ANNALES

DE

L'INSTITUT FOURIER

Imre PATYI

Analytic cohomology of complete intersections in a Banach space

Tome 54, nº 1 (2004), p. 147-158.

<http://aif.cedram.org/item?id=AIF_2004__54_1_147_0>

© Association des Annales de l'institut Fourier, 2004, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (http://aif.cedram.org/), implique l'accord avec les conditions générales d'utilisation (http://aif.cedram.org/legal/). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Ann. Inst. Fourier, Grenoble
54, 1 (2004), 147–158

ANALYTIC COHOMOLOGY OF COMPLETE INTERSECTIONS IN A BANACH SPACE

by Imre PATYI

1. Introduction.

Working towards building an analog in infinite dimensions of classical Stein theory, Lempert in [L3, L5] made it possible to prove vanishing theorems for sheaf cohomology over pseudoconvex open subsets of a large class of Banach spaces.

THEOREM 1.1 (Lempert, [L3, Thm. 0.1]). — *Let X be a Banach space with a countable unconditional basis, $\Omega \subset X$ pseudoconvex open, F a Fréchet space, $\mathcal{O}^F$ the sheaf of germs of holomorphic functions $X \to F$. Then $H^q(\Omega, \mathcal{O}^F) = 0$ for all $q \geqslant 1$.*

His methods turned out to be applicable in broader contexts as well; see [L1–L8; P1–P6]. In this paper as a first excursion beyond domains we study the analytic cohomology of smooth complete intersections (*i.e.*, the simplest complex Banach manifolds after domains), and prove Theorems 1.2, 1.3, and 1.4 below.

THEOREM 1.2. — *Let X, Ω be as in Theorem 1.1, $E \to \Omega$ a holomorphic Banach vector bundle, $f \in \mathcal{O}(\Omega, E)$ a holomorphic global section*

Keywords: Analytic cohomology – Complete intersections.
Math. classification: 32L30 – 32L10 – 46G20.

of E, $A = \{x \in \Omega : f(x) = 0\}$ an analytic subset of Ω, U open with $A \subset U \subset \Omega$. Then there is a pseudoconvex open ω such that $A \subset \omega \subset U$.

Theorem 1.2 says that analytic subsets of Ω that are defined by global equations have arbitrarily small pseudoconvex open neighborhoods. It applies, in particular, to any (possibly singular) hypersurface $A \subset \Omega$, since any hypersurface can be described as the zero locus of a holomorphic global section of a holomorphic line bundle on Ω.

THEOREM 1.3. — *Let X, Ω be as in Theorem 1.1, $\Omega = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_k = M$ complex Banach submanifolds, and assume that M_j is a hypersurface in M_{j-1} for $j = 1, \ldots, k$. Then $H^q(M, \mathcal{O}^L) = 0$ for $q \geqslant 1$ and any holomorphic line bundle $L \to M$.*

See Theorem 6.2 for a stronger result in a slightly different set-up.

THEOREM 1.4. — *Let X, Ω, M be as in Theorem 1.3, Z a Banach space, I^Z the sheaf of germs of holomorphic functions $\Omega \to Z$ that vanish on M. Then*

(a) *$H^q(\Omega, I^Z) = 0$ for all $q \geqslant 1$.*

(b) *Any $g \in \mathcal{O}(M, Z)$ extends to a $G \in \mathcal{O}(\Omega, Z)$.*

Theorem 1.2 is proved as a consequence of another contribution [L7] of Lempert on plurisubharmonic domination, while the proofs of Theorems 1.3 and 1.4 are by induction and rely on Theorems 1.1, 1.2, as well as [P2, Thm. 1.4].

Conventions. — For a sheaf $\mathcal{S}$ of Abelian groups over a paracompact Hausdorff space Ω denote by $H^{\geqslant 1}(\Omega, \mathcal{S})$ the direct product of the cohomology groups $H^q(\Omega, \mathcal{S})$ for $q \geqslant 1$. In this paper we mean by a *complex Banach manifold* a paracompact Hausdorff space with an atlas of biholomorphically related charts onto open sets in the model Banach space. (There is another notion of complex Banach manifold in [L1] in which one calls complex Banach manifold a C^∞-smooth Banach manifold endowed with a C^∞-smooth formally integrable almost complex structure. The two definitions agree in finite dimensions as the classical Newlander–Nirenberg theorem shows, while they disagree in certain Banach spaces; see [P1].) In the same vein, we mean by a *holomorphic Banach vector bundle* over a complex Banach manifold (the latter in the above sense) a locally trivial fiber bundle defined via holomorphic local trivializations, and a holomorphic gluing cocycle with values in $\mathrm{GL}(Z)$, where the Banach space Z is the fiber type of our Banach vector bundle. (Again there is a more general notion of holomorphic Banach vector bundle in [L1], but we stick to the simpler special case as above.)

2. Pseudoconvex neighborhood bases.

This section proves Theorem 1.2; resume its notation. Let $(Z, \|\cdot\|_Z)$ be the fiber type of E, E_x the fiber over $x \in \Omega$. We quote the following.

THEOREM 2.1 (Lempert, [L7, Thm. 1.1]). — *Let X, Ω be as in Theorem 1.1, $\varphi : \Omega \to \mathbb{R}$ locally upper bounded. Then there is a plurisubharmonic $\psi : \Omega \to \mathbb{R}$ such that $\varphi(x) < \psi(x)$ for all $x \in \Omega$. Moreover, ψ can be taken in the form $\psi(x) = \|h(x)\|$, where $h \in \mathcal{O}(\Omega, W)$ is holomorphic from Ω to a Banach space $(W, \|\cdot\|)$.*

THEOREM 2.2 (Lempert, [L7, Thm. 1.4]). — *Let X, Ω, E be as in Theorem 1.2. There is a continuous plurisubharmonic function $p : E \to [0, \infty)$ such that $p_x = p|E_x$ is a norm on the vector space E_x, and (E_x, p_x) is isomorphic to $(Z, \|\cdot\|_Z)$ as normed spaces for each $x \in \Omega$.*

For $c \in \mathbb{R}$, $g : \Omega \to [-\infty, \infty)$ denote $\{x \in \Omega : g(x) < c\}$ by $\{g < c\}$.

PROPOSITION 2.3.

(a) *Any p as in Theorem 2.2 is log plurisubharmonic.*

(b) *There is a continuous $u : \Omega \to [-1, 0]$ with $U = \{u < 0\}$.*

(c) *There is a locally upper bounded $\varphi : \Omega \to \mathbb{R}$ with*

$$\{\varphi + \log p \circ f < 0\} = U.$$

(d) *There is a continuous plurisubharmonic $\alpha : \Omega \to [-\infty, \infty)$ such that $\omega = \{\alpha < 0\}$ satisfies Theorem 1.2.*

Proof.

(a) This is because p is a homogeneous function. Let Δ_0 be an open disc in $\mathbb{C}$, $v \in \mathcal{O}(\Delta_0; E)$ a holomorphic map. We need to show that $w = p \circ v : \Delta_0 \to [0, \infty)$ is log subharmonic; w is clearly continuous. Let $\Delta \subset\subset \Delta_0$ be a disc, $q \in \mathcal{O}(\mathbb{C})$ a polynomial. We need to check that if $\log w \leqslant \mathrm{Re}(q)$ on the boundary $\partial\Delta$, then also on Δ, *i.e.*, if $|e^{-q}w| \leqslant 1$ on

$\partial\Delta$, then on Δ, too. As $|e^{-q}w| = p \circ (e^{-q}v)$ is subharmonic on Δ_0 part (a) is proved.

(b) Letting $u(x) = -\min\{1, \mathrm{dist}(x, \Omega \setminus U)\}$, $x \in \Omega$, will do.

(c) Let V be open with $A \subset V \subset \overline{V} \cap \Omega \subset U \cap \{\log p \circ f < 0\}$, and define φ by

$$\varphi(x) = \begin{cases} -1, & \text{if} \quad x \in V \\ u(x) - \log p_x(f(x)), & \text{if} \quad x \in \Omega \setminus V. \end{cases}$$

We check that this φ will do. Our φ is locally upper bounded: Suppose for a contradiction that $x_n \to x$ in Ω and $\varphi(x_n) \to +\infty$. Then $\log p_{x_n}(f(x_n)) \to -\infty$, and $x \in A$, and $x_n \in V$ for all n large enough, *i.e.*, $\varphi(x_n) = -1$; contradiction. Take an x with $\varphi(x) + \log p_x(f(x)) < 0$, we check that $x \in U$. Indeed, if $x \in V$, then $x \in V \subset U$. If $x \in \Omega \setminus V$, then $0 > \varphi(x) + \log p_x(f(x)) = u(x)$, *i.e.*, $x \in U$ by (b). Let x be in U, we show that $\varphi(x) + \log p_x(f(x)) < 0$. Indeed, if $x \in V$, then $\varphi(x) = -1$ and $\log p_x(f(x)) < 0$. If $x \in U \setminus V$, then $\varphi(x) + \log p_x(f(x)) = u(x) < 0$. Part (c) is proved.

(d) Lempert's theorem on plurisubharmonic domination (Theorem 2.1) gives a continuous plurisubharmonic $\psi : \Omega \to \mathbb{R}$ with $\psi > \varphi$. We check that $\alpha = \psi + \log p \circ f$ will do. Indeed, $\alpha : \Omega \to [-\infty, \infty)$ is continuous and plurisubharmonic by (a), and ω is then pseudoconvex open; $A \subset \omega$ since $\alpha = -\infty$ on A. To see that $\omega \subset U$, let $x \in \omega \setminus A$, *i.e.*, $0 > \alpha(x) = \psi(x) + \log p_x(f(x)) > \varphi(x) + \log p_x(f(x))$. Then $x \in U$ by (c). The proof of Proposition 2.3, and with it that of Theorem 1.2, is complete. □

3. Line bundles and Chern classes.

This section recalls briefly the classical relationship of holomorphic line bundles and their Chern classes.

PROPOSITION 3.1.

(a) *Let M be a complex Banach manifold and assume that $H^{\geqslant 1}(M, \mathcal{O}) = 0$. Then the group $H^1(M, \mathcal{O}^*)$ of isomorphism classes of holomorphic line bundles on M is isomorphic to $H^2(M, \mathbb{Z})$ via the Chern map.*

(b) *Let Ω be a paracompact Hausdorff space, $A \subset \Omega$ closed, $\mathbb{Z}$ the sheaf of locally constant $\mathbb{Z}$-valued functions. Then for each $n \in H^q(A, \mathbb{Z})$,*

$q \geqslant 0$, there are an open set W with $A \subset W \subset \Omega$, and an $\tilde{n} \in H^q(W,\mathbb{Z})$ with $\tilde{n}|A = n$.

(c) *Let X, Ω be as in Theorem 1.1, M a complex submanifold of Ω as in Theorem 1.2, $L \to M$ a holomorphic line bundle. Suppose that $H^{\geqslant 1}(M,\mathcal{O}) = 0$. Then there are an ω pseudoconvex open and a holomorphic line bundle $\Lambda \to \omega$ with $M \subset \omega \subset \Omega$ and $\Lambda|M$ holomorphically isomorphic to L.*

Proof.

(a) This is a classical consequence of the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$, where the second map is inclusion and the third is $f \mapsto e^{2\pi i f}$.

(b) Let n be represented by a cocycle of a locally finite open covering $\mathfrak{U}$ of A. Refining $\mathfrak{U}$ if necessary we find an open covering $\mathfrak{V}$ of A relative to which the components of n are globally constant $\mathbb{Z}$-valued functions. For each $V \in \mathfrak{V}$ choose an open $\tilde{V}$ in Ω with $\tilde{V} \cap A = V$. Define $\tilde{n}_{\tilde{V}_0\ldots\tilde{V}_q} = n_{V_0\ldots V_q}$ on $\tilde{V}_0 \cap \ldots \cap \tilde{V}_q$, and let $\tilde{\mathfrak{V}} = \{\tilde{V} : V \in \mathfrak{V}\}$, $G = \bigcup \tilde{\mathfrak{V}}$; $A \subset G$ and G is open in Ω. Then $\tilde{n}$ is a $\mathbb{Z}$-valued q-cochain on G indexed by $\tilde{\mathfrak{V}}$, and $\tilde{n}|A = n$. Let $\mathfrak{W}$ be an open covering of G which is a locally finite refinement of $\tilde{\mathfrak{V}}$ such that the components $(\delta\tilde{n})_{W_0\ldots W_{q+1}}$ of the coboundary $\delta\tilde{n}$ of $\tilde{n}$ are globally constant $\mathbb{Z}$-valued functions on an open neighborhood of $\overline{W_0 \cap \ldots \cap W_{q+1}}$. Let $F = \bigcup\{\overline{W_0 \cap \ldots \cap W_{q+1}} : W_0, \ldots, W_{q+1} \in \mathfrak{W}, \overline{W_0 \cap \ldots \cap W_{q+1}} \cap A = \emptyset\}$; obviously we have $F \cap A = \emptyset$. Being a union of a locally finite family of closed sets, F is closed in Ω. Let $W = G \setminus F$. Then W is open in Ω with $A \subset W$, and $(\delta\tilde{n})|W = 0$. To see the latter look at a component $(\delta\tilde{n})_{W_0\ldots W_{q+1}}$ at a point $x \in \overline{W_0 \cap \ldots \cap W_{q+1}} \cap W$. If $\overline{W_0 \cap \ldots \cap W_{q+1}} \cap A = \emptyset$, then $x \in F$, and $x \notin W$. So $\overline{W_0 \cap \ldots \cap W_{q+1}} \cap A \neq \emptyset$, thus $(\delta\tilde{n})_{W_0\ldots W_{q+1}} = 0$ since $(\delta\tilde{n})|A = \delta(\tilde{n}|A) = \delta n = 0$. As we have found an open neighborhood W of A in Ω such that $\tilde{n}$ is a cocycle on W, and $\tilde{n}|A$ represents n on A, the proof of (b) is complete.

(c) Let $n \in H^2(M,\mathbb{Z})$ be the Chern class of L. By (b) extend n to $\tilde{n} \in H^2(W,\mathbb{Z})$ where $M \subset W \subset \Omega$ is open. Theorem 1.2 gives a pseudoconvex open ω with $M \subset \omega \subset W$. Theorem 1.1 and (a) give a holomorphic line bundle $\Lambda \to \omega$ with Chern class $\tilde{n}$. Then $\Lambda|M$ is isomorphic to L by (a) since they have the same Chern class. The proof of Proposition 3.1 is complete. □

4. A lemma.

This section proves an auxiliary vanishing result, Proposition 4.2, that will be used in the proofs of Theorems 1.3 and 1.4.

Let Ω be a complex Banach manifold, $L \to \Omega$ a holomorphic line bundle with defining cocycle $(g_{UV}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$, Z a Banach space. Denote by $L \otimes Z$ the Banach vector bundle over Ω with fiber type Z and defining cocycle $(g_{UV} \operatorname{Id}_Z) \in Z^1(\mathfrak{U}, \mathcal{O}\mathrm{GL}(Z))$, where $\operatorname{Id}_Z : Z \to Z$ is the identity. For a complex Banach submanifold M of Ω denote by $I_M^{L\otimes Z}$ the sheaf of germs of sections of $\mathcal{O}^{L\otimes Z}$ over Ω that vanish on M.

PROPOSITION 4.1. — *Let X, Ω be as in Theorem 1.1, $L \to \Omega$ a holomorphic line bundle, Z a Banach space. Then*

(a) $H^{\geqslant 1}(\Omega, \mathcal{O}^L) = 0$, *and*

(b) $H^{\geqslant 1}(\Omega, \mathcal{O}^{L\otimes Z}) = 0$.

Proof. — Part (a) is [P2, Thm. 1.4], while (b) can be proved with a slight modification of that proof. They also follow from [P5, Thm. 1.2 (b)] for $X = c_0$ or $X = \ell_p$, $1 \leqslant p < \infty$, and from Theorem 6.1 in general.

Let X, Ω be as in Theorem 1.1, Z a Banach space, $M_0 \supset M_1 \supset \ldots \supset M_k$ submanifolds of $\Omega = M_0$. Suppose that the ideal sheaf $I_j^{\mathbb{C}}$ of M_j in M_{j-1} is an invertible sheaf isomorphic to $\mathcal{O}^{L_j}$, where $L_j \to M_{j-1}$ is a holomorphic line bundle, $j = 1, \ldots, k$.

Let $L \to M_{j-1}$ be a holomorphic line bundle, then the sheaf $I_j^{L\otimes Z}$ of germs of all holomorphic sections $M_{j-1} \to L \otimes Z$ that vanish on M_j is isomorphic to $\mathcal{O}^{L_j \otimes L \otimes Z}$, $j = 1, \ldots, k$.

PROPOSITION 4.2. — *Resume the above notation. For all $\Omega' \subset \Omega$ pseudoconvex open, and $j = 1, \ldots, k$ the following hold:*

(a) *There are a pseudoconvex open neighborhood $\omega_j \subset \Omega'$ of $M_j \cap \Omega'$, a holomorphic vector bundle $E_j \to \omega_j$ that is the sum of j holomorphic line bundles, and a holomorphic section $g_j \in \mathcal{O}(\omega_j, E_j)$ such that $M_j \cap \Omega' = \{g_j = 0\}$.*

(b) *For any open ω_0 with $M_j \cap \Omega' \subset \omega_0 \subset \Omega'$ there is a pseudoconvex open ω with $M_j \cap \Omega' \subset \omega \subset \omega_0$.*

(c) *The restriction map $\mathcal{O}(M_{j-1} \cap \Omega', L \otimes Z) \to \mathcal{O}(M_j \cap \Omega', L \otimes Z)$ is surjective for all holomorphic line bundles $L \to M_{j-1} \cap \Omega'$.*

(d) $H^{\geqslant 1}(M_j \cap \Omega', \mathcal{O}) = 0$.

(e) $H^{\geqslant 1}(M_j \cap \Omega', \mathcal{O}^{L \otimes Z}) = 0$ *for all holomorphic line bundles* $L \to M_j \cap \Omega'$.

Proof. — As (b) follows from (a) via Theorem 1.2, it remains to prove (a), (c), (d), and (e), which we do by induction on j. The proofs of (d) and (e) are essentially the same, but to prove (e) we require (d), too.

Let $j = 1$, and begin our induction proof.

(a) Let M_1 be defined in M_0 as the zero set $M_1 = \{h = 0\}$ of a section $h \in \mathcal{O}(M_0, L_1)$. Then letting $\omega_1 = \Omega'$, $E_1 = L_1$, $g_1 = h$, will do.

As Proposition 4.1 (b) shows that

$$0 = H^1(\Omega', \mathcal{O}^{L_1 \otimes L \otimes Z}) \cong H^1(\Omega', I_1^{L \otimes Z})$$

the surjectivity (c) of $\mathcal{O}(\Omega', L \otimes Z) \to \mathcal{O}(M_1 \cap \Omega', L \otimes Z)$ follows.

(d) We have an exact sequence $0 \to \mathcal{O}^{L_1} \to \mathcal{O} \to \mathcal{O}_{M_1} \to 0$ of analytic sheaves over Ω'. By the associated long exact sequence in cohomology we have $0 = H^q(\Omega', \mathcal{O}_{M_1}) \cong H^q(M_1 \cap \Omega', \mathcal{O})$ for $q \geqslant 1$ since $H^{\geqslant 1}(\Omega', \mathcal{O}) = H^{\geqslant 1}(\Omega', \mathcal{O}^{L_1}) = 0$ by Proposition 4.1.

(e) Part (d) and Proposition 3.1 (c) give a pseudoconvex open set ω, and a holomorphic line bundle $\Lambda \to \omega$ with $M_1 \cap \Omega' \subset \omega \subset \Omega'$, and $\Lambda | M_1 \cap \Omega'$ isomorphic to L. We may replace L by Λ and show that $H^{\geqslant 1}(M_1 \cap \Omega', \mathcal{O}^{\Lambda \otimes Z}) = 0$. We have an exact sequence $0 \to \mathcal{O}^{L_1 \otimes \Lambda \otimes Z} \to \mathcal{O}^{\Lambda \otimes Z} \to \mathcal{O}^{\Lambda \otimes Z}_{M_1 \cap \omega} \to 0$ of analytic sheaves over ω. By the associated long exact sequence in cohomology we have $0 = H^q(\omega, \mathcal{O}^{\Lambda \otimes Z}_{M_1 \cap \omega}) \cong H^q(M_1 \cap \Omega', \mathcal{O}^{L \otimes Z})$ for $q \geqslant 1$ since $H^{\geqslant 1}(\omega, \mathcal{O}^{L_1 \otimes \Lambda \otimes Z}) = H^{\geqslant 1}(\omega, \mathcal{O}^{\Lambda \otimes Z}) = 0$ by Proposition 4.1.

Induction step. Suppose that Proposition 4.2 is proved for $1, 2, \ldots, j-1$, and let us prove it for j.

(a) By (d) for $j-1$ Proposition 3.1 (c) gives a pseudoconvex open ω_j with $M_{j-1} \cap \Omega' \subset \omega_j \subset \omega_{j-1}$, and a holomorphic line bundle $\Lambda_j \to \omega_j$ such that $\Lambda_j = L_j$ on $M_{j-1} \cap \Omega'$. Let M_j be defined in M_{j-1} as the zero set $M_j = \{h = 0\}$ of a section $h \in \mathcal{O}(M_{j-1}, L_j)$. We see by (c) for $j-1, j-2, \ldots, 1$ that h extends to an $h \in \mathcal{O}(\omega_j, \Lambda_j)$. Then letting $E_j = E_{j-1} \oplus \Lambda_j$, $g_j = (g_{j-1}, h) \in \mathcal{O}(\omega_j, E_j)$ completes the proof of (a).

As part (e) of the case $j-1$ gives $0 = H^1(M_{j-1} \cap \Omega', \mathcal{O}^{L_j \otimes L \otimes Z}) \cong H^1(M_{j-1} \cap \Omega', I_j^{L \otimes Z})$ the surjectivity (c) of $\mathcal{O}(M_{j-1} \cap \Omega', L \otimes Z) \to \mathcal{O}(M_j \cap \Omega', L \otimes Z)$ follows.

(d) We have an exact sequence $0 \to \mathcal{O}^{L_j} \to \mathcal{O} \to \mathcal{O}_{M_j} \to 0$ of analytic sheaves over $M_{j-1} \cap \Omega'$. By the associated long exact sequence in cohomology we have $0 = H^q(M_{j-1} \cap \Omega', \mathcal{O}_{M_j}) \cong H^q(M_j \cap \Omega', \mathcal{O})$ for $q \geqslant 1$ since $H^{\geqslant 1}(M_{j-1} \cap \Omega', \mathcal{O}^{L_j}) = H^{\geqslant 1}(M_{j-1} \cap \Omega', \mathcal{O}) = 0$ by (e) of case $j-1$.

(e) Part (d) and Proposition 3.1 (c) give a pseudoconvex open set ω and a holomorphic line bundle $\Lambda \to \omega$ with $M_j \cap \Omega' \subset \omega \subset \Omega'$, and $\Lambda | M_j \cap \Omega'$ isomorphic to L. We may replace L by Λ and show that $H^{\geqslant 1}(M_j \cap \Omega', \mathcal{O}^{\Lambda \otimes Z}) = 0$. We have an exact sequence $0 \to \mathcal{O}^{L_j \otimes \Lambda \otimes Z} \to \mathcal{O}^{\Lambda \otimes Z} \to \mathcal{O}^{\Lambda \otimes Z}_{M_j \cap \omega} \to 0$ of analytic sheaves over $M_{j-1} \cap \omega$. By the associated long exact sequence in cohomology we have $0 = H^q(M_{j-1} \cap \omega, \mathcal{O}^{\Lambda \otimes Z}_{M_j \cap \omega}) \cong H^q(M_j \cap \Omega', \mathcal{O}^{\Lambda \otimes Z})$ for $q \geqslant 1$ since $H^{\geqslant 1}(M_{j-1} \cap \omega, \mathcal{O}^{L_j \otimes \Lambda \otimes Z}) = H^{\geqslant 1}(M_{j-1} \cap \omega, \mathcal{O}^{\Lambda \otimes Z}) = 0$ by (e) of case $j-1$. This completes the proof of the induction step and that of Proposition 4.2. □

5. The proofs of Theorems 1.3 and 1.4.

To complete the proofs we need the following proposition.

PROPOSITION 5.1. — *Let X, Ω, Z, M be as in Theorem 1.4, $L \to \Omega$ a holomorphic line bundle, $I_M^{L \otimes Z}$ as in §4. Then*

(a) $H^{\geqslant 1}(\Omega, I_M^{L \otimes Z}) = 0$.

(b) *Any section $g \in \mathcal{O}(M, L \otimes Z)$ extends to a section $G \in \mathcal{O}(\Omega, L \otimes Z)$.*

Proof. — Proposition 4.2 with $\Omega' = \Omega$ implies that

$$H^q(M_{j-1}, I_j^{L \otimes Z}) \cong H^q(M_{j-1}, \mathcal{O}^{L_j \otimes L \otimes Z}) = 0$$

for $q \geqslant 1$, $j = 1, \ldots, k$. The case $q = 1$, $j = k, \ldots, 1$ of the above proves (b). Looking at the short exact sequence $0 \to I_M^{L \otimes Z} \to \mathcal{O}^{L \otimes Z} \to \mathcal{O}_M^{L \otimes Z} \to 0$ of sheaves over Ω, and noting that $H^q(\Omega, \mathcal{O}^{L \otimes Z}) = H^q(M, \mathcal{O}^{L \otimes Z}) = 0$ for $q \geqslant 1$ (by Proposition 4.2) together with (b) imply (a) and complete the proof of Proposition 5.1, and that of Theorem 1.4. □

Proof of Theorem 1.3. — Letting $\Omega' = \Omega$, $j = k$ in Proposition 4.2(e) completes the proof of Theorem 1.3. □

6. Recent developments.

Several months later than this paper was first written some new developments took place. Using his plurisubharmonic domination in [L7] Lempert found a much more flexible way of exhausting domains, and proved the following.

THEOREM 6.1 (Lempert, [L8, Thm. 1.1]). — *Let X be a Banach space with a countable unconditional basis, $\Omega \subset X$ pseudoconvex open, $E \to \Omega$ a holomorphic Banach vector bundle. Then $H^q(\Omega, \mathcal{O}^E) = 0$ for all $q \geqslant 1$.*

This Theorem 6.1 has the following implication in our topic of complete intersections in domains.

THEOREM 6.2. — *Let X, Ω be as in Theorem 6.1, Z a Banach space, $f \in \mathcal{O}(\Omega, Z)$ not identically zero on any open subset of Ω, $M = \{f = 0\} \subset \Omega$ the zero set of f. Assume that the Fréchet differential $df(x) \in \mathrm{Hom}(X, Z)$ is an epimorphism with split kernel for all $x \in M$. Then*

(a) *There is a holomorphic retraction $r : \omega \to M$ of a pseudoconvex open ω with $M \subset \omega \subset \Omega$ onto M.*

(b) *$H^q(M, \mathcal{O}^E) = 0$ for all $q \geqslant 1$ and any holomorphic Banach vector bundle $E \to M$.*

Note that Lempert also applies his Theorem 6.1 to prove a holomorphic retraction theorem [L8, Thm. 1.4] in a different context.

Proof of Theorem 6.2.

(a) As the condition $df(x)$ be surjective with a split kernel is an open condition by the inverse function theorem in Banach spaces, Theorem 1.2 gives a pseudoconvex open ω_1 with $M \subset \omega_1 \subset \Omega$ such that $df(x)$ is surjective with split kernel for $x \in \omega_1$. Thus we have a locally split exact sequence

$$0 \to K \to \omega_1 \times X \to \omega_1 \times Z \to 0 \tag{6.1}$$

of Banach vector bundles over ω_1, where the second map is inclusion, and the third is $(x, \xi) \mapsto (x, df(x)\xi)$. We also have another locally split exact sequence

$$0 \to TM \to M \times X \to NM \to 0 \tag{6.2}$$

of Banach vector bundles over M, where TM is the holomorphic tangent bundle of M regarded as a subbundle of the trivial bundle $M \times X$, and NM is the holomorphic normal bundle of M in Ω. We claim that (6.1) restricted to M is an exact sequence isomorphic to (6.2). To see this we only need to check that $K_x = \operatorname{Ker} df(x)$, $x \in M$, is the same subspace of X as T_xM. Clearly, $T_xM \subset K_x$ for $x \in M$, and since after suitable local biholomorphisms near any point $x_0 \in \omega_1$ our f (and df) can be represented by a linear epimorphism $X \to Z$, the condition $\{f = 0\} = M$ implies $T_xM = K_x$ for $x \in M$ as well.

We now show that (6.2) splits holomorphically over M. To this end it is enough to show that (6.1) splits holomorphically over ω_1. The latter follows from the splitting criterion $H^1(\omega_1, \operatorname{Hom}(\omega_1 \times Z, K)) = 0$, which in turn is a special case of Lempert's Theorem 6.1.

Following Docquier–Grauert in [DG] a holomorphic splitting of (6.2) yields in a standard way also in our infinite dimensional setting an open ω_2 with $M \subset \omega_2 \subset \omega_1$, and a holomorphic retraction $r : \omega_2 \to M$ onto M; see the proof of [L8, Thm. 1.4]. As Theorem 1.2 gives a pseudoconvex open ω with $M \subset \omega \subset \omega_2$, the proof of (a) is complete.

(b) Let $r : \omega \to M$ be a holomorphic retraction as in (a), r^*E the pullback of E, $i : M \to \omega$ the inclusion; $r \circ i = \mathrm{Id}$ on M. Since

$$H^q(M,E) \xrightarrow{r^*} H^q(\omega, r^*E) \xrightarrow{i^*} H^q(M,E)$$

is the identity map $i^*r^* = (r \circ i)^* = \mathrm{Id}^* = \mathrm{Id}$, and the middle group is zero by Theorem 6.1, the proof of Theorem 6.2 is complete. □

PROPOSITION 6.3. — *Let X, Ω, M be as in Theorem 6.2, $E \to \Omega$ a holomorphic Banach vector bundle, I the sheaf of germs of holomorphic sections of $E \to \Omega$ that vanish on M. Then*

(a) *$H^q(\Omega, I) = 0$ for $q \geqslant 2$.*

(b) *If $E \to \Omega$ is holomorphically trivial, then there is a pseudoconvex open ω with $M \subset \omega \subset \Omega$, and $H^q(\omega, I) = 0$ for $q \geqslant 1$.*

Proof.

(a) Looking at the long cohomology sequence induced by the short exact sequence $0 \to I \to \mathcal{O}^E \to \mathcal{O}^E_M \to 0$ of sheaves over Ω, we see that $H^q(\Omega, I) \cong H^{q-1}(\Omega, \mathcal{O}^E_M)$ for $q \geqslant 2$ since $H^{\geqslant 1}(\Omega, \mathcal{O}^E) = 0$ by Theorem 6.1. As $H^q(\Omega, \mathcal{O}^E_M) \cong H^q(M, \mathcal{O}^E)$, and the latter is zero by Theorem 6.2 (b), part (a) is proved.

(b) Let ω be as in Theorem 6.2 (a), then the restriction map $\mathcal{O}(\Omega, E) \to \mathcal{O}(M, E)$ is surjective. Arguing as in (a) completes the proof of Proposition 6.3. □

Note that there is a hope that the proof of Proposition 6.3 (a) will be achieved soon also for the most important value $q = 1$.

We have seen that some vanishing theorems can be proved for the simplest complex Banach manifolds. As it is currently unknown, it would be important to find out which class of complex Banach manifolds could play a role similar to that of the Stein manifolds in finite dimensions. The methods of this paper may be relevant for treating some kind of singular complete intersections, too.

Acknowledgement. — The author is grateful to the Mathematics Department of the University of California at Riverside for their hospitality and excellent working conditions, and to the referee for numerous improvements to the exposition.

BIBLIOGRAPHY

[DG] F. Docquier, H. Grauert, Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann., 140 (1960), 94–123.

[L1] L. Lempert, The Dolbeault complex in infinite dimensions I, J. Amer. Math. Soc., 11 (1998), 485–520.

[L2] L. Lempert, The Dolbeault complex in infinite dimensions II, J. Amer. Math. Soc., 12 (1999), 775–793.

[L3] L. Lempert, The Dolbeault complex in infinite dimensions III, Invent. Math., 142 (2000), 579–603.

[L4] L. Lempert, Approximation de fonctions holomorphes d'un nombre infini de variables, Ann. Inst. Fourier (Grenoble), 49-4 (1999), 1293–1304.

[L5] L. Lempert, Approximation of holomorphic functions of infinitely many variables II, Ann. Inst. Fourier (Grenoble), 50-2 (2000), 423–442.

[L6] L. Lempert, Analytic cohomology in Fréchet spaces, Communications in Analysis and Geometry, to appear.

[L7] L. Lempert, Plurisubharmonic domination, J. Amer. Math. Soc., to appear.

[L8] L. Lempert, Vanishing cohomology for holomorphic vector bundles in a Banach setting, Asian J. Math., to appear.

[P1] I. Patyi, On the $\bar\partial$-equation in a Banach space, Bull. Soc. Math. France, 128 (2000), 391–406.

[P2] I. Patyi, Analytic cohomology vanishing in infinite dimensions, Ph. D. Thesis, Purdue University (2000).

[P3] I. PATYI, On a splitting problem, Bull. Sci. Math., 126 (2002), 631–636.

[P4] I. PATYI, On the Oka principle in a Banach space I, Math. Ann., 326 (2003), 417–441.

[P5] I. PATYI, On the Oka principle in a Banach space II, Math. Ann., 326 (2003), 443–458.

[P6] I. PATYI, Cohomological characterization of pseudoconvexity in a Banach space, Math. Z., 245 (2003), 371–386.

Manuscrit reçu le 9 avril 2003,
révisé le 8 juillet 2003,
accepté le 8 septembre 2003.

Imre PATYI,
Department of Mathematics
University of California at Riverside
Riverside, CA 92521-0135 (USA).
imrepatyi@member.ams.org