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On projective tori varieties whose defining ideals have minimal generators of the highest degree


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ON PROJECTIVE TORIC VARIETIES
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Introduction.

Sturmfels asked in [S2] whether a nonsingular projective toric variety should be defined by only quadrics if it is embedded by global sections of an ample line bundle. An evidence has been obtained by Koelman [K3] before Sturmfels asked the question. Koelman showed that projective toric surfaces are defined by binomials (differences of two monomials) of degree at most three ([K1] and [K2]) and obtained a criterion when the surface needs defining equations of degree three ([K3]). He used combinatorics of plane polygons.

Sturmfels showed in [S1] that for projectively normal toric varieties of dimension $n$, the defining ideals have minimal generators consisting of elements of degree at most $n + 1$ (Theorem 13.14 in [S1]). There are examples showing that this bound is optimal. In this paper we give a generalization of [K3] to higher dimensions, that is, we give a criterion for the ideals defining projectively normal toric varieties of dimension $n$ to be generated by elements of degree less than $n + 1$. Bruns, Gubeladze and Trung [BGT] also give a generalization of the results of [K3].

Keywords: Toric varieties – Convex polytopes – Generators of ideals.
A toric variety is a normal algebraic variety with an algebraic action of an algebraic torus of the same dimension of the variety and a dense orbit. Let $X$ be a projective toric variety of dimension $n$ and $T \cong (\mathbb{C}^*)^n$ the algebraic torus acting on $X$. Let $M = \text{Hom}_{gr}(T, \mathbb{C}^*)$ be the group of characters, which is isomorphic to $\mathbb{Z}^n$. For $m \in M$, we denote $e(m)$ the corresponding character of $T$. Let $L$ be an ample line bundle on $X$. Then there exist an integral convex polytope $P$ in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and an isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} C e(m),$$

where an integral convex polytope is the convex hull of a finite number of elements of $M$. Let $R(X, L) := \bigoplus_{l \geq 0} H^0(X, L^\otimes_l)$ be the homogeneous coordinate ring of $X$. Then we have an isomorphism

$$R(X, L) \cong \bigoplus_{l \geq 0} \left( \bigoplus_{m \in (lP \cap M)} C e(m) \right).$$

This is a normal polytopal semigroup ring in the sense of [BGT]. If $L$ is normally generated in the sense of Mumford [M], that is, $L$ satisfies the conditions that it is very ample and that the image of $X$ in $\mathbb{P}(H^0(X, L)^*)$ is projectively normal, then $R = R(X, L)$ is generated by its degree one elements. In this case, $R$ is a quotient ring of the polynomial ring $S = \text{Sym} H^0(X, L)$. Let $I$ be the ideal of $S$ with $R \cong S/I$. We call $I$ the defining ideal of $(X, L)$, or of the polytopal semigroup ring of $P$.

In general an ample line bundle $L$ on a projective toric variety of dimension $n$ is not very ample for $n > 2$. On the other hand, $L^\otimes i$ is normally generated for $i \geq n - 1$ ([EW]), and the defining ideal of $(X, L^\otimes i)$ is generated by quadrics for $i \geq n$ ([BGT], [NO]), or for $i = n - 1$ and $n \geq 3$ ([Og]). The normal generation of $L$ is equivalent to the condition for the corresponding integral convex polytope $P$ that for all positive integers $l$, each element $x$ in $(lP) \cap M$ can be expressed as a sum $x = m_1 + \cdots + m_l$ of $l$ elements of $P \cap M$. We call $P$ is normally generated if $P$ satisfies this condition. When $n = 2$, all ample line bundles on projective toric surfaces are normally generated. This is one of difficulties arising in generalization of Koelman’s result [K3] to higher dimensions by using combinatorics of polytopes.

We employ a method of algebraic geometry. Specifically, we consider the case of curves which are complete intersections of hyperplane sections and use regular ladders of Fujita [Fj].

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THEOREM 1. — Let $P$ be an integral convex polytope of dimension $n \geq 2$. Assume that $P$ is normally generated. Then the defining ideal of the polytopal semigroup ring of $P$ has generators of degree $n + 1$ if and only if $P$ is an $n$-simplex with standard facets and containing lattice points in its interior.

We may restate Theorem 1 in terms of algebraic geometry. It is convenient for the readers because we shall prove a part of Theorem by using algebraic geometry.

THEOREM 1'. — Let $X$ be a projective toric variety of dimension $n \geq 2$ and let $L$ a very ample line bundle on $X$ which defines an embedding of $X$ as a projectively normal variety. Let $P$ be the integral convex polytope of dimension $n$ determined by the global sections of $L$. The defining ideal of $X$ needs elements of degree $n + 1$ as generators if and only if $P$ is an $n$-simplex with standard facets and containing lattice points in its interior.

One half of Theorem is given by Proposition 1.3, which says that if $P$ has only $n + 1$ lattice points in the boundary and if it contains at least one lattice point in the interior then the defining ideal needs elements of degree $n + 1$ as generators. We can easily see that if $P$ contains only $n + 1$ lattice points then $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$. Thus another half of Theorem is that if $P$ contains more than $n + 1$ lattice points in the boundary then the defining ideal has generators of degree at most $n$, which is given by Theorem 2.6.

We know that if $X$ is nonsingular, then $P$ is simplicial and for each vertex $v_0$ there are $n$ edges $\mathbb{R}_{\geq 0}v_i$ ($i = 1, \ldots, n$) meeting at $v_0$ such that $\{v_1-v_0, \ldots, v_n-v_0\}$ is a basis of the lattice $\mathbb{Z}^n$. If, in addition, the boundary of $P$ contains only $n + 1$ lattice points, then $P$ contains no lattice point in its interior, that is, $P$ is a standard $n$-simplex. Hence it does not satisfy the condition of Theorem. Thus we have a weak answer to Sturmfels' question.

COROLLARY 1. — For a nonsingular projectively normal toric variety of dimension $n \geq 2$, its defining ideal embedded by global sections of an ample line bundle has generators of degree at most $n$.

Next consider the case that $P$ is an integral $n$-simplex, that is, $P = \text{Conv}\{u_0, u_1, \ldots, u_n\}$ with $u_0, u_1, \ldots, u_n \in \mathbb{Z}^n =: M$. Let $M'$ be the sublattice of $M$ generated by $u_1 - u_0, \ldots, u_n - u_0$. Then $P$ is a standard $n$-simplex with respect to $M'$. Hence $(P, M')$ defines the projective $n$-space $(\mathbb{P}^n, \mathcal{O}(1))$. From this consideration we see that the toric variety $X$ defined...
by $P$ is a quotient of the projective $n$-space by a finite abelian group $M/M'$. A weighted projective space $\mathbb{P}(q_0, q_1, \ldots, q_n)$ has the same form $\mathbb{P}^n/((\mathbb{Z}/q_0) \times \cdots \times (\mathbb{Z}/q_n))$. If all facets of $P$ are standard $(n-1)$-simplices, then all $n$ elements of $\{q_0, q_1, \ldots, q_n\}$ coincide, hence $\mathbb{P}(q_0, q_1, \ldots, q_n) \cong \mathbb{P}^n$. Thus it does not satisfy the condition of Theorem.

**COROLLARY 2.** — The defining ideals of projectively normal weighted projective $n$-spaces have generators of degree at most $n$.

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1. Polarized toric varieties.

First we mention the facts about toric varieties needed in this paper following Oda’s book [Od], or Fulton’s book [Fl].

Let $N$ be a free $\mathbb{Z}$-module of rank $n$, $M$ its dual and $<, >: M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic $n$-torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \text{Hom}_{\mathbb{gr}}(T_N, \mathbb{C}^*)$ is the character group of $T_N$. For $m \in M$ we denote $e(m)$ the corresponding character of $T_N$. Let $\Delta$ be a complete finite fan of $N$ consisting of strongly convex rational polyhedral cones $\sigma$, that is, there exist a finite number of elements $v_1, v_2, \ldots, v_s$ in $N$ such that

$$\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N \text{emb}(\Delta) := \cup_{\sigma \in \Delta} U_\sigma$ of dimension $n$ (see Section 1.2 [Od], or Section 1.4 [Fl]). Here $U_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of $\sigma$ with respect to the pairing $<, >$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \text{Spec} \mathbb{C}[M]$ is the unique dense $T_N$-orbit. We note that a toric variety is always normal.

Let $L$ be an ample $T_N$-equivariant invertible sheaf on $X$. Then the polarized variety $(X, L)$ corresponds to an integral convex polytope. We call the convex hull in $M_{\mathbb{R}}$ of a finite subset $\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbb{R}}$ of a finite subset.
{u_0,u_1,\ldots,u_r} \subset M an integral convex polytope in M_\mathbb{R}. The correspondence is given by the isomorphism

\[ H^0(X,L) \cong \bigoplus_{m \in P \cap M} \mathbb{C} e(m), \]

where \( e(m) \) are considered as rational functions on \( X \) because they are functions on the open dense subset \( T_N \) of \( X \) (see Section 2.2 [Od], or Section 3.5 [Fl]).

Let \( P_1 \) and \( P_2 \) be integral convex polytopes in \( M_\mathbb{R} \). Then we can consider the Minkowski sum \( P_1 + P_2 := \{x_1 + x_2 \in M_\mathbb{R}; x_i \in P_i \ (i = 1, 2)\} \) and the multiplication by scalars \( rP_1 := \{rx \in M_\mathbb{R}; x \in P_1 \} \) for a positive real number \( r \). If \( l \) is a natural number, then \( lP_1 \) coincides with the \( l \) times sum of \( P_1 \), i.e., \( lP_1 = \{x_1 + \cdots + x_l \in M_\mathbb{R}; x_1, \ldots, x_l \in P_1 \} \). The \( l \)-th tensor power \( L^\otimes l \) corresponds to the convex polytope \( lP := \{lx \in M_\mathbb{R}; x \in P \} \). Moreover the multiplication map

\[ H^0(X,L^\otimes l) \otimes H^0(X,L) \to H^0(X,L^\otimes (l+1)) \]

transforms \( e(u_1) \otimes e(u_2) \) for \( u_1 \in lP \cap M \) and \( u_2 \in P \cap M \) to \( e(u_1 + u_2) \) through the isomorphism (1.1). Therefore the equality \((lP \cap M) + (P \cap M) = (l + 1)P \cap M\) is equivalent to the surjectivity of (1.2).

In this article we assume that \( L \) is normally generated, that is, the multiplication map (1.2) is surjective for all \( l \geq 1 \), hence, it is very ample. In terms of polytopes, the normal generation of \( L \) means that the equality

\[ (lP \cap M) + (P \cap M) = (l + 1)P \cap M \]

holds for all positive integers \( l \). It is also equivalent to the condition that for all \( l \geq 1 \), and for any \( v \in lP \cap M \), there exist \( l \) elements \( u_1, \ldots, u_l \) of \( P \cap M \) with \( v = u_1 + \cdots + u_l \). From this reason we may call \( P \) to be normally generated if it satisfies (1.3) for all positive integers \( l \).

Let \( P \cap M = \{u_0, u_1, \ldots, u_r\} \). By the assumptions we have the embedding by global sections of \( L \);

\[ \Phi : X \to \mathbb{P}(H^0(X,L)^*) \cong \mathbb{P}^r. \]

Let \( Z_0, Z_1, \ldots, Z_r \) be the homogeneous coordinates of \( \mathbb{P}^r \). Then \( \Phi \) is defined by \( Z_i = e(u_i) \) for \( i = 0, 1, \ldots, r \). Set \( R := \bigoplus_{l \geq 0} R_l = \bigoplus_{l \geq 0} H^0(X,L^\otimes l) \) and \( S := \bigoplus_{l \geq 0} S_l = \mathbb{C}[Z_0, Z_1, \ldots, Z_r] \). Then we define a surjective ring homomorphism \( \varphi : S \to R \) by \( \varphi(\prod_i Z_i^{a_i}) = e(\sum_i a_i u_i) \). Let \( I \) be the kernel of \( \varphi \). Then we see that \( I_0 = I_1 = \{0\} \) for the graded ideal \( I = \bigoplus_{l \geq 0} I_l \). We call \( I \) the defining ideal of \( X \) in \( \mathbb{P}(H^0(X,L)^*) \).
LEMMA 1.1 (Eisenbud-Sturmfels [ES]). — The defining ideal $I$ is generated by binomials, that is, the differences of two monomials.

For a proof see Proposition 2.3 in [ES].

PROPOSITION 1.2 (Sturmfels [Sl]). — Let $L$ be a normally generated ample line bundle on a projective toric variety $X$ of dimension $n$. Then every minimal generator of the ideal defining $X$ in $\mathbb{P}(H^0(X, L)^*)$ has degree at most $n + 1$.

For a proof see Theorem 13.14 in [Sl].

PROPOSITION 1.3. — Let $P = \text{Conv}\{u_0, u_1, \ldots, u_n\}$ be an integral $n$-simplex such that the equality (1.3) holds for all positive integers $l$. We assume that the boundary of $P$ contains only $n + 1$ lattice points, and that $P$ contains at least one lattice point in its interior. Then the defining ideal $I$ needs an element of degree $n + 1$ as a generator.

Proof. — By a suitable affine translation of $M$ we may assume $u_0 = 0$. Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of $M$. The very ampleness of $L$ says that the set of all lattice points in the cone $\sigma^V = \mathbb{R}_{\geq 0}u_1 + \cdots + \mathbb{R}_{\geq 0}u_n$ is generated by $P \cap M$ as a semigroup. In other words, every lattice point in $\sigma^V \cap M$ can be written as a sum of elements in $P \cap M$ with positive integer coefficients. Since the lattice points of the face cone $\tau_n^V := \mathbb{R}_{\geq 0}u_1 + \cdots + \mathbb{R}_{\geq 0}u_{n-1}$ of $\sigma^V$ are also generated by $\text{Conv}\{u_0, u_1, \ldots, u_{n-1}\} \cap M = \{u_0, u_1, \ldots, u_{n-1}\}$ as a semigroup, we may set $u_1 = e_1, \ldots, u_{n-1} = e_{n-1}$. This shows that every facet of $P$ is a standard $(n - 1)$-simplex. Set $u_n = \sum_{i=1}^{n} a_i e_i$ with integer coefficients. By a change of bases we may set all $a_i \geq 0$. Since $\dim P = n$, we have $a_n > 0$. Moreover we may assume that $u_{n+1} := \sum_{i=1}^{n} e_i$ is in the interior of $P$. Then we have

\begin{align*}
(1.4) \quad a_i &< a_n \quad \text{for } i = 1, \ldots, n - 1, \\
(1.5) \quad (n - 2)a_n &< a_1 + \cdots + a_{n-1} - 1.
\end{align*}

By componentwise description with respect to the basis of $M$, we have

\begin{align*}
u_1 + \cdots + u_n &= (a_1 + 1, \ldots, a_{n-1} + 1, a_n) \\
&= u_{n+1} + (a_1, \ldots, a_{n-1}, a_n - 1).
\end{align*}

Since $(a_1, \ldots, a_{n-1}, a_n - 1)$ is contained in $nP$ from (1.4) and (1.5), there exist $v_{n+2}, \ldots, v_{2n+1}$ in $P \cap M$ such that

\begin{equation}
(1.6) \quad (a_1, \ldots, a_{n-1}, a_n - 1) = v_{n+2} + \cdots + v_{2n+1}.
\end{equation}
Corresponding to the relation \( u_0 + u_1 + \cdots + u_n = v_{n+2} + \cdots + v_{2n+1} \), we obtain a binomial \( B := Z_0 Z_1 \cdots Z_{n+1} - Y_{n+2} \cdots Y_{2n+1} \), where \( Y_j = e(v_j) \in \{ Z_0, \ldots, Z_r \} \). Since \((a_1, \ldots, a_{n-1}, a_n - 1)\) is not contained in \((n - 1)P\) from (1.5), none of \( v_{n+2}, \ldots, v_{2n+1} \) coincides with \( u_0 \). If we assume \( Y_{n+2} = Z_1 \), that is, \( v_{n+2} = u_1 \), then from (1.4) we have \((a_1 - 1, a_2, \ldots, a_{n-1}, a_n - 1) \notin (n - 1)P\), which contradicts (1.6). Hence we see that the binomial \( B \) is irreducible.

Next we assume \( B = X_1 B_1 + \cdots + X_s B_s \) with binomials \( B_i \in I_n \) of degree \( n \) and \( X_i \in \{ Z_0, \ldots, Z_r \} \). If we write binomials \( B_i \) as the difference of two monomials \( B_i = M_1^{(i)} - M_2^{(i)} \), then we have \( X_1 M_1^{(1)} = Y_{n+1} \cdots Y_{2n+1} \) and \( X_1 M_2^{(1)} = X_2 M_1^{(2)} - \cdots - X_s M_2^{(s)} = Z_0 Z_1 \cdots Z_n \). We note that for a binomial \( B_i = M_1^{(i)} - M_2^{(i)} \) we have \( \varphi(M_1^{(i)}) = \varphi(M_2^{(i)}) \in nP \cap M \). If we assume \( X_s = Z_0 \), then we have \( M_2^{(s)} = Z_1 \cdots Z_n \) and
\[
\varphi(M_1^{(s)}) = \varphi(M_2^{(s)}) = (a_1 + 1, \ldots, a_{n-1} + 1, a_n) = u_1 + \cdots + u_n \in \partial(nP).
\]

Since \( M_1^{(s)} \) is a monomial of degree \( n \), it is defined by the finite set \( \{ w_1, \ldots, w_n \} \subset P \cap M \) with \( w_1 + \cdots + w_n = u_1 + \cdots + u_n \). From the assumption of very ampleness, \( \{ u_2 - u_1, \ldots, u_n - u_1 \} \) is a basis of the sublattice of \( M \) contained in the affine subspace spanned by \( \{ u_1, \ldots, u_n \} \).

Since the expression \((w_1 - u_1) + \cdots + (w_n - u_1) = (w_2 - u_1) + \cdots + (w_n - u_1)\) is unique, we have \( \{ w_1, \ldots, w_n \} = \{ u_1, \ldots, u_n \} \), that is, \( M_1^{(s)} = M_2^{(s)} \). This implies \( B_s = 0 \). If we assume \( X_s = Z_i \) for some \( i = 1, \ldots, n \), then we can easily see that \( M_1^{(s)} = M_2^{(s)} \), hence \( B_s = 0 \) from the same reason.

This implies that \( B \notin S_1 I_n \).

Remark. — If \( P = \text{Conv}\{ u_0, u_1, \ldots, u_n \} \) does not contain any lattice point in the interior and if \( P \) satisfies the equality (1.3) for all positive integers \( l \), then from the proof of Proposition 1.3 we may set \( u_0 = 0 \), \( u_i = e_i \) for \( i = 1, \ldots, n - 1 \) and \( u_n = \sum_{i=1}^{n} a_i u_i \) with \( a_i \geq 0 \) and \( a_n > 0 \) after a suitable affine transformation of \( M \). Since \( P \cap M = \{ u_0, \ldots, u_n \} \) generates the set of all lattice points in the cone \( \mathbb{R}_{>0} P \) with the apex \( u_0 = 0 \), we see that \( a_n = 1 \). By a change of basis of \( M \), we may set \( u_n = e_n \). Thus \((X, L) \cong (\mathbb{P}^n, O(1))\).

Abe [A] constructs infinitely many examples of integral 3-simplices whose defining ideals need elements of degree 4 as generators. Here we give a part of them.

Example 1.4. — Let \( l \) be a positive integer and set \( M = \mathbb{Z}^3 \).
Let $u_0 = 0, u_1 = (1, 0, 0), u_2 = (0, 1, 0)$ and let $u_3 = (1, 1, 1), u_4 = (3, 3, 4), \ldots, u_{l+3} = (2l+1, 2l+1, 3l+1)$. Set $P_l = \text{Conv}\{u_0, u_1, u_2, u_{l+3}\}$, a 3-simplex. Then $P_l$ contains the lattice points $u_3, \ldots, u_{l+2}$ as its interior points. The volume of $P_l$ is $(3l + 1)/3!$. $P_l$ is the union of four 3-simplices with the common vertex $u_3$. Since $P_l$ is the union of $P_{l-1}$ and three 3-simplices with the common vertex $u_{l+2}$, we see that $P_l$ is divided into the union of $3l + 1$ integral 3-simplices, which means that every 3-simplex appearing in the decomposition has volume $1/3!$, hence the polytope $P_l$ has a unimodular triangulation. From Proposition 1.2.2 in [BGT], $P_l$ is normally generated. From Proposition 1.3 we see that $P_l$ defines a projectively normal toric variety of dimension 3 whose defining ideal needs elements of degree 4 as generators.

2. Characterization.

We consider an integral curve $C$ defined by the intersection of general hyperplane sections $Y_1, \ldots, Y_{n-1}$ of the linear system $|L|$, i.e., $C := \cap_{i=1}^{n-1} Y_i$. Set $L_C = L|C$, the restriction of $L$ to the curve $C$. From easy calculation, we see that

(2.1) $h^0(C, L_C) = h^0(X, L) - n + 1 = \mathbb{Z} P \cap M - n + 1,$

(2.2) $h^1(C, L_C^\otimes n-2) = h^n(X, L^{-1}) = h^0(X, \omega_X \otimes L) = \mathbb{Z} \text{ Int } P \cap M,$

(2.3) $h^1(C, L_C^\otimes i) = 0$ for all $i \geq n - 1$.

Hence we have $h^0(L_C) - h^1(L_C^\otimes n-2) = \mathbb{Z} \mathbb{P} \cap M - n + 1 \geq 2$.

**Lemma 2.1** (Iitaka [I]). — Let $D$ be a Cartier divisor on an integral complete curve $C$ with the properties that the invertible sheaf $\mathcal{O}_C(D)$ is generated by global sections and that the morphism $\Phi_D$ associated to $D$ is birational. Assume that $h^0(C, \mathcal{O}_C(D)) = l + 1 \geq 4$. Then we have an effective divisor $G$ satisfying

(1) $\deg G = l - 1,$

(2) $h^0(C, \mathcal{O}_C(D - G)) = 2,$

(3) the line bundle $\mathcal{O}_C(D - G)$ is generated by global sections and $h^1(C, \mathcal{O}_C(D - G)) = h^1(C, \mathcal{O}_C(D)).$

For a proof we may see Lemma 3.16 in [I]. Unfortunately it is written in Japanese. Hence we give an outline of a proof.
Outline of Proof. We use an induction on $l$. The image $W = \Phi_D(C)$ is a curve in $\mathbb{P}^l$ and is not contained in any hyperplane. Take general points $p, q$ on $W$ so that the line in $\mathbb{P}^l$ through $p$ and $q$ meets $W$ at only two points. These points are nonsingular points of $W$ and the map $\Phi_D$ has an inverse on an open subset containing these points. Set $P_1 = \Phi_D^{-1}(p)$ and $P_2 = \Phi_D^{-1}(q)$. Then $\mathcal{O}_C(D - (P_1 + P_2))$ is generated by global sections. Let $D' := D - P_1$. Then $\mathcal{O}_C(D')$ is generated by global sections and the map $\Phi_{D'}$ is birational.

On the other hand, we have $h^0(\mathcal{O}_C(D')) = h^0(\mathcal{O}_C(D)) - 1 = l$. By the assumption of induction for $D'$ we have a divisor $G'$. Set $G = G' + P_1$. Then this divisor $G$ satisfies (1), (2) and (3).

When $l = 3$, we set $G = P_1 + P_2$. By Riemann-Roch Theorem we have $h^1(\mathcal{O}_C(D)) = h^1(\mathcal{O}_C(D - G))$.

Remark. We note that the divisor $D$ given in Lemma 2.1 consists of general $l - 1$ points on the curve $C$.

A very ample invertible sheaf $L$ on a projective variety $X$ defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X, L)^*) = \mathbb{P}^l$. Set $M_L := \Phi_L^* \Omega_{\mathbb{P}^l}(1)$ so that there exists the following exact sequence of vector bundles:

(2.4) 
\[ 0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0. \]

Taking wedge product in (2.4) and twisting by $L^{\otimes k-1}$, we obtain an exact sequence

(2.5) 
\[ 0 \to \wedge^2 M_L \otimes L^{\otimes k-1} \to \wedge^2 H^0(X, L) \otimes \mathcal{O} \to M_L \otimes L^{\otimes k} \to 0. \]

**Lemma 2.2** (Green-Lazarsfeld [GL]). Assume that $L$ is normally generated. Let $k_0$ be an integer such that the maps induced by (2.5)

(2.6) 
\[ \sigma_k : \wedge^2 H^0(L) \otimes H^0(L^{\otimes k-1}) \to H^0(M_L \otimes L^{\otimes k}) \]

are surjective for all $k \geq k_0$. Then every minimal generator of the homogeneous ideal defining $X$ in $\mathbb{P}^l$ has degree $k_0$ or less.

In our situation we shall show $k_0 = n$ for $(X, L) = (C, L_C)$.

**Proposition 2.3.** Let $L_C$ be a very ample line bundle on an integral complete curve $C$ and let $n \geq 2$ an integer with $H^1(C, L_C^{\otimes i}) = 0$ for $i \geq n - 1$. Then we have $H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes i}) = 0$ for $i \geq n$. Furthermore if we have the inequality $h^0(L_C) - h^1(L_C^{\otimes n-2}) \geq 3$ for $n \geq 2$, then we have $H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes n-1}) = 0$. 

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Proof. —  When \( l := \deg(L_C) - 1 = 2 \), from the condition we have
\[ h^1(L_C^{\otimes n-2}) = 0. \]
Since \( \dim M_{L_C} = 2 \), we have \( \wedge^2 M_{L_C} \cong L_C^{-1} \) from the
sequence (2.4), hence, we have \( H^1(C, \wedge^2 M_{L_C} \otimes L_C^{\otimes 1}) \cong H^1(C, L_C^{\otimes n-1}) = 0 \)
for \( i \geq n - 1 \).

When \( l \geq 3 \), we can apply Lemma 2.1 to \( L_C = \mathcal{O}_C(D) \). Then we have the following commutative diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & M_{L_C(-G)} & H^0(L_C(-G)) \otimes \mathcal{O}_C & L_C(-G) \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & M_{L_C} & H^0(L_C) \otimes \mathcal{O}_C & L_C \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Sigma_G & H^0(L_C|G) \otimes \mathcal{O}_C & L_C|G \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Here we write as \( \Sigma_G \) the kernel of \( H^0(L_C|G) \otimes \mathcal{O}_C \rightarrow L_C|G \). Since
\( h^0(L_C(-G)) = 2 \), the vector bundle \( M_{L_C(-G)} \cong L_C^{-1}(G) \) is a line bundle.
And since \( G \) is a general divisor of degree \( l - 1 \), we may write \( G = \sum_{i=1}^{l-1} P_i \),
hence, we have \( \Sigma_G \cong \bigoplus_{i=1}^{l-1} \mathcal{O}_C(-P_i) \). Thus we have the exact sequence

(2.7) \[ 0 \rightarrow L_C^{-1}(G) \rightarrow M_{L_C} \rightarrow \bigoplus_{i=1}^{l-1} \mathcal{O}_C(-P_i) \rightarrow 0. \]

Taking wedge product in (2.7) and twisting by \( L_C^{k-1} \), we obtain an exact sequence

(2.8) \[ 0 \rightarrow \bigoplus_{i=1}^{l-1} L_C^{\otimes k-2}(G - P_i) \rightarrow \wedge^2 M_{L_C} \otimes L_C^{\otimes k-1} \rightarrow \bigoplus_{i<j} L_C^{\otimes k-1}(-P_i - P_j) \rightarrow 0. \]

Since \( h^1(L_C^{\otimes k-1}) = 0 \) for \( k \geq n \) and since \( P_i \) are general, we have that
\[ h^1(L_C^{\otimes k-1}(-P_i - P_j)) = h^1(L_C^{\otimes k-1}) = 0 \] for \( k \geq n \) and that \( h^1(L_C^{\otimes k-2}(G - P_i)) = h^1(L_C^{\otimes k-2}) = 0 \) for \( k \geq n + 1 \). Hence we have \( H^1(\wedge^2 M_{L_C} \otimes L_C^{\otimes 1}) = 0 \)
for \( i \geq n \).

Next set \( k = n - 1 \). If \( h^1(L_C^{\otimes n-2}(G - P_i)) = 0 \), then the proof of the
proposition is completed. Suppose that \( h^1(L_C^{\otimes n-2}(G - P_i)) > 0 \). Since the
divisor \( G - P_i \) consists of general \( l - 2 \) points, then we have
\[
h^1(L_C^{\otimes n-2}(G - P_i)) = h^1(L_C^{\otimes n-2}) - \deg(G - P_i) = h^1(L_C^{\otimes n-2}) - (l - 2)
= h^1(L_C^{\otimes n-2}) - (l + 1) + 3 = h^1(L_C^{\otimes n-2}) - h^0(L_C) + 3.
\]
The assumption \( h^0(L_C) - h^1(L_C^{\otimes n-2}) \geq 3 \) implies the inequality \( 0 \geq h^1(L_C^{\otimes n-2}(G - P_i)) \), which is a contradiction. Hence we have \( h^1(L_C^{\otimes n-2}(G - P_i)) = 0 \).

**Corollary 2.4.** — Let \( L_C \) be a normally generated ample line bundle on an integral complete curve \( C \). If \( h^1(L_C^{\otimes i}) = 0 \) for \( i \geq n - 1 \) and if \( h^0(L_C) - h^1(L_C^{\otimes n-2}) \geq 3 \) for \( n \geq 2 \), then the defining ideal of \( C \) in \( \mathbb{P}(H^0(C, L_C)^*) \) has generators of degree at most \( n \).

**Proof.** — From Proposition 2.3, we have the surjectivity of the map \( \sigma_n \) of (2.6). Thus the statement follows from Lemma 2.2.

**Lemma 2.5 (Fujita [Fj]).** — Let \( Y \) be an irreducible member of \( |L| \) with \( H^0(X, L) \rightarrow H^0(Y, L_Y) \) surjective. Let \( \delta \in H^0(X, L) \) be the class corresponding to \( Y \), and let \( \xi_\alpha \) (\( \alpha = 1, \ldots, k \)) be homogeneous elements of the graded ring \( R(X, L) := \oplus_{\alpha \geq 0} H^0(X, L^{\otimes \alpha}) \) with \( \deg \xi_\alpha = d_\alpha \) and let \( \eta_\alpha \) be the restriction of \( \xi_\alpha \) to \( R(Y, L_Y) = \oplus_{\alpha \geq 0} H^0(Y, L_Y^{\otimes \alpha}) \). Suppose that \( \{\eta_1, \ldots, \eta_k\} \) generates \( R(Y, L_Y) \). Let \( g_i (i = 1, \ldots, l) \) be homogeneous polynomials in \( k \) variables \( Y_1, \ldots, Y_k \) with \( \deg Y_i = d_i \).

Suppose that all relations among \( \{\eta_\alpha\} \) in \( R(Y, L_Y) \) are derived from \( g_i(\eta_1, \ldots, \eta_k) = 0, \ldots, g_l(\eta_1, \ldots, \eta_k) = 0 \). Then there exist \( l \) homogeneous polynomials \( f_1, \ldots, f_l \) in \( k + 1 \) variables \( X_0, X_1, \ldots, X_k \) with \( \deg X_0 = 1, \deg X_i = d_i \) for \( i = 1, \ldots, k \) such that \( f_i(0, Y_1, \ldots, Y_k) = g_i(Y_1, \ldots, Y_k) \) for \( i = 1, \ldots, k \) and that all relations among \( \delta, \xi_1, \ldots, \xi_k \) in \( R(X, L) \) are derived from \( f_1(\delta, \xi_1, \ldots, \xi_k) = 0, \ldots, f_k(\delta, \xi_1, \ldots, \xi_k) = 0 \).

For a proof see Propositions 2.2 and 2.4 in [Fj].

**Theorem 2.6.** — Let \( P \) be an integral convex polytope of dimension \( n \) satisfying (1.3) for all positive integers \( l \). We assume that the boundary of \( P \) contains at least \( n + 2 \) lattice points. Then the defining ideal \( I \) has generators of degree at most \( n \).

**Proof.** — Let \( C \) be an integral curve defined by the intersection of \( n - 1 \) general hyperplane sections of the linear system \( |L| \). Then the condition \( h^1(L_C^{\otimes n-2}) - h^0(L_C) \geq 3 \) is equivalent to the condition \( \delta P \cap M \geq n + 2 \) from the equalities (2.1) and (2.2). From Corollary 2.4 we have the statement of the theorem for the integral complete curve \( C \) in \( \mathbb{P}(H^0(C, L_C)^*) \).
Let $D$ be a general member of the linear system $|L|$. Then $D$ is irreducible and reduced, and the restriction map $H^0(X, L) \to H^0(D, L|_D)$ is surjective from the vanishing of cohomologies: We have $H^i(X, L^\otimes j) = 0$ for $0 < i < n$ and all $j$, and $H^n(X, L^\otimes j) = 0$ for all $j \geq 0$. Thus we have a sequence $X = D_n \supset D_{n-1} \supset \cdots \supset D_1 = C$ with $\dim D_j = j$, $D_{j-1} \in |L|_{D_j}$ and the surjective restriction $H^0(D_j, L|_{D_j}) \to H^0(D_{j-1}, L|_{D_{j-1}})$. This sequence is called regular ladder in [Fj]. By applying Lemma 2.5 to a regular ladder of $(X, L)$, we have that every minimal generator of the homogeneous ideal defining $X$ in $\mathbb{P}(\Gamma(X, L)^*)$ has degree $n$ or less.

**BIBLIOGRAPHY**


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