n - m = 0$, then the corollary is true.

Let $n - m \geq 1$. We choose $u_1, \ldots, u_n \in V$ such that $\varphi(v_i, u_j) = \delta_{ij}$. We put $\tilde{u}_i = v_i$, if $i \neq m + 1$ and $\tilde{u}_{m+1} = v_{m+1} - \sum_{i=1}^{m} \varphi(v_i, v_{m+1})u_i$. Clearly $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ is a lift of $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ because $\varphi(v_i, v_{m+1}) \in M$ for all $1 \leq i \leq m$. Moreover $\varphi(\tilde{v}_i, \tilde{v}_j) = 0$ for all $1 \leq i, j \leq m + 1$. This proves the corollary. \hfill $\square$

**Corollary 17.** $\text{Sp}(V)$ acts transitively on $X$.

**Proof.** Let $L_0, L_1 \in X$. There are $\overline{L}_2, \overline{L}_3 \in \overline{X}$ such that $\overline{L}_0 \cap \overline{L}_2$
and $\overline{L}_1 \cap \overline{L}_3$. Let $L_0, L_1, L_2, L_3$ be lifts of $\overline{L}_0, \overline{L}_1, \overline{L}_2, \overline{L}_3$ in $X$. Clearly $L_0 \cap L_2$ and $L_1 \cap L_3$. We choose $\{v_1, \ldots, v_{2n}\}$ and $\{u_1, \ldots, u_{2n}\}$ two basis of $V$ such that $\{v_1, \ldots, v_n\} \subset L_0$, $\{v_{n+1}, \ldots, v_{2n}\} \subset L_1$, $\{u_1, \ldots, u_n\} \subset L_2$, $\{u_{n+1}, \ldots, u_{2n}\} \subset L_3$ and $\varphi(v_i, v_{n+j}) = \varphi(u_i, u_{n+j}) = \delta_{ij}$ for all $1 \leq i, j \leq n$. Now we consider $\alpha \in \text{GL}(V)$ such that $\alpha(v_i) = u_i$, for $1 \leq i \leq 2n$. Clearly $\alpha \cdot L_0 = L_1$ and $\varphi(\alpha(x), \alpha(y)) = \varphi(x, y)$ for all $x, y \in V$. Hence $\alpha \in \text{Sp}(V)$.

Now we consider $(L_1, L_2, L_3) \in X^3$ such that $L_i \cap L_j$ for $i \neq j$. We define $\psi : L_1 \oplus L_2 \oplus L_3 \rightarrow V$ by $\psi(v_1, v_2, v_3) = v_1 + v_2 + v_3$. Then $\psi$ is surjective and $\mathcal{K}_{123} = \ker \psi$ is free of dimension $n$. We define the quadratic form $q : \mathcal{K}_{123} \rightarrow A$ by $q(v_1, v_2, v_3) = \varphi(v_1, v_2)$. Then $q$ is a non-degenerate quadratic form and the Maslov index of $(L_1, L_2, L_3)$, denoted by $m(L_1, L_2, L_3)$, is the class of $q$ in $W(A)$.

In comparison with [3], we do not define the Maslov index for all $(L_1, L_2, L_3)$ in $X^3$, but, using theorem 5, we obtain (Theorem 24) an extension

$$0 \rightarrow I^2(A) \rightarrow \text{Sp}(V) \rightarrow \text{Sp}(V) \rightarrow 1$$

as in Theorem 2.2 of [3].

**Proposition 18.** — Let $(L_0, L_1, L_2, L_3) \in X^4$ such that $L_i \cap L_j$ for $i \neq j$. Then $m(L_1, L_2, L_3) - m(L_0, L_2, L_3) + m(L_0, L_1, L_3) - m(L_0, L_1, L_2) = 0$.

**Proof.** — The proof is exactly the same as in the proof of Proposition 1.2 of [3].

**Lemma 19.** — Let $A$ be a local ring such that $|A/\mathfrak{m}| \geq m$. Then, given $m$ Lagrangians $L_0, L_1, \ldots, L_m$, there exists a Lagrangian $L$ such that $L \cap L_i$, for $0 \leq i \leq m$.

**Proof.** — It follows from Corollary 16 that we just need to prove this lemma when $A = K$ a field.

Assume the dimension of $V$ is 2. Then $K$ has more than $m$ 1-dimensional subspaces and the lemma is true. We prove the lemma by induction on $\dim V$.

We show that there exists $v \in V$, $v \notin \cup_{i=0}^{m} L_i$. This is proved if $|K| = \infty$. Suppose $|K| = q$. Then a space of dimension $l$ has cardinality $q^l$. This means that $|\cup_{i=0}^{m} L_i| \leq (m + 1)q^m < q^{2m} = |V|$.
Let $V_1 = v_1^\perp$ and $\overline{V}_1 = V_1/\langle v \rangle$. Let $\overline{L}_i$ be the image of $L_i \cap V_1$ in $\overline{V}_1$. Then $\{\overline{L}_i \mid 0 \leq i \leq m\}$ are Lagrangians in $\overline{V}_1$. By induction on the dimension of $V$, there is a Lagrangian $\overline{L}$ in $\overline{V}_1$ such that $\overline{L} \cap \overline{L}_i$, for $0 \leq i \leq m$. We consider $L$ the subspace of $V_1$ of dimension $n$ such that $L/\langle v \rangle = \overline{L}$. Then $L$ is a Lagrangian in $V$ and $L \cap L_i$, $0 \leq i \leq m$. \hfill \Box

**Corollary 20.** Let $A$ be a local ring such that $|A/M| \geq m$. We fix $L_0 \in X$ and we consider $Y_{L_0} = \{g \in \text{Sp}(V) \mid g \cdot L_0 \cap L_0\}$. Then $Y_{L_0}$ is $m$-dense.

**Proof.** We first remark that, if $L_1 \cap L_2$, then $g \cdot L_1 \cap g \cdot L_2$, for $g \in \text{Sp}(V)$ and $L_1, L_2 \in X$. Let $g_1, \ldots, g_m \in \text{Sp}(V)$. By the previous lemma there is an $L \in X$ transversal to $g_i \cdot L_0$, for $1 \leq i \leq m$. We choose $g \in \text{Sp}(V)$ such that $g \cdot L_0 = L$. Then $g \cdot L_0 \cap g_i \cdot L_0$, $1 \leq i \leq m$. This means that $g_i^{-1}g \in Y_{L_0}$. But $g = g_i g_i^{-1}g$, hence $g \in (g_1 \cdot Y_{L_0}) \cap \ldots \cap (g_m \cdot Y_{L_0})$. \hfill \Box

Now we fix $L_0 \in X$ and define $c : (Y_{L_0})^2_{\text{gen}} \to W(A)$ as follows:

$$c(g_1, g_2) = m(L_0, g_1 \cdot L_0, g_1 g_2 \cdot L_0).$$

**Proposition 21.** Let $A$ be a local ring such that $|A/M| \geq 6$. Then $c$ is a 2-cocycle for $Y_{L_0}$ which defines a central extension

$$(*) \quad 0 \to W(A) \to \widehat{\text{Sp}(V)} \to \text{Sp}(V) \to 1$$

This extension is independent of the choice of $L_0$.

Note that $A/M$ is a field. Hence $|A/M| \geq 6$ implies that $|A/M| \geq 7$.

**Proof.** Let $L_1, L_2, L_3 \in X$ such that $L_i \cap L_j$, for $i \neq j$. We remark that $m(L_1, L_2, L_3) = m(g \cdot L_1, g \cdot L_2, g \cdot L_3)$, for $g \in G$. It then follows that $c$ is a 2-cocycle for $Y_{L_0}$. Hence, using Theorem 5 and Corollary 20, we see that $c$ induces $(*)$.

We are now left with proving that $(*)$ is independent of the choice of $L_0$.

Let $L_1 \in X$. We consider $c'$, the 2-cocycle for $Y_{L_1}$ defined by

$$c'(g_1, g_2) = m(L_1, g_1 \cdot L_1, g_1 g_2 \cdot L_1).$$

We choose $g \in G$ such that $g \cdot L_0 = L_1$. Let $(g_1, g_2) \in (Y_{L_1})^2_{\text{gen}}$. We have that

$$c'(g_1, g_2) = m(L_1, g_1 \cdot L_1, g_1 g_2 \cdot L_1) = m(g \cdot L_0, g g^{-1} g_1 g \cdot L_0, g g^{-1} g_1 g_2 g \cdot L_0) = m(L_0, g^{-1} g_1 g \cdot L_0, g^{-1} g_1 g_2 g \cdot L_0) = c(g^{-1} g_1 g, g^{-1} g_2 g).$$

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Hence the proposition follows from Corollary 13.

In the last part of this paper we will prove that $c$ can be reduced to

$$
\tilde{c} : (Y_{L_0})^2_{\text{gen}} \rightarrow I^2(A).
$$

We consider the map $t : Y_{L_0} \rightarrow W(A)$, defined by $t(g) = \langle \text{id}_n \rangle$, where $\text{id}_n$ denotes the bilinear space $(A^n, \tau_n)$ defined by

$$
\tau_n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1y_1 + \ldots + x ny_n.
$$

Let $(g_1, g_2) \in (Y_{L_0})^2_{\text{gen}}$. We put $c'(g_1, g_2) = c(g_1, g_2) - t(g_1) - t(g_2) + t(g_1 g_2)$.

**Lemma 22.** — $c'$ is a 2-cocycle for $Y_{L_0}$ and $c'((Y_{L_0})^2_{\text{gen}}) \subset I(A)$.

Let $L, L_0 \in X$ such that $L \cap L_0$. We choose $B = \{v_1, \ldots, v_n\}$ a basis of $L$ and $B_0 = \{u_1, \ldots, u_n\}$ a basis of $L_0$. $M((L, B), (L_0, B_0))$ denotes the matrix $M((L,B),(L_0,B_0))$ is in $GL_n(A)$ because $L \cap L_0$.

**Proposition 23.** — Let $(L_1, L_2, L_3) \in X^3$ such that $L_i \cap L_j$ for $i \neq j$. We choose $B_1 = \{v_1, \ldots, v_n\}$ a basis of $L_1$, $B_2 = \{u_1, \ldots, u_n\}$ a basis of $L_2$ and $B_3 = \{w_1, \ldots, w_n\}$ a basis of $L_3$. Then

$$
\partial(m(L_1, L_2, L_3)) = (-1)^{n(n-1)/2} \cdot \overline{\text{det}}(M_{23}) \cdot \overline{\text{det}}(M_{13})^{-1} \cdot \overline{\text{det}}(M_{12}),
$$

where $M_{ij}$ denotes the matrix $M((L_i, B_i), (L_j, B_j))$, the map $\partial : W(A) \rightarrow A^*/(A^*)^2$ denotes the signed determinant and $\overline{\text{det}}$ denotes the homomorphism between $GL_n(A)$ and $A^*/(A^*)^2$ induced by the determinant.

**Proof.** — The proof is exactly the same as the first part of the proof of Proposition 2.1 of [3].

We fix $L_0 \in X$. Let $B_0 = \{v_i \mid 1 \leq i \leq n\}$ be a basis of $L_0$. Then $g \cdot B_0 = \{g \cdot v_i \mid 1 \leq i \leq n\}$ is a basis of $g \cdot L_0$, for $g \in \text{Sp}(V)$. We consider the map $t_{L_0} : Y_{L_0} \rightarrow I(A)$, defined by

$$
t_{L_0}(g) = \langle \text{det} \left( M((L_0, B_0), (g \cdot L_0, g \cdot B_0)) \right), (-1)^{n(n-1)/2} \rangle.
$$

Let $(g_1, g_2) \in (Y_{L_0})^2_{\text{gen}}$. We let $\overline{c}(g_1, g_2) = c'(g_1, g_2) - t_{L_0}(g_1) - t_{L_0}(g_2) + t_{L_0}(g_1 g_2)$.

**Theorem 24.** — Let $A$ be a local ring such that $|A/\mathfrak{M}| \geq 7$. Then $\overline{c}$ is a 2-cocycle for $Y_{L_0}$ which induces a central extension

$$
0 \rightarrow I^2(A) \rightarrow \text{Sp}(V) \rightarrow \text{Sp}(V) \rightarrow 1.
$$
Acknowledgments. — I would like to thank Dr. Gael Collinet for the helpful discussions and for having send me his thesis [2]. In his paper I have in particular found many useful informations about the study of the acyclicity of semi-simplicial sets.

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Manuscrit reçu le 13 novembre 2003,
accepté le 13 janvier 2004.

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