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Mapping class group and the Casson invariant


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MAPPING CLASS GROUP AND THE
CASSON INVARIANT

by Bernard PERRON

We use freely notations and results of [Pe].

0. Introduction.

0.1. — Let $S_g$ (resp. $S_{g,1}$) be a closed oriented surface (resp. with one boundary component) of genus $g$. Let $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) denote the mapping class group of $S_g$ (resp. $S_{g,1}$), that is the group of isotopy classes of homeomorphisms of $S_g$ (resp. $S_{g,1}$); in this case we consider homeomorphisms equal to the identity on the boundary, the isotopies being also identity on $\partial S_{g,1}$. Since $S_{g,1}$ can be seen as a submanifold of $S_g$ such that $S_g - S_{g,1} = D^2$, we have a natural (surjective) map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ by extending a homeomorphism of $S_{g,1}$ by identity on the 2-disc $D^2$.

0.2. — Consider the standard embedding of $S_g$ in $\mathbb{R}^3$ given by Figure 0.1 and let $H_g$ denote the oriented handlebody of genus $g$ bounded by $S_g$. Let $i_g : S_g \to S_g$ be a homeomorphism which exchanges $x_i$ and $y_i$ ($i = 1, 2, \ldots, g$), where $x_i$ and $y_i$ are the oriented circles defined by Figure 0.1. One can take for $i_g$ the composition $\rho_1 \circ \cdots \circ \rho_g$ where

$$\rho_i = D(x_i)D(y_i)D(x_i)$$

($D(x)$ is the Dehn twist along the circle $x$). Then $H_g \cup_{i_g} (-H_g)$ is homeomorphic to $\mathbb{S}^3$, $(-H_g)$ being the handlebody $H_g$ with opposite orientation.

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Figure 0.1

0.3. — For \( f \in \mathcal{M}_{g,1} \) let \( \bar{f} \) denote its extension (by identity) to \( S_g \). Let \( M_{\bar{f}} \) denote the 3-manifold obtained by gluing two copies of \( H_g \) by the homeomorphism \( i_g \circ \bar{f} \). It is obvious that, if \( f \) induces the identity at the homological level (e.g. \( f \) belongs to the Torelli group \( \mathcal{I}_{g,1} \) of \( S_{g,1} \)), then \( M_{\bar{f}} \) is a \( \mathbb{Z} \)-homology sphere. Morita [Mol] shows that any \( \mathbb{Z} \)-homology sphere is homeomorphic to \( M_{\bar{f}} \) for some \( f \) belonging to \( \mathcal{I}_{g,1} = \mathcal{M}(3) \subset \mathcal{I}_{g,1} \), where \( \mathcal{M}(3) \) is defined in [J1] (see also [Pe], Lemma 3.4).

Let \( \mathcal{N}_{g,1} \) (resp. \( \mathcal{N}_{g,1}^t \)) denote the subgroup of \( \mathcal{M}_{g,1} \) consisting of homeomorphisms \( f \) of \( S_{g,1} \) such that \( \bar{f} \) extends to a homeomorphism of \( H_g \) (resp. \( S^3 - H_g \)).

It is well-known that if \( f, g \in \mathcal{M}_{g,1} \) are such that \( f = \xi g \eta \), with \( \xi \in \mathcal{N}_{g,1}^t \) and \( \eta \in \mathcal{N}_{g,1} \), then the manifolds \( M_{\bar{f}} \) and \( M_{\bar{g}} \) are homeomorphic.

0.4. — Now, for any \( \mathbb{Z} \)-homology sphere \( \Sigma \), Casson [C] (see also [GM]) defines an invariant belonging to \( \mathbb{Z} \), denoted by \( \lambda(M) \). This allows us to define a map \( \lambda^* : \mathcal{I}_{g,1} \to \mathbb{Z} \) by setting

\[
\lambda^*(f) = \lambda(M_{\bar{f}}).
\]

0.5. — We want to express \( \lambda^*(f) \) using Johnson’s homomorphisms (see [Pe], Chap. 4). Recall from [Mo1], §1, or [Pe], 6.2, that \( T \) denotes the subgroup of \( (\wedge^2 H) \otimes H \otimes H \) (where \( H = H_1(S_{g,1}; \mathbb{Z}) \)) generated by elements of the following form

\[
(a \wedge b)^2 = a \wedge b \otimes a \wedge b \quad \text{and} \quad (a \wedge b) \leftrightarrow (c \wedge d)
\]

where

\[
(a \wedge b) \leftrightarrow (c \wedge d) = (a \wedge b) \otimes (c \wedge d) + (c \wedge d) \otimes (a \wedge b),
\]

\[
a \wedge b = a \otimes b - b \otimes a \quad \text{when} \quad a \wedge b \in H \otimes H.
\]
Then Morita [Mo 1], § 4, defines a homomorphism \( \theta_0 : T \to \mathbb{Z} \) by setting
\[
\theta_0((a \wedge b)^2) = \ell(a, a) \ell(b, b) - \ell(a, b) \ell(b, a),
\]
\[
\theta_0(a \wedge b \leftrightarrow c \wedge d) = \ell(a, c) \ell(b, d) + \ell(c, a) \ell(d, b) - \ell(a, d) \ell(b, c) - \ell(d, a) \ell(c, b)
\]
where
\[
\ell(a, b) = \text{link}(a, b^+)
\]
is defined as follows. Let \( S_g \) be standardly embedded in \( \mathbb{R}^3 \) (Figure 0.1), \( \nu \) a non singular normal vector field on \( S_g \), pointing outside \( H_g \). For \( b \in H = H_1(S_g; \mathbb{Z}) \), let \( b^+ \) be the 1-chain pushed out of \( S_g \) along \( \nu \). Then \( \ell(a, b) \) is the linking number in \( \mathbb{R}^3 \) of \( a \) and \( b^+ \). It is easy to see that
\[
\theta_0(a_i \wedge a_j \leftrightarrow b_i \wedge b_j) = 1 \quad (i \neq j)
\]
and \( \theta_0 = 0 \) for the other basis elements of \( T \).

0.6. — Recall the main result of [J 3], Theorem 5: the subgroup
\( T_{g, 1} = \mathcal{M}(3) \subset T_{g, 1} \) is normally generated by the Dehn twists \( D(f_1) \) and \( D(f_2) \), where the circles \( f_1, f_2 \) are defined by Figure 0.1. So any element \( f \) of \( T_{g, 1} \) can be written, up to order
\[
\left( \prod_{i=1}^n \varphi_i D(f_1)^{\varphi_i} \varphi_i^{-1} \right) \left( \prod_{j=1}^m \psi_j D(f_2)^{\psi_j} \psi_j^{-1} \right) \quad (\varphi_i, \psi_j \in \mathcal{M}_{g, 1}).
\]

Our first main result is:

**Theorem 0.1.** — For \( f \in T_{g, 1} \) we have
\[
\lambda^*(f) = -\frac{1}{12} \theta_0(\sigma \circ A'_2(f)) + \frac{1}{3} \sum_{j=1}^m e_j
\]
where \( A'_2 \) (resp. \( \sigma \)) is the map
\[
A'_2 : \mathcal{M}_{g, 1} \xrightarrow{A_2} (\otimes^2 H) \otimes H \otimes H \xrightarrow{\pi \otimes \text{id}} (\wedge^2 H) \otimes H \otimes H
\]
defined in [Pe, (6.1)], resp. \( \sigma : (\wedge^2 H) \otimes H \otimes H \to T \) is the map defined by
\[
\sigma(a \wedge b \otimes c \otimes d) = a \wedge b \leftrightarrow c \wedge d
\]
(see [Pe], (7.1)).

**Corollary 0.2.** — The map \( \delta : T_{g, 1} \to \mathbb{Z} \) defined by \( \delta(f) = \sum_{i=1}^m e_j \)
is a well-defined homomorphism, where up to order,
\[
f = \left( \prod_{i=1}^n \varphi_i D(f_1)^{\varphi_i} \varphi_i^{-1} \right) \left( \prod_{j=1}^m \psi_j D(f_2)^{\psi_j} \psi_j^{-1} \right).
\]
0.7. Remark. — The above homomorphism $\delta : T_{g,1} \to \mathbb{Z}$ has a more intrinsic definition. In fact, Morita [Mo1], §5, using Meyer’s 2-cocycle (see [Mel]), defines a map $d : \mathcal{M}_{g,1} \to \mathbb{Z}$, such that, when restricted to $T_{g,1}$, it becomes a homomorphism and such that if $\psi \in T_{g,1}$ is a Dehn twist along a simple closed curve in $S_{g,1}$ bounding a surface of genus $h$, then $d(\psi) = 4h(h - 1)$.

It follows from the definition of $\delta$, that $d|_{T_{g,1}} = 8\delta$. So Theorem 0.1 becomes:

$$\lambda^*(f) = -\frac{1}{12}\theta_0(\sigma \circ A^*_2(f)) + \frac{1}{24}d(f) \quad \text{for } f \in T_{g,1}.$$ 

0.8. Remark. — Formula above is a rephrasing (in a simpler way) of Morita’s formula [Mo1], Theorem 6.1:

$$\lambda^*(f) = \left(\theta_0 + \frac{1}{3}d\right)(\tau_3(f)) + \frac{1}{24}d(f) \quad \text{for } f \in T_{g,1}$$

(here $\tau_3(f) \in \overline{T}$ is the third Johnson’s homomorphism, and $\overline{T}$ a certain quotient of $T$).

0.9. — Next we want to compute $\lambda^*(f)$ for any $f \in T_{g,1}$, so extending the formula of Theorem 0.1 (or equivalently Morita’s formula (0.8)). For any $f \in \mathcal{M}_{g,1}$, set

$$\Delta(f) = -\frac{1}{12}\theta_0(\sigma \circ A^*_2(f)) + \frac{1}{24}d(f).$$

For $f \in T_{g,1}$, defined in [Pe], Corollary 4.5, an element $A_1(f) \in \overline{\Lambda^3H} \otimes \Lambda^3H$, where $\overline{\Lambda^3H}$ is the injective image of the homomorphism $\Lambda^3H \to \overline{\Lambda^3H}$ given by $x_1 \wedge x_2 \wedge x_3 \mapsto \sum_{\sigma \in G_3} \varepsilon(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$, where $G_3$ is the group of permutations of the set $\{1, 2, 3\}$. Moreover $A_1(f) = -\tau_2(f)$ where $\tau_2$ is the second Johnson’s homomorphism. Write for $f \in T_{g,1}$:

$$A_1(f) = \sum_{1 \leq i < j < k \leq g} a^f_{ijk} a_i \wedge a_j \wedge a_k + \sum_{1 \leq i < j < k \leq g} \beta^f_{ijk} b_i \wedge b_j \wedge b_k + R^f,$$

where $(a_i, b_i, i = 1, \ldots, g)$ is the symplectic basis of $H = H_1(S_{g,1}; \mathbb{Z})$, respectively equal to the homology class of the oriented circles $x_i, y_i$ of Figure 0.1, and $R^f$ is a sum of terms of the form $a \wedge b \wedge b$ and $a \wedge a \wedge b$. This basis verifies $a_i \cdot a_j = b_i \cdot b_j = 0, a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$ (the Kronecker symbol). Then we have the following:

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THEOREM 0.3. — Let \( f \in \mathcal{I}_{g,1} \), with \( A_1(f) \in \wedge^3 H \subset \otimes^3 H \) written as above. Then

\[
\lambda^*(f) = \Delta(f) + \sum_{1 \leq i < j < k \leq g} \alpha_{ijk}^f \beta_{ijk}^f.
\]

The proof of Theorem 0.3 will use the main result of [Mo2], Theorem 4.3:

THEOREM 0.4. — For \( f, g \in \mathcal{I}_{g,1} \) we have

\[
\lambda^*(fg) = \lambda^*(f) + \lambda^*(g) + 2 \sum_{1 \leq i < j < k \leq g} \beta_{ijk}^f \alpha_{ijk}^g.
\]

(By aware that in [Mo2], the role of \( \beta_{ijk} \) and \( \alpha_{ijk} \) have been interchanged.)

0.10. — Finally we want to restrict our attention to a special subgroup of \( \mathcal{M}_{g,1} \), the hyperelliptic mapping class group, denoted by \( \mathcal{H}_{g,1} \). This is the subgroup of \( \mathcal{M}_{g,1} \) generated by the Dehn twists along the circles \( x_1, \ldots, x_g, y_1, C_1, \ldots, C_{g-1} \) defined by Figure 0.1. Remark that these circles are invariant by the symmetry \( s_g \) along the axis \( x'x \) of Figure 0.1.

In [PV], it is proved that \( \mathcal{H}_{g,1} \) is isomorphic to the usual braid group \( B_{2g+1} \). The isomorphism can be described as follows.

Let \( \{ \sigma_i; i = 1, \ldots, 2g \} \) be the canonical generators of \( B_{2g+1} \): send \( \sigma_{2i} \) on \( D(x_i) \) (\( i \leq g \)), \( \sigma_1 \) on \( D(y_1) \) and \( \sigma_{2i+1} \) on \( D(C_i) \) (\( 1 \leq i \leq g-1 \)).

Moreover a homeomorphism \( f \in \mathcal{M}_{g,1} \) belongs to \( \mathcal{H}_{g,1} \) if and only if \( f \) commutes (up to isotopy) with the symmetry \( s_g \).

LEMMA 0.5. — The second Johnson’s homomorphism

\[
\tau_2 = -\frac{1}{6} \bar{A}_1|_{\mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}} : \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} \to \wedge^3 H
\]

is zero. So \( \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} = \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} \) (see [J1] for the definition of \( \tau_2 \) and [Pe], (4.6), for the definition of \( \bar{A}_1 \)).

Remark. — \( \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} \), when identified to \( B_{2g+1} \cap \mathcal{I}_{g,1} \), is the kernel of the reduced Burau representation (see [B], §3.3) when evaluated at \( t = -1 \).
0.11. — Finally, we have a simple geometric interpretation of the mapping \( d : \mathcal{M}_{g,1} \to \mathbb{Z} \), when restricted to \( \mathcal{H}_{g,1} \). As we have seen above, \( d \) is the core of Casson’s invariant, and quoting Morita, \( d \) is a rather mysterious invariant. Morita [Mo6] gives a geometric interpretation of \( d(\varphi) \) in terms of Hirzebruch’s signature defect of the mapping torus of \( \varphi \), with respect to a certain canonical framing of it tangent bundle. But it seems that this interpretation does not help for computations. Pitsch [Pi] gives a purely cohomological construction of the mapping \( d \).

Our last result gives a very simple geometric interpretation of \( d \), when restricted to \( \mathcal{H}_{g,1} \), using a nice formula of Gambaudo-Ghys [GG].

**Proposition 0.6.** — For \( f \in \mathcal{H}_{g,1} \cong B_{2g+1} \), we have the following formula

\[
d(f) = 3s(\widehat{f}) + u(f) + 2\pi(f)
\]

where:

1) \( \widehat{f} \) is the link in \( \mathbb{R}^3 \) obtained by closing the braid \( f \), and \( s \) is the classical signature of a link.

2) \( u(f) = B_0(f)(\delta_g) \cdot \delta_g \), where \( (\cdot) \) is the symplectic intersection form on \( H = H_1(S_{g,1};\mathbb{Z}) \), \( B_0(f) \) is the isomorphism induced by \( f \) at the homological level and \( \delta_g = (g-1)a_g + (g-2)a_{g-1} + \cdots + a_2 \in H \), where \( a_j \) is the homology class of the circle \( x_j \) of Figure 0.1.

3) \( \pi : B_{2g+1} \to \mathbb{Z} \) is the abelianization homomorphism sending each generator \( \sigma_i (i = 1, \ldots, 2g) \) on \( 1 \in \mathbb{Z} \).

**Theorem 0.7.** — One has \( d(D(x_i)) = d(D(y_i)) = 2 \) for \( i = 1, \ldots, g \) and \( d(D(C_i)) = 3 \) for \( i = 1, \ldots, g-1 \), where \( x_i, y_i, C_i \) are the circles defined by Figure 0.1.

**Notation.** — In the remainder of this paper, for \( f \in \mathcal{M}_{g,1} \), we will denote by \( f_* \) the isomorphism of \( H = H_1(S_{g,1};\mathbb{Z}) \) induced by \( f \). We used the notation \( B_0(f) \) instead of \( f_* \) in [Pe].

1. The mapping \( d : \mathcal{M}_{g,1} \to \mathbb{Z} \).

1.1. — We use the notations of [Pe], Chapter 3. For \( f \) in \( \mathcal{M}_{g,1} \), let \( B(f) \) denote the Fox matrix of \( f \) (Definition 3.1 of [Pe]). This belongs
to $\text{GL}_2g(\mathbb{Z}[\Gamma])$ where $\Gamma = \pi_1(S_{g,1}, \ast)$. Applying the abelianization homomorphism $\Gamma \to H$, we get a matrix $B(f)^{ab} \in \text{GL}_2g(\mathbb{Z}[H])$. We set

$$\tilde{k}(f) = \det[B(f)^{ab}].$$

**Lemma 1.1.** — $\tilde{k}$ is a crossed homomorphism, that is satisfies

$$\tilde{k}(fg) = \tilde{k}(f) \times f_\ast \tilde{k}(g) \in \mathbb{Z}[H],$$

where $f, g \in M_{g,1}$ and $\times$ is the operation in $\mathbb{Z}[H]$ induced by the law in $H$.

**Proof.** — From Lemma 3.2 of [Pe], we see that

$$B(fg)^{ab} = B(f)^{ab} \times f_\ast [B(g)^{ab}],$$

where $f_\ast (a_{ij}) = (f_\ast (a_{ij}))$. Lemma 1.1 follows. \hfill $\square$

Recall that we have defined in (0.9) an embedding $\wedge^3H \xrightarrow{i} \otimes^3H$, whose image is denoted by $\hat{\wedge}^3H$. We also have the canonical projection $\pi: \otimes^3H \to \wedge^3H$. It is obvious that $\pi \circ i = 6 \text{id}(\wedge^3H)$. On $\wedge^3H$ we define the contraction map $C: \wedge^3H \to H$ by the formula

$$C(a \wedge b \wedge c) = 2[(b \cdot c)a + (c \cdot a)b + (a \cdot b)c].$$

In [Pe], Chapter 4, we have defined a map $A_1: M_{g,1} \to \wedge^3H$ such that $A_1|_{T_{g,1}}$ is a homomorphism whose image is $\hat{\wedge}^3H$. Also, in [Pe], (4.6), we have set $A_1 = \pi \circ A_1$. The maps $A_1, \tilde{A}_1$ satisfy the following property (crossed product, see Lemma 4.1 of [Pe]):

$$A_1(fg) = A_1(f) + f_\ast A_1(g), \quad \tilde{A}_1(fg) = \tilde{A}_1(f) + f_\ast \tilde{A}_1(g).$$

We then have:

**Lemma 1.2.** — One has:

(a) For $f \in M_{g,1}$, $\tilde{k}(f)$ belongs to $H$ (a priori, it belongs to $\mathbb{Z}[H]$).

(b) For $f \in T_{g,1}$, $\tilde{k}(f) = C(A_1(f))$ ( $A_1(f) \in \hat{\wedge}^3H$).

(c) For $f \in M_{g,1}$, $\tilde{k}(f) = \frac{1}{6} C(\tilde{A}_1(f))$.

**Proof.** — By Lemma 1.1, $\tilde{k}(f)$ is a unit of $\mathbb{Z}[H]$, sent by the augmentation homomorphism $\varepsilon: \mathbb{Z}[H] \to \mathbb{Z}$ onto $\det(B_0(f)) = 1$. So $\tilde{k}(f)$ belongs to $H$, proving (a). Point (b) is proved in Proposition 6.15 of [Mo5] (remark that $A_1 = -\tau_2$, where $\tau_2$ is the second Johnson’s homomorphism, Proposition 4.4 of [Pe]).
1.2. — To prove point (c) of Lemma 1.1, using computations in the proof of Proposition 5.1 of [Pe] (or the proof of Proposition 6.15 of [Mo5]), we can verify that \( \tilde{k}(D(x_i)) = \tilde{k}(D(y_i)) = 0 \) and \( \tilde{k}(D(c_i)) = b_{i+1} - b_i \), using additive notations for \( H \). On the other hand, again using computations in the proof of Proposition 5.1 of [Pe], we have

\[
\tilde{A}_1(D(x_i)) = \tilde{A}_1(D(y_i)) = 0, \quad \tilde{A}_1(D(c_i)) = -3(a_i + a_{i+1}) \wedge b_{i+1} \wedge b_i.
\]

By the definition of \( C \), point (c) is true for \( D(x_i), D(y_i), D(C_i) \). Since these Dehn twists generate \( M_{g,1} \), and since \( \tilde{k} \) and \( \frac{1}{6} C \circ \tilde{A}_1 \) are both crossed products (that is satisfy \( \varphi(fg) = \varphi(f) + f_\ast \varphi(g) \)), point (c) follows. \( \square \)

Remark. — In the remainder of this paper, considering Lemma 1.2 (a), formula of Lemma 1.1 will be written \( \tilde{k}(fg) = \tilde{k}(f) + f_\ast \tilde{k}(g) \) where + is the law in \( H \).

1.3. — Now we can define a 2-cocycle on \( M_{g,1} \) with values in \( \mathbb{Z} \) (the action of \( M_{g,1} \) on \( \mathbb{Z} \) being trivial):

\[
c(f, g) = \tilde{k}(f^{-1}) \cdot \tilde{k}(g) = -f^{-1}_\ast \cdot (\tilde{k}(f)) \cdot \tilde{k}(g) = -\tilde{k}(f) \cdot f_\ast (\tilde{k}(g))
\]

where \( (,) \) is the symplectic form on \( H \).

Remark that this 2-cocycle coincides with the 2-cocycle of Morita [Mo1], §5, since \( \tilde{k}(f) = k(f^{-1}) \) by definition.

1.4. — We now come to Meyer 2-cocycle on the symplectic group \( \text{Sp}(2g, \mathbb{Z}) \) (see [Me1] or [Me2]). For a pair of symplectic matrices \( A, B \in \text{Sp}(2g, \mathbb{Z}) \), define a \( \mathbb{R} \)-vector space \( V_{A,B} \) by

\[
V_{A,B} = \{ (x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} ; (A^{-1} - I)x + (B - I)y = 0 \}.
\]

Consider the (possibly degenerated) symmetric bilinear form:

\[
\langle , \rangle_{A,B} : V_{A,B} \times V_{A,B} \rightarrow \mathbb{R}
\]

given by \( \langle (x_1, y_1), (x_2, y_2) \rangle = (x_1 + y_1) \cdot (I - B)y_2 \) where \( (,) \) is the symplectic form, whose matrix is \( \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). Then we set:

\[
\tau(A, B) = \text{signature}(V_{A,B}; \langle , \rangle_{A,B}).
\]
Lemma 1.3 (see [Me1], [Me2]). — The signature 2-cocycle satisfies the following properties:

1) \( \tau(A, B) + \tau(AB, C) = \tau(A, BC) + \tau(B, C) \) (2-cocycle property),
2) \( \tau(A, I) = \tau(A, A^{-1}) = 0 \),
3) \( \tau(A, B) = \tau(B, A) \),
4) \( \tau(A^{-1}, B^{-1}) = -\tau(A, B) \),
5) \( \tau(CAC^{-1}, CBC^{-1}) = \tau(A, B) \).

This defines a 2-cocycle on \( \mathcal{M}_{g,1} \) via the representation

\[
B_0 : \mathcal{M}_{g,1} \rightarrow \text{Sp}(2g, \mathbb{Z}).
\]

The remarkable fact, noted in [Mol], §5, is that the 2-cocycle \( c + 3T \) on \( \mathcal{M}_{g,1} \) is in fact a coboundary. So there exists a 1-cochain \( d : \mathcal{M}_{g,1} \rightarrow \mathbb{Z} \) (necessarily unique, since \( \mathcal{M}_{g,1} \) is perfect for \( g \geq 3 \) by [Po]) such that \( \delta d = c + 3T \).

1.5. — The mapping \( d \) satisfies the following properties.

Proposition 1.4 (see [Mol], Proposition 5.1). — For any \( f, g \in \mathcal{M}_{g,1} \) we have:

(i) \( d(fg) = d(f) + d(g) + \tilde{k}(f) \cdot f_*(\tilde{k}(g)) - 3\tau(f_*, g_*) \),
(ii) \( d(f^{-1}) = -d(f) \),
(iii) \( d(fgf^{-1}) = d(g) + \tilde{k}(f) \cdot [f_*g_*^{-1}(\tilde{k}(g)) + f_*(\tilde{k}(g)) - f_*g_*f_*^{-1}(\tilde{k}(f))] \).

Having in mind that \( \tilde{k}(\alpha) = k(\alpha^{-1}) \) this is exactly Proposition 5.1 of [Mol].

Proposition 1.5 (see [Mol], Proposition 5.3). — Let \( \mathcal{T}_{g,1} = \mathcal{M}(3) \) be the subgroup of \( \mathcal{M}_{g,1} \) generated by all Dehn twists on bounding simple closed curves. Then the mapping \( d|_{\mathcal{T}_{g,1}} : \mathcal{T}_{g,1} \rightarrow \mathbb{Z} \) is a homomorphism. Moreover if \( f \in \mathcal{T}_{g,1} \) is a Dehn twist on a bounding simple closed curve of genus \( h \), then \( d(f) = 4h(h - 1) \).

1.6. — In Chapter 3, we will need the following result.

Lemma 1.6. — 1) Let \( u \) be the simple closed curve given by Figure 1.1. Then \( d(D(u)) = d(D(x_2)) + 4 \), where \( x_2 \) is the curve defined by Figure 0.1.

2) One has \( d(D(y_2')) = d(D(y_2)) + 4 \), where \( y_2, y_2' \) are curves defined by Figure 0.1.
It is easy to see that

\[ u = D(y'_2) \circ D(y_2)^{-1}[x_2]. \]

Setting \( \delta = D(y_2) \circ D(y'_2)^{-1} \), we then have \( D(u) = \delta^{-1} \circ D(x_2) \circ \delta \). By (1.2), \( \tilde{k}(D(x_2)) = 0 \). By Proposition 1.4 (iii), we get

\[ d(D(u)) = d(D(x_2)) - \tilde{k}(\delta^{-1}) \cdot D(u) \cdot \tilde{k}(\delta^{-1}). \]

Since \( \delta \in \mathcal{I}_{g,1} \), by Lemma 1.1, Lemma 1.2 (b) and Lemma 4B of [J1], we have

\[ \tilde{k}(\delta^{-1}) = -\tilde{k}(\delta) = C(a_1 \wedge b_1 \wedge b_2) = 2b_2. \]

So \( d(D(u)) = d(D(x_2)) - 4b_2 \cdot u_*(b_2) \). But \( u_*(b_2) = |u| + b_2 = a_2 + b_2 \) (where \( |u| \) denotes the homology class of \( u \)) and the result follows.

\section{1.7. Proof of 2).} Let \( s_g \) be the symmetry of \( S_{g,1} \) along the axis \( x'x \) (Figure 0.1). Let \( S'_{g,1} \) be the surface obtained from \( S_{g,1} \) by adding the collar \( \partial S_{g,1} \times [0, 1] \) along \( \partial S_{g,1} \times \{0\} \). Extend the map \( s_g : S_{g,1} \to S_{g,1} \) by the map \( S : \partial S_{g,1} \times [0, 1] \to \partial S_{g,1} \times [0, 1] \) defined by \( S(e^{i\theta}, t) = (e^{i\theta + \pi(1-t)}, t) \), for \( e^{i\theta} \in S^1 \approx \partial S_{g,1} \) and \( t \in [0, 1] \). Then the map \( s_g \bigcup S \) represents an element of \( M_{g,1} \), denoted \( \Delta_g^2 \).

It is well known that \( \Delta_g^2 \) can be expressed as the composition

\[ [D(y_1)D(x_1)D(C_1)D(x_2) \cdots D(C_{2g-1})D(x_g)]^{2g+1} \]

(see Figure 0.1 for the definition of \( y_i, x_i, C_j \)).
The reason for the notation \( \Delta_g^2 \) is that \( \Delta_g^2 \) is the square of a homeomorphism \( \Delta_g \in \mathcal{M}_{g,1} \) which will be used later.

Since \( y_2' = \Delta_g^2(y_2) \) we have \( D(y_2') = \Delta_g^2 D(y_2) (\Delta_g^2)^{-1} \). Using again (1.2) and Proposition 1.4 (iii) we obtain

\[
d(D(y_2')) = d(D(y_2)) - \tilde{k}(\Delta_g^2) \cdot D(y_2')(\tilde{k}(\Delta_g^2)).
\]

We will see that \( \tilde{k}(\Delta_g^2) = 2[(g-1)a_g + (g-2)a_{g-1} + \cdots + a_2] \) in Chapter 4, §4.2. The Dehn twist \( D(y_2') \) act non trivially only on \( a_2 \) by

\[
D(y_2')(a_2) = a_2 + [y_2'] = a_2 - b_2.
\]

So \( D(y_2')(\tilde{k}(\Delta_g^2)) = \tilde{k}(\Delta_g^2) - 2b_2 \). Lemma 1.6, 2) follows. \( \square \)

2. Proof of Theorem 0.1.

Theorem 0.1 depends on two results of Morita.

**Proposition 2.1** (see [Mol], Proposition 3.5). — The mapping

\[
\lambda^*/T_{g,1}: T_{g,1} \longrightarrow \mathbb{Z}
\]

defined in 0.4 is a homomorphism.

**Proposition 2.2** (see [Mol], Proposition 4.5). — Let \( \psi \in T_{g,1} \) be a Dehn twist along a bounding simple closed curve \( \gamma \) of genus \( h \) of \( S_{g,1} \). Let \( (u_1, \ldots, u_h; v_1, \ldots, v_h) \) be a symplectic basis of the homology of the compact surface bounded by \( \gamma \). Then

\[
\lambda^*(\psi) = -\theta_0\left(\sum_{i=1}^{h} u_i \wedge v_i\right) ^2,
\]

where \( \left(\sum_{i=1}^{h} u_i \wedge v_i\right)^2 \) is seen in \( T \) (see 0.3) and \( \theta_0 \) has been defined in 0.5.

2.1. — Let \( f_h \) be the simple closed curve of genus \( h \), given by Figure 0.1. By a fundamental result of Johnson [J3], Theorem 5, any element \( f \) of \( T_{g,1} \) can be written, up to order, as

\[
\left(\prod_{i=1}^{n} \varphi_i D(f_1)^{e_i} \varphi_i^{-1}\right) \left(\prod_{j=1}^{m} \psi_j D(f_2)^{\epsilon_j} \psi_j^{-1}\right), \quad \varphi_i, \psi_j \in \mathcal{M}_{g,1}.
\]
By Corollary 4.3 and Lemma 6.2 of [Pe] we have

\[ A'_2(\varphi_i D(f_1)^{\varepsilon_i} \varphi_i^{-1}) = \varepsilon_i \varphi_i( A'_2( D(f_1))) = 3\varepsilon_i \varphi_i([a_1 \wedge b_1]^2) \in T, \]

\[ A'_2(\psi_j D(f_2)^{\varepsilon_j} \psi_j^{-1}) = e_j [3\psi_j((a_1 \wedge b_1 + a_2 \wedge b_2)^2) - \psi_j(s_1)] \in T, \]

where \( s_1 \) is the following element of \( T \) (see [Pe], (6.2))

\[ s_1 = (a_1 \wedge b_1) \leftrightarrow (a_2 \wedge b_2) - (a_1 \wedge a_2) \leftrightarrow (b_1 \wedge b_2) + (a_1 \wedge b_2) \leftrightarrow (b_1 \wedge a_2). \]

Using Propositions 2.1 and 2.2 we obtain

\[ \lambda^*(f) = -\frac{1}{3} \theta_0(A'_2(f)) - \frac{1}{3} \sum_{j=1}^m e_j \theta_0(\psi_j(s_1)). \]

We claim that \( \theta_0(\psi_j(s_1)) = -1 \). To see this, set \( a'_1 = \psi_j(a_1) \) and \( b'_1 = \psi_j(b_1) \). By the definition of \( \theta_0 \) given in 0.5 and the well-known formula \( \ell(u, v) - \ell(v, u) = -u \cdot v \) we find

\[ \theta_0(\psi_j(s_1)) = \ell(a'_1, b'_1)(a'_2 \cdot b'_2) - \ell(b'_1, a'_1)(a'_2 \cdot b'_2). \]

Since the symplectic form \( (, ) \) is invariant by elements of \( \mathcal{M}_{g,1} \), it follows that

\[ \theta_0(\psi_j(s_1)) = (b'_1 \cdot a'_1) \times (a'_2, b'_2) = (b_1 \cdot a_1)(a_2 \cdot b_2) = -1. \]

Now recall that we have defined in [Pe], (7.1), a homomorphism \( \sigma: (\wedge^2 H) \otimes H \otimes H \rightarrow T \) by setting \( \sigma((a \wedge b) \otimes c \otimes d) = (a \wedge b) \leftrightarrow (c \wedge d) \in T \). When restricted to \( T \), \( \sigma|_T \) is \( 4 \text{id}_T \). So we have proved Theorem 0.1.

Corollary 0.2 is obvious, since \( \lambda^*(f) \) and \( A'_2(f) \) do not depend on a particular writing of \( f \) as a product (up to order) \( \prod_i \varphi_i D(f_1)^{\varepsilon_i} \varphi_i^{-1} \cdot \prod_j (\psi_j D(f_2)^{\varepsilon_j} \psi_j^{-1}) \).

As observed in 0.7, the homomorphism \( \delta: \mathcal{T}_{g,1} \rightarrow \mathbb{Z} \) defined by \( \delta(f) = \sum_{j=1}^m e_j \) is the restriction of \( \frac{1}{8} d \), where \( d \) is the map defined in Chapter 1 (use Proposition 1.5).

**Corollary 2.3.** — For \( f \in \mathcal{T}_{g,1} \) we have

\[ \lambda^*(f) = -\frac{1}{12} \theta_0(\sigma \circ A'_2(f)) + \frac{1}{24} d(f). \]
2.2. Remark. — This is a reformulation of a formula of Morita [Mo1], Theorem 6.1, put in a simpler way. Morita’s formula is
\[ \lambda^*(f) = \left( \theta_0 + \frac{1}{3} \tilde{d} \right)(\tau_3(f)) + \frac{1}{24} d(f), \]
where \( \tau_3 : T_{g,1} \to \overline{T} = T/T_0 \) is the third Johnson homomorphism. Here \( T_0 \) is the subgroup of \( T \) generated by elements of the form \((u \land v) \leftrightarrow (w \land t) - (u \land w) \leftrightarrow (v \land t) \leftrightarrow (v \land w)\) (remark that \( s_1 \in T_0 \)). Since the homomorphism \( \theta_0 : T \to \mathbb{Z} \) does not factor through \( \overline{T} \), Morita has to correct \( \theta_0 \) by a homomorphism \( d : T \to \mathbb{Z} \) such that \( \theta_0 + \frac{1}{3} \tilde{d} \) factors through \( \overline{T} \). The main advantage of the method of [Pe] is that we have an invariant \( \sigma \circ A_2'(f) \) at the \( T \) level, and so a unified formula.

Of course we have
\[ \left( \theta_0 + \frac{1}{3} \tilde{d} \right)(\tau_3(f)) = -\frac{1}{12} \theta_0[\sigma \circ A_2'(f)] \]
since we have proved in [Pe], Lemma 7.1, that \( p \circ \sigma \circ A_2' = -12\tau_3 \), where \( p : T \to \overline{T} = T/T_0 \) is the canonical projection.

3. Proof of Theorem 0.3.

3.1. — For \( f \) belonging to \( \mathcal{I}_{g,1} \), set
\[ G(f) = \lambda^*(f) - \Delta(f) - \sum_{1 \leq i < j < k \leq g} \alpha_{ijk}^f \beta_{ijk}^f \in \mathbb{Q} \]
(the difference between the first and second member of the desired equality of Theorem 0.3). Thus, we have to show that \( G = 0 \) on \( \mathcal{I}_{g,1} \). We will first prove that \( G : \mathcal{I}_{g,1} \to \mathbb{Q} \) is a homomorphism. For this we need some computations.

3.2. — Let \( \mathcal{W}_a, \mathcal{W}_{ab}, \mathcal{W}_b \) denote the subgroups of \( \wedge^3 H \) (\( \simeq \wedge^3 H \), see 0.9) generated respectively by \( \{a_i \land a_j \land a_k\} \) (a only), \( \{c \land a_i \land b_j\} \) (at least one \( a \) and one \( b \)), \( \{b_i \land b_j \land b_k\} \) (b only). Of course we have a decomposition
\[ \wedge^3 H = \mathcal{W}_a \oplus \mathcal{W}_{ab} \oplus \mathcal{W}_b \]

Notation. — Set, for \( f \in \mathcal{I}_{g,1}, f_1 = A_1(f) \in \wedge^3 H \) (see [Pe], Corollary 4.5) and decompose \( f_1 \) as \( f_{1a} + f_{1ab} + f_{1b} \), where \( f_{1a} \in \mathcal{W}_a, f_{1ab} \in \mathcal{W}_{ab} \) and \( f_{1b} \in \mathcal{W}_b \).
3.3. — In [Pe], Lemma 4.2, we have defined a bilinear map $F : (\otimes^3 H) \otimes (\otimes^3 H) \to \otimes^4 H$ by setting $F = C_{34} - T_{23} \otimes C_{35}$ where

$$C_{34}(x_1 \otimes \cdots \otimes x_6) = (x_3 \cdot x_4)x_1 \otimes x_2 \otimes x_5 \otimes x_6,$$

$$C_{35}(x_1 \otimes \cdots \otimes x_6) = (x_3 \cdot x_5)x_1 \otimes x_2 \otimes x_4 \otimes x_6,$$

$$\tau_{23}(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 \otimes x_3 \otimes x_2 \otimes x_4.$$ 

We also consider the map $\tilde{\sigma} = \sigma \circ (\pi \otimes \text{id}) : \otimes^4 H \xrightarrow{\pi \otimes \text{id}} \wedge^2 H \otimes H \otimes H \xrightarrow{\sigma} T$, where $\pi$ is the canonical projection and $\sigma$ is the map defined in Theorem 0.1. We will need to compute $\theta_0 \circ \tilde{\sigma} \circ F$ on the subspace $(\wedge^3 H) \otimes (\wedge^3 H)$ of $\otimes^6 H$.

3.4. — Recall (see 0.5) that $\theta_0 : T \to \mathbb{Z}$ is defined by $\theta_0(a_i \wedge a_j \leftrightarrow b_i \wedge b_j) = 1$ for $i, j \in \{1, \ldots, g\}$, $i \neq j$ and $\theta_0 = 0$ on the other basis elements of $T$.

Two subspaces $A$, $B$ of $\wedge^3 H$ are said to be orthogonal for $\theta_0 \circ \tilde{\sigma} \circ F$ if $\theta_0 \circ \tilde{\sigma} \circ F(\alpha, \beta) = 0$ for any $(\alpha, \beta) \in A \times B \cup B \times A$.

**Lemma 3.1.**

1) The subspace $\mathcal{W}_a$ (resp. $\mathcal{W}_b$) is orthogonal to $\mathcal{W}_a \oplus \mathcal{W}_{ab}$ (resp. $\mathcal{W}_b \oplus \mathcal{W}_{ab}$).

2) If the sets of indices $\{i, j, k\}$, $\{i', j', k'\}$ are different,

$$\theta_0 \circ \tilde{\sigma} \circ F(a_i \wedge a_j \wedge a_k, b_{i'} \wedge b_{j'} \wedge b_{k'}) = \theta_0 \circ \tilde{\sigma} \circ F(b_{i'} \wedge b_{j'} \wedge b_{k'}, a_i \wedge a_j \wedge a_k) = 0.$$

3) For $i, j, k$ different,

$$\theta_0 \circ \tilde{\sigma} \circ F(a_i \wedge a_j \wedge a_k, b_i \wedge b_j \wedge b_k) = -\theta_0 \circ \tilde{\sigma} \circ F(b_i \wedge b_j \wedge b_k, a_i \wedge a_j \wedge a_k) = 12.$$

4) For $\alpha, \beta \in \mathcal{W}_{ab}$, we have $\theta_0 \circ \tilde{\sigma} \circ F(\alpha, \beta) = \frac{1}{2} C(\alpha) \cdot C(\beta)$ where $C$ is the contraction on $\wedge^3 H$ defined by $C(x \wedge y \wedge z) = 2[(x \cdot y)z + (y \cdot z)x + (z \cdot x)y]$, and $(., .)$ the symplectic intersection form on $H$.

3.5. **Proof.** — By definition of $F$ and $\theta_0$, it is clear that

$$\theta_0 \circ \tilde{\sigma} \circ F(c_1 \wedge c_2 \wedge c_3, d_1 \wedge d_2 \wedge d_3) = 0$$

unless the set $\{c_i, d_j ; i, j = 1, 2, 3\}$ is the union of three pairs $\{a_k, b_k\}$, $k \in \{1, 2, \cdots, g\}$. This proves points 1) and 2).
3.6. — The construction of the term corresponding to \( C_{34} \) in 
\[ \theta_0 \circ \sigma \circ F(a_i \wedge a_j \wedge a_k, b_i \wedge b_j \wedge b_k) \] 
is non zero only when a term \( a \) on the left 
is coupled with a term \( b \) on the right with same index. This contribution 
is easily seen to be equal to 12.

The contribution of the term \(-\tau_{23} \circ C_{35}\) is easily seen to be zero. This 
proves 3).

3.7. — To prove 4), we have only to consider the case \( \alpha = a_i \wedge b_j \wedge b_k \) 
and \( \beta = b_j \wedge a_j \wedge a_k \), and the case when \( \alpha \) and \( \beta \) are permuted. This amounts 
to exchange \( a \) and \( b \): this changes the sign of \( \theta_0 \circ \sigma \circ F(\alpha, \beta) \) and \( \frac{1}{2} \mathcal{C}(\alpha) \cdot \mathcal{C}(\beta) \) 
since (.) is antisymmetric.

3.8. — Using 3.4 we have only to consider the following three cases:
1) \( \theta_0 \circ \sigma \circ F(a_i \wedge a_j \wedge b_k, b_i \wedge b_j \wedge a_k) \) for \( i, j, k \) distinct;
2) \( \theta_0 \circ \sigma \circ F(a_i \wedge a_j \wedge b_j, b_i \wedge b_j \wedge a_k) \) for \( i, j, k \) distinct;
3) \( \theta_0 \circ \sigma \circ F(a_i \wedge a_j \wedge b_j, b_i \wedge b_j \wedge a_j) \) for \( i, j \) distinct.

3.9. — The contribution of the term corresponding to \( C_{34} \) in each 
case is non zero only if the \( b \) term on the left is coupled with the \( a \) term on 
the right with same index. This contribution is respectively \(-4, 0, -4\).

3.10. — The contribution of the term corresponding to \(-\tau_{23} \circ C_{35}\) 
in each case is non zero only if an \( a \) term on the left is coupled with the \( b \) 
term on the right with same index. For the cases 1), 2), 3), the contribution 
is respectively 4, \(-2\), 2.

Then point 4) of Lemma 3.1 follows immediately. \(\square\)

3.11. — We are now ready to prove:

Lemma 3.2. — \( G: I_{g,1} \to \mathbb{Q} \) is a homomorphism, equal to 0 on 
\( I_{g,1} = \mathcal{M}(3) \).

Proof. — By definition, for \( f, g \in I_{g,1} \)
\[ G(fg) = \lambda^*(fg) - (\Delta(fg)) - \sum_{1 \leq i < j < k \leq g} \alpha^f_{ijk} \beta^g_{ijk}. \]
By Theorem 0.4, \( \lambda^*(fg) = \lambda^*(f) + \lambda^*(g) + 2 \sum_{1 \leq i < j < k \leq g} \beta^f_{ijk} \alpha^g_{ijk} \).
By additivity of \( A_1 \) on \( I_{g,1} \), \( \alpha^f_{ijk} + \alpha^g_{ijk} \) and \( \beta^f_{ijk} + \beta^g_{ijk} \).
3.12. — From the properties of $A'_2$ (see [Pe], Lemma 4.2) and $d$ (see Proposition 1.4) we have:

$$\Delta(f g) = -\frac{1}{12} (\theta_0 \circ \sigma \circ A'_2(f g)) + \frac{1}{24} d(f g)$$

$$= -\frac{1}{12} \theta_0 \circ \tilde{\sigma} [A_2(f g)] + \frac{1}{24} d(f g) \quad \text{(see Theorem 0.1)}.$$  

$$= -\frac{1}{12} \theta \tilde{\sigma} [A_2(f) + A_2(g) + F(A_1(f), A_1(g))]$$

$$+ \frac{1}{24} [d(f) + d(g) + \tilde{k}(f) \cdot \tilde{k}(g)]$$

$$= \Delta(f) + \Delta(g) - \frac{1}{12} \theta_0 \circ \tilde{\sigma} \circ F(A_1(f), A_1(g)) + \frac{1}{24} \tilde{k}(f) \cdot \tilde{k}(g).$$

So $G(f g) - G(f) - G(g)$ is equal to

$$\sum_{1 \leq i < j < k \leq g} \beta_{ijk}^f \alpha_{ijk}^g - \alpha_{ijk}^f \beta_{ijk}^g + \frac{1}{12} \theta_0 \circ \tilde{\sigma} \circ F(A_1(f), A_1(g)) - \frac{1}{24} \tilde{k}(f) \cdot \tilde{k}(g).$$

3.13. — With the notations of 3.2 and Lemma 3.1 we get

$$\theta_0 \circ \tilde{\sigma} \circ F(A_1(f), A_1(g)) = \theta_0 \circ \tilde{\sigma} \circ F(f_{1a} + f_{1b} + f_{1ab}, g_{1a} + g_{1b} + g_{1ab})$$

$$= \theta_0 \circ \tilde{\sigma} \circ F(f_{1a}, g_{1b}) + \theta_0 \circ \tilde{\sigma} \circ F(f_{1b}, g_{1a})$$

$$+ \theta_0 \circ \tilde{\sigma} \circ F(f_{1ab}, g_{1ab}).$$

By definition of $f_{1a}$, $\alpha_{ijk}^f$ and Lemma 3.1:

$$\theta_0 \circ \tilde{\sigma} \circ F(f_{1a}, g_{1b}) = 12 \sum_{1 \leq i < j < k \leq g} \alpha_{ijk}^f \beta_{ijk}^g,$$

$$\theta_0 \circ \tilde{\sigma} \circ F(f_{1b}, g_{1a}) = -12 \sum_{1 \leq i < j < k \leq g} \beta_{ijk}^f \alpha_{ijk}^g,$$

$$\theta_0 \circ \tilde{\sigma} \circ F(f_{1ab}, g_{1ab}) = \frac{1}{2} C(f_{1ab}) \cdot C(g_{1ab}).$$

By the definition of $C$ (Lemma 3.1) and Lemma 1.2 it is easy to see that $C(f_{1ab}) = C(f_1) = \tilde{k}(f)$ and $C(g_{1ab}) = C(g_1) = \tilde{k}(g)$.

This proves that $G(f g) - G(f) - G(g) = 0$.

The last part of Lemma 3.2 follows from Theorem 0.1.

3.14. — By Lemma 3.2, the homomorphism $G$ factors through a homomorphism $G : \wedge^3 H \to \mathbb{Q}$ such that $G = G \circ A_1$, because of the exact sequence

$$1 \to M(3) = T_{g,1} \to T_{g,1} \xrightarrow{A_1} \wedge^3 H \to 1.$$
3.15. — So, to prove that $G = 0$, it is sufficient to show that $\mathcal{G} = 0$. For this purpose we will study some symmetries of $G$.

**Lemma 3.3.** — Let $S_{g,1} \subset \mathbb{R}^3$ be the surface of genus $g$, with one boundary component, standardly embedded in $\mathbb{R}^3$ as shown by Figure 3.2 below. Then:

(a) For each pair $(s,t)$ ($1 \leq s < t \leq g$), there exists a homeomorphism $\rho_{s,t} \in \mathcal{N}_{g,1} \cap \mathcal{N}_{g,1}' \subset \mathcal{M}_{g,1}$ (see 0.3 for the definition of $\mathcal{N}_{g,1}$ and $\mathcal{N}_{g,1}'$), which exchanges the handles $h_s$ and $h_t$ of Figure 3.2. More precisely, at the fundamental group level the action of $\rho_{s,t}$ is

$$\rho_{s,t}(z_i) = \alpha(s,t,z_i) z_{(s,t)(i)} \alpha(s,t,z_i)^{-1}$$

where $z_i = x_i$ or $y_i$, $(s,t)$ is the transposition of $s$ and $t$, and $\alpha(s,t,z_i)$ is a product of commutators.

(b) $A_1(\rho_{s,t}) = 0$ and $\tilde{k}(\rho_{s,t}) = 0$, for any pair $(s,t)$.

3.16. — Proof: It is easy to construct an isotopy $\tau_{i,u}$ ($i = 1, 2$, $u \in [0, 1]$) of $\mathbb{R}^3$ fixed outside a big compact set such that $\tau_i = \tau_{i,1}$ has the following properties:

(i) $\tau_i$ leaves invariant the surface $S_{2,1}$ of genus 2 of Figure 3.1;
(ii) $\tau_i|_{S_{2,1}}$ is the identity outside the disk $2D_i$;
(iii) $\tau_i|_{D_i \cup h_i}$ is the rotation of angle $\pi$ around the axis $Z_i$.

3.17. — It is also easy to construct an isotopy $\rho_u'$ ($u \in [0, 1]$) of $\mathbb{R}^3$, fixed outside a big compact of $\mathbb{R}^3$ such that $\rho_1'$ verifies:

(i) $\rho'_1$ leaves the surface $S_{2,1}$ of Figure 3.1 invariant;
(ii) $\rho'_1|_{S_{2,1}}$ is the identity on $\partial(2D)$;
(iii) $\rho'_1|_{D_1 \cup h_1 \cup h_2}$ is the rotation of angle $\pi$ around the axis $Z$.

3.18. — Now set $\rho = \tau_1 \circ \tau_2 \circ \rho_1$. Then $\rho$ is time 1 of an isotopy of $\mathbb{R}^3$ which has the following properties:

(i) $\rho$ leaves invariant the surface $S_{2,1}$;
(ii) $\rho$ is the identity on $\partial S_{2,1}$;
(iii) $\rho$ exchanges the handles $h_1$ and $h_2$. 
At the fundamental group level, \( \rho \) satisfies the formula of Lemma 3.3, with \( s = 1 \) and \( t = 2 \). We can even arrange things such that

\[
\rho(x_1) = f_1^{-1} x_2 f_1, \quad \rho(x_2) = x_1, \quad \rho(y_1) = f_1^{-1} y_2 f_1, \quad \rho(y_2) = y_1
\]

where \( f_1 \) is the homotopy class of \( 2D_1 \) (with suitable orientation and path).

3.19. — Now let \((s, t)\) be a pair of integers such that \(1 \leq s < t \leq g\), and \( \gamma \) be an embedded circle on \( S_{g,1} \) surrounding only the feet of the handles \( h_s, h_u \) (see Figure 3.2). Then \( \gamma \) is the boundary of a genus 2 subsurface \( \Sigma \) of \( S_{g,1} \). Then there is an isotopy \( H_u (u \in [0,1]) \) of \( \mathbb{R}^3 \) such that \( H_1 \) preserves \( S_{g,1} \) and sends \( \Sigma \) onto \( S_{2,1} \) seen as a subsurface of \( S_{g,1} \). Then \( \rho_{s,t} = H_1^{-1} \circ \rho_1 \circ H_1 \) satisfies point 1) of Lemma 3.3.

3.20. — Then, using the definition of \( A_1 \) (see [Pe], Chapter 4) and the fact that \( \alpha(s, t, z_1) \) is a product of commutators, it is easy to see that \( A_1(\rho_{s,t}) = 0 \). Then \( \tilde{k}(\rho_{s,t}) = 0 \) by Lemma 1.2 (c). This finishes the proof of Lemma 3.3.
Lemma 3.4. — For \( \varphi \in \mathcal{I}_{g,1} \), \( f \in \mathcal{M}_{g,1} \) such that \( A_1(f) = 0 \) (and so \( \tilde{k}(f) = 0 \)) we have:

1) \( A_2(f \varphi f^{-1}) = f_* \cdot A_2(\varphi) \);
2) \( d(f \varphi f^{-1}) = d(\varphi) \).

Proof. — Part 1) comes from Lemma 7.1 of [Pe] and 2) from Proposition 1.4.

Lemma 3.5. — For \( f \in \mathcal{I}_{g,1} \) we have

1) \( G(\rho_{st} f \rho_{st}^{-1}) = G(f) \);
2) \( G = G \circ (\rho_{st*}) \) where \( \rho_{st*} = B_0(\rho_{st}) \) and \( \rho_{st*} \) stands for the action of \( \rho_{st*} \) on \( \wedge^3 H \).

Proof. — By Lemma 3.4 we have \( A_2(\rho_{st} f \rho_{st}^{-1}) = \rho_{st*} A_2(f) \). Therefore

\[
\Delta(\rho_{st} \circ f \circ \rho_{st}^{-1}) = -\frac{1}{12} \theta_0(\rho_{st*} \sigma(A_2(f))) + \frac{1}{24} d(f).
\]

Since the effect of \( \rho_{st*} \) on \( H \) is to permute \( a_s \) with \( a_t \) and \( b_s \) with \( b_t \), it follows from the definition of \( \theta_0 \) (see 0.5) that \( \Delta(\rho_{st} \circ f \circ \rho_{st}^{-1}) = \Delta(f) \).

Since \( \rho_{st} \in \mathcal{N}_{g,1} \cap \mathcal{N}_{g,1}^t \) (Lemma 3.3), from 0.3, it follows that

\[
\lambda^*(\rho_{st} \circ f \circ \rho_{st}^{-1}) = \lambda^*(f).
\]

On the other hand, it is easy to see that

\[
\sum_{1 \leq i < j < k \leq g} \alpha_{ijk}^f \beta_{ijk}^f = \sum_{1 \leq i < j < k \leq g} \alpha_{ijk}^g \beta_{ijk}^g,
\]

where \( g = \rho_{st} \circ f \circ \rho_{st}^{-1} \).

This proves point 1) of Lemma 3.5. Point 2) follows from the definition of \( G \) and the formula: \( A_1(\rho_{st} \circ f \circ \rho_{st}^{-1}) = \rho_{st*} A_1(f) \) (see [Pe], Lemma 4.1).

From Lemma 3.5, we deduce that \( G(a_i \wedge a_j \wedge b_k) = G(a_j \wedge a_i \wedge b_k) = 0 \), for \( k \neq i, j \). We have the same result by replacing \( a \) by \( b \), and when we have three \( a \) or three \( b \). So, to prove that \( G \) is identically 0 on \( \wedge^3 H \), we have just to show that \( G(a_1 \wedge a_2 \wedge b_1) = G(a_1 \wedge b_1 \wedge b_2) = 0 \).

3.21. Computation of \( G(a_1 \wedge a_2 \wedge b_1) \).

Lemma 3.6.

1) \( G(a_1 \wedge a_2 \wedge b_1) = G(D(x_2)^{-1}D(u)) \), where \( x_2 \) and \( u \) are the simple closed curves given by Figure 3.3;

2) \( G(a_1 \wedge a_2 \wedge b_1) = 0 \).
Proof. — The same letter \( u, x, y, f \) will denote either the closed path or the element of the fundamental group \( \Gamma = \pi_1(S_{g,1}, \ast) \), equipped with paths as indicated in Figure 3.3.

3.22. — Then straightforward computations show that

\[
f_2 = [y_2, x_2] \cdot [y_1, x_1] \in \Gamma
\]

(where \([a, b]\) denotes the commutator \(aba^{-1}b^{-1}\)) and \( u = x_2f_2 \in \Gamma \)

\[
[D(x_2)^{-1}D(u)](x_1) = u x_1 u^{-1}, \quad [D(x_2)^{-1}D(u)](y_1) = u y_1 u^{-1},
\]

\[
[D(x_2)^{-1}D(u)](x_2) = x_2, \quad [D(x_2)^{-1}D(u)](y_2) = f_2 x_2 y_2 x_2^{-1},
\]

(composition of paths is written from left to right).

3.23. — This proves that \( D(x_2)^{-1} D(u) \in N_{g,1} \cap I_{g,1} \) by a result of \([G]\), Theorem 10.1, which says that a homeomorphism of \( S_{g,1} \) belongs to \( N_{g,1} \) if and only if it leaves the normal subgroup of \( \Gamma \) generated by \( \{y_1, \ldots, y_g\} \) invariant. This implies that \( \lambda^*(D(x_2)^{-1} D(u) = 0 \).

3.24. — Using Chapter 3 of \([Pe]\) we find:

\[
A_1(D(x_2)^{-1}D(u)) = \begin{pmatrix} -a_2 & 0 & 0 & -b_1 \\ a_1 & 0 & b_1 & 0 \\ 0 & 0 & -a_2 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{2g}(H)
\]

\( = a_1 \wedge a_2 \wedge b_1 \in \widetilde{H}^3 \subset \otimes^3 H \) (using the identification \( \mathcal{M}_{2g}(H) \cong H \otimes (H \otimes H) \); see Lemma 1.1 of \([Pe]\)). This proves point 1) of Lemma 3.6.

3.25. — Remark that in order to compute \( \theta_0 \circ \sigma \circ A_2'(f) \), it is only necessary to know the terms of the matrix \( A_2'(f) \) on the ascending diagonal:
this comes from the remark of 1.3 of [Pe] and the fact that \( \theta_0 \) is non-zero only on terms such as \( a_i \land a_j \leftrightarrow b_i \land b_j \). By 3.22 we find

\[
A'_2(D(x_2)^{-1}D(u)) = \begin{pmatrix}
\times & \times & \times & 2b_1 \land a_1 \\
\times & \times & 2a_2 \land b_1 - b_2 \land b_1 & \times \\
\times & 0 & \times & \times \\
a_2 \land a_1 & \times & \times & \times
\end{pmatrix}
\]

belonging to \( M_{2g}(\wedge^2 H) \simeq \wedge^2 H \otimes H \otimes H \) (by Lemma 1.1 of [Pe]). Applying the homomorphisms \( \sigma : \wedge^2 H \otimes H \otimes H \to T \) (defined in Theorem 0.1) and \( \theta_0 : T \to \mathbb{Z} \) (see 0.5), we find that \( \theta_0(\sigma \circ A'_2(D(x_2)^{-1}D(u))) = 2 \).

By Lemmas 1.3, 1.6, Proposition 1.4 and the fact that \( \tilde{k}(D(x_2)) = 0 \) we get \( d(D(x_2)^{-1}D(u)) = d(D(u)) - d(D(x_2)) = 4 \). This finishes the proof of Lemma 3.6. \( \square \)

### 3.26. Computation of \( G(a_1 \land b_1 \land b_2) \).

**Lemma 3.7.**

1) \( G(a_1 \land b_1 \land b_2) = -G(D(y_2)D(y'_2)^{-1}) \), where \( y_2, y'_2 \) are defined by Figure 0.1.

2) \( G(a_1 \land b_1 \land b_2) = 0 \).

### 3.27. — Proof: by Proposition 4.4 of [Pe] and [J1], Lemma 4.B, we have

\[
A_1(D(y_2)D(y'_2)^{-1}) = -\tau_2(D(y_2)D(y'_2)^{-1}) = -a_1 \land b_1 \land b_2.
\]

Moreover, \( D(y_2)D(y'_2)^{-1} \in \mathcal{N}_{g,1} \cap \mathcal{I}_{g,1} \), since \( y_2, y'_2 \) bound a 2-disc in the handlebody \( H_g \). Set \( \delta = D(y_2)D(y'_2)^{-1} \). Then we have

\[
\begin{align*}
y'_2 &= x_2y_2^{-1}x_2^{-1}[y_1, x_1] \in \Gamma, \quad \delta(x_1) = y'_2^{-1}x_1y'_2, \\
\delta(y_1) &= y'_2^{-1}y_1y'_2, \quad \delta(x_2) = [x_1, y_1]x_2, \\
\delta(y_2) &= y_2.
\end{align*}
\]

Then \( A'_2(D(y_2)D(y'_2)^{-1}) \in M_{2g}(\wedge^2 H) \) is equal to

\[
\begin{pmatrix}
\times & \times & \times & 0 \\
\times & \times & b_2 \land b_1 & \times \\
\times & 2a_1 \land b_1 & \times & \times \\
2b_2 \land a_1 + a_1 \land a_2 & \times & \times & \times
\end{pmatrix}.
\]
As in the proof of Lemma 3.6, using the isomorphism $\mathcal{M}_{2g}(\wedge^2 H) \cong (\wedge^2 H) \otimes H \otimes H$, we find that $\theta_0(\sigma \circ A_2(D(y_2)D(y'_2)^{-1})) = -2$.

Also using properties of $d$, we can show (using Lemma 1.6) that $d(D(y_2)D(y'_2)^{-1}) = d(D(y_2)) - d(D(y'_2)) = -4$. This shows that $G(D(y_2)D(y'_2)^{-1}) = 0$.

This finishes the proof of Proposition 0.3.

Chapter 4. The hyperelliptic mapping class group.

4.1. Proof of Lemma 0.5.

In 0.10 we have defined the hyperelliptic mapping class group as the subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ generated by the Dehn twists along the curves $y_1, x_1, \ldots, x_g, C_1, C_2, \ldots, C_{g-1}$ of Figure 0.1. In [PV] we have described an isomorphism between the usual braid group $B_{2g+1}$ and $\mathcal{H}_{g,1}$ as follows: let $\{\sigma_1, \ldots, \sigma_{2g}\}$ be the standard generators of $B_{2g+1}$. Then send $\sigma_{2i}$ ($i = 1, \ldots, g$) onto $D(x_i)$, $\sigma_1$ onto $D(y_1)$ and $\sigma_{2i+1}$ ($i = 1, \ldots, g-1$) onto $D(C_i)$.

Moreover an element $f \in \mathcal{M}_{g,1}$ belongs to $\mathcal{H}_{g,1}$ if and only if $f$ commutes (up to isotopy) with the symmetry $s_g$ along the axis $x'x$ of Figure 0.1. This symmetry can be seen as an element of $\mathcal{M}_{g,1}$ (in fact of $\mathcal{H}_{g,1} \simeq B_{2g+1}$), after composition with a half-twist (see 1.7). As an element of $\mathcal{H}_{g,1}$, $s_g$ is represented by

$$\Delta_g^2 = (\sigma_1\sigma_2\cdots\sigma_{2g})^{2g+1} = (D(y_1)D(x_1)D(C_1)D(x_2)\cdots D(C_{g-1})D(x_g))^{2g+1}.$$

The reason of the notation $\Delta_g^2$ is that $\Delta_g^2$ as an element of $B_{2g+1}$ is the square of $\Delta_g = (\sigma_1\sigma_2\cdots\sigma_{g-1})(\sigma_1\sigma_2\cdots\sigma_{g-1})\cdots(\sigma_1\sigma_2)(\sigma_1\sigma_2)$ (see [B], §2.3).

Lemma 4.1. — For $\beta \in \mathcal{H}_{g,1}$ we have:

(i) $2A_1(\beta) = A_1(\Delta_g^2) - \beta_* A_1(\Delta_g^2) \in \otimes^3 H$;

(ii) $2\tilde{k}(\beta) = \tilde{k}(\Delta_g^2) - \beta_* \tilde{k}(\Delta_g^2) \in H$, where $\beta_* = B_0(\beta)$ is the isomorphism induced by $\beta$ at the homological level (see remark below).

Proof. — By Lemma 4.1 of [Pe] we have

$$A_1(\beta \Delta_g^2) = A_1(\beta) + \beta_* A_1(\Delta_g^2), \quad A_1(\Delta_g^2 \beta) = A_1(\Delta_g^2) + (\Delta_g^2)_* A_1(\beta).$$
But \((\Delta^2_g)_* = -\text{id}_H\) since it represents the symmetry along the axis \(x'x\). Point (i) follows from the fact that \(\beta\) and \(\Delta^2_g\) commutes. Point (ii) is proved in the same way, using Lemma 1.1.

Lemma 0.5 is a direct corollary of Lemma 4.1.

4.3. Remark. — For \(\beta \in \mathcal{H}_{g,1} \simeq B_{2g+1}\), \(\beta_0 = B_0(\beta)\) is a conjugate of the reduced Burau representation of \(B_{2g+1}\) evaluated at \(t = -1\) (see [B], §3.2). In fact if we write the matrix of \(D(y_1), D(x_i), D(C_j)\) in the basis \(([y_1], [x_1], [C_1], \ldots, [x_i], [C_i], \ldots, [x_{2g}]\) of \(H\) (where \([\ ]\) represents the homology class), we find exactly the Burau representation when \(t = -1\).

As a consequence \(\mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}\) is identified with the kernel of the Burau representation of \(B_{2g+1}\) when \(t = -1\).

4.2. Computation of \(\tilde{k}(\Delta^2_g)\).

Lemma 4.2. — One has \(\tilde{k}(\Delta^2_g) = 2[(g-1)a_g + (g-2)a_{g-1} + \cdots + a_2]\), where \(a_i\) is the homology class of the oriented circle \(x_i\) (see Figure 0.1).

Proof. — Recall that \(\Delta^2_g\) is the symmetry of \(S_{g,1}\) along the axis \(x'x\), followed by a half twist. We have first to find the effect of \(\Delta^2_g\) on the generators \(x_i, y_i(i = 1, 2 \cdots g)\) of \(\Gamma = \pi_1(S_{g,1}, *)\).

The image by \(\Delta^2_g\) of the oriented curve \(y_i\) equipped with path \(\gamma_i\) is the oriented curve \(y'_i\) of Figure 4.1 equipped with the path \(\gamma'_i\).

Figure 4.1

4.4. — A careful inspection of Figure 4.1 shows that

\[\Delta^2_g(y_i) = x_g^{-1} \cdots x_{i+1}^{-1}(f_iy_i^{-1})x_{i+1} x_g\]

where \(f_i = [y_i, x_i][y_{i-1}, x_{i-1}] \cdots [y_1, x_1]\). We set

\[\beta_i = x_g^{-1} x_{g-1}^{-1} \cdots x_{i+1}^{-1},\]

for \(1 \leq i < g\) and \(\beta_g = 1\), so that we get \(\Delta^2_g(y_i) = \beta_if_iy_i^{-1}\beta_i^{-1}\).
4.5. — The image by $\Delta_g^2$ of the oriented curve $x_i$ (with path $\mu_i$) is $x_i^{-1}$ with path $\mu'_i$ given by Figure 4.2.

![Figure 4.2](image_url)

We finally get $\Delta_g^2(x_i) = \alpha_i x_i^{-1} \alpha_i^{-1}$ where $\alpha_i = x_g^{-1} \cdots x_{i+1}^{-1} y_i = \beta_i y_i$.

Recall that $\bar{k}(\Delta_g^2) = \det(B(\Delta_g^2)^{ab})$. By Lemma 1.2, a), we know that $\bar{k}(\Delta_g^2)$ belongs to $H$ (rather than $\mathbb{Z}[H]$). Let $\alpha = \sum_{i=1}^{2g} n_i c_i$ be any element of $H$. Written multiplicatively it becomes $c_1^{n_1} \cdots c_{2g}^{n_{2g}}$. Applying the commutative Magnus representation (see [Pe], 2.6) and taking the term of degree one, we recover the additive writing of $\alpha$. So $1 + \bar{k}(f)$ is equal to the 1-jet (in the sense of Definition 2.1 of [Pe]), of the determinant of $(B_0(f) + B_1(f))^{ab} = B_0(f) + B_1(f)$.

Since $B_0(\Delta_g^2) = -I$, we obtain that $\bar{k}(\Delta_g^2) = -\text{trace}(B_1(\Delta_g^2))$ using properties of the determinant. Now, by 4.4 and 4.5, trace $(B_1(\Delta_g^2))$ is equal to

$$\text{degree 1 term of } \sum_{j=1}^{g} \frac{\partial \Delta_g^2(x_j)}{\partial x_j} + \frac{\partial \Delta_g^2(y_j)}{\partial y_j}$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} \frac{\partial \alpha_j}{\partial x_j} (1 - x_j) - x_j \alpha_j^{-1}$$

$$+ \sum_{j=1}^{g} \frac{\partial \beta_j}{\partial y_j} (1 - y_j) + \left( \frac{\partial f_j}{\partial y_j} - y_j \right) \beta_j^{-1}$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} -x_j \alpha_j^{-1} + \left( \frac{\partial f_j}{\partial y_j} - y_j \right) \beta_j^{-1} \text{ since } \frac{\partial \alpha_j}{\partial x_j} = \frac{\partial \beta_j}{\partial y_j} = 0$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} -x_j y_j^{-1} x_{j+1} \cdots x_g$$

$$+ (1 - x_j^{-1}) x_{j+1} \cdots x_g - y_j x_{j+1} \cdots x_g$$

$$= \sum_{j=1}^{g} -a_j + \cdots + a_j - b_j + a_j - (a_j + \cdots + a_{j+1} + b_j) = -2 \sum_{j=1}^{g} (\sum_{i=j+1}^{g} a_i).$$

This proves Lemma 4.2. \[\square\]
4.3. A formula of Gambaudo and Ghys for the signature of a link.

4.6. Let $\beta \in B_n$ (the usual braid group with $n$ strings) and $\bar{\beta}$ be the link in $R^3$ obtained by closing $\beta$ according to Figure 4.3:

![Figure 4.3](image)

For a link $k$ in $S^3$, we recall the definition of the signature of $k$, denoted by $s(k)$ (see [GL], §2). Let $V$ be an orientable surface embedded in $S^3$, bounded by $k$. Let $N$ denote a closed tubular neighbourhood of $V$: this is a I-bundle $(I \simeq [0,1])$ over $V$. Let $\bar{V}$ denote the corresponding $\partial I$-bundle and let $\tau : H_1(V) \to H_1(\bar{V})$ be the transfer map. Then define the bilinear form $G_V$ on $H_1(V)$ by: $G_V(\alpha, \beta) = \text{linking number of } (\alpha, \tau(\beta))$. It is shown in [GL] that $G_V$ is symmetric. Then $s(k)$ is the signature of $G_V$.

**Proposition 4.3** (see [GG], Theorem 1.1). Let $\alpha, \beta \in B_{2g+1}$. Then

$$s(\alpha\beta) = s(\alpha) + s(\beta) - \tau(\alpha_*,\beta_*)$$

where $\tau$ is the Meyer 2-cocycle defined in 1.3, associated to $\alpha, \beta$ identified to elements of $H_{g,1} (\simeq B_{2g+1})$, by 4.1.

4.4. Proof of proposition 0.6.

4.7. We now restrict $d$ to $H_{g,1}$. The signature defines a map $s : H_{g,1} \to \mathbb{Z}$ by setting $s(\alpha) = s(\tilde{\alpha})$. From Proposition 1.4 and Proposition 4.3, the mapping $d - 3s : H_{g,1} \to \mathbb{Z}$ satisfies

$$(d - 3s)(\alpha\beta) = (d - 3s)(\alpha) + (d - 3s)(\beta) + \bar{k}(\alpha) \cdot \alpha_* \bar{k}(\beta).$$

In cohomological terms, the 1-chain $d - 3s$ on $H_{g,1}$ has it coboundary equal to the 2-cocycle $c$ on $H_{g,1}$ defined by $c(\alpha, \beta) = -\bar{k}(\alpha) \cdot \alpha_* \bar{k}(\beta)$. Using Lemma 4.1, this 2-cocyle is given by

$$c(\alpha, \beta) = -\frac{1}{4} \bar{k}(\Delta_g^2) \cdot [\alpha_* \bar{k}(\Delta_g^2) - \alpha_\ast \beta_\ast \bar{k}(\Delta_g^2) + \beta_\ast \bar{k}(\Delta_g^2)].$$

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4.8. — Let be the 1-cochain \( u(\alpha) = \frac{1}{4} \alpha \cdot \tilde{k}(\Delta_g^2) \cdot \tilde{k}(\Delta_g^2) \). Obviously \( u \) satisfies \( u(\alpha \beta) = u(\alpha) + u(\beta) - c(\alpha, \beta) \) by the above formula. Remark that \( u \) takes a priori its values in \( \frac{1}{4} \mathbb{Z} \). But we have seen in Lemma 4.2 that \( \tilde{k}(\Delta_g^2) = 2\delta_g \) where \( \delta_g = (g - 1)a_g + \cdots + a_2 \). So \( u(\alpha) = \alpha \cdot (\delta_g) \cdot \delta_g \) belongs to \( \mathbb{Z} \).

4.9. — By 4.7 the mapping \( d - 3s - u : \mathcal{H}_{g,1} \to \mathbb{Z} \) is a homomorphism. From the presentation of \( \mathcal{H}_{g,1} \cong B_{2g+1} \), it is well known that the abelianization of \( B_{2g+1} \) is isomorphic to \( \mathbb{Z} \), the canonical homomorphism \( \pi : B_{2g+1} \to \mathbb{Z} \) sending each generator \( \sigma_i \) \( (i = 1, \ldots, g) \) onto \( 1 \in \mathbb{Z} \). So there exists an integer \( n_0 \in \mathbb{Z} \) such that \( d - 3s - u = n_0 \pi \).

4.10. — To determine the value of \( n_0 \), it is enough to evaluate the two terms of the equality above on the element \( D(f_1) \) \( (f_1 \) is the circle defined by Figure 0.1). The Dehn twist \( D(f_1) \) is known to be equal to \( (D(x_1)D(y_1))^6 \cong (\sigma_2 \sigma_1)^6 \in \mathcal{H}_{g,1} \cong B_{2g+1} \).

By Corollary 0.2, \( d(D(f_1)) = \delta(D(f_1)) = 0 \). Since \( D(f_1) \) belongs to \( \mathcal{I}_{g,1} \cong \mathcal{M}(3) \subset \mathcal{I}_{g,1} \), \( u(D(f_1)) = 0 \). Then formula 4.9 gives

\[
3s((\sigma_2 \sigma_1)^6) = 3s((\hat{\sigma_2 \sigma_1})^6) = n_0 \pi(D(f_1)) = 12n_0.
\]

Claim: \( s((\hat{\sigma_2 \sigma_1})^6) = -8 \) and so \( n_0 = 2 \). We can compute the signature of the link \( (\sigma_2 \sigma_1)^6 \) by the method of [GL], using the diagram of \( (\sigma_2 \sigma_1)^6 \) given by the braid \( (\hat{\sigma_2 \sigma_1})^6 \), or use the formula of Proposition 4.3.

**Corollary 4.4.** — The mapping \( d \) takes the following values on the Lickorish generators of \( \mathcal{M}_{g,1} \):

1) \( d(D(y_1)) = d(D(x_i)) = 2 \), for \( i = 1, \ldots, g \).
2) \( d(D(C_j)) = 3 \), for \( i = 1, \ldots, g - 1 \).
3) \( d(D(y_i)) = 2 \), for \( i = 2, \ldots, g \).
4) \( d(D(y_i')) = 6 \), for \( i = 2, \ldots, g \).

(The circles \( x_i, y_i, y_i', C_j \) are defined by Figure 0.1.)

4.11. Remark. — A priori \( d \) depends on the choice of the symplectic basis \( \{a_i, b_j; i = 1, \ldots, g\} \). Morita [Mo1] proved that \( d/\mathcal{I}_{g,1} \) is independant of the choices.

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Proof. — It is easy to see that \( u(D(y_i)) = u(D(x_i)) = 0 \) for \( i = 1, \ldots, g \) (\( u \) is defined in 4.8). So \( d(D(y_1)) = d(D(x_i)) = 2 \) by Proposition 0.6.

Since, for \( i = 1, \ldots, g \), the circle \( C_i \) cuts transversally in one point the circle \( x_i \), the corresponding Dehn twists \( D(C_i) \) and \( D(x_i) \) satisfy the usual braid relation, which is equivalent to

\[
D(C_i) = D(x_i)D(C_i)D(x_i)D(C_i)^{-1}D(x_i)^{-1}.
\]

By Lemma 1.1, Proposition 1.4 (iii) and the fact that \( \tilde{k}(D(x_i)) = 0 \), we have

\[
d(D(C_i)) = d(D(x_i)) - D(x_i) \cdot \tilde{k}(D(C_i)) \cdot D(x_i) \cdot \tilde{k}(D(C_i)).
\]

By [Pe], 5.2, and Lemma 1.2 we get \( \tilde{k}(D(C_i)) = b_{i+1} - b_i \). Point 2) follows.

Point 3) follows from the same type of argument, using the fact that the circle \( y_i \) cuts transversally in one point the circle \( x_i \). Point 4) follows from Lemma 1.6 (ii).

\[
\□
\]

4.12. Remark. — Corollary 4.4 contradicts an affirmation of [Mo1], §5, line above Proposition 5.1, which says that the values of \( d \) on the Lickorish generators is 3. This affirmation cannot be true, since we have proved above that \( d(D(C_i)) = d(D(x_i)) + 1 \).

Corollary 4.5. — Let \( \beta \in H_{g,1} \cap I_{g,1} \) (which is equivalent to say, by Remark 4.3, that \( \beta \) belongs to the kernel of the Burau representation when \( t = -1 \)). Then the Casson invariant of the homology-sphere \( M_\beta \) is given by

\[
\lambda^*(\beta) = \lambda(M_\beta) = \frac{1}{12} \left( \pi(\beta) - \theta_0(\sigma \circ A'_2(\beta)) \right) + \frac{1}{8} s(\tilde{\beta}),
\]

where \( \pi \) and \( s \) are defined in Proposition 0.6.

The object of the next proposition is to describe geometrically the 3-manifold \( M_\beta \) when \( \beta \in H_{g,1} \simeq B_{2g+1} \). For a braid \( \gamma \in B_{2n} \) denote by \( \tilde{\gamma} \) the closure by “plats” (see Figure 4.4).

Then we have:

Proposition 4.6. — For \( \beta \in H_{g,1} \simeq B_{2g+1} \), the 3-manifold \( M_\beta \) is homeomorphic to the 2-fold covering of \( S^3 \) branched along the link \( \tilde{\gamma} \) where \( \gamma = \sigma_2 \cdots \sigma_{2g+1} \beta \) (remark that \( \gamma \) belongs to \( B_{2g+2} \)).
Proof. — Recall the construction of the 3-manifold $M_\beta$. Let $S_{g,1} \subset S_g$ be standardly embedded in $\mathbb{R}^3 \subset \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, bounding the handlebody $H_g$. Set $H'_g = \mathbb{S}^3 - H_g$. Let $x'x$ be the symmetry axis of $H_g$, intersecting $S_g$ (resp. $H_g$) at points $P_i$, $i = 1, \ldots, 2g + 2$ (resp. segments $\alpha_i = [P_{2i-1}, P_{2i}]$, $i = 1, \ldots, g + 1$).

Denote by $C$ the circle of $\mathbb{S}^3$ defined by $C = (x'x) \cup \{\infty\}$, and let $\{\gamma_i ; i = 1, \ldots, g + 1\}$ the trace of $C$ on $H'_g$. More precisely $\gamma_i = [P_{2i}, P_{2i+1}]$ for $1 \leq i \leq g$ and $\gamma_{g+1}$ is the segment of $C$ with ends $P_{2g+2}, P_1$ containing $\infty \in \mathbb{S}^3$ (see Figure 4.5).

The quotient of $H_g$ (resp. $H'_g$) by the symmetry along $xx'$ (resp $C$) is homeomorphic to a 3-ball $B$ (resp. $B'$). The image of the fixed points $\{P_i ; i = 1, \ldots, 2g + 2\}$ are denoted $\{Q_i ; i = 1, \ldots, 2g+2\}$. The image of the set of fixed points $\{\alpha_i ; i = 1, \ldots, g+1\}$ (resp. $\{\gamma_i ; i = 1, 2, \ldots, g + 1\}$) are denoted $\{\overline{\alpha}_i\} \subset B$ (resp $\{\overline{\gamma}_i\} \subset B'$). They are arcs in the interior of $B$ (resp $B'$) whose ends are the points $\{Q_i\}$ (see Figure 4.6).

The mapping $H_g \xrightarrow{\pi} B$ (resp. $H'_g \xrightarrow{\pi'} B'$) is the 2-fold cyclic covering ramified along the arcs $\{\overline{\alpha}_i\}$ (resp. $\{\overline{\gamma}_i\}$).

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By definition, an element $\beta$ of $\mathcal{M}_{g,1}$ belongs to the hyperelliptic mapping class group $\mathcal{H}_{g,1} \simeq B_{2g+1}$ if $\beta$ is the lift of the braid $\beta \in B_{2g+1}$ by the 2-fold cyclic covering $\pi_{|S_{g,1}} : \partial \mathcal{H}_g \to \partial B - D^2$ where $D^2$ is a small disc centered at $P_{2g+2}$.

Then $M_{\beta}$ is the 2-fold cyclic covering over $B \cup_{\beta} B'$, where $\beta$ is seen as a homeomorphism of $\partial B - D^2$ (we extend it by identity on $D^2$) leaving $\{Q_i; \ i = 1, \ldots, g - 1\}$ globally invariant and fixing $P_{2g+2}$, the set of ramification being $\{\bigcup_i \overline{a}_i\} \cup \{\bigcup_i \overline{\gamma}_i\}$. Equivalently, if $\beta \in \beta_{2g+1}$ is represented by $2g + 1$ strings in $(\partial B - D^2) \times [0,1]$, let $\beta'$ be the $2g + 2$ strings of $\partial B \times [0,1]$ obtained by adding the trivial string $P_{2g+2} \times [0,1]$. Let $\mathcal{L}$ be the link in $S^3 = B^3 \cup B^3$ obtained from the braid $\beta'$ by Figure 4.7 or Figure 4.8.

Clearly the link $\mathcal{L}$ is isotopic to the link $\mathcal{L}'$ of Figure 4.9.

The link $\mathcal{L}'$ is obtained from the braid $\gamma = \sigma_2 \cdots \sigma_{g+1} \beta$ by closing by plats. This concludes the proof of Proposition 4.6.

**Corollary 4.7.** Let $\beta \in B_{2g+1} \cap \mathcal{I}_{g,1}$ and $\gamma$ the braid of $B_{2g+2}$ given by $\gamma = \sigma_2 \cdots \sigma_{g+1} \beta$. Denote by $M^{(2)}(\hat{\gamma})$ the two-fold cyclic covering of $S^3$ ramified along $\hat{\gamma}$. Then the Casson invariant $\lambda(M^{(2)}(\hat{\gamma}))$ is given by

$$\lambda(M^{(2)}(\hat{\gamma})) = \frac{1}{12} \left( \pi(\beta) - \theta_0(\sigma \circ A'_2(\beta)) \right) + \frac{1}{8} s(\beta).$$

**Proof.** This follows from Corollary 4.5 and Proposition 4.6.
Final remark. — One should compare formula of Corollary 4.7 with Mullins formula [Mu] giving the Casson invariant of the 2-fold cyclic
covering of \(S^3\) ramified along a link \(L\)

\[
\lambda_2'(L) = \frac{-(dV_L/dt)(-1)}{12V_L(-1)} + \frac{1}{8}s(L),
\]

where \(V_L(t)\) is the Jones polynomial of \(L\) (be aware that Mullins formula in Theorem 5.1 of [Mu] gives two times Casson’s invariant).

**BIBLIOGRAPHY**


