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Stratification theory from the Newton polyhedron point of view

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A stratification of a variety $V$ is an expression of $V$ as the disjoint union of a locally finite set of connected analytic manifolds, called strata, such that the frontier of each stratum is the union of a set of lower-dimensional strata. The most important notion in stratification theory is the regularity condition between strata. The notion of $(w)$-regularity introduced by Verdier in [15] plays a very important role in the study of algebraic and analytic varieties. Moreover, he showed that the $(w)$-regularity condition implies the Whitney $(b)$-regularity condition. The $(c)$-regularity, defined by K. Bekka in [2], is weaker than the Whitney $(b)$-regularity, and he showed that the $(c)$-regularity condition implies topological triviality. In this paper, we will investigate these regularity conditions relative to a Newton filtration in terms of the defining equations of the strata. The article is organized as follows. In Section 1 we present a characterization for Bekka’s $(c)$-regularity condition. Next we give a criterion for regularity conditions in terms of the defining equations of the strata, following [1] we introduce a pseudo-metric adapted to the Newton polyhedron in Section 2. Using this construction we obtain versions relative to the Newton filtration of the Fukui-Paunescu Theorem (Theorem 4 below). In this approach it is possible to consider a version relative to a Newton filtration of the $(w)$-regularity condition. We show that this
condition implies the \((c)\)-regularity condition. In Section 3, using the criterion of the regularity condition given in Section 2, we prove that the J. Damon and T. Gaffney condition in ([5], Theorem 1) implies the \((w)\)-regularity condition related to the Newton polyhedron.

Since complex varieties can be considered as real varieties, we shall only consider the real case.

**Notation.** — To simplify the notation, we will adopt the following conventions: for a function \(g(x,t)\), we denote by \(\partial g\) the gradient of \(g\) and by \(\partial_x g\) the gradient of \(g\) with respect to the variables \(x\). For a non zero vector \(v\) of \(\mathbb{R}^n\), we denote by \(L(v)\) the line spanned by \(v\). Also, let \(\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, \text{ each } x_i > 0, i = 1, \ldots, n\}\) and \(Q^n_+ = Q^n \cap \mathbb{R}^n_+\), \(Z^n_+ = Z^n \cap Q^n_+\).

Let \(\varphi, \psi: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)\) be two functions. We say that \(|\varphi(x)| \lesssim |\psi(x)|\) if there exists a constant \(C\) such that \(|\varphi(x)| \leq C|\psi(x)|\). We write \(|\varphi| \sim |\psi|\) if \(|\varphi(x)| \lesssim |\psi(x)|\) and \(|\psi(x)| \lesssim |\varphi(x)|\). Finally, \(|\varphi(x)| \ll |\psi(x)|\) when \(x\) tends to \(x_0\) means \(\lim_{x\to x_0} \frac{\varphi(x)}{\psi(x)} = 0\).

1. Stratification.

In this section, we recall some definitions about stratification. The stratification theory has been introduced by H. Whitney [16] and R. Thom [13].

Let \(M\) be a smooth manifold, and let \(X, Y\) be smooth submanifolds of \(M\) such that \(Y \subseteq \overline{X}\) and \(X \cap Y = \emptyset\).

(i) (Whitney \((a)\)-regularity)

\((X,Y)\) is \((a)\)-regular at \(y_0 \in Y\) if:

for each sequence of points \(\{x_i\}\) which tends to \(y_0\) such that the sequence of tangent spaces \(\{T_{x_i}X\}\) tends in the Grassman space of \((\dim X)\)-planes to some plane \(\tau\), then \(T_{y_0}Y \subset \tau\). We say \((X,Y)\) is \((a)\)-regular if it is \((a)\)-regular at any point \(y_0 \in Y\).

(ii) (Bekka \((c)\)-regularity)

Let \(\rho\) be a smooth non-negative function such that \(\rho^{-1}(0) = Y\). \((X,Y)\) is \((c)\)-regular at \(y_0 \in Y\) for the control function \(\rho\) if:

for each sequence of points \(\{x_i\}\) which tends to \(y_0\) such that the sequence of tangent spaces \(\{\text{Ker}(d\rho(x_i) \cap T_{x_i}X\}\) tends in the Grassman space of \((\dim X - 1)\)-planes to some plane \(\tau\), then \(T_{y_0}Y \subset \tau\). \((X,Y)\)
is \((c)\)-regular at \(y_0\) if it is \((c)\)-regular for some control function \(\rho\). We say \((X, Y)\) is \((c)\)-regular if it is \((c)\)-regular at any point \(y_0 \in Y\).

### 1.1. A criterion for \((c)\)-regularity.

We suppose now that \(M = \mathbb{R}^{n+m}\) and \(0 \in Y \subset \overline{X} - X\) (the regularity conditions are defined locally). Modulo an analytic transformation of \(\mathbb{R}^{n+m}\) near \(0\), if necessary, we may assume that \(Y\) coincides with its tangent space \(T_0Y\). Let \((x, t) = (x_1, \ldots, x_n, t_1, \ldots, t_m)\) denote a system of coordinates of \(\mathbb{R}^{n+m}\). For notational convenience we also use \(x_{n+s} = t_s\). We assume that

\[
Y = \{(x, t) \in \mathbb{R}^{n+m} \mid x_1 = \ldots = x_n = 0\}.
\]

Then we can characterize \((c)\)-regularity as follows:

**Theorem 1.** The pair \((X, Y)\) is \((c)\)-regular at \(0\) for the control function \(\rho\) if and only if \((X, Y)\) is \((a)\)-regular at \(0\) and \(|\partial_t(\rho|_X)(x, t)| \ll |\text{grad } (\rho|_X)(x, t)|\) as \((x, t) \in X\) and \((x, t) \to 0\).

The following proof is inspired by the proof of Bekka-Koike ([3], Theorem 2.4)

**Proof.** At first, we have the following equality:

\[
T_{(x,t)}X = (\text{Ker } d\rho(x, t) \cap T_{(x,t)}X) \oplus K_{(x,t)},
\]

where \(K_{(x,t)} = (\text{Ker } d\rho(x, t) \cap T_{(x,t)}X)^\perp \cap T_{(x,t)}X = L(\partial_t(\rho|_X)(x, t))\) i.e., a line spanned by the gradient of the function \(\rho|_X\).

\((\Rightarrow)\) Let \((x_i, t_i)\) be a sequence of points \(X\) which tends to \(0\) such that \(T_{(x_i, t_i)}X\) tends to some \((\text{dim } X)\)-dimensional space \(\tau\). Taking a subsequence if necessary we can suppose that \(\text{Ker } d\rho(x_i, t_i) \cap T_{(x_i, t_i)}X\) tends to some \((\text{dim } X - 1)\)-dimensional space \(\tau'\) and \(K_{(x_i, t_i)}\) tends to some one-dimensional space \(L\). By Bekka \((c)\)-regularity \(\{0\} \times \mathbb{R}^m \subset \tau'\). Since \(\text{Ker } d\rho(x_i, t_i) \cap T_{(x_i, t_i)}X \subset T_{(x_i, t_i)}X\) and \(K_{(x_i, t_i)}\) is orthogonal to \(\text{Ker } d\rho(x_i, t_i) \cap T_{(x_i, t_i)}X\), we have \(\{0\} \times \mathbb{R}^m \subset \tau\) and \(L\) is orthogonal to \(\{0\} \times \mathbb{R}^m\) which means \((X, Y)\) is \((a)\)-regular at \(0\) and \(|\partial_t(\rho|_X)(x_i, t_i)| \ll |\partial_t(\rho|_X)(x_i, t_i)|\).

\((\Leftarrow)\) Let \((x_i, t_i)\) be a sequence of points \(X\) which tends to \(0\) such that \(\text{Ker } d\rho(x_i, t_i) \cap T_{(x_i, t_i)}X\) tends to some \((\text{dim } X - 1)\)-dimensional space \(\tau\).
When passing to a subsequence one can suppose that all the $T_{(x, t)} X$ have the same dimension (dim $X$), and that this sequence of space converges to some space $\tau'$ and $K_{(x, t)}$ tends to some one-dimensional space $L$. By the Whitney (a)-regularity $\{0\} \times \mathbb{R}^m \subset \tau'$. Since $|\partial_t(\rho_{(x)}(x, t, t))| \ll |\partial(\rho_{(x)}(x, t, t))|$, which implies $L \subset \mathbb{R}^n \times \{0\}$, $L$ is orthogonal to $\{0\} \times \mathbb{R}^m$. Hence we have $\{0\} \times \mathbb{R}^m \subset \tau$.

This completes the proof of the theorem. 

1.2. Ratio test conditions and $(w)$-regularity.

For $X, Y$ as above, we say $X$ is $(r)$-regular (resp. $(w)$-regular) over $Y$ at 0, if for any unit vector $v$ tangent to $Y$

$$|\pi_p(v)||x(t)| \ll |x| \text{ as } p = (x, t) \in X \text{ and } (x, t) \to 0$$

(resp. $|\pi_p(v)| \lesssim |x|$ when $p = (x, t) \in X$ near 0) where $\pi_p$ denotes the orthogonal projection of $\mathbb{R}^{n+m}$ to the normal space of $X$ at $p \in X$. We can find a lot of information about this in [6, 8, 14].

Let $F: (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)$ be an analytic map-germ. We denote by $V_F$ the variety of the zero locus of $F$. One can note that $\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m\}$ gives a stratification of $V_F$ around $\{0\} \times \mathbb{R}^m$. Hereafter, we will assume that

$$X = F^{-1}(0) - \{0\} \times \mathbb{R}^m \text{ and } Y = \{0\} \times \mathbb{R}^m.$$ 

Setting $F := (F_1, \ldots, F_p)$, assume that the Jacobi matrix of $F$ has rank $k$ on $X$ near 0, where $k \leq p$ is the codimension of $X$ in $\mathbb{R}^{n+m}$. We note that the normal space to $X$ is generated by the gradient of the functions $F_j (j = 1, \ldots, p)$ at each $P \in X$ near 0. Let us recall some definitions and notations, used by Fukui and Paunescu in [6].

Let $j_1, \ldots, j_k$ be integers with $1 \leq j_1 < \cdots < j_k \leq p$. We set $J = \{j_1, \ldots, j_k\}$, $F_J = (F_{j_1}, \ldots, F_{j_k})$ and

$$dF_J = dF_{j_1} \wedge \cdots \wedge dF_{j_k}, \text{ where } dF_j = \sum_{i=1}^{n+m} \frac{\partial F_j}{\partial x_i} dx_i,$$

$$d_x F_J = d_x F_{j_1} \wedge \cdots \wedge d_x F_{j_k}, \text{ where } d_x F_j = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i} dx_i,$$

and we define $d^x F_J$ by $dF_J = d_x F + d^x F_J$. 

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For $I \subset \{1, \ldots, n\}$, $S \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, p\}$ with $\#I + \#S = \#J = k$, we set $\frac{\partial F_j}{\partial (x_{i_1}, t_{s_1})}$ to be the Jacobian of $F_j$ with respect to the variables $x_i$ ($i \in I$), and $t_s$ ($s \in S$). When $S = \emptyset$, we simply denote it by $\frac{\partial F_j}{\partial x_i}$. We then define $\|dF\|$, $\|d_x F\|$ and $\|d^x F\|$ by the following formulae:

\[
\|dF\|^2 = \sum_j \|dF_j\|^2 \quad \text{where} \quad \|dF_j\|^2 = \sum_{I, S} \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right|^2,
\]

\[
\|d_x F\|^2 = \sum_j \|d_x F_j\|^2 \quad \text{where} \quad \|d_x F_j\|^2 = \sum_I \left| \frac{\partial F_j}{\partial x_I} \right|^2,
\]

\[
\|d^x F\|^2 = \sum_j \|d^x F_j\|^2 \quad \text{where} \quad \|d^x F_j\|^2 = \sum_{I, S: S \neq \emptyset} \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right|^2.
\]

For a matrix $M$ we denote by $|M|$ the absolute value of its determinant.

Then we have a simple criterion for the regularity conditions of $\Sigma(V_F)$ as follows:

**Theorem 2.** — For $X$, $Y$ as above, we have the following equivalences

(i) $(X, Y)$ is $(a)$-regular at 0 if and only if $\|d^x F\| \ll \|dF\|$ when $(x, t) \to 0$ on $X$.

(ii) $(X, Y)$ is $(r)$-regular at 0 if and only if $|x| \|d_x F\|$ when $(x, t) \to 0$ on $X$.

(iii) $(X, Y)$ is $(w)$-regular at 0 if and only if $\|d^x F\| \lesssim |x| \|d_x F\|$ holds on $X$ near 0.

(iv) $(X, Y)$ is $(c)$-regular at 0 for the function $\rho$ if and only if $\|d^x F\| \ll \|dF\|$ and $|\partial_t \rho_{|X}| \ll \frac{\|dF \wedge d\rho\|}{\|dF\|}$ as $(x, t) \in X$, $(x, t) \to 0$.

Here, $\|dF \wedge d\rho\|^2 = \sum_j \|dF_j \wedge d\rho\|^2$.

**Proof.** — Since (i), (ii) and (iii) have already been obtained in [6], we only have to prove (iv). Indeed, following ([6], lemma 1.4), one get that the orthogonal projection $\pi$ of $v \in T_{(x,t)}M$ to the tangent space $T_{(x,t)}X$ is expressed by the following form:

\[
\pi(v) = \sum_{i=1}^{n+m} \frac{\sum_j (dF_j \wedge dx_i, dF_j \wedge v)}{\|dF\|^2} \frac{\partial}{\partial x_i}.
\]
Since $\partial \rho_{|X} = \pi(\partial \rho)$, we can easily see that $\langle \partial \rho_{|X}, \partial \rho \rangle = \frac{\|dF \wedge d\rho\|^2}{\|dF\|^2}$, but $\partial \rho = \partial \rho_{|X} + \partial \rho_{|N}$ (where $N$ denotes the normal space to $X$), which implies

$$\|\partial \rho_{|X}\|^2 = \langle \partial \rho_{|X}, \partial \rho \rangle = \frac{\|dF \wedge d\rho\|^2}{\|dF\|^2}.$$  

Hence, we can deduce from Theorem 1 that (iv) holds. $\square$

We next state one sufficient condition for (c)-regularity.

**Corollary 3.** — Suppose that $\partial_t \rho = 0$, then $X$ is (c)-regular over $Y$ at 0, if

$$\|d^x F\| \ll \frac{\|dF \wedge d\rho\|}{|\partial \rho|} \quad \text{as} \quad (x, t) \in X, \quad (x, t) \to 0.$$  

Note that when $p = k = 1$, this inequality is a necessary condition for (c)-regularity.

**Proof.** — It is trivial that (1.3) implies $(X, Y)$ is (a)-regular at 0. We first remark, by (1.1) the following equality:

$$\partial_t \rho_{|X} = \sum_j \langle dF_j \wedge dt_j, dF_j \wedge d\rho \rangle \frac{\partial}{\partial t_j} \frac{\|dF\|^2}{\sqrt{\|dF\|^2}} = \sum_{i=1}^n \frac{\partial \rho}{\partial x_i} \sum_j \langle dF_j \wedge dt_j, dF_j \wedge dx_i \rangle \frac{\partial}{\partial t_j}.$$  

Then, by Cauchy-Schwartz inequality, we have

$$|\partial_t \rho_{|X}| \lesssim \frac{|\partial \rho| \|d^x F\|}{\|dF\|} \quad \text{for} \quad j = 1, \ldots, m.$$  

We now assume (1.3). We then have $|\partial_t \rho_{|X}| \ll \frac{\|dF \wedge d\rho\|}{\|dF\|}$ as $(x, t) \in X$, $(x, t) \to 0$. It follows from the equivalence in (iv) of Theorem 2 that $(X, Y)$ is (c)-regular at 0. $\square$

**2. (w)-regularity and (c)-regularity relative to the Newton filtration.**

Let us recall some basic definitions and properties of the Newton filtration (see [1, 5, 7] for details). Let $A \subseteq \mathbb{Q}^d_+$. A Newton polyhedron
\( \Gamma_+(A) \subset \mathbb{R}^n \) is defined by \( \{ \text{the convex closure of } A + \mathbb{R}^n_+ \} \). The Newton boundary of \( A \), \( \Gamma(A) \) is the union of the compact faces of \( \Gamma_+(A) \). We let \( \mathcal{F}(A) \) denote the union of the top dimensional faces of \( \Gamma(A) \). The Newton vertex \( \text{Ver}(A) \) is defined by \( \{ \alpha : \alpha \text{ is vertex of } \Gamma(A) \} \). \( A \) is called convenient if the intersection of \( \Gamma_+(A) \) with each coordinate axis is non-empty. Throughout, we suppose that \( A \) is convenient.

From the Newton polyhedron, we construct the Newton filtration. We first observe that by the convenience assumption on \( A \), any face \( F \in \mathcal{F}(A) \), \( \dim F = n - 1 \). So let \( w_F \) be the unique vector of \( \mathbb{Q}^n_+ \) such that \( F = \{ b \in \Gamma_+(A) : \langle b, w_F \rangle = 1 \} \). We can suppose that the vertices of \( A \) are sufficiently close to the origin so that all the \( w_F \in \mathbb{Z}^n_+ \). We will suppose henceforth that \( A \) satisfies this property. Then, we construct the following map \( \phi : \mathbb{R}^n_+ \to \mathbb{R}_+ \). The restriction of \( \phi \) to each cone \( C(F) \) (where \( C(F) \) denotes the cone of half-rays emanating from 0 and passing through \( F \)) is defined as follows:

\[
\phi_{|C(F)}(\alpha) = \langle \alpha, w_F \rangle, \quad \text{for all } \alpha \in C(F).
\]

We extend this map to \( \mathbb{R}^n_+ \) as follows:

\[
(2.1) \quad \phi(\alpha) = \min \{ \langle \alpha, w_F \rangle : F \in \mathcal{F}(A) \}, \quad \text{for all } \alpha \in \mathbb{R}^n_+.
\]

The map \( \phi \) is linear on each cone \( C(F) \) (where \( F \in \mathcal{F}(A) \)), and the value of \( \phi \) along each point over \( \Gamma(A) \) is equal to 1 and \( \phi(\mathbb{Z}^n_+) \subset \mathbb{Z}_+ \). This is called the Newton filtration induced by \( A \).

For any monomial \( x^\alpha \), we define \( \text{fil}(x^\alpha) = \phi(\alpha) \). This extends to a filtration on the ring \( \mathcal{C}_n \) of analytic function germs : \( (\mathbb{R}^n,0) \to (\mathbb{R},0) \) (via Taylor expansion) by defining

\[
(2.2) \quad \text{fil} \left( \sum c_\alpha x^\alpha \right) = \min \{ \phi(\alpha) : c_\alpha \neq 0 \}.
\]

We denote the set of \( g \) with \( \text{fil}(g) \geq l \) in \( \mathcal{C}_n \) by \( \mathcal{A}_l \). The number \( \text{fil}(g) \) will be also called the level of \( g \) with respect to \( A \).

Now we introduce the control functions associated to \( A \) as follows:

\[
(2.3) \quad \rho(x) = \left( \sum_{\alpha \in \text{Ver}(A)} x^{2p\alpha} \right)^{\frac{1}{2p}} \quad \text{and} \quad \bar{\rho}(x) = \sum_{\alpha \in \text{Ver}(A)} x^{2p\alpha},
\]

where \( p \) a positive integer. Moreover if \( p \) is big enough (it suffices, for example, that \( p\alpha \in \mathbb{Z}^n_+ \)), \( \bar{\rho} \) will be \( C^w \).

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Note that for an element $g = \sum c_\alpha x^\alpha \in C_n$, the support of $g$ is $\text{supp}(g) = \{ \alpha : c_\alpha \neq 0 \}$; it is clear that $g \in \mathcal{A}_t$ if and only if $\text{supp}(g) \subseteq \Gamma_+(t\mathcal{A})$ which is also equivalent to $|g| \lesssim \rho^t$ (see [1, 5] for details). Thus $\mathcal{A}_t$ can be written as

$$
(2.4) \quad \mathcal{A}_t = \{ g \in C_n : \text{supp}(g) \subseteq \Gamma_+(t\mathcal{A}) \} = \{ g \in C_n : |g| \lesssim \rho^t \}.
$$

We say that an analytic function germ $g \in C_n$ is an $\mathcal{A}$-form of degree $d$ if $\text{supp}(g) \subseteq \Gamma(d\mathcal{A})$ (i.e., $g \in \mathcal{A}_d \setminus \mathcal{A}_{d+1}$). Furthermore, for $f \in C_n$, we denote the Taylor expansion of $f(x)$ at the origin by $\sum_{\nu} c_\nu x^\nu$. Setting

$$
H_j(x) = \sum_{\nu \in \Gamma(j\mathcal{A})} c_\nu x^\nu, \quad j \in \mathbb{Z}_+,
$$

we can write $f(x) = \sum_j H_j(x)$ (Newton filtration), where $H_j$ is $\mathcal{A}$-form of degree $j$. Also if $\#\mathcal{F}(\mathcal{A}) = 1$, we can replace the Newton filtration associated with $\mathcal{A}$ by the weighted filtration associated to $w^F$. Moreover, if $w^F = (1, \ldots, 1)$, this Newton filtration coincides with the usual filtration.

### 2.1. Compensation factor.

Let $\rho_i : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a continuous function. We say that $\rho_i$ is the $i$th compensation factor associated with $\mathcal{A}$ if for each $g \in C_n$, we have that $|\rho_i \partial_{x_i} g| \lesssim \rho^{\tilde{\rho_i}(g)}$. Next we give some examples of compensation factors associated with $\mathcal{A}$.

(i) Here, we have the trivial example for the compensation factors, given by

$$
\rho_i(x) = x_i \quad \text{for} \quad i = 1, \ldots, n.
$$

(ii) Let $L_j = L(x_j)$ denote the $x_j$-axis. We then put $\alpha^j = L_j \cap \Gamma(\mathcal{A})$ for $j = 1, \ldots, n$ (the axial vertices of $\Gamma(\mathcal{A})$). We define the weight of the variable $x_i$, $\mathcal{A}(i) = \mathcal{A}(x_i) = \max\{ w_i^F : F \in \mathcal{F}(\mathcal{A}) \}$. We may introduce the compensation factors as follows:

$$
\rho_i(x) = \left( x_i^{\frac{2_\mathcal{A}}{\mathcal{A}(i)}} + \sum_{\alpha \in \text{Ver}(\mathcal{A}) \setminus \{ \alpha^i \}} x^{2_\mathcal{A}_\alpha} \right)^{\frac{\mathcal{A}(i)}{2_\mathcal{A}}}, \quad i = 1, \ldots, n.
$$

It is easy to check that these functions $\rho_i$ are compensation factors associated with $\mathcal{A}$ (see [1, 11] for details).
(iii) The following compensation factors are inspired by the work of Damon-Gaffney in [5]. For all integers $l \geq 0$, we let

$$R_{l,i} = \{ \alpha \in \mathbb{Q}_+^n : \langle \alpha, w^F \rangle \geq l + w_i^F, \forall F \in \mathcal{F}(A) \} \quad \text{for } i = 1, \ldots, n.$$  

We may introduce the compensation factors as follows:

$$\rho_{l,i}(x) = \left( \sum_{\alpha \in Var(R_{l,i})} \frac{x^{2\alpha}}{\rho^{2l}} \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n.$$ 

It is easy to see that for any integers $l \geq 0$, we have that $\rho_{l,i}(x) \leq \rho^{m_i}(x)$, where $m_i = \min_{F \in \mathcal{F}(A)} \{ w_i^F \}$, which implies that $\rho_{l,i}$ is continuous at the origin. On the other hand, by the construction of $\rho_{l,i}$ we can deduce that $|\rho_{l,i} \partial_x g| \leq \rho^{\text{fil}(g)}$ for all $g \in C_n$. Hence, we get that these functions $\rho_{l,i}$ are compensation factors associated with $A$.

**Observation.** We should note that in the case where $\# \mathcal{F}(A) = 1$ (i.e., weighted filtration associated with $w = (w_1, \ldots, w_n)$), the natural choice of compensation factor is that given by L. Paunescu in [10] as follows:

$$\rho_i = \rho^{w_i} \quad \text{for } i = 1, \ldots, n.$$ 

Moreover, for any other compensation factors $\xi_1, \ldots, \xi_n$ associated with the weighted filtration, we have that $\xi_i \leq \rho^{w_i}, i = 1, \ldots, n$. Unfortunately, in the general case we have not succeeded in finding the best compensation factors $\rho_1, \ldots, \rho_n$ such that for any other compensation factors $\xi_1, \ldots, \xi_n$, we have that $\xi_i \leq \rho_i$. However, for each $\gamma \in \mathbb{Q}_+^n$ such that the monomial $x^\gamma$ is $i$th compensation factor, we have $|x^\gamma| \leq \rho_{l,i}$, where $\rho_{l,i}$ are the compensation factors defined in (iii).

Now we fix the compensation factors $\rho_i$ for $i = 1, \ldots, n$ relative to the Newton filtration, and consider the singular metric of $M = \mathbb{R}^{n+m}$ defined by

$$\langle p_i(x) \frac{\partial}{\partial x_i}, p_j(x) \frac{\partial}{\partial x_j} \rangle = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

$$\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \rangle = 0 \quad \text{and} \quad \langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \rangle = \delta_{i,j}.$$ 

Here, $(x, t) = (x_1, \ldots, x_n, t_1, \ldots, t_p)$ denotes a system of coordinates of $\mathbb{R}^{n+m}$. By elementary calculation we have

$$\langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \rangle = \rho_{I} := \rho_{i_1} \cdots \rho_{i_k}.$$ 

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2.2. \((w)\)-regularity associated with \(\mathcal{A}\).

Let \(F : (\mathbb{R}^n \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m) \to (\mathbb{R}^p, 0)\) be analytic. We next assume that

\[
(2.6) \quad Y = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^m : x_1 = \cdots x_n = 0\} \quad \text{and} \quad X = F^{-1}(0) - Y.
\]

Setting \(F := (F_1, \ldots, F_p)\), assume that the Jacobi matrix of \(F\) has rank \(k\) on \(X\) near \(0\), where \(k \leq p\) is the codimension of \(X\) in \(\mathbb{R}^{n+m}\). We note that the normal space of \(X\) is generated by the gradient of the functions \(F_j\) \((j = 1, \ldots, p)\) at each \(P \in X\) near \(0\). Following [6], we define \(\|dF\|_{\mathcal{A}}, \|d_x F\|_{\mathcal{A}}, \|d^2 F\|_{\mathcal{A}}\) and \(\mathcal{D}_{\mathcal{A}}(\ell)\) by the following formulae:

\[
(2.7) \quad \|dF\|_{\mathcal{A}}^2 = \sum_j \|dF_j\|_{\mathcal{A}}^2 \quad \text{where} \quad \|dF_j\|_{\mathcal{A}}^2 = \sum_{I,S} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2,
\]

\[
\|d_x F\|_{\mathcal{A}}^2 = \sum_j \|d_x F_j\|_{\mathcal{A}}^2 \quad \text{where} \quad \|d_x F_j\|_{\mathcal{A}}^2 = \sum_I \left( \rho_I \left| \frac{\partial F_j}{\partial x_I} \right| \right)^2,
\]

\[
\|d^2 F\|_{\mathcal{A}}^2 = \sum_j \|d^2 F_j\|_{\mathcal{A}}^2 \quad \text{where} \quad \|d^2 F_j\|_{\mathcal{A}}^2 = \sum_{I,S: S \neq \emptyset} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2
\]

and

\[
(2.8) \quad \mathcal{D}_{\mathcal{A}}(\ell) = \sum_j \sum_{I,S: \#S = \ell} \left( \rho_I \left| \frac{\partial F_j}{\partial (x_I, t_S)} \right| \right)^2 \quad \text{where} \quad \rho_I = \prod_{i \in I} \rho_i.
\]

We first remark that \(\langle dF, dF \rangle = \|dF\|_{\mathcal{A}}^2\) and \(\langle d_x F, d_x F \rangle = \|d_x F\|_{\mathcal{A}}^2\).

Now using the above construction, we state the version relative to the Newton filtration of the Fukui-Paunescu Theorem ([6], Theorem 2.1).

**Theorem 4.** — The following conditions are equivalent

(i) \(\mathcal{D}_{\mathcal{A}}(m) \preceq \mathcal{D}_{\mathcal{A}}(m-1) \preceq \cdots \preceq \mathcal{D}_{\mathcal{A}}(1) \preceq \mathcal{D}_{\mathcal{A}}(0)\) holds on \(X\) near \(0\).

(ii) \(\|d^2 F\|_{\mathcal{A}} \preceq \|d_x F\|_{\mathcal{A}}\) holds on \(X\) near \(0\).

(iii) For any \(C^1\)-functions \(\varphi_j\) \((j = 1, \ldots, p)\) near \(0\), and \(s = 1, \ldots, m\),

\[
\left| \sum_{j=1}^p \frac{\partial F_j}{\partial t_s} \right| \preceq \sum_{i=1}^n \rho_i \left| \sum_{j=1}^p \frac{\partial F_j}{\partial x_i} \right| \quad \text{holds on} \quad X \quad \text{near} \quad 0.
\]
(iv) For \( J \subset \{1, \ldots, p\} \), \( I = \{i_1, \ldots, i_{k-1}\} \subset \{1, \ldots, n\} \) with \( 1 \leq i_1 < \cdots < i_{k-1} \leq n \), \( s = 1, \ldots, m \),
\[
\rho_I \left| \frac{\partial F_J}{\partial (x_I, t_s)} \right| \lesssim \|d_x F\|_\mathcal{A} \quad \text{holds on } X \text{ near } 0.
\]

(v) For \( J \subset \{1, \ldots, p\} \), \( i = 1, \ldots, n \), \( s = 1, \ldots, m \),
\[
\|dF_J \wedge dx_i, dF_J \wedge dt_s\| \lesssim \rho_i \|d_x F\|^2_\mathcal{A} \quad \text{holds on } X \text{ near } 0.
\]

(vi) For some positive \( C^1 \)-functions \( \phi_J \) on \( X \) with \( J \subset \{1, \ldots, p\} \), \( i = 1, \ldots, n \), \( s = 1, \ldots, m \),
\[
\left| \sum_J \phi_J (dF_J \wedge dx_i, dF_J \wedge dt_s) \right| \lesssim \rho_i \sum_J \phi_J \|d_x F\|^2_\mathcal{A} \quad \text{holds on } X \text{ near } 0.
\]

**Proof.** — The proof is similar to that of Fukui-Paunescu in [6]; it is enough to replace the \( \|x\|_w \) (resp. \( \|x\|_{w^F} \)) in the proof of Theorem 2.1 [6] by the \( \rho_i \) (resp. \( \rho_I \)). \( \square \)

We say that \( X \) is \((w)\)-regular over \( Y \) at 0 with respect to \( \mathcal{A} \) (or \( w^A \)-regular), if one of the above equivalent conditions holds. When \( \#\mathcal{F}(\mathcal{A}) = 1 \), we find that \( \rho_i(x) = \rho_{w^F}(x) \) for \( i = 1, \ldots, n \), hence our \((w^A)\)-regularity reduces to the weighted \((w)\)-regularity (see [6]). Moreover, if \( w^F = (1, \cdots, 1) \), these coincide with the usual \((w)\)-regularity (Verdier’s regularity).

We shall prove the following theorem.

**Theorem 5.** — For \( X, Y \) as above, if \((X, Y)\) is \((w^A)\)-regular, then \((X, Y)\) is \((c)\)-regular for the control function \( \bar{\rho} \) (we recall that \( \bar{\rho}(x) = \sum_{\alpha \in \text{Ver}(\mathcal{A})} x^{2p_{\alpha}} \)).

**Remark 6.** — The converse of the theorem is false in general: (Kuo’s example [8])

\[
F(x, y, t) = y^2 - tx^2 - x^5, \quad X = \{y^2 = tx^2 + x^5\} - \{0\} \times \mathbb{R} \quad \text{and} \quad Y = \{0\} \times \mathbb{R}.
\]

We consider the usual filtration \( (\mathcal{A} = \{(1, 0); (0, 1)\}) \). It is easy to see that \((X, Y)\) is \((c)\)-regular at 0 for the control function \( \bar{\rho}(x, y) = x^2 + y^2 \), but that \((X, Y)\) is not Verdier \((w)\)-regular at 0 (see [14] for details).
As an immediate corollary we have

**Corollary 7.** Let \( f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \), \( t \in \mathbb{R}^m \) be a family of weighted homogeneous polynomials defining an isolated singularity at the origin. We set \( F(x, t) = f_t(x) \), then the stratification \( \Sigma(V_F) \) is \((c)\)-regular. (we again recall that \( \Sigma(V_F) = \{ F^{-1}(0) - \{0\} \times \mathbb{R}^m, \{0\} \times \mathbb{R}^m \} \))

**Proof.** Let us put \( X = F^{-1}(0) - \{0\} \times \mathbb{R}^m \) and \( Y = \{0\} \times \mathbb{R}^m \). Consider the weighted filtration associated with \( \mathcal{A} = \{ (\frac{1}{w_1}, 0, \cdots, 0), \ldots, (0, \cdots, 0, \frac{1}{w_n}) \} \) such that \( f_t \) is a weighted homogeneous polynomial with the weight \( w = (w_1, \cdots, w_n) \in \mathbb{Z}^n_+ \). Now from the Theorem 5, it is enough to show that \((X, Y)\) is \((w^A)\)-regular, that is,

\[
|\partial_t F| \lesssim \|d_x F\|_A \quad \text{holds on} \quad X \text{ near } Y.
\]

Since \( f_t \) defines an isolated singularity at the origin, we can see that \( \|d_x F\|_A^2 = \sum_{i=1}^{n} (\rho^{x_i} \frac{\partial F}{\partial x_i})^2 \) is not zero outside the origin, and this implies our inequality. \( \square \)

**Corollary 8.** Let \( f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \), \( t \in \mathbb{R}^m \) be a real analytic family non-degenerate (in the sense of Kouchnirenko [7]) and \( \Gamma(f_t) = \Gamma(f_0) \), then the stratification \( \Sigma(V_F) \) is \((c)\)-regular.

**Proof.** By standard argument, based on the curve selection lemma, we can see that

\[
|\partial_t F| \lesssim \sum_{\alpha \in \text{Ver}(\Gamma(f_0))} |x^\alpha| \lesssim \sum_{i=1}^{n} |x_i \frac{\partial F}{\partial x_i}|.
\]

Therefore, \((X, Y)\) is \((w^A)\)-regular for any Newton filtration. In particular, \((X, Y)\) is usual \((w)\)-regular (Verdier’s regular). \( \square \)

Before starting the proofs of the above results, we will first illustrate these results with several examples.

**Example 9 (Briançon-Speder family [4]).** Let \( f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0) \), \( t \in J = [-1, 1] \), be a family of weighted homogeneous polynomials defined by

\[
f_t(x, y, z) = z^5 + t y^6 z + x y^7 + x^{15}.
\]

We set \( F(x, t) = f_t(x) \), \( Y = \{0\} \times J \) and \( X = F^{-1}(0) - Y \). It is easy to check that \( |\partial_t F| \lesssim \|d_x F\|_A \) holds on \( X \) near 0, where \( A = \).
\{(1,0,0), \(0, \frac{1}{2},0\), \(0,0, \frac{1}{3}\)\}. Thus, by Theorem 5, we have that \((X,Y)\)
is \((c)\)-regular for the function \(\overline{p}(x,y,z) = x^{12} + y^6 + z^4\). (It is well known that \(f_t\) is not Whitney regular and not usual \((w)\)-regular).

**Example 10 (Oka family [9]).** — Let \(f_t : (\mathbb{R}^3,0) \to (\mathbb{R},0), \ t \in J = [-1,1]\), be a family of polynomial functions defined by

\[f_t(x,y,z) = x^8 + y^{16} + z^{16} + t x^5 z^2 + x^3 y z^3.\]

We set \(F(x,t) = f_t(x), \ Y = \{0\} \times J \ , \ X = F^{-1}(0) - Y \) and

\[A = \left\{ \left( \frac{1}{2},0,0 \right), \ (0,1,0), \ (0,0,1), \ \left( \frac{5}{16},0,\frac{1}{8} \right) \right\}.\]

It is not hard to see that the inequality \(|\partial_t F|^2 \leq \|d_x F\|^2_A = \sum_{i=1}^n (\rho_i \frac{\partial F}{\partial x_i})^2\) holds on \(X\) near \(Y\), where \(\rho_i\) denotes the \(i\)th compensation factor of type (ii) as defined in 2.1. It follows from Theorem 5 that \((X,Y)\) is \((c)\)-regular for the control function \(\overline{p}(x,y,z) = x^{16} + y^{32} + z^{32} + x^{10} z^4\).

### 2.3. Proof of Theorem 5.

In order to show this theorem we need the following lemma.

**Lemma 11.**

1. \(|d\overline{p}|_A \leq \overline{p}(x), \ x \text{ near } 0,\)
2. \(\overline{p} \ll \|dF \wedge d\overline{p}\|_{dF}\) when \((x,t) \to 0\) on \(X\).

**Proof.** — We first recall that:

\[\|d\overline{p}\|^2_A = \sum_{i=1}^n \left( \rho_i \frac{\partial \overline{p}}{\partial x_i} (x) \right)^2.\]

Therefore, (1) is a simple consequence of the construction of the compensation factors and the control functions.

Let us observe that, by (1.2) we have \(|\partial \overline{p}|_X| = \|dF \wedge d\overline{p}\|_{dF}\). On the other hand, \(\partial \overline{p} = \partial \overline{p}|_X + \partial \overline{p}|_N\) (where \(N\) denotes the normal space to \(X\)). Since \(N\) is generated by the gradients of \(F_j (j = 1, \ldots, p)\), we have that \(\partial \overline{p}|_X = \partial \overline{p} + \eta_1 \partial F_1 + \cdots + \eta_p \partial F_p\). After this, (2) in the lemma, follows from the following more general proposition.
PROPOSITION 12. — Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^r, 0)$, $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two germs of analytic maps, setting $f : = (f_1, \ldots, f_r)$. Then there exists a real constant $C$ such that for $p \in f^{-1}(0)$, and sufficiently close to the origin,

$$|g(p)| \leq C \left| p \right| \inf_{(\eta_1, \ldots, \eta_r) \in \mathbb{R}^r} |\eta_1 \partial f_1(p) + \cdots + \eta_r \partial f_r(p) + \partial g(p)|.$$

We note that if $r = 1$, one finds Theorem 1.1 of Adam Parusiński [12]. Moreover, the proof of this proposition is similar to that of Theorem 1.1 in [12] (we omit the details).

Now we are ready to prove Theorem 5. We assume that $(X, Y)$ is $(w^A)$-regular at $0$. By inequality (iii) in Theorem 4, we have

$$\left| \frac{\partial F_J}{\partial (x_1, t_\mathcal{S})} \right| \lesssim \sum_{i=1}^n \rho_i \left| \frac{\partial F_J}{\partial (x_1, t_\mathcal{S}, x_i)} \right| \quad \text{on } X \text{ near } 0,$$

where $\mathcal{S} \subset S$ such that $\# \mathcal{S} = \# S - 1$. Thus we obtain $\|d^2 F\| \ll \|dF\|$ when $(x, t) \to 0$ on $X$ (i.e., $(X, Y)$ is $(a)$-regular at $0$), and so by Theorem 2, we only have to prove that:

$$|\partial t \bar{\rho}| \ll \frac{\|d^2 F \wedge d\bar{\rho}\|}{\|dF\|} \quad \text{as } (x, t) \in X, (x, t) \to 0.$$

We first remark, by (1.1) the following equality:

$$|\partial t \bar{\rho}| \big| \big| = \left| \sum_{i=1}^n \sum_{I, S} \frac{\partial (F_{J, t_\eta})}{\partial (x_1, t_\mathcal{S}, t_\eta)} \frac{\partial (F_{J, \bar{\rho}})}{\partial (x_1, t_\mathcal{S}, x_\eta)} \right|,$$

and hence

$$|\partial t \bar{\rho}| \big| \big| \lesssim \left| \sum_{i=1}^n \sum_{I, S} \frac{\partial (F_{J, \bar{\rho}})}{\partial (x_1, t_\mathcal{S}, x_\eta)} \right|.$$

According to the inequality in (iii) of Theorem 4, we have

$$\left| \frac{\partial (F_{J, \bar{\rho}})}{\partial (x_1, t_\mathcal{S}, t_\eta)} \right| \lesssim \sum_{i=1}^n \rho_i \left( \left| \frac{\partial (F_{J, \bar{\rho}})}{\partial (x_1, t_\mathcal{S}, x_i)} \right| + \left| \frac{\partial \bar{\rho}}{\partial x_i} \right| \left| \frac{\partial F_J}{\partial (x_1, t_\mathcal{S})} \right| \right).$$
Thus, we obtain
\[
\left| \frac{\partial(F, \tilde{p})}{\partial(x, t)} \right| \lesssim \|d\tilde{p}\|_A \|dF\| + \sum_{i=1}^{n} \rho_i \|dF \wedge d\tilde{p}\|
\]
and, using (2.13), we obtain
\[
(2.14) \quad |\partial_i \tilde{p}_x| \lesssim \|d\tilde{p}\|_A + \sum_{i=1}^{n} \rho_i \frac{\|dF \wedge d\tilde{p}\|}{\|dF\|} \quad \text{on } X \text{ near } 0.
\]
It follows from Lemma 11 that (2.12) holds. This completes the proof of Theorem 5.

3. The Damon-Gaffney condition and (c)-regularity.

In this section we describe some definitions and notations used by Damon-Gaffney in [5].

Given a Newton filtration $A$ as above. We extend this filtration on the ring $C_{x,t}$ of formal power series in the variables $x_1, \ldots, x_n; t_1, \ldots, t_m$ around the origin by defining
\[
\text{fil} \left( \sum c_{\nu}(t)x^\nu \right) = \min \{ \phi(\nu) : c_{\nu}(t) \neq 0 \}.
\]
Let $g = \sum c_{\nu}(t)x^\nu$ be a series in $C_{x,t}$, the support of $g$, denoted by supp$(g)$, is the set of points $\nu \in \mathbb{Z}^n_+$ such that $c_{\nu}(t) \neq 0$. We denote the set of $g$ with fil$(g) \geq l$ in $C_{x,t}$ by $A_{l,x,t}$. It is not difficult to see the following equality:
\[
A_{l,x,t} = \{ g \in C_{x,t} : \text{supp}(g) \subset \Gamma_+(lA) \} = \{ g \in C_{x,t} : |g| \leq \rho^l \}.
\]
We say that level $A_l$ of the Newton filtration is fit if all the vertices of $\phi^{-1}(l)$ are lattice points of $\mathbb{R}^n_+$. This says that $l \text{Ver}(A) = \text{Ver}(lA) \in \mathbb{Z}^n_+$ (because of the linearity of the Newton filtration on cones). For $A_l$ which is fit, we let
\[
\text{ver}(A_l) = \{ x^\beta : \beta \text{ is a vertex of } \phi^{-1}(l) \} = \{ x^{l\alpha} : \alpha \in \text{Ver}(A) \}.
\]
We also let
\[
V_{l,x,t} = \left\{ \zeta \in A_{l+1,x,t} : \zeta(A_{k,x,t}) \subset A_{l+k,x,t} \right\},
\]
with $A_{t+1,x,t}\{\partial/\partial x_i\}$ denoting the $A_{t+1,x,t}$-module generated by the $\partial/\partial x_i$, $i = 1, \ldots, n$. Finally, for an element $g \in C_{x,t}$, we let $V_{i,x,t}(g) = \{\zeta(g) : \zeta \in V_{i,x,t}\}$.

Now we can announce the Damon-Gaffney Theorem.

**THEOREM 13 (Damon-Gaffney [5]).** — Let $f : (\mathbb{R}^{n+m}, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic deformation of a germ $f_0 : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ (i.e., $f \in C_{x,t}$). Then a sufficient condition that $f$ be a topologically trivial deformation is that there exists a fit $A_i$ so that

\begin{equation}
ver(A_i) \cdot \frac{\partial f}{\partial t_j} \subset V_{i,x,t}(f), \ j = 1, \ldots, m.
\end{equation}

We will call condition (3.5) the Damon-Gaffney condition. Next, our principal goal will be to show that this condition implies a (w)-regularity condition relative to the Newton filtration, hence, these deformations will, in fact, satisfy the Bekka condition.

Given an analytic function $f \in C_{x,t}$, we define

$$\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m) = \{\mathbb{R}^n \times \mathbb{R}^m - f^{-1}(0), \ f^{-1}(0) - \{0\} \times \mathbb{R}^m, \ \{0\} \times \mathbb{R}^m\},$$

which gives a stratification of $\mathbb{R}^n \times \mathbb{R}^m$ around $\{0\} \times \mathbb{R}^m$. Then, we have

**THEOREM 14.** — For $f \in C_{x,t}$, if there is a positive integer $l$ such that

$$ver(A_l) \cdot \frac{\partial f}{\partial t_j} \subset V_{i,x,t}(f), \ j = 1, \ldots, m \ (The \ Damon-Gaffney \ condition),$$

then the stratification $\Sigma_f(\mathbb{R}^n \times \mathbb{R}^m)$ is $(c)$-regular.

**Proof.** — Let us put $ver(A_l) = \{x^\alpha\}$ then we get the following expression:

$$x^\alpha \frac{\partial f}{\partial t_j} = \sum_{i=1}^n \xi_{ij}^{(\alpha)} \frac{\partial f}{\partial x_i} = \xi_j^{(\alpha)}(f),$$

and summing over $x^\alpha \in ver(A_l)$ we obtain

\begin{equation}
\left( \sum_{\alpha \in Ver(lA)} |x^\alpha| \right) \left| \frac{\partial f}{\partial t_j} \right| \lesssim \sum_{i=1}^n \left( \sum_{\alpha \in Ver(lA)} |\xi_{ij}^{(\alpha)}| \right) \left| \frac{\partial f}{\partial x_i} \right|.
\end{equation}
Since $\text{Ver}(lA) = l\text{Ver}(A)$, which means $\rho^l \sim \sum_{\alpha \in \text{Ver}(lA)} |x^\alpha|$. Then we let

$$\xi'_i = \sum_{j=1}^{m} \sum_{\alpha \in \text{Ver}(lA)} \rho^{-1}\xi^{(\alpha)}_{ij} \quad \text{for} \quad i = 1, \ldots, n.$$ 

It follows from (3.6) that $|\partial_t f|^2 \leq \sum_{i=1}^{n} (\xi'_i \partial f)_{xi}^2$, and so by Theorem 5, it is sufficient to show that these $\xi'_i$ are compensation factors associated with $A$. Indeed, for any $g \in C_n$, we have from the filtration properties of the $\xi^{(\alpha)}_{ij}$ that

$$\text{fil}(\xi^{(\alpha)}_{ij}(g)) = \text{fil}(\xi^{(\alpha)}_{ij} \partial_x g) \geq \text{fil}(g) + l$$

which means

$$|\xi^{(\alpha)}_{ij} \partial_x g| \leq \rho^{l+\text{fil}(g)}.$$ 

Therefore, for $i = 1, \ldots, n$,

$$|\xi'_i \partial_x g| \leq \rho^{\text{fil}(g)}.$$ 

This completes the proof of the Theorem

**Remark 15.** We observe that $\zeta = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \in V_{l,x,i}$ if and only if $\text{supp}(\xi_i) \subset R_{l,i}$ (we recall that $R_{l,i} = \{ \alpha \in \mathbb{Q}_+^n : \langle \alpha, w^F \rangle \geq l + w^F_i, \forall F \in \mathcal{F}(A) \}$) which is also equivalent to $|\xi_i| \leq \sum_{\alpha \in \text{Ver}(R_{l,i})} |w^\alpha|$. Hence, the Damon-Gaffney condition implies a $(w^A)$-regularity condition with $\rho_{l,i}$ as compensation factors, where $\rho_{l,i}$ denotes the $i$th compensation factor of type (iii) as defined in 2.1.

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