



ANNALES

DE

L'INSTITUT FOURIER

Valentin BLOMER

Non-vanishing of class group L -functions at the central point

Tome 54, n° 4 (2004), p. 831-847.

http://aif.cedram.org/item?id=AIF_2004__54_4_831_0

© Association des Annales de l'institut Fourier, 2004, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

NON-VANISHING OF CLASS GROUP L -FUNCTIONS AT THE CENTRAL POINT

by Valentin BLOMER

1. Introduction.

The behaviour of L -functions in the critical strip has received a lot of attention from the first proof of the prime number theorem up to now. Especially the value at the central point is an important property and subject of intensive studies. By now there are numerous results that many members – in some cases even a positive proportion – of a certain family of L -functions do not vanish at the central point, see e.g. [1, 3, 7, 9, 10, 11, 13, 15, 17] and the references given there. This is of interest in various aspects such as the Birch-Swinnerton-Dyer conjecture, the Siegel zero (see [7]) and the theory of modular forms of half-integral weight (see [16, 18]). A large number of old and new results around this theme can be found in [12].

In this article we consider L -functions attached to class group characters of an imaginary quadratic field. Let $K = \mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic field of discriminant $-D$. We denote its class group by \mathcal{C} and write $h = |\mathcal{C}|$. Each character $\chi \in \hat{\mathcal{C}}$ gives rise to an L -function

$$L_K(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s},$$

the summation being taken over all nonzero integral ideals \mathfrak{a} . The series is absolutely convergent for $\Re s > 1$, and the L -function can be extended to the entire complex plane having a functional equation that relates s to $1 - s$. For real characters $L_K(s, \chi)$ is the product of two Dirichlet L -functions, while for complex characters $L_K(s, \chi)$ comes from the cusp form $\sum_{\mathfrak{a}} \chi(\mathfrak{a})e(zN\mathfrak{a})$ of weight 1 for $\Gamma_0(D)$ and character χ_D .

We consider only the central point $s = 1/2$ although the method works for every point on the critical line $\Re s = 1/2$. We shall obtain the following result.

THEOREM. — *There is an absolute constant $c > 0$ such that*

$$(1.1) \quad \frac{1}{h} \#\{\chi \in \hat{\mathcal{C}} \mid L_K(1/2, \chi) \neq 0\} \geq c \prod_{p|D} \left(1 - \frac{1}{p}\right)$$

for sufficiently large D .

Assuming the Riemann hypothesis for the $L_K(s, \chi)$, we have of course $L_K(\sigma, \chi) > 0$ for $\sigma > 1/2$ and $L_K(1/2, \chi) \geq 0$. Since we appeal to Siegel's lower bound for $L(1, \chi_{-D})$, we do not know how large D must be chosen to ensure the validity of (1.1). The constant c is in principle computable, but we did not make any effort to do so because our method yields only a very small value for c , something around 10^{-6} .

According to general conjectures on zeros of L -functions [8] coming from random matrix theory we would expect that 100 percent of the $L_K(s, \chi)$ do not vanish at $s = 1/2$. In fact, the present family of L -functions has been studied by Fouvry and Iwaniec [6] who showed under the Riemann hypothesis for the $L_K(s, \chi)$ that the distribution of low-lying zeros is governed by the symplectic group.

2. Preliminaries.

We introduce some notation which will be kept fixed throughout the paper. The letter p is always reserved for (rational) prime numbers. Let $\chi_{-D} = \left(\frac{\cdot}{-D}\right)$ be the Kronecker symbol, and let w be the number of units in $\mathbb{Q}(\sqrt{-D})$, i.e. $w = 2$ for $D > 4$. Let $0 < \eta < 1/4$ be a parameter to be chosen later and define

$$\mathcal{Q} := \{q \leq D^\eta \mid \mu^2(q) = 1, (p \mid q \Rightarrow \chi_{-D}(p) = 1)\}.$$

For $q \in \mathcal{Q}$ and $\chi \in \hat{\mathcal{C}}$ we write as in [5]

$$\gamma_q(\chi) = 2^{-\omega(q)} \sum_{\mathfrak{q}: N\mathfrak{q}=q} \chi(\mathfrak{q}).$$

For the proof of the theorem we compare different weighted averages over the set of L -functions for $\mathbb{Q}(\sqrt{-D})$. We consider

$$\mathcal{L}_1 := \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \left(\sum_{q \in \mathcal{Q}} \lambda(q) \gamma_q(\chi) \right) L_K(1/2, \chi)$$

and

$$\mathcal{L}_2 := \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \left(\sum_{q \in \mathcal{Q}} \lambda(q) \gamma_q(\chi) \right)^2 |L_K(1/2, \chi)|^2$$

where the $\lambda(q)$'s are suitably chosen real numbers, see (4.5) and (4.7) below. By the Cauchy-Schwarz inequality we have

$$(2.1) \quad \#\{\chi \in \hat{\mathcal{C}} \mid L_K(1/2, \chi) \neq 0\} \geq h \mathcal{L}_1^2 \mathcal{L}_2^{-1}.$$

The sum $\sum \lambda(q) \gamma_q(\chi)$ is supposed to work as a mollifier and to smooth out irregularities in the behaviour of the $L_K(1/2, \chi)$ so that not too much is lost when we apply the Cauchy-Schwarz inequality.

It is *a priori* not clear whether the set \mathcal{Q} contains elements other than 1, so the character sum may be trivial if the class number is extraordinarily small and there are only few or no small split primes. However, it turns out that the smaller the class number is, i.e. the less effective the mollifier works, the less variation exists, roughly speaking, in the values of $L_K(1/2, \chi)$. Thus our method works even in the improbable case of an exceptionally small class number, and we need not to appeal to any unproven hypothesis.

To estimate the right side of (2.1) we need asymptotic formulae for

$$\mathcal{L}_1^{(q)} := \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \gamma_q(\chi) L_K(1/2, \chi)$$

and

$$\mathcal{L}_2^{(q_1, q_2)} := \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \gamma_{q_1}(\chi) \gamma_{q_2}(\chi) |L_K(1/2, \chi)|^2.$$

This will be done in Section 3 and relies ultimately on deep results from spectral theory of modular forms and equidistribution properties of Heegner points ([4, 5]).

3. The first and second mollified moment.

It is not hard to obtain an asymptotic expansion for $\mathcal{L}_1^{(q)}$. As in [5], section 3, we see

$$\frac{w}{h} \sum_{\chi \in \hat{C}} \chi(\mathfrak{q}) L_K(s, \chi) = \frac{\zeta(2s)}{(N\mathfrak{q})^s} + \frac{\zeta(2-2s)\Gamma(1-s)}{\Gamma(s)(N\mathfrak{q})^{1-s}} \left(\frac{\sqrt{D}}{2\pi}\right)^{1-2s} + O_s(e^{-\sqrt{D}/N\mathfrak{q}}).$$

The main term on the right-hand side has a removable singularity at $s = 1/2$. We obtain

$$(3.1) \quad \mathcal{L}_1^{(q)} = \frac{1}{w\sqrt{q}} \left(\log\left(\frac{\sqrt{D}}{q}\right) + c_0 \right) + O(e^{-\sqrt{D}/q})$$

where $c_0 = \gamma - \log(8\pi)$ (and γ is Euler’s constant).

We proceed to establish an asymptotic formula for $\mathcal{L}_2^{(q_1, q_2)}$. The analysis is similar as in [5], sections 4 and 6, but we need a more precise result here. Let

$$(3.2) \quad \Phi(s, \chi) := \left(\frac{\sqrt{D}}{2\pi}\right)^s \Gamma(s) L_K(s, \chi)$$

be the L -function with its local factor at infinity. It turns out to be more convenient to replace $L_K(s, \chi)$ by

$$\Phi(s, \chi) + \delta_\chi \frac{h}{ws(1-s)} = \int_1^\infty (u^{s-1} + u^{-s}) \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \exp\left(-\frac{2\pi u N \mathfrak{a}}{\sqrt{D}}\right) du$$

where $\delta_\chi = 1$ if $\chi = 1$ or else it vanishes. By Burgess’ estimate [2],

$$(3.3) \quad L(1/2 + it, \chi_{-D}) \ll_{t,\varepsilon} D^{3/16+\varepsilon},$$

we obtain

$$(3.4) \quad \mathcal{L}_2^{(q_1, q_2)} = 2D^{-1/2}(\mathcal{M}^{(q_1, q_2)} - 16hw^{-2}) + O(D^{-1/16+\varepsilon})$$

where

$$(3.5) \quad \mathcal{M}^{(q_1, q_2)} := \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \gamma_{q_1}(\chi) \gamma_{q_2}(\chi) |\Phi(1/2, \chi) + 4\delta_\chi h w^{-1}|^2$$

$$= 4 \int_1^\infty \int_1^\infty (u_1 u_2)^{-1/2} \mathcal{M}^{(q_1, q_2)}(u_1, u_2) du_1 du_2$$

with

$$\mathcal{M}^{(q_1, q_2)}(u_1, u_2)$$

$$= \frac{1}{h} \sum_{\chi \in \hat{\mathcal{C}}} \gamma_{q_1}(\chi) \gamma_{q_2}(\chi) \sum_{\mathfrak{a}_1} \sum_{\mathfrak{a}_2} \chi(\mathfrak{a}_1) \bar{\chi}(\mathfrak{a}_2) \exp\left(-\frac{2\pi(u_1 N \mathfrak{a}_1 + u_2 N \mathfrak{a}_2)}{\sqrt{D}}\right).$$

We write

$$(3.6) \quad d := (q_1, q_2), \quad q := \frac{q_1 q_2}{d^2}$$

and \sim for the ideal equivalence. Note that d and q are coprime by the definition of \mathcal{Q} . By orthogonality we see for $q_1, q_2 \in \mathcal{Q}$

$$\mathcal{M}^{(q_1, q_2)}(u_1, u_2)$$

$$= 2^{-\omega(q_1) - \omega(q_2)} \sum_{N \mathfrak{d}_1 = d} \sum_{N \mathfrak{d}_2 = d} \sum_{N \mathfrak{q} = q} \sum_{\mathfrak{a}_1} \sum_{\mathfrak{a}_2 \sim \mathfrak{a}_1 \mathfrak{d}_1 \mathfrak{d}_2 \mathfrak{q}} \exp\left(-\frac{2\pi(u_1 N \mathfrak{a}_1 + u_2 N \mathfrak{a}_2)}{\sqrt{D}}\right)$$

$$= 2^{-\omega(dq)} \sum_{r|d} 2^{-\omega(r)} \sum_{N \mathfrak{r} = r} \sum_{N \mathfrak{q} = q} \sum_{\mathfrak{a}_1} \sum_{\mathfrak{a}_2 \sim \mathfrak{a}_1 \mathfrak{r}^2 \mathfrak{q}} \exp\left(-\frac{2\pi(u_1 N \mathfrak{a}_1 + u_2 N \mathfrak{a}_2)}{\sqrt{D}}\right).$$

We write $\mathfrak{a}_1 \mathfrak{r}^2 \mathfrak{q} = l \mathfrak{a}$ with $l \in \mathbb{N}$ and a primitive ideal \mathfrak{a} , i.e. an ideal with $n \nmid \mathfrak{a}$ for all $n > 1$). If we use for the moment the abbreviation $E := \exp\left(-\frac{2\pi}{\sqrt{D}}\left(\frac{u_1 l^2}{r^2 q} N \mathfrak{a} + u_2 N \mathfrak{a}_2\right)\right)$, we obtain

$$\mathcal{M}^{(q_1, q_2)}(u_1, u_2) = 2^{-\omega(dq)} \sum_{r|d} 2^{-\omega(r)} \sum_{N \mathfrak{r} = r} \sum_{N \mathfrak{q} = q} \frac{2}{w} \sum_{l \in \mathbb{N}} \sum_{\substack{\mathfrak{a} \text{ primitive} \\ \mathfrak{r}^2 \mathfrak{q} | l \mathfrak{a}}} \sum_{\mathfrak{a}_2 \sim \mathfrak{a}} E$$

$$= 2^{-\omega(dq)} \frac{2}{w} \sum_{r|d} 2^{-\omega(r)} \sum_{l \in \mathbb{N}} \sum_{\substack{\mathfrak{a} \text{ primitive} \\ N \mathfrak{a} \equiv 0 \left(\frac{r^2 q}{(r^2 q, l)}\right)}} \sum_{\substack{(\mathfrak{r}, \mathfrak{q}): N \mathfrak{r} = r, N \mathfrak{q} = q \\ \mathfrak{r}^2 \mathfrak{q} | \mathfrak{a}}} \sum_{\mathfrak{a}_2 \sim \mathfrak{a}} E$$

$$= 2^{-\omega(d)} \frac{2}{w} \sum_{r|d} \sum_{l \in \mathbb{N}} 2^{-\omega\left(\frac{r^2q}{(r^2q,l)}\right)} \sum_{\substack{\mathfrak{a} \text{ primitive} \\ N\mathfrak{a} \equiv 0 \pmod{\frac{r^2q}{(r^2q,l)}}}} \sum_{\mathfrak{a}_2 \sim \mathfrak{a}} E$$

since $(r, q) = 1$. We also used $(\mathfrak{r}^2\mathfrak{q}, \bar{\mathfrak{r}}^2\bar{\mathfrak{q}}) = 1$ so that $N(\mathfrak{r}^2\mathfrak{q}, l) = (r^2q, l)$. For a primitive ideal $\mathfrak{a} = \left(N\mathfrak{a}, \frac{b+i\sqrt{D}}{2}\right)$ let $z_{\mathfrak{a}} = \frac{b+i\sqrt{D}}{2N\mathfrak{a}}$ be the Heegner point. Since \mathfrak{a}^{-1} is generated by 1 and $z_{\mathfrak{a}}$, the last two sums equal

$$\frac{1}{w} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \sum_{\substack{\mathfrak{a} \text{ primitive} \\ N\mathfrak{a} \equiv 0 \pmod{\frac{r^2q}{(r^2q,l)}}}} \exp\left(-\frac{2\pi N\mathfrak{a}}{\sqrt{D}} \left(\frac{u_1 l^2}{r^2q} + u_2 |m + n z_{\mathfrak{a}}|^2\right)\right).$$

We call the contribution from the terms with $n = 0$ the *diagonal part* $\mathcal{M}_0^{(q_1, q_2)}(u_1, u_2)$. Pulling out the greatest common divisor of l and m we see by Mellin inversion

$$\begin{aligned} (3.7) \quad & \mathcal{M}_0^{(q_1, q_2)}(u_1, u_2) \\ &= 2^{-\omega(d)} \frac{2}{w} \sum_{r|d} \sum_{\substack{l, m \in \mathbb{N} \\ (l, m) = 1}} 2^{-\omega\left(\frac{r^2q}{(r^2q,l)}\right)} \sum_{N\mathfrak{a} \equiv 0 \pmod{\frac{r^2q}{(r^2q,l)}}} \exp\left(-\frac{2\pi N\mathfrak{a}}{\sqrt{D}} \left(\frac{u_1 l^2}{r^2q} + u_2 m^2\right)\right) \\ &= 2^{-\omega(d)} \frac{2}{w} \sum_{r|d} \frac{1}{2\pi i} \int_{(2)} \Phi(v, 1) \sum_{(l, m) = 1} \left(\left(\frac{u_1 l^2}{r^2q} + u_2 m^2\right) \frac{r^2q}{(r^2q, l)}\right)^{-v} dv \end{aligned}$$

where Φ is given by (3.2). Using the Mellin inversion formula for the Euler beta-functor, the sum over l and m can be rewritten as

$$\begin{aligned} (3.8) \quad & \sum_{l \in \mathbb{N}} \left(\frac{r^2q}{(r^2q, l)}\right)^{-v} \sum_{\substack{m \in \mathbb{N} \\ (l, m) = 1}} \frac{1}{\Gamma(v)} \frac{1}{2\pi i} \int_{(1)} \Gamma(z)\Gamma(v-z) \left(\frac{u_1 l^2}{r^2q}\right)^{-z} (u_2 m^2)^{z-v} dz \\ &= \frac{1}{\Gamma(v)} \frac{1}{2\pi i} \int_{(1)} \Gamma(z)\Gamma(v-z) u_1^{-z} u_2^{z-v} (r^2q)^{z-v} \sum_{l \in \mathbb{N}} \frac{(r^2q, l)^v}{l^{2z}} \sum_{\substack{m \in \mathbb{N} \\ (m, l) = 1}} \frac{1}{m^{2v-2z}} dz. \end{aligned}$$

Here and in the following we may change the order of integration and summation because of absolute convergence. The double sum in (3.8) equals

$$\zeta(2v-2z) \sum_{l \in \mathbb{N}} \frac{(r^2q, l)^v \prod_{p|l} (1-p^{2z-2v})}{l^{2z}} = \zeta(2v-2z) \frac{\zeta(2z)}{\zeta(2v)} T(r^2q; z, v)$$

where

$$T(n; z, v) = \prod_{p^\gamma \parallel n} \frac{1 - p^{-2z}}{1 - p^{-2v}} \left(1 + (1 - p^{2z-2v}) \left(\sum_{k=1}^{\gamma-1} \frac{p^{kv}}{p^{2kz}} + \sum_{k=\gamma}^{\infty} \frac{p^{\gamma v}}{p^{2kz}} \right) \right),$$

i.e.

$$(3.9) \quad T(r^2q; z, v) = \prod_{p|q} \frac{p^v + p^{2v-2z}}{p^v + 1} \prod_{p|r} \frac{p^{3v-4z} + p^v - p^{v-2z} + p^{2v-2z}}{p^v + 1}$$

for coprime, squarefree integers q, r . By (3.5) and (3.7) we thus obtain as the diagonal contribution $\mathcal{M}_0^{(q_1, q_2)}$ to $\mathcal{M}^{(q_1, q_2)}$

$$\begin{aligned} \mathcal{M}_0^{(q_1, q_2)} &= 4 \int_1^\infty \int_1^\infty (u_1 u_2)^{-1/2} 2^{-\omega(d)} \frac{2}{w} \sum_{r|d} \frac{1}{2\pi i} \int_{(2)} \Phi(v, 1) \times (3.8) \, dv du_1 du_2 \\ &= 2^{-\omega(d)} \frac{8}{w} \sum_{r|d} \frac{1}{2\pi i} \int_{(2)} \left(\frac{\sqrt{D}}{2\pi r^2 q} \right)^v \frac{\zeta(v)}{\zeta(2v)} L(v, \chi_D) R(v) dv \end{aligned}$$

with

$$(3.10) \quad R(v) := \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma(z)\Gamma(v-z)\zeta(2z)\zeta(2(v-z))(r^2q)^z T(r^2q; z, v)}{(z - \frac{1}{2})(v - z - \frac{1}{2})} dz.$$

The integrand in (3.10) has a double pole at $z = 1/2$, and we decompose $R(v) = R_1(v) + R_2(v)$ where

$$R_1(v) := \operatorname{res}_{z=\frac{1}{2}} \frac{\Gamma(z)\Gamma(v-z)\zeta(2z)\zeta(2(v-z))(r^2q)^z T(r^2q; z, v)}{(z - \frac{1}{2})(v - z - \frac{1}{2})}$$

and $R_2(v)$ is given by the integral in (3.10) on the line $\Re z = \sigma$ with $0 < \sigma < \min(1/2, \Re v - 1/2)$. Observe that

$$(3.11) \quad R_1(v) = \frac{\pi(r^2q)^{1/2} T(r^2q; 1/2, 1)}{2(v-1)^3} + \dots$$

has a triple pole at $v = 1$ while R_2 is holomorphic in $\Re v > 1/2$. We evaluate $\mathcal{M}_0^{(q_1, q_2)}$ by moving the integration to the line $\Re v = 1/2 + \varepsilon$. Here we have $R_1(v) + R_2(v) \ll (r^2q)^{1/2+\varepsilon}$, hence we obtain by the Burgess inequality (3.3)

$$\begin{aligned}
 \mathcal{M}_0^{(q_1, q_2)} &= 2^{-\omega(d)} \frac{8}{w} \sum_{r|d} \left(\operatorname{res}_{v=1} \left(\frac{\sqrt{D}}{2\pi r^2 q} \right)^v \frac{\zeta(v)}{\zeta(2v)} L(v, \chi_D) R_1(v) \right. \\
 (3.12) \quad &\quad \left. + \frac{3\sqrt{D}}{\pi^3 r^2 q} L(1, \chi_D) R_2(1) \right) + O(D^{7/16+\varepsilon}).
 \end{aligned}$$

To evaluate the residue, we need the derivatives of $T(r^2q; z, v)$ at $z = 1/2$ and $v = 1$. We write $l(p) := (\log p)(1 + O(1/p))$. It is easy to see that

$$(3.13) \quad T(p^\gamma; 1/2, 1)^{-1} \frac{\partial^k}{\partial z^k} \frac{\partial^m}{\partial v^m} T(p^\gamma; 1/2, 1) = c(m, k, \gamma) (l(p))^{m+k}$$

for $m, k \in \mathbb{N}_0$ with a certain constant $c(m, k, \gamma) \in \mathbb{R}$. (We have $c(m, k, \gamma) = (-2)^k (\gamma + 1)^{-1} \sum_{\nu=1}^\gamma \nu^{m+k}$ if $m + k \geq 1$, but this is of little interest here.) For $k \in \mathbb{N}$ we write

$$S_k(n) = \sum_{p|n} l(p)^k.$$

Let \mathcal{E}_j be the finite set of functions $\psi(n) = \prod_{k=1}^j S_k(n)^{\nu_k}$ with $\sum_{k=1}^j k\nu_k = j$ (e.g. $\mathcal{E}_3 = \{S_1(n)^3, S_1(n)S_2(n), S_3(n)\}$). Let $\mathcal{F}_0 = \{1\}$ and \mathcal{F}_j ($j \geq 1$) be the \mathbb{Q} -vectorspace generated by \mathcal{E}_j . Then we obtain for squarefree n by (3.13) and the rule for differentiating a product

$$(3.14) \quad n^{\gamma/2} T(n^\gamma; 1/2, 1)^{-1} \frac{\partial^k}{\partial z^k} \frac{\partial^m}{\partial v^m} n^{\gamma(z-v)} T(n^\gamma; z, v) \Big|_{(z,v)=(\frac{1}{2}, 1)} = \rho_{m,k,\gamma}(n)$$

with $\rho_{m,k,\gamma} \in \mathcal{F}_{m+k}$.

The off-diagonal part, $\mathcal{M}_+^{(q_1, q_2)} := \mathcal{M}^{(q_1, q_2)} - \mathcal{M}_0^{(q_1, q_2)}$, can be evaluated exactly as in [5], sections 8 - 14. We get

$$(3.15) \quad \mathcal{M}_+^{(q_1, q_2)} = 16hw^{-2} + 2^{-\omega(d)} \frac{12h}{\pi w^2} \sum_{r|d} \frac{I_{r^2q}}{\sqrt{r^2q}} + O((q_1 q_2)^6 D^{13/28+\varepsilon})$$

where

$$I_{r^2q} = \frac{16}{2\pi i} \int_{(\frac{1}{2})} \left(\frac{\sqrt{r^2q}}{2\pi} \right)^v \Gamma(v) \zeta(v) \zeta(v+1) T_1(r^2q; v) v^{-2} dv$$

and (cf. [5], (11.5))

$$(3.16) \quad T_1(r^2q; v) = \prod_{p|q} \frac{p + p^{1-v}}{p + 1} \prod_{p|r} \frac{p + p^{1-v} + p^{1-2v} - p^{-v}}{p + 1}$$

for coprime squarefree integers q, r . Note that in [5], (11.16) a factor 4 is missing. We shall see in a moment that the terms for I_{r^2q} and $R_2(1)$ cancel completely. This lucky fact might indicate that there is a more elementary way of computing $\mathcal{L}_2^{(q_1, q_2)}$.

LEMMA 3.1. — We have

$$\frac{24L(1, \chi_{-D})}{\pi^3 w} \frac{R_2(1)}{r^2 q} + \frac{12h}{\pi w^2} \frac{I_{r^2q}}{\sqrt{r^2q}} = 0.$$

Proof. — By the class number formula we have to show

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(\frac{1}{4})} \frac{\Gamma(z)\Gamma(1-z)\zeta(2z)\zeta(2-2z)(r^2q)^{z-\frac{1}{2}} T(r^2q; z, 1)}{\pi (\frac{1}{2} - z)^2} dz \\ &= \frac{4}{2\pi i} \int_{(\frac{1}{2})} \frac{\Gamma(v)\zeta(v)\zeta(1+v)(r^2q)^{v/2} T_1(r^2q; v)}{(2\pi)^v v^2} dv. \end{aligned}$$

We write $\theta(z) = \pi^{-z}\Gamma(z)\zeta(2z)$ and use the duplication formula of the Γ -function. Then the above is equivalent to

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(\frac{1}{4})} \frac{\theta(z)\theta(1-z)(r^2q)^{z-\frac{1}{2}} T(r^2q; z, 1)}{(\frac{1}{2} - z)^2} dz \\ &= \frac{2}{2\pi i} \int_{(\frac{1}{2})} \frac{\theta(\frac{v}{2})\theta(\frac{v+1}{2})(r^2q)^{v/2} T_1(r^2q; v)}{v^2} dv. \end{aligned}$$

Now we observe that $(r^2q)^{-v/2} T(r^2q; \frac{1-v}{2}, 1) = (r^2q)^{v/2} T_1(r^2q; v)$ by (3.9) and (3.16), and $\theta(z) = \theta(\frac{1}{2} - z)$ by the functional equation of the ζ -function. Thus the lemma follows from a change of variables $z \mapsto \frac{1-z}{2}$ in the first integral.

Collecting (3.4), (3.12), (3.14), (3.15), and Lemma 3.1, we finally arrive at the following result.

PROPOSITION 3.2. — For coprime, squarefree integers q, r let

$$\alpha(r^2q) := \tau(r^2q)^{-1}T(r^2q; 1/2, 1) = \prod_{p|q} \frac{p}{p+1} \prod_{p|r} \frac{p - \frac{1}{3}}{p+1}.$$

Then

$$\begin{aligned} \mathcal{L}_2^{(q_1, q_2)} &= 2^{-\omega(d)} \sum_{r|d} \frac{\alpha\tau(r^2q)}{\sqrt{r^2q}} \sum_{j_1+\dots+j_4 \leq 3} c_{j_1, \dots, j_4} (\log D)^{j_1} L^{(j_2)}(1, \chi_{-D}) \rho_{j_3}(q) \tilde{\rho}_{j_4}(r) \\ &\quad + O(D^{-1/28+12\eta+\epsilon}) \end{aligned}$$

where d and q are given by (3.6), $c_{j_1, \dots, j_4} \in \mathbb{R}$ and $\rho_j, \tilde{\rho}_j \in \mathcal{F}_j$.

The constants occuring in the main term can be calculated explicitly. However, we only need to know the following which can directly be seen from (3.11) and (3.12) : There is a positive constant $C > 0$ such that

$$(3.17) \quad c_{3,0,0,0} = \frac{1}{8}C, \quad c_{2,1,0,0} = \frac{3}{4}C, \quad c_{1,2,0,0} = \frac{3}{2}C, \quad c_{0,3,0,0} = C.$$

4. End of the proof.

In this section we choose the variables $\lambda(q)$ as to maximize the right-hand side of (2.1). To this end, we try to diagonalize the quadratic form \mathcal{L}_2 . For $d \in \mathbb{N}$ let

$$\mathcal{Q}_d := \{q \in \mathbb{N} \mid dq \in \mathcal{Q}\} = \{q \in \mathcal{Q} \mid (q, d) = 1, q \leq d^{-1}D^\eta\}.$$

We start with the following lemma.

LEMMA 4.1. — Let $\rho \in \mathcal{F}_j$.

a) For $(q_1, q_2) = 1$ we have $\rho(q_1q_2) = \sum_{i=0}^j \rho'_i(q_1)\rho'_{j-i}(q_2)$ with $\rho'_i \in \mathcal{F}_i$.

b) We have $\sum_{f|g} \frac{\rho(f)}{f} \ll_j (\log \log 3g)^{j+1}$.

c) The Dirichlet convolution $\mu * \rho$ is supported on integers with at most j prime divisors and satisfies $\sum_{n \leq x} \frac{|\mu * \rho(n)|}{n} \ll_j (\log x)^j$.

Proof. — The first part of the lemma is clear since $S_j(q_1q_2) = S_j(q_1) + S_j(q_2)$ for $(q_1, q_2) = 1$. For the second part we note that

$$\begin{aligned} \sum_{f|g} \frac{\rho(f)}{f} &\ll \sum_{f|g} \frac{1}{f} \left(\sum_{p|f} \log p \right)^j \\ &\leq \sum_{p_1|g} \dots \sum_{p_j|g} \frac{\log p_1 \dots \log p_j}{[p_1, \dots, p_j]} \sum_{f|g} \frac{1}{f} \ll (\log \log 3g)^{j+1} \end{aligned}$$

where $[p_1, \dots, p_j]$ denotes the least common multiple of p_1, \dots, p_j .

The last part follows similarly by pulling out the sums over the primes in $\sum_{d|n} \mu(n/d)\rho(d)$ and observing that $\sum_{p_1 \dots p_k | d} \mu(n/d)$ vanishes unless n is the squarefree kernel of $p_1 \dots p_k$.

By the first part of the lemma we find that

$$\begin{aligned} &\sum_{q_1, q_2 \in \mathcal{Q}} \lambda(q_1)\lambda(q_2)2^{-\omega((q_1, q_2))} \sum_{r|(q_1, q_2)} \frac{\alpha\tau\left(r^2 \frac{q_1q_2}{(q_1, q_2)^2}\right)}{\sqrt{r^2 \frac{q_1q_2}{(q_1, q_2)^2}}} \rho_{j_3}\left(\frac{q_1q_2}{(q_1, q_2)^2}\right) \tilde{\rho}_{j_4}(r) \\ &= \sum_{d \in \mathcal{Q}} \sum_{\substack{q_1, q_2 \in \mathcal{Q}_d \\ (q_1, q_2) = 1}} \lambda(dq_1)\lambda(dq_2)2^{-\omega(d)} \sum_{r|d} \frac{\alpha\tau(r^2q_1q_2)}{\sqrt{r^2q_1q_2}} \rho_{j_3}(q_1q_2)\tilde{\rho}_{j_4}(r) \\ &= \sum_{i_1+i_2=j_3} \sum_{d \in \mathcal{Q}} \sum_{\substack{q_1, q_2 \in \mathcal{Q}_d \\ (q_1, q_2) = 1}} \frac{\alpha\tau\rho'_{i_1}(q_1)\alpha\tau\rho'_{i_2}(q_2)}{\sqrt{q_1q_2}} \lambda(dq_1)\lambda(dq_2)2^{-\omega(d)} \sum_{r|d} \frac{\alpha\tau(r^2)\tilde{\rho}_{j_4}(r)}{r} \\ &= \sum_{i_1+i_2+i_3=j_3} \sum_{d \in \mathcal{Q}} \sum_{f \in \mathcal{Q}_d} \frac{\mu\alpha^2\tau^2\rho'''_{i_1}(f)}{f} \sum_{q_1, q_2 \in \mathcal{Q}_{df}} \frac{\alpha\tau\rho''_{i_2}(q_1)\alpha\tau\rho''_{i_3}(q_2)}{\sqrt{q_1q_2}} \\ &\quad \lambda(dfq_1)\lambda(dfq_2)2^{-\omega(d)} \sum_{r|d} \frac{\alpha\tau(r^2)\tilde{\rho}_{j_4}(r)}{r} \end{aligned}$$

with $\rho'_i, \rho''_i, \rho'''_i \in \mathcal{F}_i$. In the last step we have used that $\sum_{f|(q_1, q_2)} \mu(f) = 0$ unless $(q_1, q_2) = 1$. Let

$$(4.1) \quad \xi_j(g) := \sum_{q \in \mathcal{Q}_g} \frac{\alpha(q)\tau(q)\rho''_j(q)}{\sqrt{q}} \lambda(gq)$$

and

$$(4.2) \quad h_{j_1, j_2}(g) := \sum_{df=g} \frac{\mu(f)\alpha^2(f)\tau^2(f)\rho'''_{j_1}(f)}{f} 2^{-\omega(d)} \sum_{r|d} \frac{\alpha(r^2)\tau(r^2)\tilde{\rho}_{j_2}(r)}{r}.$$

Note that $h_{0,0}$ is a multiplicative function with $h_{0,0}(p) = \frac{(p-1)^3}{2p(p+1)^2}$. With these definitions the main term of \mathcal{L}_2 equals

$$(4.3) \quad \sum_{j_1+\dots+j_6 \leq 3} c_{j_1, j_2, j_3+j_5+j_6, j_4} (\log D)^{j_1} L^{(j_2)}(1, \chi_{-D}) \sum_{g \in \mathcal{Q}} h_{j_3, j_4}(g) \xi_{j_5}(g) \xi_{j_6}(g).$$

We shall see below that for large D the diagonal form

$$(4.4) \quad \sum_{j_1+j_2=3} c_{j_1, j_2, 0, 0} (\log D)^{j_1} L^{(j_2)}(1, \chi_{-D}) \sum_{g \in \mathcal{Q}} h_{0,0}(g) \xi_0(g)^2$$

is a fairly good approximation to (4.3). For $q \in \mathcal{Q}$ we have by (4.1)

$$(4.5) \quad \lambda(q) = \sum_{f \in \mathcal{Q}_q} \frac{\mu(f)\alpha(f)\tau(f)}{\sqrt{f}} \xi_0(fq),$$

hence by (3.1) (for $\lambda(q) \ll 1$, say)

$$(4.6) \quad \begin{aligned} \mathcal{L}_1 &= \frac{1}{w} \sum_{q \in \mathcal{Q}} \sum_{f \in \mathcal{Q}_q} \frac{\mu\alpha\tau(f)}{\sqrt{fq}} \xi_0(fq) \left(\log \left(\frac{\sqrt{D}}{q} \right) + c_0 \right) + O \left(Qe^{-\frac{\sqrt{D}}{Q}} \right) \\ &= \frac{1}{w} \sum_{g \in \mathcal{Q}} \frac{\xi_0(g)}{\sqrt{g}} \sum_{fq=g} \mu\alpha\tau(f) \left(\log \left(\frac{\sqrt{D}f}{g} \right) + c_0 \right) + O \left(Qe^{-\frac{\sqrt{D}}{Q}} \right) \\ &= \frac{1}{w} \sum_{g \in \mathcal{Q}} \frac{\mu\alpha_1(g)(\log(\sqrt{D}g) + c_0) + O(\sum_{p|g} \frac{\log p}{p})}{\sqrt{g}} \xi_0(g) + O \left(Qe^{-\frac{\sqrt{D}}{Q}} \right) \end{aligned}$$

where $\alpha_1(g) = \prod_{p|g} \frac{p-1}{p+1}$. Comparing (4.4) and (4.6) it seems sensible to choose

$$(4.7) \quad \xi_0(g) := \frac{\mu(g)\alpha_1(g)(\log(\sqrt{D}g) + c_0)}{(\log D)^3 h_{0,0}(g) \sqrt{g}}$$

for $g \in \mathcal{Q}$ so that indeed $\lambda(q) \ll q^{-\frac{1}{2}+\varepsilon}$ and

$$(4.8) \quad \mathcal{L}_1 \gg \frac{1}{(\log D)^3} \sum_{g \in \mathcal{Q}} \frac{\alpha_1^2(g)(\log(\sqrt{D}g) + c_0)^2}{gh_{0,0}(g)}$$

for sufficiently large D . Next we estimate \mathcal{L}_2 . The contribution of the error term is negligible if η is sufficiently small :

$$(4.9) \quad \sum_{q_1, q_2 \in \mathcal{Q}} \lambda(q_1)\lambda(q_2) D^{-1/28+12\eta+\varepsilon} \ll D^{-1/28+13\eta+\varepsilon}.$$

By (4.2), Lemma 4.1b and the Cauchy-Schwarz inequality we see

$$(4.10) \quad h_{j_1, j_2}(g) \ll 2^{-\omega(g)} \sum_{f|g} \frac{\tau^3(f)\rho'''_{j_1}(f)}{f} \sum_{r|g} \frac{\tau(r^2)\tilde{\rho}_{j_2}(r)}{r} \\ \ll h_{0,0}(g)(\log \log 3g)^A$$

for a certain constant A depending on j_1 and j_2 . Furthermore, from (4.1), (4.5), (4.7) and Lemma 4.1c we infer

$$(4.11) \quad \xi_j(g) = \sum_{q \in \mathcal{Q}_g} \frac{\alpha\tau\rho''_j(q)}{\sqrt{q}} \sum_{f \in \mathcal{Q}_{gq}} \frac{\mu\alpha\tau(f)}{\sqrt{f}} \xi_0(fgq) \\ = \sum_{n \in \mathcal{Q}_g} \frac{\alpha\tau(n)}{\sqrt{n}} (\mu * \rho''_j)(n)\xi_0/ng \ll \eta^j \xi_0(g)(\log D)^j$$

where the implied constant is absolute and in particular independent of η . Let $\varepsilon_j := \text{sign } L^{(j)}(1, \chi_{-D})$. Then by (4.3), (4.9), (4.10), (4.11) and Siegel's lower bound for $L(1, \chi_{-D})$ we obtain for sufficiently small η and sufficiently large D

$$(4.12) \quad \mathcal{L}_2 \leq \sum_{j_1+j_2=3} \left(c_{j_1, j_2, 0, 0} + \varepsilon_{j_2} \frac{C}{20} \right) (\log D)^{j_1} L^{(j_2)}(1, \chi_{-D}) \sum_{g \in \mathcal{Q}} h_{0,0}(g)\xi_0(g)^2$$

with C as in (3.17) and

$$(4.13) \quad \sum_{g \in \mathcal{Q}} h_{0,0}(g)\xi_0(g)^2 = \frac{1}{(\log D)^6} \sum_{g \in \mathcal{Q}} \frac{\alpha_1^2(g)(\log(\sqrt{D}g) + c_0)^2}{gh_{0,0}(g)}.$$

To handle the higher derivatives of $L(s, \chi_{-D})$ we appeal to the following lemma.

LEMMA 4.2. — For $x \geq D^{1/2+\varepsilon}$ we have

$$(4.14) \quad L''(1, \chi_{-D}) \leq P_2(\log x)L(1, \chi_{-D}) - 2\gamma L'(1, \chi_{-D}) + O(D^{\frac{1}{4}}x^{-\frac{1}{2}+\varepsilon})$$

and

$$(4.15) \quad L'''(1, \chi_{-D}) \leq 2P_3(\log x)L(1, \chi_{-D}) + 3\tilde{P}_2(\log x)L'(1, \chi_{-D}) + O(D^{\frac{1}{4}}x^{-\frac{1}{2}+\varepsilon}),$$

where P_2, \tilde{P}_2, P_3 are monic polynomials of degree 2, 2, 3, respectively.

Proof. — Let $\zeta(s)L(s, \chi_{-D}) = \sum \frac{r(n)}{n^s}$. Since $r(n) \geq 0$, we clearly have

$$0 \leq \sum_{n \leq x} \frac{r(n)}{n} \left(\log \frac{x}{n} \right)^{j+1} \leq \log x \sum_{n \leq x} \frac{r(n)}{n} \left(\log \frac{x}{n} \right)^j$$

for $j \in \mathbb{N}_0$ and $x \geq 1$. The lemma follows by evaluating these terms for $x \geq D^{1/2+\varepsilon}$: From [14], Lemma 1, we cite

$$\sum_{n \leq x} \frac{r(n)}{n} = L'(1, \chi_{-D}) + (\log x + \gamma)L(1, \chi_{-D}) + O(x^{-1/2+\varepsilon} D^{1/4}),$$

while for $j \geq 1$ we use the inversion formula

$$\sum_{n \leq x} \frac{r(n)}{n} \left(\log \frac{x}{n} \right)^j = \frac{1}{2\pi i} \int_{(2)} \zeta(s+1)L(s+1, \chi_{-D}) \frac{x^s}{s^{j+1}} ds.$$

Shifting the contour to $\Re s = -1/2$ and using standard (convexity) estimates for the growth of $\zeta(s)$ and $L(s, \chi_{-D})$ on the critical line, we see

$$\sum_{n \leq x} \frac{r(n)}{n} \left(\log \frac{x}{n} \right)^j = \operatorname{res}_{s=0} \zeta(s+1)L(s+1, \chi_{-D}) \frac{x^s}{s^{j+1}} + O\left(x^{-1/2+\varepsilon} D^{1/4}\right)$$

for all $j \geq 1$ where

$$\begin{aligned} & \operatorname{res}_{s=0} \zeta(s+1)L(s+1, \chi_{-D}) \frac{x^s}{s^2} \\ &= L(1, \chi_{-D}) \left(\frac{1}{2} (\log x)^2 + \gamma \log x + c_1 \right) + L'(1, \chi_{-D}) (\log x + \gamma) \\ & \quad + \frac{1}{2} L''(1, \chi_{-D}), \\ & \operatorname{res}_{s=0} \zeta(s+1)L(s+1, \chi_{-D}) \frac{x^s}{s^3} \\ &= L(1, \chi_{-D}) \left(\frac{1}{6} (\log x)^3 + \frac{\gamma}{2} (\log x)^2 + c_1 \log x + c_2 \right) \\ & \quad + L'(1, \chi_{-D}) \left(\frac{1}{2} (\log x)^2 + \gamma \log x + c_1 \right) + \frac{1}{2} L''(1, \chi_{-D}) (\log x + \gamma) \\ & \quad + \frac{1}{6} L'''(1, \chi_{-D}) \end{aligned}$$

for certain constants c_1, c_2 . This gives the Lemma.

Now we choose $x = D^{3/5}$ in (4.14) and $x = D$ in (4.15). Together with (3.17), (4.12), and (4.13) we obtain

$$(4.16) \quad \mathcal{L}_2 \ll \left(3 \frac{L(1, \chi_{-D})}{(\log D)^3} + 4 \frac{L'(1, \chi_{-D})}{(\log D)^4} \right) \sum_{g \in \mathcal{Q}} \frac{\alpha_1^2(g)(\log(\sqrt{D}g) + c_0)^2}{gh_{0,0}(g)},$$

so that by (4.8) and (4.16)

$$(4.17) \quad \begin{aligned} \mathcal{L}_1^2 \mathcal{L}_2^{-1} &\gg \left(\frac{3}{4} L(1, \chi_{-D}) \log D + L'(1, \chi_{-D}) \right)^{-1} \sum_{g \in \mathcal{Q}} \frac{\alpha_1^2(g)(\log(\sqrt{D}g) + c_0)^2}{gh_{0,0}(g)(\log D)^2} \\ &\gg \left(\frac{3}{4} L(1, \chi_{-D}) \log D + L'(1, \chi_{-D}) \right)^{-1} \sum_{\substack{g \in \mathcal{Q} \\ (6, g) = 1}} \frac{\alpha_2(g)\tau(g)}{g} \end{aligned}$$

where $\alpha_2(g) = \prod_{p|g} \frac{p}{p-1}$. Our mollifier has only length D^η with small η . To evaluate the sum on the right side of (4.17) in terms of $L(1, \chi_{-D})$ and $L'(1, \chi_{-D})$ we would therefore need a kind of Lindelöf hypothesis. However, to keep the proof unconditional, we proceed as follows. For $\Re s > 1$ we have

$$\begin{aligned} \sum_{p|g \Rightarrow \chi_{-6^2 D}(p) = 1} \frac{\alpha_2(g)\tau(g)\mu^2(g)}{g^s} &= \prod_{\substack{\chi_{-D}(p) = 1 \\ p > 3}} \left(1 + \frac{2\alpha_2(p)}{p^s} \right) \\ &= \zeta(s)L(s, \chi_{-D}) \prod_{p|D} \left(1 - \frac{1}{p^s} \right) H(s) \end{aligned}$$

where H is holomorphic in $\Re s \geq 1/2 + \varepsilon$ and bounded uniformly in D . Let

$$\mathcal{Q}^* := \{g \leq D \mid \mu^2(g) = 1, (p \mid g \Rightarrow p > 3 \text{ and } \chi_{-D}(p) = 1)\}.$$

Then we find by contour integration and Siegel’s theorem

$$\begin{aligned} &\sum_{g \in \mathcal{Q}^*} \frac{\alpha_2(g)\tau(g)}{g} \\ &= H(1) \prod_{p|D} \left(1 - \frac{1}{p} \right) \left(L(1, \chi_{-D}) \left((\log D) + O\left(1 + \sum_{p|D} \frac{\log p}{p} \right) \right) + L'(1, \chi_{-D}) \right) \\ &\gg \prod_{p|D} \left(1 - \frac{1}{p} \right) \left(\frac{3}{4} L(1, \chi_{-D}) (\log D) + L'(1, \chi_{-D}) \right), \end{aligned}$$

hence

$$(4.18) \quad \mathcal{L}_1^2 \mathcal{L}_2^{-1} \gg \prod_{p|D} \left(1 - \frac{1}{p}\right) \left(\sum_{g \in \mathcal{Q}^*} \frac{\alpha_2(g)\tau(g)}{g}\right)^{-1} \sum_{\substack{g \in \mathcal{Q} \\ (6,g)=1}} \frac{\alpha_2(g)\tau(g)}{g}.$$

Let \mathcal{P} be any set of primes in the interval $[1, x]$, $P := \prod_{p \in \mathcal{P}} p$, and β be a multiplicative function supported on squarefree numbers. Suppose that $0 \leq \beta(p) < p$ for all p . Then we have for $p \notin \mathcal{P}$

$$\sum_{\substack{n \leq x \\ (n, pP)=1}} \frac{\beta(n)}{n} = \sum_{\substack{n \leq x \\ (n, P)=1}} \frac{\beta(n)}{n} - \sum_{\substack{n \leq x/p \\ (n, P)=1}} \frac{\beta(pn)}{pn} \geq \left(1 - \frac{\beta(p)}{p}\right) \sum_{\substack{n \leq x \\ (n, P)=1}} \frac{\beta(n)}{n}.$$

Inductively we find

$$\prod_{\substack{p \leq x \\ p \notin \mathcal{P}}} \left(1 - \frac{\beta(p)}{p}\right)^{-1} \geq \sum_{\substack{n \leq x \\ (n, P)=1}} \frac{\beta(n)}{n} \geq \prod_{p \in \mathcal{P}} \left(1 - \frac{\beta(p)}{p}\right) \sum_{n \leq x} \frac{\beta(n)}{n}.$$

Now we choose $\mathcal{P} = \{p \leq D^\eta \mid \chi_{-D}(p) \neq 1\}$ and $\beta(n) = \alpha_2(n)\tau(n)\mu^2(n)$ if $(6, n) = 1$, $\beta(n) = 0$ if $(6, n) > 1$. Then we obtain

$$\begin{aligned} & \left(\sum_{g \in \mathcal{Q}^*} \frac{\alpha_2(g)\tau(g)}{g}\right)^{-1} \sum_{\substack{g \in \mathcal{Q} \\ (6,g)=1}} \frac{\alpha_2(g)\tau(g)}{g} \\ & \geq \prod_{\substack{p \leq D \\ p \notin \mathcal{P}}} \left(1 - \frac{\alpha_2\tau(p)}{p}\right) \prod_{p \in \mathcal{P}} \left(1 - \frac{\alpha_2\tau(p)}{p}\right) \sum_{n \leq D^\eta} \frac{\alpha_2\tau(n)}{n} \gg \eta^2 \end{aligned}$$

by standard estimates. Inserting this into (4.18) completes the proof.

BIBLIOGRAPHY

- [1] D. BUMP, S. FRIEDBERG and J. HOFFSTEIN, Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L -functions and their derivatives, *Ann. of Math.* (2) 131 (1990), 53-127.
- [2] D.A. BURGESS, On character sums and L-series, *Proc. London Math. Soc.* (2) 12 (1962), 193-206.
- [3] J.-M. DESHOULLERS and H. IWANIEC, The nonvanishing of Rankin-Selberg zeta-functions at special points, *Contemp. Math.* 53 (1986) 51-95.
- [4] W. DUKE, Hyperbolic distribution problems and half-integral weight Maass forms, *Invent. Math.* 112 (1988), 73-90.

- [5] W. DUKE, J. FRIEDLANDER and H. IWANIEC, Class group L -functions, Duke Math. J. 79 (1995), 1-56.
- [6] E. FOUVRY and H. IWANIEC, Low-lying zeros of dihedral L -functions, Duke Math. J. 116 (2003), 189-217.
- [7] H. IWANIEC and P. SARNAK, The non-vanishing of central values of automorphic L -functions and the Landau-Siegel zero, Isr. J. Math. 120 (2000), 155-177.
- [8] N. KATZ and P. SARNAK, Zeros of zeta-functions and symmetry, Bull. AMS 36 (1999), 1-26.
- [9] E. KOWALSKI, P. MICHEL and J. VANDERKAM, Mollification of the fourth moment of automorphic L -functions and arithmetic applications, Invent. Math. 142 (2000), 95-151.
- [10] E. KOWALSKI, P. MICHEL and J. VANDERKAM, Non-vanishing of high derivatives of automorphic L -functions at the center of the critical strip, J. Reine Angew. Math. 526 (2000), 1-34.
- [11] M.R. MURTY and V.K. MURTY, Mean values of derivatives of modular L -series, Ann. of Math. (2) 133 (1991), 447-475.
- [12] M.R. MURTY and V.K. MURTY, Non-vanishing of L -functions and applications, Progress in Mathematics 157, Birkhäuser, Basel, 1997.
- [13] A. PERELLI and J. POMYKALA, Averages over twisted elliptic L -functions, Acta Arith. 80 (1997), 149-163.
- [14] J. PINTZ, Elementary methods in the theory of L -functions II, Acta Arith. 31 (1976), 273-306.
- [15] D. ROHRLICH, Nonvanishing of L -functions for $GL(2)$, Invent. Math. 97 (1989), 383-401.
- [16] G. SHIMURA, On modular forms of half-integral weight, Ann of Math. (2) 97 (1973), 440-481.
- [17] K. SOUNDARARAJAN, Non-vanishing of quadratic Dirichlet L -functions at $s = \frac{1}{2}$, Ann. of Math. (2) 152 (2000), 447-488.
- [18] J.-L. WALDSPURGER, Sur le coefficients de Fourier des formes modulaires de poids demi-entier, J.Math.Pures Appl. 60 (1981), 375-484.

Manuscrit reçu le 6 juin 2003,
accepté le 15 janvier 2004.

Valentin BLOMER,
University of Toronto
Department of Mathematics
100 St. George Street
Toronto, Ontario (Canada) M5S 3G3.
vblomer@math.toronto.edu