ANNALES

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Tome 54, nº 3 (2004), p. 773-786.

<http://aif.cedram.org/item?id=AIF_2004__54_3_773_0>

Ann. Inst. Fourier, Grenoble
54, 3 (2004), 773–786

LOCAL WELL-POSEDNESS FOR THE INCOMPRESSIBLE EULER EQUATIONS IN THE CRITICAL BESOV SPACES

by Yong ZHOU

1. Introduction.

In this paper, we consider the incompressible Euler equations in $\mathbb{R}^N$, $N \geqslant 3$,

$$\begin{cases} \dfrac{\partial v}{\partial_t} + (v \cdot \nabla)v + \nabla P = f \\ \operatorname{div} v = 0 \\ v(x, t = 0) = v_0(x), \end{cases} \tag{1.1}$$

where $v(x,t) \in \mathbb{R}^N$ stands for the velocity field, $P(x,t)$ is the pressure, while $f(x,t)$ is the force, which will be assumed as zero just for simplicity. Our main results can be gone through for any $f \in L^1(0,T; B^{N/p+1}_{p,1})$.

For the local well-posedness of the sysytem (1.1), we mention the following results. Given $v_0 \in H^m(\mathbb{R}^N)$, $m > N/2+1$, Kato [9] proved local existence and uniqueness for a solution belonging to $C([0,T]; H^m(\mathbb{R}^N))$ with $T = T(\|v_0\|_{H^m})$. Later on, many various function spaces (see [3], [4], [5], [10], [13]) are used to establish the local existence and uniqueness for the incompressible Euler equations. For example, $W^{s,p}(\mathbb{R}^N)$ with $s > N/p+1$, $1 < p < \infty$ is used in [10] and $F^s_{p,q}$ for $s > N/p+1$, $1 < p < \infty$, $1 < q < \infty$ is used in [3]. In particular, Vishik [17] showed the (global) well-posedness

Keywords: Well-posedness – Euler equations – Besov spaces.
Math. classification: 76D03 – 35Q35 – 46E35.

for 2-D incompressible Euler equations in the critical (borderline) Besov spaces $B^{2/p+1}_{p,1}(\mathbb{R}^2)$. Later, Vishik [18] proved the existence ($N = 2$) and uniqueness ($N \geqslant 2$) result for (1.1) with initial vorticity belonging to a space of Besov type. The purpose of this paper is to establish local well-posedness of (1.1) in $\mathbb{R}^N$, for any $N \geqslant 3$.

THEOREM 1.1. — *Let $1 < p < \infty$. Given any $v_0 \in B^{N/p+1}_{p,1}(\mathbb{R}^N)$, there exists a $T = T(\|v_0\|_{B^{N/p+1}_{p,1}})$ and a unique solution $(v, \nabla P)$ to (1.1) such that*

$$v(x,t) \in C([0,T]; B^{N/p+1}_{p,1}) \quad \text{and} \quad \nabla P \in L^1(0,T; B^{N/p+1}_{p,1}). \tag{1.2}$$

The maximum local existence time, say T^*, is called the lifespan of the solution. If T^* is finite, we say that the solution blows up at time T^*. In our case it is $\limsup_{t\to T^*} \|v(.,t)\|_{B^{N/p+1}_{p,1}} = \infty$. Beal, Kato and Majda [1] established the following blow-up criterion for the smooth solution $v(x,t)$ to (1.1), $v(x,t) \in C([0,T]; H^m(\mathbb{R}^N))$ with $m > N/2+1$ as

$$\int_0^{T^*} \|\omega(.,t)\|_{L^\infty} dt = \infty,$$

where $\omega = \operatorname{curl} v$ is the vorticity field. Later on, some refined results were proved in [11], [12]. The blow-up criterion for our case reads

THEOREM 1.2. — *The local solution constructed in Theorem 1.1 blows up at time T^* if and only if*

$$\int_0^{T^*} \|\omega(.,t)\|_{\dot{B}^{N/p}_{p,1}} dt = \infty.$$

2. Littlewood-Paley decomposition and Besov spaces.

We start by recalling the Littlewood-Paley decomposition of temperate distributions. Let $\mathcal{S}$ be the class of Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, the Fourier transform is defined as

$$\mathcal{F}(f) = \hat{f} = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x) dx.$$

One can extend $\mathcal{F}$ and $\mathcal{F}^{-1}$ to $\mathcal{S}'$ in the usual way, where $\mathcal{S}'$ denotes the set of all tempered distributions. Let $\phi \in \mathcal{S}$ satisfying

$$\operatorname{Supp} \hat{\phi} \subset \left\{\xi : \frac{5}{6} \leqslant |\xi| \leqslant \frac{12}{5}\right\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \hat{\phi}\left(2^{-j}\xi\right) = 1,$$

for $\xi \neq 0$. Setting $\hat{\phi}_j = \hat{\phi}(2^{-j}\xi)$, in other words, $\phi_j(x) = 2^{jN}\phi(2^j x)$, for any $f \in \mathcal{S}'$, we define

$$\Delta_j f = \phi_j * f \quad \text{and} \quad S_j f = \sum_{k \leqslant j-1} \phi_k * f. \tag{2.1}$$

Then the homogeneous Besov semi-norm $\|f\|_{\dot{B}^s_{p,q}}$ and Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ are defined by

DEFINITION 2.1 [14], [16]. — *For $-\infty < s < \infty$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, set*

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jqs}\|\Delta_j f\|^q_{L^p}\right)^{1/q} & \text{if } q \in (0,\infty), \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

$$\|f\|_{\dot{F}^s_{p,q}} = \begin{cases} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{jqs}|\Delta_j f|^q\right)^{1/q}\right\|_{L^p} & \text{if } q \in (0,\infty), \\ \left\|\sup_{j\in\mathbb{Z}}(2^{js}|\Delta_j f|)\right\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

The space $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ are quasi-normed spaces with the above quasi-norm given by Definition 2.1. For $s > 0$, $(p,q) \in (1,\infty) \times [1,\infty]$, we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ and inhomogeneous Triebel-Lizorkin space norm $\|f\|_{F^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}, \quad \|f\|_{F^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}. \tag{2.2}$$

The inhomogeneous Besov and Triebel-Lizorkin spaces are Banach spaces equipped with the norm $\|f\|_{B^s_{p,q}}$ and $\|f\|_{F^s_{p,q}}$ respectively.

Let us now state some classical results.

LEMMA 2.2 [14], [16] (Bernstein's Lemma). — *Assume that $k \in \mathbb{Z}^+$, $f \in L^p$, $1 \leqslant p \leqslant \infty$, and $\operatorname{Supp}\hat{f} \subset \{2^{j-2} \leqslant |\xi| < 2^j\}$, then there exists a constant $C(k)$ such that the following inequality holds.*

$$C(k)^{-1}2^{jk}\|f\|_{L^p} \leqslant \|D^k f\|_{L^p} \leqslant C(k)2^{jk}\|f\|_{L^p}.$$

For any $k \in \mathbb{Z}^+$, there exists a constant $C(k)$ such that the following inequality is true.

$$C(k)^{-1}\|D^k f\|_{\dot{B}^s_{p,q}} \leqslant \|f\|_{\dot{B}^{s+k}_{p,q}} \leqslant C(k)\|f\|_{\dot{B}^s_{p,q}} \tag{2.3}$$

$$C(k)^{-1}\|D^k f\|_{\dot{F}^s_{p,q}} \leqslant \|f\|_{\dot{F}^{s+k}_{p,q}} \leqslant C(k)\|f\|_{\dot{F}^s_{p,q}}. \tag{2.4}$$

LEMMA 2.3 [14], [16] (Embeddings). — *Let $p \in (1,\infty)$, then*

$$B^{N/p}_{p,1} \hookrightarrow L^\infty.$$

PROPOSITION 2.4 (Product). — *Let $s > 0$, $1 < p < \infty$. If f and g belong to $\dot{B}^s_{p,1} \cap L^\infty$, then fg is in $\dot{B}^s_{p,1}$, and*

$$\|fg\|_{\dot{B}^s_{p,1}} \leqslant C\left(\|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,1}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,1}}\right). \tag{2.5}$$

In particular, for $f, g \in \dot{B}^{N/p}_{p,1}$, there holds

$$\|fg\|_{\dot{B}^{N/p}_{p,1}} \lessdot C\|f\|_{\dot{B}^{N/p}_{p,1}}\|g\|_{\dot{B}^{N/p}_{p,1}}. \tag{2.6}$$

This proposition will be showed in the appendix.

LEMMA 2.5 [7] (Commutator). — *Suppose that $s \in (-N/p-1, N/p]$. Then for $f \in \dot{B}^{N/p+1}_{p,1}$ and $g \in \dot{B}^s_{p,1}$, we have*

$$\|[f,\Delta_j]g\|_{L^p} \leqslant C_j 2^{-j(s+1)}\|f\|_{\dot{B}^{N/p+1}_{p,1}}\|g\|_{\dot{B}^s_{p,1}}$$

with $\sum_{j\in\mathbb{Z}} C_j \leqslant 1$.

3. Proof of Theorem 1.1.

In this section, C denotes a absolute constant, which maybe different from line to line.

Consider the following linear system

$$\begin{cases} \partial_t v + (w\cdot\nabla)v + \nabla P = 0 \\ \operatorname{div} v = 0 \\ v(x,t=0) = v_0(x). \end{cases} \tag{3.1}$$

We have the following local existence theorem for (3.1), which will be proved in the appendix.

PROPOSITION 3.1. — *Assume that $\operatorname{div} w = 0$, $w \in L^\infty(0,T; B^{N/p+1}_{p,1})$, for some $T > 0$. Then for any $v_0 \in B^{N/p+1}_{p,1}$, $\operatorname{div} v_0 = 0$, there exists a unique solution $v \in C(0,T; B^{N/p+1}_{p,1})$ to (3.1). And consequently, ∇P can be determined uniquely.*

In order to prove the existence part of the main theorem, we consider the following approximate linear iteration system for (1.1)

$$\begin{cases} \partial_t v^{n+1} + v^n \cdot \nabla v^{n+1} + \nabla P^{n+1} = 0, \\ \operatorname{div} v^{n+1} = \operatorname{div} v^n = 0, \\ v^{n+1}(x,t=0) = v^{n+1}(0) = S_{n+1} v_0, \end{cases} \tag{3.2}$$

where $v^0 = 0$. In [3], Chae used a similar (not same) iterative system to construct the local solution. But unfortunately, the linear system (3.32)–(3.33) on page 671 of [3] is not solvable, since the system itself lacks consistence.

If we have the uniform estimate for the sequence v^n by induction, which satisfies the conditions in Proposition 3.1, then the system (3.2) can be solved with solution v^{n+1}.

Uniform estimates.

First multiply (3.2) coordinate by coordinate with $|v_l^{n+1}|^{p-2} v_l^{n+1}$, where v_l^{n+1} is the l-th coordinate of the vector field v^{n+1}, and integrate over $\mathbb{R}^N$. Taking the divergence free property of v^n into account, we have

$$\frac{1}{p}\frac{d}{dt}\|v_l^{n+1}\|_{L^p}^p \leqslant \|\nabla P^{n+1}\|_{L^p}\|v_l^{n+1}\|_{L^p}^{p-1}$$

therefore,

$$\frac{d}{dt}\|v^{n+1}\|_{L^p} \leqslant \|\nabla P^{n+1}\|_{L^p}. \tag{3.3}$$

Note that for $p = 2$, multiplying (3.2) by v^{n+1}, and integrating by parts, we have

$$\sup_{0\leqslant t\leqslant T} \|v(x,t)^{n+1}\|_{L^2} \leqslant \|v^{n+1}(0)\|_{L^2}.$$

Now taking Δ_j on (3.2), we get

$$\partial_t \Delta_j v^{n+1} + v^n \cdot \nabla \Delta_j v^{n+1} + \nabla \Delta_j P^{n+1} = [v^n, \Delta_j]\nabla v^{n+1}. \tag{3.4}$$

Multiplying (3.4) coordinate by coordinate with $|\Delta_j v_l^{n+1}|^{p-2}\Delta_j v_l^{n+1}$, and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\Delta_j v_l^{n+1}\|_{L^p}^p &\leqslant \big\|[v^n,\Delta_j]\nabla v_l^{n+1}\big\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\quad + \|\Delta_j \nabla P^{n+1}\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\leqslant CC_j 2^{-j(N/p+1)}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot B^{N/p+1}_{p,1}}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1} \\ &\quad + \|\Delta_j \nabla P^{n+1}\|_{L^p}\|\Delta_j v_l^{n+1}\|_{L^p}^{p-1}. \end{aligned} \tag{3.5}$$

Then apply $2^{j(N/p+1)}$ on (3.5) and take summation,

$$\frac{d}{dt}\|v^{n+1}\|_{\dot{B}^{N/p+1}_{p,1}} \leqslant C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot{B}^{N/p+1}_{p,1}} + \|\nabla P^{n+1}\|_{\dot{B}^{N/p+1}_{p,1}} \tag{3.6}$$

Now we turn our attention to the estimates for ∇P^{n+1}. Taking divergence on the both sides of (3.2), it follows that

$$-\Delta P^{n+1} = \operatorname{div}(v^n \cdot \nabla v^{n+1}),$$

thus

$$\partial_i\partial_j P^{n+1} = R_i R_j \operatorname{div}(v^n \cdot \nabla v^{n+1}). \tag{3.7}$$

Thanks to the divergence free property of v^n, we obtain

$$\begin{aligned}\operatorname{div}(v^n \cdot \nabla v^{n+1}) = \sum_{k,l=1}^{N} \partial_k(v_l^n \partial_l v_k^{n+1}) &= \sum_{k,l=1}^{N} \partial_k\partial_l(v_l^n v_k^{n+1}) \\ &= \sum_{k,l=1}^{N} \partial_l(\partial_k v_l^n v_k^{n+1}).\end{aligned}$$

For $1 < p < \infty$, it was proved [14], [16] that $\dot{F}^0_{p,2} = L^p$ and R_i is bounded from L^p into itself [8], [15]. Due to Bernstein's lemma, we have

$$\begin{aligned}\|\nabla P^{n+1}\|_{L^p} = \|\nabla P^{n+1}\|_{\dot{F}^0_{p,2}} &\leqslant C\sum_{i,j=1}^{N}\|\partial_i\partial_j P^{n+1}\|_{\dot{F}^{-1}_{p,2}} \\ &\leqslant C\|\operatorname{div}(v^n\cdot\nabla v^{n+1})\|_{\dot{F}^{-1}_{p,2}} \\ &\leqslant C\sum_{k,l=1}^{N}\|\partial_l(\partial_k v_l^n v_k^{n+1})\|_{\dot{F}^{-1}_{p,2}} \leqslant C\sum_{k,l=1}^{N}\|\partial_k v_l^n v_k^{n+1})\|_{L^p} \\ &\leqslant C\|\nabla v^n\|_{L^\infty}\|v^{n+1}\|_{L^p} \leqslant C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|v^{n+1}\|_{L^p}.\end{aligned} \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned}\|\nabla P^{n+1}\|_{\dot{B}^{N/p+1}_{p,1}} &\leqslant C\sum_{i,j=1}^{N}\|\partial_i\partial_j P^{n+1}\|_{\dot{B}^{N/p}_{p,1}} \\ &\leqslant C\sum_{i,j,k,l=1}^{N}\|R_iR_j\partial_k v_l^n\partial_l v_k^{n+1}\|_{\dot{B}^{N/p}_{p,1}} \\ &\leqslant C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|v^{n+1}\|_{\dot{B}^{N/p+1}_{p,1}}\end{aligned} \tag{3.9}$$

where we used that R_i is bounded from $\dot{B}^s_{p,q}$ into itself [8].

Combining (3.3), (3.6), (3.8) and (3.9),

$$\frac{d}{dt}\|v^{n+1}\|_{B^{N/p+1}_{p,1}} \leqslant C\|v^n\|_{B^{N/p+1}_{p,1}}\|v^{n+1}\|_{B^{N/p+1}_{p,1}}. \tag{3.10}$$

Note that although the above constants C maybe depend on N and p, it is nothing to do with n, therefore we can obtain uniform estimates by induction.

In fact, suppose that the initial datum v_0 satisfies $\|v_0\|_{B^{N/p+1}_{p,1}} \leqslant C_1/2$, then the following inequality holds

$$\|v^{n+1}\|_{L^\infty(0,T_1;B^{N/p+1}_{p,1})} \leqslant C_1, \tag{3.11}$$

for all $n \geqslant 0$, provided that T_1 (independent of n) is sufficiently small.

(3.11) can be showed easily by mathematical induction. First, it is true for $n = 0$. Suppose (3.11) holds for n, we want to prove it is true for $n+1$. It follows from (3.10) that

$$\|v^{n+1}\|_{L^\infty(0,T;B^{N/p+1}_{p,1})} \leqslant \frac{C_1}{2} \exp\left(\int_0^T \|v^n(.,t)\|_{\dot{B}^{N/p+1}_{p,1}} dt \right) \leqslant \frac{C_1}{2} \exp(CTC_1).$$

Hence, (3.11) holds, if we choose T_1 so small that

$$\exp(CT_1C_1) \leqslant 2.$$

Moreover, T_1 is independent of n.

Convergence.

To prove the convergence, it is sufficient to estimate the difference of the iteration. Take the difference between the equation (3.2) for the $(n+1)$-th step and the n-th step, and set

$$u^{n+1} = v^{n+1} - v^n, \quad \Pi^{n+1} = P^{n+1} - P^n,$$

then we get the equation as follows

$$\begin{cases} \partial_t u^{n+1} + v^n \cdot \nabla u^{n+1} + u^n \cdot \nabla v^n + \nabla \Pi^{n+1} = 0, \\ \operatorname{div} u^{n+1} = \operatorname{div} v^n = 0, \\ u^{n+1}(x, t=0) = u^{n+1}(0) = \Delta_n v_0. \end{cases} \tag{3.12}$$

Just as what done for v^{n+1}, multiplying (3.12) coordinate by coordinate with $|u_l^{n+1}|^{p-2}u_l^{n+1}$. Thanks to Hölder's inequality, we have

$$\begin{aligned} \frac{d}{dt}\|u^{n+1}\|_{L^p} &\leqslant \|u^n \cdot \nabla v^n\|_{L^p} + \|\nabla \Pi^{n+1}\|_{L^p} \\ &\leqslant \|u^n\|_{L^p}\|\nabla v^n\|_{L^\infty} + \|\nabla \Pi^{n+1}\|_{L^p} \\ &\leqslant C\|u^n\|_{L^p}\|v^n\|_{\dot{B}^{N/p+1}_{p,1}} + \|\nabla \Pi^{n+1}\|_{L^p}. \end{aligned} \tag{3.13}$$

Taking Δ_j on (3.12), we get

$$\partial_t \Delta_j u^{n+1} + v^n \cdot \nabla \Delta_j u^{n+1} + \nabla \Delta_j \Pi^{n+1} = [v^n, \Delta_j]\nabla u^{n+1} - \Delta_j(u^n \cdot \nabla v^n). \tag{3.14}$$

Multiplying (3.14) coordinate by coordinate with $|\Delta_j u_l^{n+1}|^{p-2}\Delta_j u_l^{n+1}$, and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}\frac{d}{dt}\|\Delta_j u^{n+1}\|_{L^p} \leqslant\ & CC_j 2^{-jN/p}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}}\\ &+\|\Delta_j\nabla\Pi^{n+1}\|_{L^p}+\|\Delta_j(u^n\cdot\nabla v^n)\|_{L^p}.\end{aligned}\tag{3.15}$$

Then apply $2^{jN/p}$ on (3.15) and take summation,

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}} &\leqslant C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}}+\|u^n\cdot\nabla v^n\|_{\dot B^{N/p}_{p,1}}\\ &\quad+\|\nabla\Pi^{n+1}\|_{\dot B^{N/p}_{p,1}}\\ &\leqslant C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot B^{N/p}_{p,1}}+\|u^n\|_{\dot B^{N/p}_{p,1}}\|v^n\|_{\dot B^{N/p+1}_{p,1}}\\ &\quad+\|\nabla\Pi^{n+1}\|_{\dot B^{N/p}_{p,1}}.\end{aligned}\tag{3.16}$$

Combining (3.13) and (3.16), we have

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{B^{\frac{N}{p}}_{p,1}} \leqslant\ & C\|v^n\|_{B^{\frac{N}{p}+1}_{p,1}}\|u^{n+1}\|_{B^{\frac{N}{p}}_{p,1}}+\|v^n\|_{B^{\frac{N}{p}+1}_{p,1}}\|u^n\|_{B^{\frac{N}{p}}_{p,1}}\\ &+\|\nabla\Pi^{n+1}\|_{B^{\frac{N}{p}}_{p,1}}.\end{aligned}\tag{3.17}$$

We can estimate $\nabla\Pi^{n+1}$ as follows. From the equation (3.2), it follows

$$\partial_i\partial_j\Pi^{n+1}=R_iR_j\operatorname{div}(v^n\cdot\nabla u^{n+1})+R_iR_j\operatorname{div}(u^n\cdot\nabla v^n).$$

Thanks to the divergence free of v^n, we have

$$\begin{aligned}\operatorname{div}(v^n\cdot\nabla u^{n+1}) &= \sum_{k,l=1}^N\partial_l(v_k^n\partial_k u_l^{n+1})=\sum_{k,l=1}^N\partial_l\partial_k(v_k^n u_l^{n+1})\\ &=\sum_{k,l=1}^N\partial_k(\partial_l v_k^n u_l^{n+1}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|\nabla\Pi^{n+1}\|_{L^p} &= \|\nabla\Pi^{n+1}\|_{\dot F^0_{p,2}}\leqslant C\sum_{i,j=1}^N\|\partial_i\partial_j\Pi^{n+1}\|_{\dot F^{-1}_{p,2}}\\ &\leqslant C\|\operatorname{div}(v^n\cdot\nabla u^{n+1})\|_{\dot F^{-1}_{p,2}}+C\|\operatorname{div}(u^n\cdot\nabla v^n)\|_{\dot F^{-1}_{p,2}}\\ &\leqslant C\sum_{k,l=1}^N\|\partial_l v_k^n u_l^{n+1}\|_{L^p}+C\|u^n\cdot\nabla v^n)\|_{L^p}\\ &\leqslant C\|\nabla v^n\|_{L^\infty}\|u^{n+1}\|_{L^p}+C\|u^n\|_{L^p}\|\nabla v^n\|_{L^\infty}\\ &\leqslant C\|v^n\|_{\dot B^{N/p+1}_{p,1}}\|u^{n+1}\|_{L^p}+C\|u^n\|_{L^p}\|v^n\|_{\dot B^{N/p+1}_{p,1}},\end{aligned}\tag{3.18}$$

where we used the Hölder inequality and embedding Lemma 2.3. And similarly,

$$\begin{aligned} \|\nabla\Pi^{n+1}\|_{\dot{B}^{N/p}_{p,1}} &\leqslant C\sum_{i,j=1}^{N}\|\partial_i\partial_j\Pi^{n+1}\|_{\dot{B}^{N/p-1}_{p,1}} \\ &\leqslant C\sum_{k,l=1}^{N}\|\partial_l v_k^n u_l^{n+1}\|_{\dot{B}^{N/p}_{p,1}} + C\|u^n\cdot\nabla v^n\|_{\dot{B}^{N/p}_{p,1}} \\ &\leqslant C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|u^{n+1}\|_{\dot{B}^{N/p}_{p,1}} + C\|v^n\|_{\dot{B}^{N/p+1}_{p,1}}\|u^n\|_{\dot{B}^{N/p}_{p,1}} \end{aligned} \tag{3.19}$$

where we used (2.6).

Then integrate (3.17) on the time interval (0,T) by taking (3.18) and (3.19) into account,

$$\begin{aligned} \|u^{n+1}\|_{L^\infty(0,T;B^{N/p}_{p,1})} \leqslant \|u^{n+1}(0)\|_{B^{N/p}_{p,1}} &+ CT\|v^n\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\|u^{n+1}\|_{L^\infty(0,T;B^{N/p}_{p,1})} \\ &+ CT\|v^n\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\|u^n\|_{L^\infty(0,T;B^{N/p}_{p,1})}. \end{aligned} \tag{3.20}$$

So if we choose $T_2 \leqslant T_1$ sufficiently small such that

$$CC_1T_2 \leqslant \frac{1}{4},$$

where C_1 is the constant obtained for the uniform estimate, then it follows from (3.20) that

$$\|u^{n+1}\|_{L^\infty(0,T;B^{N/p}_{p,1})} \leqslant \frac{4}{3}\|u^{n+1}(0)\|_{B^{N/p}_{p,1}} + \frac{1}{3}\|u^n\|_{L^\infty(0,T;B^{N/p}_{p,1})}. \tag{3.21}$$

Hence due to (3.21), it is clear that

$$\|u^{n+1}\|_{L^\infty(0,T;B^{N/p}_{p,1})} \to 0,$$

as n tends to infinity.

Therefore, the solution to the system (1.1) is obtained by taking the limit for the approximate sequence v^{n+1}. Moreover, from the equation, we have $v \in C(0,T;B^{N/p+1}_{p,1})$. This completes the proof of local existence part.

Uniqueness.

Suppose (v_1, P_1) and (v_2, P_2) are two solutions to (1.1) with the same initial datum. If we set $v = v_1 - v_2$ and $P = P_1 - P_2$, then we get a similar system as (3.12)

$$\begin{cases} \partial_t v + v_1\cdot\nabla v + v\cdot\nabla v_2 + \nabla P = 0, \\ \operatorname{div} v_1 = \operatorname{div} v_2 = 0, \\ v(x,t=0) = 0. \end{cases} \tag{3.22}$$

Just as what done for the convergence part for the sequences, we obtain, from (3.22),

$$\|v\|_{L^\infty(0,T;B^{N/p}_{p,1})} \leqslant CT\|v_1\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\|v\|_{L^\infty(0,T;B^{N/p}_{p,1})} + CT\|v_2\|_{L^\infty(0,T;B^{N/p+1}_{p,1})}\|v\|_{L^\infty(0,T;B^{N/p}_{p,1})}. \tag{3.23}$$

If we choose $T \leqslant \min\{T_1, T_2\}$ such that

$$CC_1T \leqslant \frac{1}{4},$$

where C_1 is the constant obtained by the existence part such that

$$\|v_1\|_{L^\infty(0,T;B^{N/p+1}_{p,1})} \leqslant C_1 \quad \text{and} \quad \|v_2\|_{L^\infty(0,T;B^{N/p+1}_{p,1})} \leqslant C_1,$$

then, (3.23) tells us, on $(0,T)$,

$$\|v\|_{L^\infty(0,T;B^{N/p}_{p,1})} \leqslant \frac{1}{2}\|v\|_{L^\infty(0,T;B^{N/p}_{p,1})},$$

this implies the uniqueness.

4. Proof of Theorem 1.2.

The proof is easy. Indeed, just as the uniform estimate which was done in section 3, we have the following estimate for the solution to (1.1).

$$\frac{d}{dt}\|v\|_{B^{N/p+1}_{p,1}} \leqslant C\|\nabla v\|_{\dot{B}^{N/p}_{p,1}}\|v\|_{B^{N/p}_{p,1}} + \|\nabla P\|_{B^{N/p+1}_{p,1}}. \tag{4.1}$$

On the other hand, the pressure can be estimated as

$$\|\nabla P\|_{B^{N/p+1}_{p,1}} \leqslant C\|\nabla v\|_{\dot{B}^{N/p}_{p,1}}\|v\|_{B^{N/p+1}_{p,1}}. \tag{4.2}$$

Therefore, it follows from (4.1) and (4.2) that

$$\|v(.,t)\|_{B^{N/p+1}_{p,1}} \leqslant \|v_0\|_{B^{N/p+1}_{p,1}} \exp\left(\int_0^t C\|\nabla v(.,\tau)\|_{\dot{B}^{N/p}_{p,1}} d\tau\right). \tag{4.3}$$

Then use the known fact that

$$\nabla v = \mathcal{P}(\omega) + A\omega,$$

where $\mathcal{P}$ is a singular integral operator homogeneous of degree $-N$ and A is a constant matrix. By the boundedness of the singular integral operator [8], we have

$$\|\nabla v\|_{\dot{B}^{N/p}_{p,1}} \leqslant C\|\omega\|_{\dot{B}^{N/p}_{p,1}}. \tag{4.4}$$

So Theorem 1.2 follows from (4.3) and (4.4).

5. Appendix.

Proof of Proposition 2.4. — We use Bony's decomposition [2], [5] to present the product as

$$fg = T_f g + T_g f + R(f,g),$$

where

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g R(f,g) = \sum_{j\in\mathbb{Z}} \Delta_j f(\Delta_{j-1} + \Delta_j + \Delta_{j+1}) g.$$

By compactness of the supports of the series of Fourier transform, for any u, v,

$$\Delta_k \Delta_l u \equiv 0, \quad |k-l| \geqslant 2 \quad \text{and} \quad \Delta_k(S_{q-1} u \Delta_q v) = 0, \quad \text{if } |k-q| \geqslant 5,$$

it follows that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} 2^{js} \|\Delta_j T_f g\|_{L^p} &= \sum_{j\in\mathbb{Z}} 2^{js} \sum_{|j-j'|\leqslant 4} \|\Delta_j(S_{j'-1} f \Delta_{j'} g)\|_{L^p} \\
&\leqslant C \sup_q \|S_q f\|_{L^\infty} \sum_{j'\in\mathbb{Z}} 2^{js} \|\Delta_{j'} g\|_{L^p} \qquad (5.1)\\
&\leqslant C \|g\|_{L^\infty} \|f\|_{\dot{B}^s_{p,1}}.
\end{aligned}$$

Similarly,

$$\|T_g f\|_{\dot{B}^s_{p,1}} \leqslant C \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,1}}. \qquad (5.2)$$

It follows from Bony's formula that

$$\begin{aligned}
\Delta_j R(f,g) &= \sum_{\max\{i',j'\}\geqslant j-3, |i'-j'|\leqslant 1} \Delta_j(\Delta_{i'} f \Delta_{j'} g) \\
&= \sum_{j'\geqslant j-4} \sum_{|i'-j'|\leqslant 1} \Delta_j(\Delta_{i'} f \Delta_{j'} g).
\end{aligned}$$

Therefore, by Minkowski inequality, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} 2^{js} &\leqslant \sum_{k\geqslant -4} \sum_{m=-1}^{1} \sum_{j'\in\mathbb{Z}} 2^{(j'-k)s} \|\Delta_{j'-k}(\Delta_{j'-m} f \Delta_{j'} g)\|_{L^p} \\
&\leqslant C \sum_{k\geqslant -4} 2^{-ks} \sum_{m=-1}^{1} \sum_{j'\in\mathbb{Z}} 2^{j's} \|\Delta_{j'-m} f \Delta_{j'} g\|_{L^p} \qquad (5.3)\\
&\leqslant C \sup_q \|\Delta_q f\|_{L^\infty} \sum_{j'\in\mathbb{Z}} 2^{j's} \|\Delta_{j'} g\|_{L^p} \\
&\leqslant C \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,1}}.
\end{aligned}$$

Then (2.5) follows from (5.1), (5.2) and (5.3).

Remark 5.1. — Actually, we can prove the following Moser type inequality

$$\|fg\|_{\dot{B}^s_{p,q}} \leqslant C\left(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot{B}^s_{r_2,q}}\right),$$

provided that $f \in L^{p_1} \cap \dot{B}^s_{r_2,q}$, $s > 0$, $1 \leqslant p, q, p_1, r_2 \leqslant \infty$, $g \in L^{r_1} \cap \dot{B}^s_{p_2,q}$, $1 \leqslant r_1, p_2 \leqslant \infty$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Proof of Proposition 3.1. — The idea is to approximate (3.1) by linear transport equations. First it is easy to check that (3.1) is equivalent to the following system.

$$\begin{cases} \partial_t v + w\cdot\nabla v + \nabla P = f, \\ -\Delta P = \operatorname{div}(w\cdot\nabla v) - \operatorname{div} f, \\ v(x,t=0) = v_0(x), \quad \operatorname{div} v_0 = 0. \end{cases} \tag{5.4}$$

So we approximate (5.4) by linear transport equations

$$\begin{cases} \partial_t v^{n+1} + w\cdot\nabla v^{n+1} + \nabla P^n = f, \\ -\Delta P^n = \operatorname{div}(w\cdot\nabla v^n) - \operatorname{div} f, \\ v^{n+1}(x,t=0) = S_{n+1}v_0(x). \end{cases} \tag{5.5}$$

The existence theorem for (5.5) is well-known for each n. Just as the proof of Theorem 1.1, we should give a uniform estimates for the sequence v^{n+1} and the convergence of the corresponding sequence. In order to do so, we only need to do a priori estimates for the equivalent system (5.4).

$$\frac{d}{dt}\|v(.,t)\|_{B^{N/p+1}_{p,1}} \leqslant C\|w\|_{\dot{B}^{N/p+1}_{p,1}}\|v\|_{B^{N/p+1}_{p,1}} + \|f\|_{B^{N/p+1}_{p,1}} + \|\nabla P\|_{B^{N/p+1}_{p,1}}. \tag{5.6}$$

The estimate for the pressure is easy now,

$$\|\nabla P\|_{B^{N/p+1}_{p,1}} \leqslant C\|w\|_{\dot{B}^{N/p+1}_{p,1}}\|v\|_{\dot{B}^{N/p+1}_{p,1}} + C\|f\|_{B^{N/p+1}_{p,1}}.$$

Therefore, it follows from (5.6) that

$$\frac{d}{dt}\|v(.,t)\|_{B^{N/p+1}_{p,1}} \leqslant C\|w\|_{\dot{B}^{N/p+1}_{p,1}}\|v\|_{B^{N/p+1}_{p,1}} + C\|f\|_{B^{N/p+1}_{p,1}}. \tag{5.7}$$

Apply Gronwall inequality on (5.7), then

$$\begin{aligned} \|v(.,t)\|_{B^{N/p+1}_{p,1}} \leqslant{}& \|v_0\|_{B^{N/p+1}_{p,1}} \exp\left(\int_0^t C\|w(.,s)\|_{\dot{B}^{N/p+1}_{p,1}} ds\right) \\ &+ \int_0^t \|f(.,\tau)\|_{B^{\frac{N}{p}+1}_{p,1}} \exp\left(\int_\tau^t C\|w(.,s)\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} ds\right) d\tau. \end{aligned} \tag{5.8}$$

Since we have the a priori estimate (5.8), the existence and uniqueness of solutions for the system (5.4) can be obtained by the approximate sequence v^{n+1}, solutions to (5.5). This finishes the proof of Proposition 3.1.

6. Acknowledgment.

The author would like to express sincere gratitude to his supervisor Professor Zhouping Xin for enthusiastic guidance and constant encouragement. And thanks also to Professor Shing-Tung Yau and Professor Zhouping Xin for providing an excellent study and research environment in The Institute of Mathematical Sciences. This work is partially supported by Hong Kong RGC Earmarked Grants CUHK-4219-99P and CUHK-4279-00P.

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Manuscrit reçu le 20 juin 2003,
accepté le 18 novembre 2003.

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