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THE GENERAL DEFINITION OF THE COMPLEX MONGE-AMPÈRE OPERATOR

by Urban CEGRELL

1. Introduction.

Denote by $PSH(Q)$ the plurisubharmonic functions on $Q$ and by $PSH^{-}(Q)$ the subclass of negative functions. A set $Q \subset \mathbb{C}^n$ is said to be a hyperconvex domain if it is open, bounded, connected and if there exists $\varphi \in PSH^{-}(Q)$ such that $\{ z \in Q; \varphi(z) < -c \} \subset Q$, $\forall c > 0$. Such function is called an exhaustion function for $Q$. Throughout this paper, $Q$ will always denote a hyperconvex domain. In the first part of this paper, we consider global approximation of negative plurisubharmonic functions by decreasing sequences of negative plurisubharmonic functions that are continuous on $Q$, equal to zero on $\partial Q$ and with bounded Monge-Ampère mass. The elements of this class of functions serves as “test functions”. Theorem 2.1 below states that global approximation is possible in $PSH^{-}$. We use Theorem 2.1 to show that integration by parts is almost always allowed (Corollary 3.4). In the second part of this paper, we discuss a general definition of the complex Monge-Ampère operator. This is done by introducing a class $E$ of plurisubharmonic functions which consists of all functions that are locally equal to decreasing limits of test functions described above. The Monge-Ampère operator can be extended to $E$, and this is the most general definition if we require the operator to be continuous under decreasing limits (Theorem 4.5). In the remaining part of the paper, we study the Monge-Ampère operator using this general definition.

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2. Approximation by continuous plurisubharmonic functions.

In this section, we prove an approximation theorem for negative plurisubharmonic functions, used throughout the paper.

**Theorem 2.1.** Suppose $u \in PSH^-(\Omega)$. Then there is a decreasing sequence of functions $u_j \in PSH(\Omega) \cap C(\bar{\Omega})$ with $u_j|_{\partial \Omega} = 0$, $\forall j \in \mathbb{N}$, $\lim_{j \to +\infty} u_j(z) = u(z)$, $\forall z \in \Omega$ and $\int_{\Omega}(dd^c u_j)^n < +\infty$.

**Proof.** Denote by $h_E$ the relative extremal function for $E \subseteq \Omega$. See [17]. Then $\text{supp}(dd^c h_E)^n \subseteq \Omega$ and if $E = B$ is a ball, it follows from a theorem by Walsh [18] that $h_B$ is continuous on $\bar{\Omega}$. Thus, there is a continuous exhaustion function $v$ for $\Omega$ with $\text{supp}(dd^c v)^n \subseteq \Omega$, in particular $\int_{\Omega}(dd^c v)^n < +\infty$. See [5] and [14] for details.

For each $j \in \mathbb{N}$, choose $\{r_j\}_{j=1}^{\infty}$ a decreasing sequence such that $0 < r_j < \text{dist}\left(\left\{z \in \Omega; v(z) < -\frac{1}{2j^2}\right\}, \bar{\Omega}\right)$. Denote by $u_{r_j}$ the usual regularization of $u$, defined on $\Omega_{r_j}$ where $\Omega_m = \{z \in \Omega; \text{dist}(z, \partial \Omega) > m\}$.

Define $u_j = \sup_{j \leq m}(u_{r_m} - \frac{1}{m}, mv)$ on $\Omega_m$ and $mv$ otherwise on $\Omega$.

Note that if $d(z_0, \partial \Omega) \leq r_j$ then $d(z_0, \partial \Omega) < \text{dist}\left(\left\{v < -\frac{1}{2j^2}\right\}, \partial \Omega\right)$ so $\frac{1}{2j^2} > r_j$ and $jv(z_0) \geq \frac{1}{2j} > \frac{1}{2} \left(\geq u_{r_j}(z_0) - \frac{1}{j}\right)$. Therefore, each $\max\left(u_{r_j} - \frac{1}{j}, jv\right)$ is plurisubharmonic on $\Omega$, continuous on $\bar{\Omega}$ and equal to zero on $\partial \Omega$.

Also, since, $u_j$ is the upper envelope of continuous functions, $u_j$ is lower semicontinuous. We claim that $u_j$ is upper semicontinuous. We have

$$u_j = \sup_{j \leq m}\left(u_{r_m} - \frac{1}{m}, mv\right) = \sup_{j \leq m}\left[\max\left(u_{r_m}, mv + \frac{1}{m}\right) - \frac{1}{m}\right]$$

$$\leq \max\left\{\max_{j \leq m < k}\left[\max\left(u_{r_m}, mv + \frac{1}{m}\right) - \frac{1}{m}\right], \max\left(u_{r_k}, kv + \frac{1}{k}\right)\right\}$$

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since max\( (u_{rk}, kv + \frac{1}{k}) \) is decreasing in \( k \). Finally, since
\[
\max \left\{ \max_{j \leq m < k} \left[ \max \left( u_{rm}, mv + \frac{1}{m} \right) - \frac{1}{m} \right], \max(u_{rk}, kv + \frac{1}{k}) \right\} \\downarrow \ u_j, \quad k \to +\infty
\]
it follows that each \( u_j \) is upper semicontinuous. It is a consequence of Stokes theorem that \( \int_\Omega (dd^c u_j)^n \leq j^n \int_\Omega (dd^c v)^n < +\infty \). 

Remark. — Note that the \( u_j:s \), used in the proof above have compactly supported Monge-Ampère measures.

Remark. — There is always an exhaustion function in \( E_0 \cap C^\infty(\Omega) \). Cf. [10]. For the definition of \( E_0 \) see next section.

Remark. — In the case when \( \Omega \) is a so called B-regular domain, arbitrary continuous boundary data has been studied in [19].

3. Integration by parts.

In this section, we study "integration by parts" which is an essential tool in this paper.

We first recall the definition of the class \( E_0 \), introduced in [7]; \( E_0(\Omega) \) is the convex cone of bounded plurisubharmonic functions \( \varphi \) with \( \lim_{z \to \xi} \varphi(z) = 0, \forall \xi \in \partial \Omega \) and \( \int_\Omega (dd^c \varphi)^n < +\infty \).

**Lemma 3.1.** \( C_0^{\infty}(\Omega) \subset E_0 \cap C(\Omega) - E_0 \cap C(\Omega) \).

**Proof.** — Choose \( 0 < \psi \in E_0 \) and let \( \chi \in C_0^{\infty}(\Omega) \) be given. Choose \( m \) so large that
\[
\chi + m|z|^2 \in PSH(\Omega).
\]
Let \( a < \inf \chi < \sup_\Omega (|\chi| + m|z|^2) < b \) and define
\[
\varphi_1 = \max(\chi + m|z|^2 - b, \ M\psi)
\]
where \( M \) is so large that \( M\psi < a - b \) on the support of \( \chi \). Then \( \varphi_1 \in E_0 \) and so is \( \varphi_2 = \max(m|z|^2 - b, \ M\psi) \). This proves the lemma since \( \chi = \varphi_1 - \varphi_2 \). 

**Theorem 3.2.** — Suppose \( u, v \in PSH^-(\Omega) \), \( u \not\equiv 0 \), \( \lim_{z \to \xi} u(z) = 0, \forall \xi \in \partial \Omega \), and \( T \) a positive and closed current of bidegree \((n-1,n-1)\). Then
$\ddc u \wedge T$ is a well-defined positive measure on $\Omega$. Furthermore, if
\[
\int v \ddc u \wedge T > -\infty
\]
then $\ddc v \wedge T$ is also a well-defined positive measure on $\Omega$ and
\[
\int v \ddc u \wedge T \leq \int u \ddc v \wedge T.
\]

The following lemma is well-known, cf. [13].

**Lemma 3.3.** — If $u \in PSH^-(\Omega)$, $T$ a positive and closed current
of bidegree $(n-1,n-1)$ and if $u \in L^\infty(\Omega \setminus K)$ where $K$ is a compact subset
of $\Omega$, then $\ddc u \wedge T$ is a well-defined positive measure on $\Omega$.

**Proof.** — Note first that if $\chi \in C_0^\infty(\Omega)$ then,
\[
\int g\ddc \chi \wedge T, \ g \in C_0^\infty(\Omega)
\]
is a Radon measure with mass zero.

Therefore, $\int w\ddc \chi \wedge T$ is a positive number when
\[
0 \leq \chi \in C_0^\infty(\Omega'), \ w \in PSH \cap L^\infty(\Omega')
\]
for every $\Omega' \subset \subset \Omega$ any strictly pseudoconvex domain containing $K$. In
particular, the conclusion of Lemma 3.3 holds true if $K$ is empty. Choose
$u^1_j, u^2_j \in PSH \cap C^\infty$ in a neighborhood of $\Omega'$:
\[
u^1_j \searrow u, \quad j \to +\infty
\]
\[
u^2_j \searrow \max(u, \inf_{\xi \in \Omega \setminus K} u(\xi)), \quad j \to +\infty
\]
and $u^1_j = u^2_j$ near $\partial \Omega'$.

Then $\int_{\Omega'} \ddc (u^1_j - u^2_j) \wedge T = 0$ since $T$ is closed. Also, if $0 \leq \chi \in
C_0^\infty(\Omega'), \ \chi \equiv 1$ near $\text{supp} (u^1_j - u^2_j)$ then
\[
\int_{\Omega'} (\ddc u^1_j - \ddc u^2_j) \wedge T = \int \chi \ddc u^1_j \wedge T - \int u^2_j \ddc \chi \wedge T.
\]
So since $\lim \int_{\Omega'} \ddc \chi \wedge T$ exists, so does $\int_{\Omega'} \chi \ddc u^1_j \wedge T$. Therefore,
$\ddc u \wedge T$ is a well-defined positive measure on $\Omega$.

**Proof of Theorem 3.2.** — Suppose $u, v \in PSH(\Omega) \cap C(\Omega)$, $u = v = 0$
on $\partial \Omega$ and that
\[
\int v \ddc u \wedge T > -\infty.
\]
Then, by Theorem 3.3 in [12]

\[-\infty < \int_{\Omega} v \ddc u \wedge T \leq \int_{\{u < -\varepsilon\}} v \ddc u \wedge T \leq \int_{\{u < -\varepsilon\}} (u + \varepsilon) \ddc v \wedge T\]

so if we denote by \(\chi_\varepsilon\) the characteristic function of \(\{u \leq -\varepsilon\}\), we have that \((u + \varepsilon)\chi_\varepsilon\) decreases to \(u\) when \(\varepsilon\) decreases to zero.

Hence,

\[\int_{\Omega} v \ddc u \wedge T \leq \int_{\Omega} u \ddc v \wedge T\]

and in the same way, using that \(-\infty < \int u \ddc v \wedge T\) we find that

\[\int_{\Omega} u \ddc v \wedge T = \int_{\Omega} v \ddc u \wedge T.\]

To complete the proof of Theorem 3.2, we use Theorem 2.1 and choose

\[u_j, v_j \in \mathcal{E}_0 \cap C(\bar{\Omega}),\]

\[u_j \searrow u, v_j \searrow v, j \to +\infty.\]

Now, since

\[\int_{\Omega} v \ddc u \wedge T > -\infty.\]

we have by dominated convergence that

\[\lim_{\varepsilon \to 0} \int_{\Omega} \max(v, -\varepsilon) \ddc u \wedge T = 0.\]

Then, by Lemma 3.3 and what we have proved so far,

\[\int_{\Omega} v_k \ddc u \wedge T = \lim_{j} \int_{\Omega} v_k \ddc u_j \wedge T = \lim_{j} \int_{\Omega} u_j \ddc v_k \wedge T = \int_{\Omega} u \ddc v_k \wedge T\]

so since \(u\) is upper semicontinuous we get

\[\int_{\Omega} v \ddc u \wedge T \leq \int_{\Omega} u \ddc v \wedge T\]

which proves Theorem 3.2. 

\[\square\]

**Corollary 3.4.** — Suppose

\(u, v \in PSH(\Omega), \lim_{z \to \xi} u(z) = \lim_{z \to \xi} v(z) = 0,\)

\(\forall \xi \in \partial \Omega\) and that \(T\) is a positive closed current of bidegree \((1,1)\). If

\[\int_{\Omega} v \ddc u \wedge T > -\infty \text{ then } \int_{\Omega} v \ddc u \wedge T = \int_{\Omega} u \ddc v \wedge T.\]
Remark. — See also [6], Theorem 5.7.

Remark. — In Section 4, we will extend the result of Corollary 3.4.

4. Definition of the complex Monge-Ampère operator.

Using decreasing sequences of $C^\infty$—smooth plurisubharmonic functions, it is known from [2] how to define $(dd^cu)^n$ when $u$ is plurisubharmonic and locally bounded which is a special case of plurisubharmonic functions satisfying a certain growth condition studied in [1]. In these cases, as well as in the class $E_p$, studied in [7], the comparison principle is valid.

Leaving this principle behind, $(dd^cu)^n$ is well-defined for $u$ bounded near the boundary of its domain of definition, [13] and [16].

We are now going to consider a definition which covers all these cases.

**DEFINITION 4.1.** — Assume $u \in \text{PSH}^-(\Omega)$. We say that $u \in \mathcal{E}(\Omega)$ if to every $z_0 \in \Omega$ there is a neighborhood $\omega$ of $z_0$ in $\Omega$ and a decreasing sequence $h_j \in \mathcal{E}_0(\Omega)$ such that $h_j \searrow u$ on $\omega$ and $\sup_j \int_\Omega (dd^chu_j)^n < +\infty$.

Remark. — Since $\mathcal{E}_0(\Omega)$ is a convex cone, so is $\mathcal{E}$. See Section 2 in [7].

**THEOREM 4.2.** — Suppose $u^p \in \mathcal{E}(\Omega), 1 \leq p \leq n$. If $g_j^p \in \mathcal{E}_0(\Omega)$ decreases to $u^p$, $j \to +\infty$, then $dd^cg_j^1 \wedge dd^cg_j^2 \wedge \ldots \wedge dd^cg_j^n$ is weak*-convergent and the limit measure does not depend on the particular sequences $(g_j^p)^j=1$.

Proof. — Suppose first that $\sup_j \int (dd^cg_j^n)^n < +\infty$. Then, for $h \in \mathcal{E}_0(\Omega)$, $\int hdd^cg_j^1 \wedge dd^cg_j^2 \wedge \ldots \wedge dd^cg_j^n$ is a decreasing sequence by Corollary 3.4 and since

$$\int h(dd^cg_j^n)^n \geq (\inf_{\Omega} h) \sup_j \int (dd^cg_j^n)^n > -\infty,$$

$$\lim_{j \to \infty} \int hdd^cg_j^1 \wedge dd^cg_j^2 \wedge \ldots \wedge dd^cg_j^n$$

exists for all $h \in \mathcal{E}_0$.

By Lemma 3.1, $dd^cg_j^1 \wedge dd^cg_j^2 \wedge \ldots \wedge dd^cg_j^n$ is weak*-convergent.
If $v_j^P$ is another sequence decreasing to $u^P$, we get, again by Corollary 3.4,
\[
\int hdd^c v^1_j \land dd^c v^2_j \land \ldots \land dd^c v^n_j = \int v^1_j dd^c h \land dd^c v^2_j \land \ldots \land dd^c v^n_j \geq \int u^1 dd^c h \land dd^c v^2_j \land \ldots \land dd^c v^n_j = \lim_{s_1 \to +\infty} \int g^1_{s_1} dd^c h \land dd^c v^2_j \land \ldots \land dd^c v^n_j = \lim_{s_1 \to +\infty} \int v^2_{s_1} dd^c h \land dd^c g^1_{s_1} \land \ldots \land dd^c v^n_j \geq \ldots \geq \lim_{s_1 \to +\infty} \lim_{s_2 \to +\infty} \ldots \lim_{s_n \to +\infty} \lim_{s \to +\infty} \int hdd^c g^1_{s_1} \land dd^c g^2_{s_2} \land \ldots \land dd^c g^n_{s_n}.
\]
Therefore, $\lim_{j \to +\infty} \int hdd^c v^1_j \land dd^c v^2_j \land \ldots \land dd^c v^n_j$ exists and is minorized by
\[
\lim_{j \to +\infty} \int hdd^c g^1_j \land dd^c g^2_j \land \ldots \land dd^c g^n_j.
\]
But this is a symmetric situation so we conclude that the limits are equal.

To complete the proof it remains to remove the restriction
\[
\sup_j \left( \int (dd^c g^P_j)^n \right) < +\infty.
\]

Let $K$ be a given compact subset of $\Omega$, cover $K$ with finitely many $W_q$, $q = 1, \ldots, N$ as in the definition of $E$. Let $\{h^{pq}_j\}_{j=1}^\infty$, $1 \leq q \leq N$, $1 \leq p \leq n$ be the corresponding $h^p_j$'s and put $w^P_j = \sum_{q=1}^N h^{pq}_j$.

Then $w^P_j \in E_0, w^P_j \leq g^P_j$ on $\bigcup_{q=1}^N W_q$, and $\sup_j \int (dd^c w^P_j)^n < +\infty$. Thus, if we define $u^P_j = \max(g^P_j, w^P_j)$, $\sup_j \int (dd^c u^P_j)^n < +\infty$ and $u^P_j = g^P_j$ near $K$.

**DEFINITION 4.3.** — For $u^P \in E, 1 \leq p \leq n$, we define $dd^c u^1 \land dd^c u^2 \land \ldots \land dd^c u^n$ to be the limit measure obtained in Theorem 4.2.

**DEFINITION 4.4.** — Consider a class $\mathcal{K} (= \mathcal{K}(\Omega)) \subset PSH^-(\Omega)$ such that:

1. If $u \in \mathcal{K}$, $v \in PSH^-(\Omega)$ then $\max(u, v) \in \mathcal{K}$.
2. If $u \in \mathcal{K}, \varphi_j \in PSH^-(\Omega) \cap L^\infty_{loc}, \varphi_j \searrow u, j \to +\infty$, then $((dd^c \varphi_j)^n)_{j=1}^\infty$ is weak*-convergent.
Remark. — We will now see that $\mathcal{E}$ is the largest class for which (1) and (2) holds true. Thus, we may say that $\mathcal{E}$ is optimal in this sense.

**THEOREM 4.5.** — The class $\mathcal{E}$ has property 1. and 2. of Definition 4.4. Conversely, if $\mathcal{K}$ meets the requirements of Definition 4.4, then $\mathcal{K} \subset \mathcal{E}$.

**Proof.** — Suppose $u \in \mathcal{E}$. Then (1) holds true by Theorem 3.2. To prove (2), suppose $\varphi_j \in PSH^-(\Omega) \cap L^\infty_{loc}(\Omega)$, $\varphi_j \downarrow u$, $j \to +\infty$. If $\varphi \in \mathcal{E}_0$, then $g_j = \max(\varphi_j, m_j \varphi) \in \mathcal{E}_0$ and since $g_j \downarrow u \in \mathcal{E}$, $(dd^c g_j)^n$ is weak*-convergent by Theorem 4.2. Here, $(m_j)$ is any sequence, decreasing to $-\infty$. Hence $(dd^c \varphi_j)^n$ is weak*-convergent and therefore (2) follows.

Conversely, suppose $u \in \mathcal{K}$, $\omega$ open and relatively compact in $\Omega$. By Theorem 2.1, we can find $h_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, $h_j \downarrow u$ on $\Omega$, $j \to +\infty$.

Define

$$\tilde{h}_j = \sup\{v \in PSH^-(\Omega), v \leq h_j \text{ on } \omega\}$$

Then $\tilde{h}_j \in PSH(\Omega) \cap L^\infty$, $\text{supp}(dd^c h_j)^n \subset \bar{\omega}$ and $\tilde{h}_j \downarrow u$ on $\omega$, $\tilde{h}_j$ decreases on $\Omega$, and $\tilde{h}_j \geq u$ everywhere on $\Omega$.

Now, $u \in \mathcal{K}$ and so is $\tilde{h} = \lim_{j \to +\infty} \tilde{h}_j$ since $\tilde{h} \geq u$. Therefore, by (2), $(dd^c \tilde{h}_j)^n$ is weak*-convergent and since $\text{supp}(dd^c h_j)^n \subset \bar{\omega} \subset \subset \Omega$ it follows that $\sup_{j} \int_{\Omega} (dd^c h_j)^n < +\infty$ and we have proved that $u \in \mathcal{E}$. □

**Remark.** — Note that $PSH^-(\Omega) \cap L^\infty_{loc}(\Omega) \subset \mathcal{E}$.

**DEFINITION 4.6.** — We denote by $\mathcal{F}(\Omega)$ the subclass of functions $u$ in $\mathcal{E}(\Omega)$ such that there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \downarrow u$ on $\Omega$ and $\sup_{j} \int_{\Omega} (dd^c u_j)^n < +\infty$.

**Remark.** — It follows from Corollary 3.4 and Theorem 4.2 that integration by parts is allowed in $\mathcal{F}$.

**Remark.** — It follows from the proof of Theorem 4.5 that every $u \in \mathcal{E}(\Omega)$ is locally in $\mathcal{F}(\Omega)$; to every $u \in \mathcal{E}(\Omega)$ and every $w$, open and relatively compact in $\Omega$, there is a $u_w \in \mathcal{F}(\Omega)$ with $u = u_w$ in $w$. 

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5. The class $\mathcal{F}$.

In this section we to study $\mathcal{F}$ and prove some inequalities, a generalized comparison principle and a decomposition of $(dd^c u)^n$, $u \in \mathcal{F}$.

**Proposition 5.1.** — Suppose $u^p \in \mathcal{F}(\Omega)$, $1 \leq p \leq n$ and $h \in PSH^-(\Omega)$. If $g^p_j \in \mathcal{E}_0(\Omega)$ decreases to $u^p$, $j \to +\infty$, then

$$
\lim_{j \to +\infty} \int hdd^c g^1_j \wedge dd^c g^2_j \wedge \ldots \wedge dd^c g^n_j = \int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.
$$

**Proof.** — Since $\Omega$ is open, and since $u^p \in \mathcal{F}(\Omega)$, $1 \leq p \leq n$, $dd^c g^1_j \wedge dd^c g^2_j \wedge \ldots \wedge dd^c g^n_j$ converges weak* to $dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$ and we have

$$
+\infty > \lim_{j \to +\infty} \int dd^c g^1_j \wedge dd^c g^2_j \wedge \ldots \wedge dd^c g^n_j \geq \int dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.
$$

If $h$ is in $\mathcal{E}_0 \cap C$ then, by the proof of Theorem 4.2

$$
\int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n = \lim_{j \to +\infty} \int hdd^c g^1_j \wedge dd^c g^2_j \wedge \ldots \wedge dd^c g^n_j.
$$

Suppose now $h \in PSH^-(\Omega)$ and that $\int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n$ is finite.

For each $j$, choose $h_j \in \mathcal{E}_0 \cap C$, decreasing to $h$, $q_j$ and $s_j$ such that

$$
\int -hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n \leq \frac{1}{j} - \int h_j dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n
$$

$$
\leq \frac{2}{j} - \int h_j dd^c g^1_j \wedge \ldots \wedge dd^c g^n_{q_j} \leq \frac{2}{j} - \int hdd^c g^1_{q_j} \wedge dd^c g^2_{q_j} \wedge \ldots \wedge dd^c g^n_{q_j}
$$

$$
\leq \frac{4}{j} - \int h_{s_j} dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n
$$

Also, if $\int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n = -\infty$, then $\lim_{j \to +\infty} \int hdd^c g^1_j \wedge dd^c g^2_j \wedge \ldots \wedge dd^c g^n_j = -\infty$. □

**Corollary 5.2.** — Suppose $u^p \in \mathcal{F}(\Omega)$, $1 \leq p \leq n$ and $h \in PSH^-(\Omega)$. If $g^p_j \in \mathcal{E}_0(\Omega)$ decreases to $u^p$, $j \to +\infty$ and

$$
\int -hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n < +\infty
$$

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then \( hdd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n \) tends weak* to \( hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n \), \( j \to +\infty \).

\textbf{Proof.} — Since \(-h\) is lower semicontinuous so is \(-h\chi\) for all \( 0 \leq \chi \in C^\infty_0(\Omega) \). Hence

\[
\liminf_{j \to +\infty} \int -h\chi dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n \geq \int -h\chi dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n
\]

but

\[
\lim_{j \to +\infty} \int hdd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^n = \int hdd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^n.
\]

\( \Box \)

The following lemma is proved in [4].

\textbf{Lemma 5.4.} — If \( \varphi \in PSH^- \cap L^\infty(\Omega) \) and \( \psi \in \mathcal{F} \) then

\[
\int (-\psi)^n (dd^c \varphi)^n \leq n!(\sup(-\varphi))^{n-1} \int -\varphi (dd^c \psi)^n.
\]

We are going to prove another type of inequality, where we do not need to control the sup-norm; Theorem 5.5 and Corollary 5.6.

\textbf{Lemma 5.4.} — Suppose \( u_1, u_2 \in \mathcal{E}_0, 1 \leq p, q < n \) and \( T = -hT_1 \) where \( h, g_1, \ldots, g_n, p-q \in \mathcal{E}_0 \) and where \( T_1 = dd^c g_1 \wedge \ldots \wedge dd^c g_n, p-q \). Then

\[
\int (dd^c u_1)^p \wedge (dd^c u_2)^q \wedge T \leq \left[ \int (dd^c u_1)^{p+q} \wedge T \right]^{p/p+q} \left[ \int (dd^c u_2)^{p+q} \wedge T \right]^{q/p+q}
\]

\textbf{Proof.} — 1. We first prove the statement in the lemma when \( p=q=1 \).

\[
\int dd^c u_1 \wedge dd^c u_2 \wedge T = \int -hdd^c u_1 \wedge dd^c u_2 \wedge T_1
\]

\[
= \int -u_1 dd^c u_2 \wedge dd^c h \wedge T_1
\]

\[
= \int du_1 \wedge dd^c u_2 \wedge dd^c h \wedge T_1
\]

\[
\leq \left[ \int du_1 \wedge dd^c h \wedge T_1 \right]^{1/2} \left[ \int du_2 \wedge dd^c u_2 \wedge dd^c h \wedge T_1 \right]^{1/2}
\]

\[
= \left[ \int -u_1 dd^c u_1 \wedge dd^c h \wedge T_1 \right]^{1/2} \left[ \int -u_2 dd^c u_2 \wedge dd^c h \wedge T_1 \right]^{1/2}
\]

\[
= \left[ \int -h(dd^c u_1)^2 \wedge T_1 \right]^{1/2} \left[ \int -h(dd^c u_2)^2 \wedge T_1 \right]^{1/2}
\]
2. Assuming the lemma is proved when $p + q \leq m$, we prove it for $p + q \leq m + 1$. However, we first prove:

$$
\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T \\
\leq \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q/p+q+1} \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1}.
$$

For

$$
\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T = \int (dd^c u_2)^{p+q-1} \wedge dd^c u_1 \wedge dd^c u_2 \wedge T \\
\leq \left[ \int (dd^c u_2)^{p+q-1} \wedge dd^c u_1 \wedge dd^c u_2 \wedge T \right]^{1/p+q} \\
= \left[ \int (dd^c u_2)^{p+q-1} \wedge dd^c u_1 \wedge (dd^c u_2) \wedge T \right]^{1/p+q} \\
\leq \left[ \int (dd^c u_2)^{p+q-1} \wedge dd^c u_1 \wedge (dd^c u_2) \wedge T \right]^{1/p+q} \\
\leq \left[ \int (dd^c u_2)^{p+q-1} \wedge T \right]^{p+q-1/p+q} \left[ \int (dd^c u_1)^{p+q} \wedge dd^c u_2 \wedge T \right]^{1/p+q} \\
\leq \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q-1/p+q} \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1} \\
\leq \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q-1/p+q} \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1}.
$$

Therefore,

$$
\int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T \\
\leq \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{p+q/p+q+1} \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{1/p+q+1}.
$$

Using this, we have

$$
\int (dd^c u_1)^{p+1} \wedge (dd^c u_2)^q \wedge T = \int (dd^c u_1)^{p} \wedge (dd^c u_2)^q \wedge dd^c u_1 \wedge T \\
\leq \left[ \int (dd^c u_1)^{p+q} \wedge dd^c u_1 \wedge T \right]^{q/p+q} \left[ \int (dd^c u_2)^{p+q} \wedge dd^c u_1 \wedge T \right]^{q/p+q} \\
\leq \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{p+q/p+q+1} \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{q/p+q/1+p+q+1} \\
\leq \left[ \int (dd^c u_1)^{p+q+1} \wedge T \right]^{p+1/p+q+1} \left[ \int (dd^c u_2)^{p+q+1} \wedge T \right]^{q/p+q+1}.
$$

and the lemma is proved.

\[ \square \]

**THEOREM 5.5.** — Suppose $u_1, \ldots, u_n \in \mathcal{F}$ and $h \in \mathcal{E}_0$. Then

$$
\int -hdd^c u_1 \wedge \ldots \wedge dd^c u_n \\
\leq \left( \int -h(dd^c u_1)^n \right)^{1/n} \ldots \left( \int -h(dd^c u_n)^n \right)^{1/n}.
$$
Proof. — Using the definition of $\mathcal{F}$ and Proposition 5.1, we see that it is enough to consider the case when $u_1, \ldots, u_n \in \mathcal{E}_0$. We can then use Lemma 5.4:

$$\int -hdd^cu_1 \wedge (dd^c u_2)^{n-1} \leq \left( \int -h(d\bar{d}^cu_1)^n \right)^{1/n} \left( \int -h(dd^c u_2)^n \right)^{n-1/n}$$

so the theorem is true if $u_2 = \ldots = u_n = u$.

Assume the theorem is true for $u_{p+1} = \ldots = u_n = u$ and suppose $u_{p+2} = \ldots u_n = u$.

Then

$$\int -hdd^cu_1 \wedge \ldots \wedge dd^cu_{p+1} \wedge (dd^c u)^{n-p-1}$$

$$\leq \left( \int -h(dd^c u_{p+1})^{n-p} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_p \right)^{1/n-p}$$

$$\left( \int -h(dd^c u)^{n-p} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_p \right)^{n-p-1/n-p}$$

$$\leq \left[ \left( \int -h(dd^c u_1)^n \right)^{1/n} \ldots \right.$$  

$$\left. \left( \int -h(dd^c u_p)^n \right)^{1/n} \left( \int -h(dd^c u_{p+1})^n \right)^{n-p/n} \right]^{1/n-p}$$

$$\left[ \left( \int -h(dd^c u_1)^n \right)^{1/n} \ldots \left( \int -h(dd^c u_p)^n \right)^{1/n} \right]$$

$$\left( \int -h(dd^c u)^{n-p/n} \right)^{n-p-1/n-p}$$

$$\leq \left( \int -h(dd^c u_1)^n \right)^{1/n} \ldots \left( \int -h(dd^c u_p)^n \right)^{1/n}$$

$$\left( \int -h(dd^c u_{p+1})^n \right)^{1/n} \left( \int -h(dd^c u)^n \right)^{n-p-1/n}.$$

The theorem is proved.

\[\Box\]

**Corollary 5.6.** — Suppose $u_1, \ldots, u_n \in \mathcal{F}$. Then

$$\int dd^c u_1 \wedge \ldots \wedge dd^c u_n \leq \left( \int (dd^c u_1)^n \right)^{1/n} \ldots \left( \int (dd^c u_n)^n \right)^{1/n}.$$

**Corollary 5.7.** — Suppose $u \in \mathcal{E} (\Omega)$ and $x \in \Omega$. Then

$$2\pi \nu_u(x) \leq \left((dd^c u)^n (\{x\})\right)^{1/n}$$

where $\nu_u(x)$ is the Lelong number of $u$ at $x$. 

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Proof. — We first assume that \( u \in \mathcal{F}(B) \), \( x = 0 \). Note that
\[
\lim_{s \to 0} \int_{B(0,s)} dd^c u \wedge (dd^c \log |z|)^{n-1} = (2\pi)^n \nu_u(0).
\]
so, for \( 1 \leq r \), using Theorem 5.5,
\[
(2\pi)^n \nu_u(x) \leq \int -\max(\log |z|/r, -1)(dd^c u)^n \leq \left[ \int -\max(\log |z|/r, -1)(dd^c \log |z|)^n \right]^{1/n} \left[ \int -\max(\log |z|/r, -1)(dd^c u)^n \right]^{(n-1)/n} = (2\pi)^{-n} \left( \int -\max(\log |z|/r, -1)(dd^c u)^n \right)^{1/n}.
\]
If we now let \( r \) tend to \(+\infty\), we get the desired conclusion. In the general case, we replace \( \log |z| \) by the pluricomplex Greenfunction with pole at \( x \).

Remark. — It follows from Corollary 5.7 that if \( u \in \mathcal{E} \), then \( \{\nu_u(x) > 0\} \) is discrete.

Theorem 5.8. — Suppose \( E \) is a pluripolar set in \( \Omega \). Then there is an \( h \in \mathcal{F}_1 \) such that \( E \subset \{h = -\infty\} \).

Proof. — Recall the definition of \( \mathcal{F}_p \) from [7]: \( u \in \mathcal{F}_p \) if \( \sup \int (-u_j)^p (dd^c u_j)^n \) is finite where \( u_j \) is as in Definition 4.6. Choose a sequence of relatively compact subsets \( \theta_j \) of \( \Omega \) such that every point of \( E \) is in all but finitely many \( \theta_j \) and \( \int (dd^c h_{\theta_j})^n < \frac{1}{j} \) where \( h_{\theta_j} \) denotes the relative extremal plurisubharmonic function. By Lemma 3.9 in [7], there is a subsequence \( \theta_{K_j} \) such that \( \sum h_{\theta_{K_j}} \in \mathcal{E}_1 \). By Corollary 5.6 we can select a subsequence of this subsequence, denoted by \( h_j \), such that \( \sum h_j \in \mathcal{F} \). Hence \( \sum h_j \in \mathcal{F}_1 \) and obviously \( \sum h_j = -\infty \) on \( E \).

Example 5.9. — The function \( \log |z_2| \) is not in \( \mathcal{E}(B) \) where \( B \) is the unit ball in \( \mathbb{C}^2 \). The classical energy of \( \varphi \in \mathcal{E}_0 \) equals
\[
16 \int_B -\varphi (dd^c \varphi) \wedge dd^c (1 - |z|^2) = 16 \int (1 - |z|^2) (dd^c \varphi)^2.
\]
Since the classical energy of \( \log |z_2| \) is locally unbounded, it follows from the remark after Definition 4.6 that \( \log |z_2| \) is not an element of \( \mathcal{E}(B) \). Using this idea, a computation, performed in [8], shows that \( -(\log |z_2|)^v \) is in \( \mathcal{E}(B) \) if and only if \( 0 < v < 1/2 \).
LEMMA 5.10. — Suppose $u \in \mathcal{E}$. Then
\[
\frac{(dd^c u)^n}{(1 - u)^{2n}} \leq \left( \frac{dd^c u}{1 - u} \right)^n.
\]

Proof. — We can assume that $u \in \mathcal{F}$. Choose $u_j \in \mathcal{E}_0 \cap C(\Omega)$ decreasing to $u_j \to \infty$. Then $u_j$ decreases to $u \in \mathcal{F} \cap L^\infty(\Omega)$ and a simple calculation shows that
\[
\frac{(dd^c u_j)^n}{(1 - u_j)^{2n}} \leq \left( \frac{dd^c u_j}{1 - u_j} \right)^n.
\]
Now, $(dd^c \frac{u_j}{1 - u_j})^n$ tends weakly to $(dd^c \frac{u_j}{1 - u_j})^n$ and to prove the lemma, it is enough to prove that $(dd^c u_j)^n$ tends weakly to $(dd^c u)^n$, $j \to +\infty$.

Since $\frac{1}{(1 - u)^{2n}} - 1 \in PSH^-(\Omega)$ we can use Corollary 5.2 and find, for every fixed $p$,
\[
\left( \frac{1}{(1 - u)^{2n}} - 1 \right)(dd^c u)^n = \lim_{j \to \infty} \left( \frac{1}{(1 - u_j)^{2n}} - 1 \right)(dd^c u_j)^n \leq \lim_{j \to \infty} \left( \frac{1}{(1 - u_j)^{2n}} - 1 \right)(dd^c u_j)^n = \left( \frac{1}{(1 - u_j)^{2n}} - 1 \right)(dd^c u)^n.
\]
Letting $p$ tend to $\infty$, we get the desired conclusion. \qed

THEOREM 5.11. — Assume that $\mu$ is a positive measure on $\Omega$. Then there is a $\psi \in \mathcal{E}_0$ and a function $0 \leq f \in L^1_{\text{loc}}((dd^c \psi)^n)$ such that
\[
\mu = f(dd^c \psi)^n + \nu
\]
where $\nu$ is carried by a pluripolar set.

Furthermore, if $\mu = (dd^c u)^n$ for a function $u \in \mathcal{F}$ then $\nu$ is carried by \{u = -\infty\}.

Proof. — Using the Radon-Nikodym theorem, the first part follows from Theorem 6.3 in [7]. By Lemma 5.10,
\[
\frac{(dd^c u)^n}{(1 - u)^{2n}} \leq \left( \frac{dd^c u}{1 - u} \right)^n.
\]
In particular, $(dd^c u)^n$ has no mass on pluripolar sets. So if
\[
(dd^c u)^n = f(dd^c \psi)^n + \nu
\]
we have that \( \frac{\mu}{(1-u)^n} \) equals zero so \( \nu \) is carried by \( \{ u = -\infty \} \). \( \square \)

The following theorem, usually referred to as the comparison principle, was proved in [3] and generalized to \( \mathcal{F}_p \) in [7].

**Theorem 5.12.** — If \( u, v \in \text{PSH} \cap L^\infty(\Omega) \), \( \lim_{z \to \zeta} (u(z) - v(z)) = 0 \), \( \forall \zeta \in \partial \Omega \) and \( (dd^c u)^n \leq (dd^c v)^n \) on \( \Omega \), then \( u \geq v \) on \( \Omega \).

**Definition 5.13.** — We denote by \( \mathcal{F}^\alpha \) the subclass of functions \( \varphi \in \mathcal{F} \) such that \( (dd^c \varphi)^n \) vanishes on all pluripolar sets.

**Lemma 5.14.** — Assume that \( \mu \) is a positive measure on \( \Omega \). If \( \mu(\Omega) < +\infty \) and if \( \mu \) vanishes on all pluripolar sets, then there exists a uniquely determined function \( \varphi \in \mathcal{F}^\alpha \) such that \( (dd^c \varphi)^n = \mu \).

**Proof.** — It follows from Theorem 5.11 that there is a \( \psi \in \mathcal{E}_0 \) and a function \( 0 \leq f \in L^1((dd^c \psi)^n) \) such that \( \mu = f(dd^c \psi)^n \). By [15] (see also [7]), there is a unique solution \( g^j \in \mathcal{E}_0 \) to \( (dd^c g^j)^n = \min(f, j)(dd^c \psi)^n \) and it follows from Theorem 5.12 that \( g^j \) is a decreasing sequence. We put \( g = \lim_{j \to +\infty} g^j \) and it follows from Lemma 5.3 that \( g \) is plurisubharmonic and therefore in \( \mathcal{F} \). We will now prove that \( g \) is uniquely determined; assume \( \varphi \in \mathcal{F}^\alpha \) with \( (dd^c \varphi)^n = \mu \), we will prove that \( \varphi = g \).

Let \( s_j \) be a sequence of natural numbers and \( K_j \subset \Omega \) a fundamental sequence of compacts of \( \Omega \) with \( h_{K_j} \) continuous such that

\[
\int [(1 + h_{K_j}) - \max(\varphi/s_j, -1)](dd^c \varphi)^n \leq 1/j
\]

which is possible by monotone convergence since \( (dd^c \varphi)^n \) has no mass on pluripolar sets. Furthermore, using Proposition 5.1, we know that

\[
\lim_{j \to +\infty} \int (dd^c \max(\varphi, s_j h_{K_j}))^n = \int (dd^c \varphi)^n.
\]

Let \( s \geq s_j \) and write \( d_j = -h_{K_j} + \max(\varphi/s_j, h_{K_j}) \). We have that \( 0 \leq d_j \leq \chi_{\{ \varphi > s_j h_{K_j} \}} \) and \( \chi_{\{ \varphi > s_j h_{K_j} \}}(dd^c \max(\varphi, sh_{K_j}))^n \) is independent of \( s \) by Lemma 5.4 in [7]. This means that

\[
0 \leq d_j(dd^c \max(\varphi, sh_{K_j}))^n \leq \chi_{\{ \varphi > s_j h_{K_j} \}}(dd^c \max(\varphi, sh_{K_j}))^n
\]

so letting \( s \to +\infty \), we find, using Corollary 5.2,

\[
d_j(dd^c \varphi)^n \leq \chi_{\{ \varphi > s_j h_{K_j} \}}(dd^c \max(\varphi, s_j h_{K_j}))^n \leq (dd^c \varphi)^n.
\]
Combining these inequalities, we get
\[ d_j \frac{\min(f, p)}{f} (dd^c \max(\varphi, s_j h_{K_j}))^n \leq \frac{\min(f, p)}{f} (dd^c \varphi)^n = (dd^c g^p)^n \]
\[ \leq \frac{\min(f, p)}{f} (dd^c \max(\varphi, s_j h_{K_j}))^n + (1 - d_j) \frac{\min(f, p)}{f} (dd^c \varphi)^n. \]

Solve for \( v_j^p \in E_0, (dd^c v_j^p)^n = (1 - d_j) \frac{\min(f, p)}{f} (dd^c \varphi)^n. \)

Then it follows from Theorem 5.12 that
\[ \max(\varphi, s_j h_{K_j}) + v_j^p \leq g^p \leq w_j^p \]
where
\[ w_j^p \in E_0, (dd^c w_j^p)^n = d_j \frac{\min(f, p)}{f} (dd^c \max(\varphi, s_j h_{K_j}))^n. \]

Put \( v_j = \lim_{p \to +\infty} v_j^p \). Since \( \int (dd^c v_j)^n \leq 1/j \), it follows that \( v_j \) tends to zero, \( j \to +\infty \) so it remains to prove that \( w_j = \lim_{p \to +\infty} w_j^p \) tends weakly to \( \varphi, j \to +\infty \).

Solve for \( t_j \in E_0, (dd^c t_j)^n = (1 - d_j) (dd^c (\max(\varphi, s_j h_{K_j}))^n. \)

Then \( w_j + t_j \leq \max(\varphi, s_j h_{K_j}) \leq w_j \) and
\[ \int (dd^c t_j)^n = \int (1 - d_j) (dd^c (\max(\varphi, s_j h_{K_j}))^n \]
\[ \leq \int (1 - d_j^2) (dd^c \varphi)^n \leq 2 \int (1 - d_j) (dd^c \varphi)^n \]
\[ = 2 \int (1 + h_{K_j} - \max(\varphi/s_j, h_{K_j})) (dd^c \varphi)^n \leq 2/j. \]

Hence \( t_j \) tends weakly to zero, \( j \to +\infty \) so \( w_j \) tends weakly to \( \varphi \) which completes the proof of Lemma 5.14. \( \square \)

**Theorem 5.15.** — If \( u \in F^\alpha \) and \( v \in E \) with \( (dd^c v)^n \geq (dd^c u)^n \) then \( v \leq u. \)

**Proof.** — Without loss of generality we can assume that \( v \in F \). We know that \( (dd^c u)^n = h (dd^c \psi)^n \) for some \( \psi \in E_0, 0 \leq h \in L^1((dd^c \psi)^n) \) and
\[ (dd^c v)^n = f (dd^c w)^n + \nu \]
for some \( w \in E_0, 0 \leq f \in L^1((dd^c w)^n) \) and where \( \nu \) is carried by a pluripolar set. Since \( (dd^c v)^n \geq (dd^c u)^n \), we can assume that \( \psi = w \) and thus \( h \leq f. \) By Lemma 5.14, it is enough to show that for the unique solution \( g \) to
\((dd^c g)^n = f(dd^c w)^n\) we have that \(v \leq g\). Let \(K\) be any non-empty compact subset of \(\Omega\). We have already used that \(-h_K + \max(v/s, h_K) \leq \chi_{\{v > sh_K\}}\) so using Corollary 5.2 and Lemma 5.4 in [7], we find

\[
(-h_K + \max(v/s, h_K))(dd^c v)^n = \lim_{j \to +\infty} (-h_K + \max(v/s, h_K))(dd^c \max(v, jh_K))^n \leq \lim_{j \to +\infty} \chi_{\{v > sh_K\}}(dd^c \max(v, jh_K))^n \leq \chi_{\{v > sh_K\}}(dd^c \max(v, sh_K))^n.
\]

Hence \((-h_K + \max(v/s, h_K))(dd^c v)^n \leq \chi_{\{v > sh_K\}}(dd^c \max(v, sh_K))^n\) so \((-h_K + \max(v/s, h_K))(dd^c v)^n\) has no mass on pluripolar sets (Which we already know from Theorem 5.11). Therefore, if \(g_{s,K}\) is the unique solution in \(E_0\) to

\[(dd^c g_{s,K})^n = (-h_K + \max(v/s, h_K))(dd^c v)^n\]

then \(g_{s,K} \geq \max(v, sh_K)\) by the comparison principle and

\[(dd^c g_{s,K})^n = (-h_K + \max(v/s, h_K))(dd^c v)^n = (-h_K + \max(v/s, h_K))(dd^c g)^n.
\]

Lemma 5.14 gives that \(g_{s,K}\) decreases to \(g\) when \(s\) tends to \(+\infty\) and \(K\) increases to \(\Omega\). Thus, \(v = \lim_{s \to +\infty} \max(v, sh_K) \leq g\).

**Remark.** — It was shown in [3] that \((dd^c)^n\) is continuous under increasing sequences in \(L^\infty_{\text{loc}}\) and it can be shown that \((dd^c)^n\) is also continuous under increasing sequences in \(\mathcal{F}\). In [20] it was shown that \((dd^c)^n\) is continuous under sequences in \(PSH \cap L^\infty_{\text{loc}}\), converging in capacity and in [9], this was generalized to sequences in \(\mathcal{F}\).

### 6. The Dirichlet problem in \(\mathcal{F}\).

The Dirichlet problem for \((dd^c)^n\) on \(PSH \cap L^\infty\) was studied in [2], [11] and [15] and on \(E_p\) in [7]. Here, we consider the Dirichlet problem in \(\mathcal{F}\).

**Lemma 6.1.** — Suppose \(\psi \in \mathcal{E}_0, v \in \mathcal{F}\) where \((dd^c v)^n\) is carried by a pluripolar set. Then there is a \(g \in \mathcal{F}\) with

\[(dd^c g)^n = (dd^c \psi)^n + (dd^c v)^n.
\]

**Proof.** — By assumption and Theorem 5.11 we can assume that \((dd^c v)^n\) is carried by \(\{v = -\infty\}\).
Choose \( v_j \in \mathcal{E}_0 \cap C(\bar{\Omega}), v_j \searrow v, j \to +\infty \). Choose an increasing sequence of compact sets \( K_j \subset \{ v = -\infty \} \) such that
\[
\frac{1}{2j} + \int_{K_j} (d\overline{d}v)^n > \int_{\Omega} (d\overline{d}v)^n
\]
and then \( t_j \) such that \( K_j \subset \{ v_{t_j} < -j^3 \} \). This is possible since \( K_j \) is compact and \( K_j \subset \{ v = -\infty \} \). Then
\[
\frac{1}{2j} + \int_{\{ v_{t_j} < -j^3 \}} (d\overline{d}v)^n > \int_{\Omega} (d\overline{d}v)^n
\]
and since \( \{ v_{t_j} < -j^3 \} \) is open,
\[
\frac{1}{j} + \int_{\{ v_{t_j} < -j^3 \}} (d\overline{d}v_{s_j})^n > \int_{\Omega} (d\overline{d}v)^n
\]
for \( s_j \) large enough. But \( v_{t_j} \) is decreasing so
\[
\frac{1}{j} + \int_{\{ v_{s_j} < -j^3 \}} (d\overline{d}v_{s_j})^n \geq \int_{\Omega} (d\overline{d}v)^n.
\]
Choose now \( \chi_j \in C_0^\infty(\{ v_{s_j} < -j^2 \}), 0 \leq \chi_j \leq 1 \) such that \( \chi_j \equiv 1 \) on \( \{ v_{s_j} < -j^3 \} \). To simplify notation, we write \( v_j \) for \( v_{s_j} \).

Solve
\[
(d\overline{d}\omega_j)^n = (1 - \chi_j)(d\overline{d}v_j)^n, \quad \omega_j \in \mathcal{E}_0
\]
\[
(d\overline{d}g_j)^n = (d\overline{d}\psi)^n + (d\overline{d}v_j)^n, \quad g_j \in \mathcal{E}_0.
\]

Define
\[
g^j(z) = \sup\{ \varphi(z), \varphi \in \mathcal{E}_0, (d\overline{d}\varphi)^n \geq (d\overline{d}\psi)^n, \varphi \leq v_j \}
\]
Then we define \( g \) by \( g^j \searrow g, j \to +\infty \). By the comparison principle we have \( g_j \leq g^j \) and we claim that
\[
g^j + \frac{v_j}{j} + \omega_j \leq g_j \quad \text{if} \quad -j < \inf_{\Omega} \psi.
\]
For
1. \( g^j + \frac{v_j}{j} + \omega_j = g_j = 0 \) on \( \partial \Omega \).
2. On \( \{ \frac{v_j}{j} \leq \psi \} \) we have
\[
g^j + \frac{v_j}{j} + \omega_j \leq v_j + \frac{v_j}{j} + \omega_j \leq v_j + \psi + \omega_j \leq g_j.
\]
On the open set \( \{ \frac{v_j}{j} > \psi \} \) we have for \( -j < \inf \psi \)
\[
\frac{v_j}{j} > \psi > -j \Rightarrow v_j > -j^2 \quad \text{so} \quad \chi_j \equiv 0
\]
on this set and
\[
(dd^c g_j)^n = (dd^c \psi)^n + (dd^c v_j)^n = (dd^c \psi)^n + (1 - \chi_j)(dd^c v_j)^n
\]
\[
= (dd^c \psi)^n + (dd^c \omega_j)^n \leq (dd^c (g_j + \frac{v_j}{j} + \omega_j))^n
\]
so \( g_j + \frac{v_j}{j} + \omega_j \leq g_j \) by the comparison principle.

Integration by parts now gives that
\[
\int \varphi(dd^c (g_j + \frac{v_j}{j} + \omega_j))^n \leq \int \varphi(dd^c g_j)^n \leq \int \varphi(dd^c g_j^n), \forall \varphi \in \mathcal{E}_0.
\]
Since \((dd^c g_j)^n \sim (dd^c g_j)^n\) and \((dd^c g_j)^n \sim (dd^c \psi)^n + (dd^c v)^n, j \to +\infty, we have proved that \((dd^c g_j)^n = (dd^c \psi)^n + (dd^c v)^n\) if we prove that
\[
\int \varphi(dd^c g_j)^{n-m} \wedge (dd^c \omega_j)^m \to 0, j \to +\infty, 1 \leq m \leq n.
\]
But this follows from Theorem 5.5 so the proof of the lemma is complete.

\[ \square \]

**Theorem 6.2.** Suppose \( \mu \) is a positive measure on \( \Omega \) with finite total mass. Then \( \mu = f(dd^c \psi)^n + \nu \) where \( \psi \in \mathcal{E}_0, 0 \leq f \in L^1(dd^c \psi)^n \) and \( \nu \) is carried by a pluripolar set. If there is a \( \nu \in \mathcal{F} \) with \((dd^c \nu)^n = \nu\) then there is a \( g \in \mathcal{F} \) with \((dd^c g)^n = \mu\).

**Proof.** It follows from Lemma 5.14 and Lemma 6.1 that for each \( j \) there is a \( g_j \in \mathcal{F} \) with \((dd^c g_j)^n = \min(f,j)(dd^c \psi)^n + \nu\). It follows from the proof of Lemma 6.1 that \( g_j \geq g_{j+1} \). Since \( \int_{\Omega}(dd^c g_j)^n \leq \mu(\Omega) < +\infty \) it follows that \( \lim_{j \to +\infty} g_j \) exists and is in \( \mathcal{F} \). This completes the proof of the theorem.

\[ \Box \]

**Remark.** The theorem above generalizes results in [20] and [21].
BIBLIOGRAPHY


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