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ON SUMMABILITY OF MEASURES
WITH THIN SPECTRA

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1. Introduction.

According to the general uncertainty principle a distribution (a measure in our case) and its Fourier transform can not be both too concentrated. In particular, if the Fourier transform of a measure is supported on a set of a special form then it has no singular part. We call a set with this property a Riesz set. Many different sufficient conditions for Riesz sets are known - we refer to [M], [Sh], [A], [HJ], where the conditions for $\mathbb{T}^d$ are given - roughly speaking the set should be concentrated on a halfspace and it can not contain a line. Another sufficient condition (both for $\mathbb{R}^d$ and $\mathbb{T}^d$) is given in [R], where the set is required to be strongly antisymmetric. In the present paper we study phenomena which occur only in the non-compact setting. We give a new class of examples of Riesz sets on $\mathbb{R}^d$ which are both symmetric and also include a lot of lines.

In Section 2 we prove the following criterion inspired by the de Leeuw transference method, on which the examples of Riesz sets are based.

THEOREM 1. — Suppose that $\alpha_j K \cap \mathbb{Z}^d$ is a Riesz set in $\mathbb{Z}^d$ for every $j = 1, 2, \ldots$ for some $K \subset \mathbb{R}^d$, and a sequence $\alpha_j \to \infty$. Then $K$ is a Riesz set in $\mathbb{R}^d$.

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As a direct application of the above criterion we prove that so called $f$-poles are Riesz sets for every $f : \mathbb{R}_+ \to \mathbb{R}_+$ which decreases to 0. For any positive, decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we call a set an $f$-pole iff it is an image of the set $K_f = \{(x_1, x') \in \mathbb{R}^d : |x'| \leq f(|x_1|)\}$ under a linear transformation.

**Corollary 1.** — For each function $f : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing to 0, every $f$-pole is a Riesz set.

We also give an example of a Riesz set in $\mathbb{R}^d$ ($d \geq 2$) whose interior contains all lines in one direction except for lines passing through a set of small $(d - 1)$-dimensional Hausdorff measure, which disproves the conjecture, that a Riesz set in $\mathbb{R}^d$ can not contain a line, as it does in the $\mathbb{T}^d$ case.

The formulation of Theorem 1 is in the spirit of the criterion given by Meyer for compact group (cf. [M]). However, instead of using an argument of a topological nature, we transfer the results from tori to the Euclidean spaces.

In Section 3 we study special cases of Riesz sets for which the $L^1$-summability can be improved. It is easy to see that if the Fourier-Stieltjes transform of a measure $\mu \in M(\mathbb{R}^d)$ is supported on a set $K \subset \mathbb{R}^d$ of finite Lebesgue measure, then $\mu$ is a bounded continuous function and belongs to $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. Moreover, $||\mu||_p \leq \frac{1}{|K|^\frac{d-1}{p}} ||\mu||_M$. We consider the class of sets $K \subset \mathbb{R}^d$ such that the function assigning to $t \in \mathbb{R}$ the $(d - 1)$-dimensional Lebesgue measure of the intersection of $K$ with the hyperplane $\{x_1 = t\}$ is $L^p$-summable. We prove the following result.

**Theorem 2.** — Let $1 < p < 2$, $K \subset \mathbb{R}^d$ is a closed set and suppose that there exists $y \in \mathbb{R}^d$ such that the function

$$h(t) = m_{d-1}(K \cap \{\xi : <y, \xi> = t\}),$$

where $m_{d-1}$ is $(d - 1)$-dimensional Hausdorff measure, belongs to $L^p(\mathbb{R})$. Then any finite measure with Fourier transform supported in $K$ is locally $L^{p'}$-summable where $\frac{1}{p} + \frac{1}{p'} = 1$.

We also give in Section 3 several results about the sharpness of Theorem 2. Among them, we show that there exists a Riesz set $K$ which is not a Hardy set, i.e. there exists a summable function with Fourier-Stieltjes transform supported on $K$, which does not belong to the class $H^1(\mathbb{R}^d)$.  

*Annales de L'Institut Fourier*
In Section 4 we study conditions on the set of zeros of the Fourier-Stieltjes transform of a measure which imply that the measure is absolutely continuous. We call a sequence $\Lambda \subset \mathbb{R}^d$ a co-Riesz sequence iff every finite measure with Fourier-Stieltjes transform vanishing on $\Lambda$ is absolutely continuous with respect to Lebesgue measure. We prove that the co-Riesz sequences exist.

**Theorem 3.** — **No matter how slowly the sequence $\tau_n$ tends to 0, there exists a co-Riesz sequence $\Lambda$ such that** $\text{dist}(\lambda_n, \Lambda \setminus \{\lambda_n\}) > \tau_n$.

On the other hand, the fact that the sequence of differences of a sequence $\Lambda$ tends to 0, that doesn’t guarantee, that the sequence $\Lambda$ is a co-Riesz sequence. An example of such a sequence was provided to us by J.-P. Kahane.

Later in Section 4, we show that vanishing of the Fourier transform of a function on any sequence without limit points does not guarantee any additional summability of the function (compare with the Theorem 2). We also study some properties of co-Riesz sequences and formulate some problems.

In Section 5 we apply the method developed in the previous sections to co-Lebesgue sequences. A sequence $\Lambda \subset \mathbb{R}^d$ is a co-Lebesgue iff for every measure $\mu \in M(\mathbb{R}^d)$ with Fourier-Stieltjes transform vanishing on $\Lambda$, the Fourier-Stieltjes transforms of its singular and absolutely continuous parts also vanish on $\Lambda$. We establish a criterion for being co-Lebesgue and apply it to the sequences $\left( n^{1/k} \right)_{n=1}^\infty$ $(k = 2, 3, \ldots)$ and $(\log n)_{n=1}^\infty$.

**Notation.** — We denote by $\mathbb{R}^d$ the $d$-dimensional Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $| \cdot |$. By $T^d$ we denote $d$-dimensional torus identified naturally with the unit cube in $\mathbb{R}^d$. All measures are supposed to be finite Borel measures. The space of finite Borel measures of bounded total variation on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. By $\| \cdot \|$ we denote the usual norm on this space, i.e. the total variation of a measure. We denote by $\mu_s$ the part of $\mu$ singular with respect to Lebesgue measure (cf. [HR, Chapt. III, Th. 14.22]). If a measure $\mu$ is absolutely continuous with respect to Lebesgue measure $m_d$, there exists $f \in L^1(\mathbb{R}^d)$ such that $d\mu = f \, dm_d$. In this case we write for shortness $\mu \in L^1(\mathbb{R}^d)$, i.e we identify the measure with $f$. The restriction of a measure $\mu$ to a Borel set $\Omega$ is denoted by $\mu|_\Omega$. By $\hat{\mu}(\xi) = \int e^{-i2\pi \langle x, \xi \rangle} d\mu(x)$ we denote the Fourier-Stieltjes transform of the measure $\mu \in M(\mathbb{R}^d)$. For $A, B \subset \mathbb{R}^d$ by $A + B$ we denote the Minkowski sum $\{x + y : x \in A, y \in B\}$; $rA$ denotes
the set \( \{ra \in \mathbb{R}^d : a \in A \} \) \((r \in \mathbb{R})\). By \( \text{dist}(x, A) \) we denote the distance between \( x \in \mathbb{R}^d \) and the nonempty set \( A \subset \mathbb{R}^d \). The symbol \( C \) (possibly with indexes) denotes a non-negative constant which can change in value from one occurrence to another.

2. Symmetric Riesz sets.

We begin with the proof of Theorem 1.

**Proof of Theorem 1.** — Suppose that \( K \) is not a Riesz set. Then there exists \( \mu \in M(\mathbb{R}^d) \) such that supp \( \hat{\mu} \subset K \) and \( \mu_s \neq 0 \). Let us choose an integer \( j \) such that \( |\mu_s| (\alpha_j I^d) > \frac{2}{3} ||\mu_s|| \) (here \( I^d = \{ x \in \mathbb{R}^d : -\frac{1}{2} < x_k \leq \frac{1}{2} \} \)). Let \( \nu \in M(\mathbb{T}^d) \) be the measure defined by \( \nu(E) = \mu(\alpha_j E + \alpha_j Z^d) \) for \( E \subset \mathbb{T}^d \). It is easy to see that \( \hat{\nu}(\xi) = \hat{\mu}(\frac{\xi}{\alpha_j}) \) for every \( \xi \in \mathbb{Z}^d \). Since supp \( \hat{\mu} \subset K \), the Fourier-Stieltjes transform of \( \nu \) vanishes outside a Riesz subset of \( Z^d \). Hence \( \nu_s = 0 \). But \( \nu_s(E) = \sum_{\xi \in \mathbb{Z}^d} \mu_s(\alpha_j E + \alpha_j \xi) \) and therefore

\[
||\nu_s|| \geq ||(\mu|\alpha_j I^d)_s|| - ||(\mu|\mathbb{R}^d \setminus \alpha_j I^d)_s|| > \frac{1}{3} ||\mu_s|| > 0.
\]

This contradiction completes the proof. \( \square \)

Corollary 1 is a direct consequence of Theorem 1. Linear transformation preserve Riesz sets, and one always can shift the \( f \)-pole in such a way that it does not contain any line with rational points. For this shifted \( f \)-pole the sequence \( \alpha_j = j \) is the sequence required in Theorem 1.

**Example 1.** — Given \( \varepsilon > 0 \) there exists a closed symmetric (with respect to the origin) subset \( E \) of the hyperplane \( L = \{ x_1 = 0 \} \subset \mathbb{R}^d \) and a Riesz set \( K \subset \mathbb{R}^d \) such that \( m_L(L \setminus E) < \varepsilon \) and \( \mathbb{R} \times E = \{(x_1, x') : (0, x') \in E\} \subset \text{Int} \ K \).

Let \( A \subset L \) be an open symmetric set of measure \( m_L(A) < \varepsilon \) containing all the rational points in \( L \) and \( A_1 \subset A_2 \subset \ldots \subset A \) be a sequence of open symmetric sets such that \( \frac{1}{n} \mathbb{Z}^d \cap L \subset A_n \) and \( A_n \subset A \) for \( n = 1, 2, \ldots \). Then we put \( K = \mathbb{R}^d \setminus \bigcup F_n \) where

\[
F_n = \{(x_1, x') \in \mathbb{R}^d : |x_1| > \frac{n}{1 + |x'|} - 1 \text{ and } (0, x') \in A_n\}.
\]

We put \( E = L \setminus A \). Clearly \( nK \cap \mathbb{Z}^d \subset \{|x_1| \leq \frac{n^3}{n + |x'|} - n\} \) which is a finite subset of \( \mathbb{Z}^d \). Hence, by Theorem 1, \( K \) is a Riesz set. The remaining property, \( \mathbb{R} \times E \subset \text{Int} \ K \), is obvious. \( \square \)
It might happen (however we do not know it) that a strengthened form of Corollary 1 is valid: every $L^1$ function with the Fourier transform supported by an $f$-pole is better than $L^1$ integrable (e.g., belongs to some fixed Orlicz space), and this could be the reason for being a Riesz set. Theorem 2 being applied to $f$-poles supports this conjecture. The next result shows however that this possible improvement cannot be uniform for all functions $f$.

Let $\Phi$ be a Young function which defines the Orlicz norm on $\mathbb{R}^d$; we denote the corresponding Orlicz space by $L^\Phi(\mathbb{R}^d)$ (cf. [RR]). We say that a function $f$ belongs to $L^\Phi_{\text{loc}}(\mathbb{R}^d)$ iff for every $x \in \mathbb{R}^d$ there exists a neighbourhood $U$ such that $f \cdot \chi_U \in L^\Phi(\mathbb{R}^d)$.

**Proposition 2.** — Let the Young function $\Phi$ be such that $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$. Then there exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $F \in L^1(\mathbb{R}^d)$ with the Fourier-Stieltjes transform supported on the $f$-pole $K_f$, such that $F \notin L^\Phi_{\text{loc}}(\mathbb{R}^d)$.

**Proof.** — Let $\psi \in C^\infty(\mathbb{R}^d)$ be a positive function such that $\|\psi\|_1 = 1$ and its Fourier transform $\hat{\psi}$ is positive and supported on the unit cube $I^d$. We can get such a function as the square of an $L^1$ function with smooth positive Fourier transform supported on $\frac{1}{2}I^d$. Clearly we have $\psi(x) > \sigma > 0$ for a fixed small enough constant $\sigma$ for $x \in rI^d$ for some $r > 0$. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function decreasing to 0 (to be fixed later). For $n = 1, 2, \ldots$ we define $\psi_n$ by

$$\hat{\psi}_n(x_1, x') = \hat{\psi}\left(\frac{x_1}{2^n}, \frac{(d-1)x'}{f(2^n-1)}\right),$$

where $x = (x_1, x')$ with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$.

Note that

1) $\text{supp } \psi_n \subset K_f$

2) $\psi_n \geq 0$;

3) $\psi_n > 2^n(d-1)^{-(d-1)}f^{-1}(2^n-1)\sigma$ on $E_n = 2^{-n}rI \times \left(\frac{(d-1)r}{f(2^n-1)}\right)I^{d-1}$;

4) $\|\psi_n\|_1 = 1$.

Then we put

$$F = \sum_{j=1}^\infty \frac{1}{j^2} \psi_{nj},$$

where the increasing sequence of integers $(n_j)$ will be fixed later. We are going to show that if $f$ is chosen properly then $\int_{I^d} \Phi(\alpha|F|) = \infty$ for every
\( \varepsilon > 0 \) and \( \alpha > 0. \) Put \( \phi(t) = t^{-1} \Phi(t). \) Since \( L^1(\mathbb{R}^d) \not\subset L^p(\mathbb{R}^d), \) we have \( \phi(t) \to \infty \) as \( t \to \infty. \) Since \( \Phi \) is superadditive, we have
\[
\int_{\varepsilon I^d} \Phi(\alpha|F|) \geq \sum_j \int_{\varepsilon I^d} \Phi\left(\frac{\alpha \psi_{n_j}}{j^2}\right).
\]
Thus, using properties 1) - 4), we get that for \( j \) such that \( n_j > j^2(\alpha \sigma)^{-1} (d-1)^{d-1}, \varepsilon f(2n_j^{-1}) < (d-1)r \) and \( 2n_j > \frac{r}{\varepsilon}, \)
\[
\int_{\varepsilon I^d} \Phi\left(\frac{\alpha}{j^2} \psi_{n_j}\right) \geq \int_{\varepsilon I^d \cap E_{n_j}} \Phi\left(\frac{\alpha}{j^2} 2n_j^{-1}(d-1)^{-d-1} f^{d-1}(2n_j^{-1}) \sigma\right)
\]
\[
\geq 2^{-d} \varepsilon^{d-1} \frac{r}{2n_j} \Phi\left(\frac{\alpha}{j^2} 2n_j^{-1}(d-1)^{-d-1} f^{d-1}(2n_j^{-1}) \sigma\right)
\]
\[
= 2^{-d} \varepsilon^{d-1} r \frac{\alpha}{j^2} \sigma(d-1)^{-d-1} f^{d-1}(2n_j^{-1})
\]
\[
= \phi\left(\frac{\alpha}{j^2} 2n_j^{-1}(d-1)^{-d-1} f^{d-1}(2n_j^{-1}) \sigma\right).
\]

Put now

\[
f(t) = \begin{cases} 
\max\left(\frac{1}{\log_2 t}, \phi^{-\frac{1}{2}}\left(\frac{t}{\log_2 t}\right)\right) & \text{for } t > 2^d, \\
\max\left(\frac{1}{d^2}, \phi^{-\frac{1}{2}}(\frac{2^d}{d^2})\right) & \text{for } t < 2^d.
\end{cases}
\]

Choose the sequence \( (n_j) \) such that \( \phi(2n_{j-1}^{-d}) > j^d \) and \( n_j > j^3. \)
Then, using the above estimation and the definition of \( f, \) we get for the large values of \( j \)
\[
\int_{\varepsilon I^d} \Phi\left(\frac{\alpha}{j^2} \psi_{n_j}\right) \geq 2^{-d} \varepsilon^{d-1} (d-1)^{-d-1} r \frac{\alpha}{j^2} \sigma f^{d-1}(2n_j^{-1}) \phi(2n_j^{-d})
\]
\[
\geq 2^{-d} \varepsilon^{d-1} (d-1)^{-d-1} r \alpha \sigma j^{-2} \phi^{\frac{1}{2}}(2n_j^{-d})
\]
\[
\geq 2^{-d} \varepsilon^{d-1} (d-1)^{-d-1} r \alpha \sigma \cdot j^{-1}.
\]

Hence the integral \( \int_{\varepsilon I^d} \Phi(\alpha|F|) \) is estimated from below by a tail of the divergent series.

Using now the well known fact that \( H^{1 \frac{1}{2}}(\mathbb{R}^d) \subset (L \log L)_{\text{loc}}(\mathbb{R}^d) \) (cf. [St, Chap. III.5.3]) and that the constructed function \( F \) is positive, we get as a corollary that on \( \mathbb{R}^d \) the class of Riesz sets is slightly larger that the class of Hardy sets:

**Corollary 2.** — There exists an f-pole \( K_f \subset \mathbb{R}^d \) which is not a Hardy set, i.e. there exists \( F \in L^1(\mathbb{R}^d) \) with Fourier transform supported by \( K_f \) such that \( F \not\in H^1(\mathbb{R}^d). \)
3. Proof of Theorem 2.

We can assume that \( y = (1,0,\ldots,0) \). Let \( \mu \in M(\mathbb{R}^d) \) satisfy \( \text{supp} \hat{\mu} \subset K \). For \( t > 0 \) we put \( \mu_t = \mu * P_t \), where \( \{P_t\}_{t>0} \) are Poisson kernels. Clearly \( \mu_t \in L^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) and \( \|\mu_t\|_1 \leq \|\mu\| \). It is also clear that \( \text{supp} \hat{\mu}_t \subset K \) and \( \hat{\mu}_t \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d) \). We have

\[
\mu_t(x) = \int_{\mathbb{R}^d} \hat{\mu}_t(\xi) e^{2\pi i (x_1,x')} \, d\xi \\
= \int_{-\infty}^{\infty} P(\hat{\mu}_t(\xi) e^{2\pi i (x',\xi')})(\xi_1) e^{2\pi i x_1 \xi_1} \, d\xi_1 \\
= (P(\hat{\mu}_t(\xi_1,\cdot) e^{2\pi i (x',\cdot)}))(-x_1),
\]

where

\[
P(f)(s) = \int_{\{x_1 = s\}} f(x_1,x') \, dm_{d-1}(x').
\]

Since \( |\hat{\mu}_t(\xi) e^{2\pi i (x',\xi')}| \leq \|\mu_t\|_1 \leq \|\mu\| \), and \( m_{d-1}(\{x \in K : x_1 = s\}) = h(s) \), we get

\[
\|P(\hat{\mu}_t(\xi_1,\cdot) e^{2\pi i (x',\cdot)}))\|_{L^p(d\xi_1)} \leq \|\mu\| \cdot \|h\|_p.
\]

Hence, if \( \frac{1}{p} + \frac{1}{p'} = 1 \), by the Hausdorff-Young inequality,

\[
\|(P(\hat{\mu}_t(\xi_1,\cdot) e^{2\pi i (x',\cdot)}))\|_{L^{p'}} \leq \|\mu\| \cdot \|h\|_p.
\]

Thus \( \|\mu_t(\cdot,x')\|_{p'} \leq \|\mu\| \cdot \|h\|_p \) for every \( x' \in \mathbb{R}^{d-1} \).

Let \( y = (y_1,y') \in \mathbb{R}^d \) and \( U = \mathbb{R} \times \Omega \) be an open neighborhood of \( y \) such that \( \Omega \in \mathbb{R}^{d-1} \) is an open neighborhood of \( y' \) with finite \((d-1)\)-dimensional Lebesgue measure. Then

\[
\int_U |\mu_t|^p' \, dm = \int_\Omega \int_\mathbb{R} |\mu_t(x_1,x')|^p' \, dx_1 \, dx' \\
\leq m_{d-1}(\Omega) \cdot \|\mu\|^{p'} \|h\|_p^{p'}.
\]

Hence there exists \( C > 0 \) such that for \( t > 0 \),

\[
\|\mu_t\|_{L^{p'}(U)} \leq C.
\]

By assumption \( (\mu_t)_{|U} \rightarrow \mu_{|U} \) in the *-weak topology. Since \( \|(\mu_t)_{|U}\|_{p'} \) is bounded for \( t > 0 \), we get that \( \mu_{|U} \in L^p(U) \).

The \( f \)-pole with \( f(t) = \min\{1,t^{-\frac{q-1}{q(d-1)}}\} \) is called a \( q \)-pole.

**Corollary 3.** — Let \( 2 \leq p < \infty \). If the support of the Fourier transform of a measure \( \mu \) is contained in a finite union of \( q \)-poles, where \( q > p \), then \( \mu \in L^p_{\text{loc}}(\mathbb{R}^d) \).
Corollary 3 gives another proof that q-poles are Riesz sets for \( q > 2 \). However, by applying Theorem 2, one can construct Riesz sets which do not seem to be treated by Theorem 1.

Example 2. — Let \( K \in \mathbb{R}^d \) be any q-pole \( (q > 2) \) which does not contain a line orthogonal to the first coordinate. Let \( K_n = K \cap \{ n \leq x_1 \leq n + 1 \} \). If \( (r_n)_{n=-\infty}^\infty \subset \mathbb{R}^d \) is any sequence with bounded first coordinate, then the set \( \bigcup_{n \in \mathbb{Z}} (K_n + r_n) \) satisfies the assumption of Theorem 2 for \( p > q' \).

Corollary 3 shows that every q-pole is a “local” \( \Lambda_p \) for every \( q > p \geq 2 \). The next remark shows that (a) \( \Lambda_{q,\text{loc}} \not\subset \Lambda_p \) for \( 2 < q < \infty \) and \( 1 < p < \infty \), and (b) \( \Lambda_{p,\text{loc}} \neq \Lambda_{q,\text{loc}} \) for \( p, q \geq 2 \) and \( p \neq q \).

**Proposition 3.** —

a) Let \( 1 < q < \infty \). There exists a function \( F \in L^1(\mathbb{R}^d) \) with the Fourier transform supported on a q-pole, such that \( F \notin L^p(\mathbb{R}^d) \) for any \( 1 < p < \infty \).

b) Let \( 1 < q < \infty \). There exists a function \( F \) with the Fourier transform supported on a q-pole, such that \( F \notin L^p_{\text{loc}}(\mathbb{R}^d) \) for every \( p > q \).

**Proof.** — We use the function \( F \) constructed in the proof of Proposition 2. We let \( f(t) = \min(1, t^{-\frac{q-1}{q(d-1)}}) \) and \( n_j = j \). Then

\[
\|F\|_p \geq j^{-2}\|\psi_j\|_p
\]

\[
\geq j^{-2}\left(\|E_{n_j}\| \cdot (2^j (d - 1)^{-(d-1)} f^{d-1}(2^j-1)\sigma)^p\right)^{1/p}
\]

\[
= j^{-2}r^{d/p}\sigma(2^j (d - 1)^{-(d-1)} f^{d-1}(2^j-1))^{p-1/p}
\]

\[
= j^{-2}r^{d/p}\sigma^{(p-1)/p} (d - 1)^{-(d-1)} \left(2^{j-1} \frac{p-1}{p}\right) \to \infty
\]

as \( j \to \infty \). This proves part (a). For part (b) we have for \( j \) such that \( \varepsilon f(2^j-1) < (d - 1) r \),

\[
\int_{\varepsilon I^d} |F|^p \geq j^{-2} \int_{\varepsilon I^d} |\psi_j|^p
\]

\[
\geq j^{-2}\varepsilon^{d-1} r^{p-1}(d - 1)^{-p(d-1)} \left(2^{j-1} \frac{p-1}{p}\right) \to \infty
\]

as \( j \to \infty \) for any fixed \( \varepsilon > 0 \).
4. co-Riesz sequences on $\mathbb{R}$.

We call $A = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ a co-Riesz sequence iff every measure $\mu \in M(\mathbb{R}^d)$ such that $\mu(A) = 0$ for $A \in A$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Though a number of results in this section make the impression that every sequence $\Lambda = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}$ such that $\lim_{n \to \infty} \text{dist}(\Lambda, x) = 0$ (resp. $\lim_{n \to \infty} |\lambda_n - \lambda_{n+1}| = 0$) should be a co-Riesz sequence, this is not true. That is shown by Example 3 below, which was provided by J.-P. Kahane to the previous version of this manuscript (and appears here with his kind permission). The further study of co-Riesz sequences in connection with Helson sets is provided in the forthcoming paper [W].

On the other hand we can show that a number particular sequences from this class are indeed co-Riesz. This is in the case when Theorem 1 could be applied. Note that, despite the fact that Theorem 1 is formulated for Riesz sets, which by definition are closed, it remains valid in this setting – in the proof of Theorem 1 we only use the values of the Fourier transform at the points from some special (countable) set.

Proof of Theorem 3. — Without loss of generality we can assume that $(r_j)$ is a non-increasing sequence consisting of powers of 2. Moreover we can assume that $\sum r_j = \infty$. Then we put $\lambda_n = \sum_{j=0}^{n} r_j$ for $n = 1, 2, \ldots$. It is easy to check that for every $n = 1, 2, \ldots$ the intersection $2^n(\mathbb{R} \setminus \Lambda) \cap \mathbb{Z}$ is a set contained in a halfline (i.e. bounded from above). Hence it follows from the theorem of F. and M. Riesz (cf. [HJ, 1.1.3, p.13]) and Theorem 1 that $\Lambda$ is a co-Riesz sequence. $\square$

Remark. — An obvious modification of the proof of Theorem 3 gives its analogue for several variables. Namely, one can prove that for every sequence $(r_n)$ decreasing to 0 there exists a co-Riesz sequence $\Lambda = (\lambda_n) \subset \mathbb{R}^d$ such that $\text{dist}(\lambda_n, \Lambda \setminus \{\lambda_n\}) > r_n$ for $n = 1, 2, \ldots$.

Contrasting to the example given by the Theorem 3 is the following example.

Example 3. — There exists a non co-Riesz set $\Lambda$, which is yet “thick” at infinity, i.e., $\lim_{x \to \infty} d(x, \Lambda) = 0$. 

TOME 54 (2004), FASCICULE 2
Proof. — Let us start from a sequence of Riesz products
\[ \rho_n(x) = \prod_{j=1}^{\infty} (1 + c_{j,n} \cos 2\pi 4^j x), \]
where
\[ (1) 0 < c_{j,n} < 1, \sum_{j=1}^{\infty} c_{j,n}^2 = \infty, \]
and
\[ (2) \lim_{j \to \infty} \frac{c_{j,n+1}}{c_{j,n}} = \infty \text{ for each } n. \]

We can e.g. take for \( j > 2^n, \) \( c_{j,n} = \frac{1}{\sqrt{j}} \) \( \text{for } j > 2^n, c_{j,n} = \frac{1}{2} \) otherwise. For the Riesz products the condition (1) implies that \( \rho_n \) is a singular probability measure supported by \([\frac{-1}{2}, \frac{1}{2}]\), whose Fourier-Stieltjes transform is carried by the set \( \Omega = \{ \sum_{j=0}^{J} \varepsilon_j 4^j, \varepsilon_j = 0, \pm 1, J \geq 0 \} \subset \mathbb{Z} \). We identify \( \rho_n \) with a 1-periodic measure on the real line and consider \( \sigma_n = \varphi \rho_n \), so that \( \sigma_n = \hat{\varphi} \rho_n \), where \( \varphi \in C^\infty(\mathbb{R}) \), supp \( \hat{\varphi} = [\frac{-1}{2}, \frac{1}{2}] \), \( \varphi \geq 0 \), \( \varphi \geq 0 \), and \( \varphi > 0 \) on \( (-\frac{1}{2}, \frac{1}{2}) \). Note that supp \( \sigma_n = \Omega + [-\frac{1}{2}, \frac{1}{2}] \).

Finally, we set
\[ (3) \tau_n = \frac{\sigma_n}{\|\sigma_n\|} 2^{-n}, \mu_n = \tau_n * (\frac{\delta_{\frac{n}{2}} + \delta_{-\frac{n}{2}}}{2}), \text{ and } \mu = \sum_{n=1}^{\infty} \mu_n \text{ where } \{a_n\} \text{ is a very rapidly increasing sequence of integers (to be chosen later).} \]
Observe that \( \hat{\mu}_n(\xi) = \hat{\tau}_n(\xi) \cos(a_n \xi). \)
Moreover, as all the measures in the sum \( \mu = \sum_{n=1}^{\infty} \mu_j \) are positive, the measure \( \mu \) is a singular probability measure: \( \|\mu\| = \sum_{n=1}^{\infty} \|\tau_n\| = \sum_{n=1}^{\infty} 2^{-n} = 1. \)

On each particular interval \( (\omega - \frac{1}{2}, \omega + \frac{1}{2}) \), with \( \omega = \sum_{j=1}^{J} \varepsilon_j 4^j \in \Omega, \varepsilon_j = 0, \pm 1 \), we have \( \tau_n/\tau_n' = \prod_{j=1}^{J} \frac{c_{j,n}}{c_{j,n'}} \) and \( \tau_n \) vanishes outside of these intervals. So, taking into account the condition (2), we can construct a sequence \( m_n \) such that \( \sum_{j=1}^{n} \tau_j(\xi) \leq \tau_{n+1}(\xi) \) for all \( |\xi| > m_n \).

Now we will chose inductively the sequence \( a_n \) and the sequence of sequences \( \Lambda_n \) such that
\[ (4) \langle \sum_{j=1}^{n} \mu_j \rangle(\lambda) = 0 \text{ for all } \lambda \in \Lambda_n; \]
\[ (5) \text{for any } \lambda \in \Lambda_n \cap [-m_n, m_n] \text{ the distance } \text{dist}(\lambda, \Lambda_{n+1}) < 2^{-n}; \]
and
\[ (6) \text{for all } |\xi| > m_n, \text{dist}(\xi, \Lambda_{n+1}) < 2^{-n}. \]

After such sequence \( \Lambda_n \) is constructed, we can take \( \Lambda = \lim_{n \to \infty} \Lambda_n \) (in the sense that \( \lambda \in \Lambda \) if dist(\( \lambda, \Lambda_n \)) \( \to 0 \)). The norm convergence of the sum \( \sum_{j=0}^{\infty} \mu_j \) and (4) imply \( \hat{\mu}|_{\Lambda} = 0. \) And the conditions (5) and (6) give...
that \( \text{dist}(x, \Lambda) < 2^{-n+1} \), for all \( x \in [-m_{n+1}, m_{n+1}] \setminus [-m_n, m_n] \), which completes the proof.

To construct the sequence \( \Lambda_n \) notice that outside of the intervals \((\omega - \frac{1}{2}, \omega + \frac{1}{2})\), with \( \omega \in \Omega \), all the functions \( \hat{\tau}_n \) (and so \( \hat{\rho}_n \)) vanishes, by the construction, and thus the only difficulty in the choice of \( \Lambda_n \) occurs on these intervals. Let \( s_n \) be the partial sum \( \sum_{j=1}^{n} \mu_j \). Note that, \( \hat{s}_n = \tau_1 P \), where \( P \) is a trigonometric polynomial on each of the intervals under consideration. As any trigonometric polynomial has only finitely many roots on an interval, \( \hat{s}_n \) has finitely many roots on the interval \((\omega - \frac{1}{2}, \omega + \frac{1}{2})\) for each \( \omega \). Denote the set of those roots by \( E_{n,\omega} \). It is clear, that \( \Lambda_n \cap (\omega - \frac{1}{2}, \omega + \frac{1}{2}) \subset E_{n,\omega} \). As for any \( \lambda \in E_{n,\omega} \) we can choose an interval \( I_\lambda \subset (\omega - \frac{1}{2}, \omega + \frac{1}{2}) \) which contains \( \lambda \) and has length less than \( 2^{-n} \), such that \( |\hat{s}_n(\lambda)| < \hat{\tau}_n \) on \( I_\lambda \). As there are only finitely many intervals \((\omega - \frac{1}{2}, \omega + \frac{1}{2})\) in the interval \([-M_n, \omega) \) and there are only finitely many points in each interval, there are only finitely many intervals \( I_\lambda \), with \( \lambda \in (\bigcup_{\omega} E_{n,\omega}) \cap [-m_n, m_n] \). Let \( r_n \) be the minimal length of such an interval. Then we choose \( a_{n+1} = \max\{2\pi, 2^{n+1} \pi\} \). Let \( \Lambda_{n+1} \) on the intervals \((\omega - \frac{1}{2}, \omega + \frac{1}{2})\) be the sequence of zeros of the function \( \sum_{j=1}^{n+1} \mu_j \). Notice that on each interval \( I_\lambda \) the function \( \hat{s}_n(\xi) + \hat{\tau}_{n+1}(\xi) \cos(a_{n+1}\xi) \) takes a non-positive value at the point where \( \cos(a_{n+1}\xi) = -1 \) and a non-negative value at the point where \( \cos(a_{n+1}\xi) = 1 \), as \( |\hat{s}_n(\xi)| < \hat{\tau}_{n+1} \). For the chosen value of \( a_{n+1} \) we can find all the range of values of \( \cos(a_{n+1}\xi) \) on the interval \( I_\lambda \). So there exists a root of the function \( \sum_{j=1}^{n+1} \mu_j \) (and so an element of \( \Lambda_{n+1} \)) on each interval \( I_\lambda \), for all \( \lambda \in (\bigcup_{\omega} E_{n,\omega}) \cap [-m_n, m_n] \). As the length of all the intervals \( I_\lambda \) is less than \( 2^{-n} \) this means that condition (5) is satisfied. Similar arguments show that condition (6) is satisfied as well. As condition (4) is satisfied by the construction, this complete the proof. \( \square \)

It appears that there is no estimate on the growth of the distribution of the values of a function with the Fourier transform vanishing on a sequence \( \Lambda \) (unlike the result about \( f \)-poles).

**Proposition 4.** — For every sequence \( \Lambda \subset \mathbb{R}^d \) with no limit points, and every Young function \( \Phi \) such that \( L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d) \), there exists \( f \in L^1(\mathbb{R}^d) \setminus L^\Phi_{\text{loc}}(\mathbb{R}^d) \) such that \( \hat{f}(\lambda) = 0 \) for \( \lambda \in \Lambda \).

**Proof.** — Put \( \phi(t) = t^{-1} \Phi(t) \). Since \( L^1(\mathbb{R}^d) \neq L^\Phi(\mathbb{R}^d) \), \( \phi(t) \to \infty \) as \( t \to \infty \). Let \( \psi \) be the function from the proof of Proposition 2 which is, additionally, decreasing to 0 at infinity (this requirement is equivalent to
smoothness of \( \hat{\psi} \). Let \( x_0 = (1, 0, \ldots, 0) \in \mathbb{R}^d \) and put for \( m, k \in \mathbb{Z} \)

\[
\hat{f}_{m,k}(x) = 2^{md} \psi(2^m x) - 2^{md} \psi(2^m x + 2^m k x_0). 
\]

Obviously \( \hat{f}_{m,k}(\xi) = (1 - e^{i2\pi k(\xi, x_0)}) \hat{\psi}(\xi 2^{-m}) \). Thus the Fourier transform of \( f_{m,k} \) is supported on the cube \( 2^m I^d \), and \( |\hat{f}_{m,k}| = |1 - e^{i2\pi k(\xi, x_0)}| \) for \( \xi \in \mathbb{R}^d \). Since \( \psi > 0 \) and \( \psi \) is decreasing at infinity, for every \( m \in \mathbb{Z} \) there exists \( K = K(m) \in \mathbb{Z} \) such that \( f_{m,k} \) is positive on the cube \( I^d \) and \( f_{m,k}(x) > 2^m \sigma \) for \( x \in r2^{-m} I^d \), for all \( |k| > K(m) \). Since the set \( \Lambda \cap 2^m I^d \) is finite, for every \( \varepsilon > 0 \) we can always find an (arbitrarily large) integer \( N \) such that \( \text{dist}(N(\lambda, x_0), \mathbb{Z}) < \varepsilon \) for \( \lambda \in \Lambda \cap 2^m I^d \). Hence there exists integer \( k_m \) such that we have \( \hat{f}_{m,k_m}(\lambda) < \frac{1}{M} \) for \( \lambda \in \Lambda \cap 2^m I^d \), where \( M = \#(\Lambda \cap 2^m I^d) \). Clearly \( \hat{f}_{m,k_m}(\lambda) = 0 \) for \( \lambda \in \Lambda \setminus 2^m I^d \). Put

\[
h_1 = \sum \frac{1}{n^2} f_{m_n,k_m},
\]

where the numbers \( m_n (n = 1, 2, \ldots) \) are going to be chosen later. We prove that for every \( \alpha > 0 \) the function \( \Phi(\alpha h_1) \) is not integrable on any fixed neighbourhood of the origin, say \( aI^d \) (note that we can assume that \( h_1 \) is positive in \( aI^d \)). Since \( \Phi \) is a superadditive function,

\[
\int_{aI^d} \Phi(\alpha h_1) \geq \sum \int_{aI^d} \Phi\left( \frac{\alpha}{n^2} f_{m_n,k_m} \right).
\]

If \( r2^{-m_n} < a \) we have

\[
\int_{aI^d} \Phi\left( \frac{\alpha}{n^2} f_{m_n,k_m} \right) \geq \int_{r2^{-m_n} I^d} \Phi\left( \frac{\alpha \sigma}{n^2} 2^{m_n} d \right) = 2^r \frac{\alpha \sigma}{n^2} \phi\left( \frac{\alpha \sigma}{n^2} 2^{m_n} d \right).
\]

If \( m_n \) is chosen to satisfy \( \phi(n^{-3/2} 2^{m_n} d) > n \), then for \( n \) sufficiently large

\[
\int_{aI^d} \Phi\left( \frac{\alpha}{n^2} f_{m_n,k_m} \right) \geq 2r \alpha \sigma n^{-1}.
\]

Thus \( h_1 \notin L^\Phi_{\text{loc}}(\mathbb{R}^d) \). Let us consider now the functions \( g_m \) defined by

\[
\hat{g}_m(\xi) = \sum \hat{f}_{m,k_m}(\lambda) \hat{\psi}\left( \sqrt{d} \frac{\xi - \lambda}{\tau_m} \right),
\]

where \( \tau_m = \min\{1, |\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap (2^m + 1) I^d, \lambda \neq \lambda'\} \). Then, obviously, \( \hat{g}_m(\lambda) = \hat{f}_{m,k_m}(\lambda) \) for all \( \lambda \in \Lambda \). Also

\[
\|g_m\|_1 \leq \sum_{\lambda \in \Lambda \cap 2^m I^d} |\hat{f}_{m,k_m}(\lambda)| < 1,
\]

and

\[
\|g_m\|_\infty \leq \|\hat{g}_m\|_1 \leq \sum_{\lambda \in \Lambda \cap 2^m I^d} |\hat{f}_{m,k_m}(\lambda)| \tau_m^d d^{d/2} \|\hat{\psi}\|_1 \leq \sqrt{d^{-d}} \|\hat{\psi}\|_1.
\]
Thus the function \( h_2 = \sum \frac{1}{n^s} g_{m_n} \) is bounded, summable, and \( \hat{h}_1(\lambda) = \hat{h}_2(\lambda) \) for all \( \lambda \in \Lambda \). Hence the function \( f = h_1 - h_2 \in L^1(\mathbb{R}^d) \), its Fourier transform vanishes on \( \Lambda \) and \( f \) is still not in \( L^\Phi_{\text{loc}}(\mathbb{R}^d) \).

**Remark.** — If \( \Phi(x) = x \log(x + 1) \), then one can modify the above construction to get a function which does not belong to \( H^1(\mathbb{R}) \).

**Proof.** — After some minor modification we can assume that \( h_2 \) is continuously differentiable. Indeed, during the construction, when we define numbers \( k_m \), we replace the condition \( \int_{m,k_m}^\lambda \frac{1}{M} \) by the condition \( \int_{m,k_m}^\lambda \frac{1}{(2m+1)^M} \). Then we get the estimate on the gradient of \( g_{m_n} \) in the same way as the estimate for the sup norm of \( g_{m_n} \) in the proof above. Let \( \chi \) be a smooth function supported on \( 2I^d \) such that \( \chi \equiv 1 \) on \( I^d \). Since the function

\[
h_3(x) = h_2(x)\chi(x) - h_2(x-3x_0)\chi(x-3x_0)
\]

belongs to the space \( H^1 \), the function \( f \) belongs to \( H^1 \) iff \( f + h_3 \) does. The function \( f + h_3 \) is positive on the cube \( I^d \), because it coincides there with \( h_1 \). Hence, by [St, Chapt. III.5.3], the restriction \( (f + h_3)|_{I^d} = h_1|_{I^d} \) should agree with an \( L \log L \)-summable function on every compact subset of \( I^d \), which is not the case.

In the proof of Theorem 3 the arithmetic relations between elements of \( \Lambda \) were crucial, as we used the fact that all but finitely many elements of \( \alpha^{-1}Z \) belong to \( \Lambda \). On the other hand, the set of non co-Riesz sequences is open in the following sense.

**Proposition 5.** — For any function \( f \) on \( \mathbb{R}^+ \), which decreases to 0, and any sequence \( \Lambda = (\lambda_n) \subset \mathbb{R} \) which has no limit points, there exists a sequence of positive numbers \( (r_n) \) such that for every measure \( \mu \in M(\mathbb{R}) \), for which \( \|\mu\| = 1 \) and \( \|\mu(\mathbb{R} \setminus [-r,r])\| \leq f(r) \) and any sequence \( \Lambda' = (\lambda'_n) \subset \mathbb{R} \) such that \( \lambda_n - \lambda'_n < r_n \) \( (n = 1, 2, \ldots) \), there exists a measure \( \mu' \in M(\mathbb{R}) \) such that \( \mu(\lambda_n) = \mu'(\lambda'_n) \) for \( n = 1, 2, \ldots \) and \( \mu_s = \mu'_s \).

**Proof.** — We need the following lemma.

**Lemma 1.** — Given \( c \in (0,1) \), \( r > 0 \), \( x \in \mathbb{R} \) and a measure \( \mu \in M(\mathbb{R}) \) as in the Proposition 5, there exists a measure \( \nu = \nu(c,r,x) \in M(\mathbb{R}) \) absolutely continuous with respect to Lebesgue measure, such that \( \text{supp} \ \widehat{\nu} \subset [x-2r,x+2r] \), \( \widehat{\mu}(x) = (\mu - \nu)(\lambda)(y) \) for every \( y \in (x-r,x+r) \),
and \( \|v\| < C(c + f(T)) \), where the constant \( C \) does not depend on \( c, r, x \) and \( \mu \).

We show first how Proposition 5 follows from the lemma. Let \((r_n)\) and \((c_n)\) be sequences of positive numbers such that the intervals \([\lambda_n - 2r_n, \lambda_n + 2r_n]\) are pairwise disjoint, \(\sum c_n < \infty\) and \(\sum f(c_n) < \infty\). Then \(\mu' = \mu - \sum \nu(c_n, r_n, \lambda_n)\) is a finite measure which satisfies all the requirements.

Proof of Lemma 1. — Let \(\psi \in L^1(\mathbb{R})\) be such that \(\|\psi_1\| < C\), \(\|\psi_1\| < C\), \(\psi_1(x) < \frac{C}{1 + 2}\) and \(\hat{\psi} \subset [-2, 2]\) and \(\hat{\psi}(x) = 1\) for \(x \in [-1, 1]\). Put \(\rho_R(x) = \sup_{y \in [x - R, x + R]}|\psi_1(y)|\). It is easy to see that \(\|\rho_R\| < C_1\max(1, R)\). We denote \(\psi_1(x) = t\psi(tx)\). Without loss of generality we suppose that \(x = 0\). Then we have \(\|\mu \star \psi_1\| < C_2(c + f(T))\). Indeed, since \(\widehat{\mu}(0) = 0\), we can represent \(\mu = \mu_R + \mu_{nR}\) where \(\mu_{nR}\) is supported on the interval \([-R, -R]\), \(\mu_{nR} = 0\) and \(\|\mu_{nR}\| < 2f(R)\). Then we estimate the convolution separately for \(\mu_R\) and \(\mu_{nR}\)

\[
\|\mu_R \star \psi_1\| \leq \|\mu_R\| \cdot \|\psi_1\| \leq C \cdot f(R),
\]

\[
|(\mu_R \star \psi_1)(x)| = \left| \int \psi_1(x - \cdot) \psi_1(x) \, d\mu_R \right| \leq R \sup_{y \in [x - R, x + R]} |\psi_1(y)| \cdot \|\mu_R\|.
\]

Since \(\sup_{y \in [x - R, x + R]} |\psi_1(y)| = r^2 \rho_R(rx)\) we get

\[
\|\mu_R \star \psi_1(x)\| \leq CrR\max(1, rR)\|\mu\|.
\]

Putting \(R = \frac{c}{r}\) we get the desired estimation. Hence the measure \(\nu = \mu \star \psi_1\) satisfies the conditions of the lemma. If \(\widehat{\mu}(0) \neq 0\) we put \(\nu = (\mu - (\int d\mu)\delta_0) \star \psi_1\).

Remarks. — 1) It follows from the proof that if \(\Lambda\) satisfies the assumption of Proposition 5 and \(\widehat{\mu}(\lambda) = 0\) for \(\lambda \in \Lambda\) then there exists \(\mu' \in M(\mathbb{R})\) such that \(\mu_s = \mu_s'\) and \(\widehat{\mu'}\) vanishes on some open set containing \(\Lambda\).

2) Proposition 5 can be easily extended to a multidimensional case.

The next result shows that in the previous proposition the sequence \((r_n)\) could not be chosen uniformly for all measures, without the decrease condition.

**Proposition 6.** — For every positive sequence \((r_j)\) there exist sequences \(\Lambda = (\lambda_n)\) and \(\Lambda' = (\lambda'_n)\) and \(\mu \in M(\mathbb{R})\) such that \(|\lambda_j - \lambda'_j| < r_j\) and there is no \(\mu' \in M(\mathbb{R})\) such that \(\widehat{\mu}(\lambda_n) = \widehat{\mu'}(\lambda'_n)\) for \(n = 1, 2, \ldots\).
Proof. — We set \( \Lambda' = \mathbb{Z} \). We index the sequences \( \Lambda \) and \( \Lambda' \) by integer numbers rather than natural ones. Let \( (a_n) \) be a decreasing sequence of positive numbers such that \( \sum a_n < \infty \) and \( (k_n) \) be the sequence of positive integers such that \( \sum k_n a_n^2 = \infty \). Put \( b_n = a_j \) where \( j \) is the unique index such that \( k_1 + \ldots + k_{j-1} < n \leq k_1 + \ldots + k_{j-1} + k_j \) (here we put \( k_0 = 0 \)).

Set
\[
\lambda_m = \begin{cases} 
0 & \text{if } m \neq 2^j, (j = 1, 2, \ldots) \\
{m + \frac{1}{2} \omega_j^{-1}} & \text{if } m = 2^n, \quad k_1 + \ldots + k_{j-1} < n \leq k_1 + \ldots + k_j,
\end{cases}
\]
where \( (\omega_j) \) is a sequence of positive integers satisfying for \( j = 1, 2, \ldots \)

1) \( (2\omega_j) \omega_{j+1} - 1 \);
2) \( \omega_j^{-1} < \min\{r_{2^n} : n < k_1 + \ldots + k_j\} \).

Let \( \mu_n = (2i)^{-1}(\delta_{\omega_n} - \delta_{-\omega_n}) \) and \( \mu = \sum a_n \mu_n \). We have \( \tilde{\mu}_n(t) = \sin \pi \omega_n t \). Hence \( \tilde{\mu}_j(\lambda_{2^n}) \) is positive for \( n \leq k_1 + \ldots + k_{j-1} \), equals 1 for \( k_1 + \ldots + k_{j-1} < n \leq k_1 + \ldots + k_j \) and vanishes for \( k_1 + \ldots + k_j < n \). Thus \( \tilde{\mu}(\lambda_{2^n}) > b_n \) for \( n = 1, 2, \ldots \). Clearly \( \tilde{\mu}(\lambda_n) = 0 \) for \( n \neq 2^j \), \( (j = 1, 2, \ldots) \).

Suppose to the contrary that there exists a finite measure \( \mu' \) such that \( \tilde{\mu}'(\lambda_j) = \tilde{\mu}(\lambda_j) \). By the de Leeuw transference theorem (cf. \[\text{[deL]}, \text{[StW}, \text{Chapt. VII, Th. 3.8]})}, there exists a bounded measure \( \nu \in M(\mathbb{T}) \) such that \( \|\nu\|_{M(\mathbb{T})} \leq \|\mu'\|_{M(\mathbb{R})} \) and \( \tilde{\nu}(n) = \tilde{\mu}'(n) \) for \( n = 1, 2, \ldots \). But \( \sum |\tilde{\nu}(n)|^2 = \sum b_j^2 = \infty \) which contradicts the fact that \( A = \{2^k : k = 1, 2, \ldots\} \) is a \( \Lambda_2 \) set, i.e. \( \nu \in L^2(\mathbb{T}) \) and \( \|\nu\|_2 \leq C\|\nu\|_M \) for every measure \( \nu \in M(\mathbb{T}) \) with the Fourier transform vanishing outside \( A \). \( \square \)

The above construction has one more application. We can use it to construct a sequence which does not allow co-balayage.

**Proposition 7.** — There exists a sequence \( \Lambda = (\lambda_n) \) such that
\[
\inf_{\lambda \in \Lambda} \text{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0,
\]
and measure \( \mu \in M(\mathbb{R}) \) such that there is no measure \( \mu' \in M(\mathbb{R}) \) supported on a compact set such that \( \tilde{\mu}(\lambda_n) = \tilde{\mu}'(\lambda_n) \) for \( n = 1, 2, \ldots \).

Proof. — Let \( \Lambda \) and \( \mu \) be the same as in the proof of Proposition 6 with one modification: the condition 2) on the sequence \( (\omega_j) \) is replaced by another condition

2') \( \omega_j^{-1} < j^{-1} a_j \).

Suppose that there exists \( \mu' \in M(\mathbb{R}) \) such that \( \tilde{\mu}'(\lambda_n) = \tilde{\mu}(\lambda_n) \) for \( n = 1, 2, \ldots \) and \( \text{supp} \mu' \subset [-T, T] \) for some \( T > 0 \). Then the derivative of
\( \hat{\mu} \) is bounded by \( T \cdot \|\mu'\|_M \). Hence, for sufficiently large \( n \), we have

\[
|\hat{\mu}'(2^n)| \geq |\hat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot |\lambda_{2^n} - 2^n| \\
= |\hat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot (2\omega_j)^{-1} \\
> \frac{1}{2} |\hat{\mu}'(\lambda_{2^n})| = \frac{1}{2} b_n.
\]

Thus \( \sum |\hat{\mu}'(n)|^2 = \infty \), and \( \hat{\mu}' \) vanish on \( m \neq 2^n \). We finish proceeding as in the proof of Proposition 6.

\[ \square \]

Remarks. — 1) Note that for \( \Lambda = \mathbb{Z} \), the measure \( \mu' \) with properties postulated by Proposition 7 exists, and it is supported by an interval of length 1. This is exactly what the de Leeuw theorem says:

2) We say that a sequence \( \Lambda \subset \mathbb{R} \) has de Leeuw property iff for every measure \( \mu \in M(\mathbb{R}) \), there exists a measure \( \mu' \in M(\mathbb{R}) \) with compact support, such that \( \mu'(\lambda) = \hat{\mu}(\lambda) \) for every \( \lambda \in \Lambda \). By the de Leeuw transference theorem, for every finite set \( F \subset \mathbb{R} \) and \( r \in \mathbb{R} \), any subset of the set \( F + r\mathbb{Z} \) has the de Leeuw property. We do not know whether the converse is true.

3) It is much easier to construct a sequence \( \Lambda \) without the de Leeuw property if we omit the condition \( \inf_{\lambda \in \Lambda} \text{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0 \). Moreover, every sequence \( \Lambda \) which contains an increasing subsequence \( (x_n) \) such that \( \lim x_n = \infty \) and \( \lim (x_{2n} - x_{2n+1}) = 0 \), has not de Leeuw property. Indeed, let \( \nu \in L^1(\mathbb{R}) \) be a measure with Fourier transform supported on the interval \([-1, 1]\) such that \( \hat{\nu}(0) = 1 \) and let \( \nu_r \in M(\mathbb{R}) \) be defined by \( \hat{\nu}_r(t) = \hat{\nu}(\frac{t}{r}) \). Passing, if necessary, to a subsequence we can assume that \( \sum r_n^{1/2} < \infty \) where \( r_n = x_{2n+1} - x_{2n} < x_{2n} - x_{2n-1} \) for \( n = 1, 2, \ldots \). Put \( \mu = \sum r_n^{1/2} \nu_r e^{2\pi i x_{2n} t} \). Then we have \( \|\mu\| < \|\nu\| \cdot \sum r_n^{1/2} < \infty \), \( \hat{\mu}(x_{2n}) = r_n^{1/2} \) and \( \hat{\mu}(x_{2n+1}) = 0 \) for \( n = 1, 2, \ldots \). Hence the supremum of the derivative of \( \hat{\mu} \) on the interval \((x_{2n}, x_{2n+1})\) is greater than \( r_n^{-1/2} \). Therefore the derivative of \( \hat{\mu} \) is unbounded, which means that \( \mu \) is not compactly supported.

5. co-Lebesgue sequences.

We call the sequence \( \Lambda \subset \mathbb{R}^d \) a co-Lebesgue sequence iff for every measure \( \mu \in M(\mathbb{R}^d) \) such that \( \hat{\mu}(\xi) = 0 \) for \( \xi \in \Lambda \), the singular part \( \mu_s \) shares the same property, i.e. \( \hat{\mu}_s(\xi) = 0 \) for \( \xi \in \Lambda \). Clearly every co-Riesz
sequence is co-Lebesgue. A slight modification of Theorem 1 allows to state the following criterion.

**Proposition 8.** — Assume that $\Lambda \subset \mathbb{R}^d$ has the following property. For every $\xi \in \Lambda$ there exists $\alpha \in \mathbb{R}$ such that $\mathbb{Z}^d \setminus \alpha \Lambda$ is a Riesz set, and $\alpha \xi \in \mathbb{Z}^d$. Then $\Lambda$ is a co-Lebesgue sequence.

**Proof.** — Let $\xi \in \Lambda$ and $\alpha \in \mathbb{R}$ be such that $\alpha \xi \in \mathbb{Z}^d$ and $\mathbb{Z}^d \setminus \alpha \Lambda$ is a Riesz set. Let $\nu \in M(\mathbb{T}^d)$ be the measure defined by $\nu(E) = \mu(\alpha E + \alpha \mathbb{Z}^d)$ for $E \subset \mathbb{T}^d$. Clearly $\nu_s(E) = \mu_s(\alpha E + \alpha \mathbb{Z}^d)$. It is easy to see that for every $k \in \mathbb{Z}^d$, 
\[ \hat{\nu}(k) = \hat{\mu}\left(\frac{1}{\alpha}k\right), \]
as well as
\[ \hat{\nu}_s(k) = \hat{\mu}_s\left(\frac{1}{\alpha}k\right). \]
Since $\hat{\mu}(\xi) = 0$ for $\xi \in \Lambda$, the Fourier transform of $\nu$ vanishes outside some Riesz subset of $\mathbb{Z}^d$. Hence, by the assumption, $\nu_s = 0$. Since $\alpha \xi \in \mathbb{Z}^d$, the above formula yields that $\mu_s(\xi) = \nu_s(\alpha \xi) = 0$. \(\square\)

**Examples 4.5. —** 4) Let $k = 2, 3, \ldots$. Then the sequence $\Lambda_k = (n^{1/k})_{n=1}^\infty \subset \mathbb{R}$ is co-Lebesgue one. Indeed, let $a \in \Lambda_k$. Then $a^k \in \mathbb{Z}$. Therefore $j^k a^k \in \mathbb{Z}$ for $j = 1, 2, \ldots$. Hence $ja \in \Lambda_k$ for $j = 1, 2, \ldots$. Therefore $\frac{1}{a} \Lambda_k \cap \mathbb{Z} = \mathbb{Z}_+$, and, by F. and M. Riesz theorem, $\mathbb{Z} \setminus (\frac{1}{a} \Lambda_k)$ is a Riesz set.

5) Let $\Lambda_0 = (\log n)_{n=1}^\infty$. If $a = \log m \in \Lambda_0$ then $na = \log m^n \in \Lambda$ for $n = 1, 2, \ldots$ and hence, similarly as in Example 4, $\mathbb{Z} \setminus (\frac{1}{a} \Lambda_0) = \mathbb{Z}_-$ is a Riesz set.

**Remarks.** — 1) In fact, Proposition 8 together with the above example give something more, namely if $\hat{\mu}(\xi) = 0$ for $\xi \in \Lambda_k$ then $\hat{\mu}_s(\xi) = 0$ for $\xi \in \Lambda_k \cup -\Lambda_k$.

2) We do not know whether $\Lambda_k$ are co-Riesz sequences.

**BIBLIOGRAPHY**


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