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BROLIN'S THEOREM FOR CURVES IN TWO COMPLEX DIMENSIONS

by C. FAVRE and M. JONSSON

Introduction.

In 1965 H. Brodin [B] proved a remarkable result about the distribution of preimages of points for polynomial maps in one variable: if $f(z) = z^d + \dots$ is a polynomial of degree $d \geq 2$, then there is a set \mathcal{E} with $\#\mathcal{E} \leq 1$ such that if $a \in \mathbb{C} \setminus \mathcal{E}$, then

$$(0.1) \quad \frac{1}{d^n} \sum_{f^n z = a} \delta_z \rightarrow \mu \quad \text{as } n \rightarrow \infty,$$

where μ is the harmonic measure on the filled Julia set of f . In particular, the limit in (0.1) is independent of a . Further, the *exceptional set* \mathcal{E} is empty unless f is affinely conjugate to $z \mapsto z^d$, in which case $\mathcal{E} = \{0\}$, the totally invariant point.

Lyubich [L] and Freire, Lopez and Mañé [FLM] later generalized Brodin's theorem to rational maps of the Riemann sphere \mathbb{P}^1 , with $\#\mathcal{E} \leq 2$.

In this paper we prove a version of Brodin's Theorem in two complex dimensions.

THEOREM A. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic mapping of algebraic degree $d \geq 2$. Then there is a totally invariant, algebraic set \mathcal{E}_1 consisting of at most three complex lines and a finite, totally invariant set \mathcal{E}_2 with the following property: If S is a positive closed current on \mathbb{P}^2 of bidegree $(1, 1)$ and unit mass such that*

- (i) S does not charge any irreducible component of \mathcal{E}_1 ;
- (ii) S has a bounded local potential at each point of \mathcal{E}_2 ;

then we have the convergence

$$(0.2) \quad \frac{1}{d^n} f^{n*} S \rightarrow T \quad \text{as } n \rightarrow \infty.$$

Here T is the Green current of f , defined as the limit (in the sense of currents) of $d^{-n} f^{n*} \omega$ as $n \rightarrow \infty$, where ω is the Fubini-Study form on \mathbb{P}^2 . See Section 1 for more details.

As a consequence we have the following result on the distribution of the preimages of curves. The space of curves in \mathbb{P}^2 of degree k may be identified with \mathbb{P}^N for some $N = N(k)$.

COROLLARY B. — *Let f be as in Theorem A and let $k \geq 1$. Let \mathcal{E}^* be the set of curves $C \in \mathbb{P}^N$ such that*

$$\frac{1}{d^{nk}} f^{nk*}[C] \rightarrow T \quad \text{as } n \rightarrow \infty.$$

Then \mathcal{E}^* is contained in an algebraic proper subset of \mathbb{P}^N .

In a similar way, the space Hol_d of holomorphic maps of \mathbb{P}^2 of degree d can be identified with a Zariski open set of some \mathbb{P}^M .

COROLLARY C. — *There exists an algebraic proper subset $\mathcal{H} \subset \text{Hol}_d$ such that for any $f \notin \mathcal{H}$ the convergence*

$$\frac{1}{d^n} f^{n*} S \rightarrow T \quad \text{as } n \rightarrow \infty$$

holds for all positive closed $(1, 1)$ currents S of unit mass.

The set \mathcal{E}_2 is defined asymptotically (see Section 3) and is nontrivial to understand. It certainly contains the following two types of points:

(α) totally invariant points on totally invariant curves;

(β) homogeneous points, that is f preserves the pencil of lines passing through the point.

It can be shown the set of such points contains at most three elements. We conjecture that these are the only points in \mathcal{E}_2 . The proof of this fact, which involves both global mapping properties on \mathbb{P}^2 and the local dynamics at a superattracting point, is postponed to a later study. Denote by $\nu(p, S)$ the Lelong number of the positive closed current S at p . We then have

THEOREM A'. — *Let f, S be as in Theorem A, and assume \mathcal{E}_2 is reduced to points of type (α) and (β). Then the following statements are equivalent:*

(1) $d^{-n} f^{n*} S \rightarrow T$ as $n \rightarrow \infty$;

(2) S does not charge any irreducible component of \mathcal{E}_1 and $\nu(p, S) = 0 \forall p \in \mathcal{E}_2$.

Remark. — After we completed this paper, V. Guedj told us he was able to obtain volume estimates which prove the implication (1) \Rightarrow (2) of Theorem A'. On the other hand, the converse implication (2) \Rightarrow (1) follows from the material we aim at developing in subsequent papers. So that Theorem A' actually holds without knowing the classification of points in \mathcal{E}_2 .

Let us finally mention another conjecture: if $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a holomorphic map of algebraic degree $d \geq 2$, and S is any positive closed (1, 1) current, then $d^{-n} f^{n*} S$ converges to a current \bar{S} satisfying the invariance relation $f^* \bar{S} = d \bar{S}$.

Results in the direction of our paper were previously obtained by Fornæss and Sibony [FS4], who proved a weaker form of Corollary C with \mathcal{H} a countable union of algebraic sets in Hol_d . Later Russakovskii and Shiffman [RS] proved a version of Corollary B: for any holomorphic mapping $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ there exists a pluripolar set $\mathcal{E}^* \subset \mathbb{P}^{2*}$ such that if L is a line in $\mathbb{P}^{2*} \setminus \mathcal{E}^*$, then $d^{-n} f^{n*}[L] \rightarrow T$ as $n \rightarrow \infty$. (Their result also applies to certain rational maps of \mathbb{P}^k , $k \geq 2$ and pullbacks of planes of higher codimension.)

Other related results include Briend and Duval [BD], who recently proved Brolin's theorem for preimages of points under holomorphic maps of \mathbb{P}^k . A version of Theorem A was proven by Fornæss and Sibony [FS1]

for Hénon maps (see also [BS]) and by Favre and Guedj [FG] for birational maps (see also [F2]).

The main ingredient in our proofs of Theorems A and A', as well as in most previous approaches, are *volume estimates*. These are much harder to prove for holomorphic mappings than for birational maps. They come in two forms.

The first type of volume estimates are dynamical and aim at bounding $\text{Vol } f^n E$ from below for any Borel set E . Such estimates are related to the rate of recurrence of the critical set. In previous work, restrictive assumptions on the dynamics were made to get the required volume estimates. A main novelty of this paper is that we are able to control volume decay for arbitrary holomorphic maps.

More precisely we show that the phase space \mathbb{P}^2 splits naturally into two parts: the exceptional set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and its complement. Outside \mathcal{E} , the critical set is not too recurrent and $\text{Vol } f^n E$ does not decay too fast. Near \mathcal{E} , on the other hand, $\text{Vol } f^n E$ may *a priori* decay quite rapidly, but precise information on the structure of \mathcal{E} allows a good understanding of the dynamics and in particular of volume decay.

To obtain this partition we first study asymptotic volume decay along orbits and relate it to the growth of two algebraic quantities: the multiplicity of the vanishing of the Jacobian determinant, and the generic rate of contraction. A key contribution of the paper is the understanding of the asymptotic behavior of these multiplicities under iteration. In particular, we characterize the locus where these asymptotic multiplicities are maximal, giving rise to the exceptional sets \mathcal{E}_1 and \mathcal{E}_2 in Theorem A. Semicontinuity properties of the multiplicities imply that these sets are algebraic and present strong recurrence properties: they are in fact totally invariant.

The second type of volume estimates involve pluripotential theory. We estimate the volume of sublevel sets of plurisubharmonic (psh) functions using the Kiselman-Skoda theorem: asymptotics of these volumes are small exactly when Lelong numbers are small.

In this context, to show that certain Lelong numbers decay under iteration, we make use of Kiselman numbers (or directional Lelong numbers). These allow us to deduce dynamical information in a neighborhood of an invariant curve from the dynamics on the curve itself. We believe this technique could prove useful in other situations, too.

We also believe our result to be true in any dimension but the description of \mathcal{E} and hence the control of decay of volumes around \mathcal{E} become much harder than in dimension 2.

The organization of this paper is as follows. We briefly recall some facts from holomorphic dynamics and reduce Theorem A to an estimate of the size of images of balls in Section 1. In Section 2, we state some pluripotential facts that we use in the paper. In particular we investigate the behavior of Kiselman numbers as one weight degenerates. The same results appeared independently in [M]. In Section 3 we define three asymptotic multiplicities related to volume decay. These multiplicities are used to define the exceptional sets \mathcal{E}_1 and \mathcal{E}_2 , and we study the latter sets in Sections 4 and 5. The next two sections are devoted to volume estimates outside $\mathcal{E}_1 \cup \mathcal{E}_2$ (Section 6) and near $\mathcal{E}_1 \setminus \mathcal{E}_2$ (Section 7). In Section 8, we show a useful technical result about Lelong numbers of pull-backs of currents near a totally invariant curve. After these estimates, we prove Theorem A and Corollaries B and C in Section 9. Finally, we prove Theorem A' and discuss the existence of totally invariant currents in Section 10.

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1. Background and reduction.

In this section we recall some known facts about holomorphic mappings of \mathbb{P}^2 ; see e.g. [FS2] for more information. We also reduce the proof of Theorem A to an estimate on the sizes of images of balls.

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. This means that $f = [P(z, w, t) : Q(z, w, t) : R(z, w, t)]$, where P , Q , and R are homogeneous polynomials of degree d with no nontrivial common zero.

Let T be a positive closed $(1, 1)$ current on \mathbb{P}^2 and take a local plurisubharmonic (psh) potential $T = dd^c u$ around $p \in \mathbb{P}^2$. One defines locally at any point in $f^{-1}\{p\}$ the positive closed $(1, 1)$ current $f^*T := dd^c(u \circ f)$. This does not depend on the choice of u and induces a continuous linear operator on the set of positive closed $(1, 1)$ currents on \mathbb{P}^2 . One can project f^* to an action f^* on $H_{\mathbb{R}}^{1,1}(\mathbb{P}^2) \simeq \mathbb{R}$. This latter is given by the multiplication by d .

Let ω be the Fubini-Study Kähler form on \mathbb{P}^2 . The positive closed currents $f^*\omega$ and $d\omega$ are cohomologous, one can hence find a continuous function u such that $f^*\omega = d\omega + dd^c u$. Iterating this equality n times yields $d^{-n} f^{n*}\omega = \omega + dd^c(\sum_{j=1}^n d^{-j} u \circ f^{j-1})$. This latter series converges uniformly on \mathbb{P}^2 to a continuous function G and one finally infers

$$(1.1) \quad \frac{1}{d^n} f^{n*}\omega \rightarrow T \quad \text{as } n \rightarrow \infty,$$

where $T := \omega + dd^c G$ is called the *Green current* of f . It satisfies the invariance property $d^{-1} f^* T = T$. Replacing ω in (1.1) by a general positive closed current S of unit mass leads to (0.2); the purpose of this paper is to investigate exactly for what currents S this convergence holds.

As stated in Theorem A, the exceptional currents (for which (0.2) may fail) will be connected with totally invariant algebraic sets, and we recall the following two results.

PROPOSITION 1.1 [FS3], [SSU], [D], [CL]. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$. Then the following holds:*

(i) *any (possibly reducible) totally invariant curve $V \supset f^{-1}V$ is a union of at most three lines; if there are three lines, then they are in generic position; further, the set of intersection points between different lines is totally invariant;*

(ii) *any finite totally invariant set $X \supset f^{-1}X$ is contained in the singular locus of the critical value set of f ; the cardinality of the union of all such sets X is bounded by a number depending only on d .*

As a start of the proof of Theorem A we will reduce (0.2) to a study of sizes of images of balls under iterates of f . Namely, suppose that S is a positive, closed current cohomologous to ω for which (0.2) fails. Then we may write

$$S = \omega + dd^c u,$$

where $u \leq 0$ is a quasi-plurisubharmonic (qpsH) function on \mathbb{P}^2 . It then follows that for all $n \geq 0$,

$$d^{-n} f^{n*} S = d^{-n} f^{n*} \omega + d^{-n} dd^c(u \circ f^n),$$

so since (0.2) fails and $d^{-n} f^{n*} \omega \rightarrow T$, we have $d^{-n} dd^c(u \circ f^n) \not\rightarrow 0$ as $n \rightarrow \infty$, which is equivalent to $d^{-n} u \circ f^n \not\rightarrow 0$ in L^1_{Loc} . By Hartog's Lemma

(see [H]) this implies that there is a ball $B \subset \mathbb{P}^2$, a constant $\alpha > 0$ and a sequence $n_j \rightarrow \infty$ such that

$$(1.2) \quad f^{n_j} B \subset \{u < -\alpha d^{n_j}\}.$$

The rest of the proof consists of showing that (1.2) is not possible if S is a current satisfying the hypotheses of Theorem A. This is done by estimating the volume of $f^{n_j} B$ from below (using dynamics) and the volume of $\{u < -\alpha d^{n_j}\}$ from above (using pluripotential theory).

2. Some pluripotential theory.

In this section we discuss some results from pluripotential theory. First we need the definition of Lelong numbers.

DEFINITION 2.1. — *Let u be a psh function near the origin in \mathbb{C}^2 . We define the Lelong number $\nu(0, u)$ of u at the origin to be the supremum of $\nu > 0$ such that*

$$u(\zeta) \leq \nu \log |\zeta| + O(1) \quad \text{as } \zeta \rightarrow 0.$$

This definition is invariant under local biholomorphisms. We will need an estimate of the volume of sublevel sets for a psh function in terms of its Lelong numbers. The following result is due to Kiselman [K2] and relies on previous work of Skoda [S].

THEOREM 2.2. — *Let $U \subset \mathbb{C}^2$ be an open set, K a compact subset of U , and u a psh function on U . For any real number $\alpha < 2(\sup_K \nu(z, u))^{-1}$ there exists a constant $C_\alpha > 0$ such that for any $t \geq 0$, the estimate*

$$(2.1) \quad \text{Vol}(K \cap \{u \leq -t\}) \leq C_\alpha \exp(-\alpha t)$$

holds.

In Section 8 we will need to work with directional Lelong numbers or *Kiselman numbers*. We refer to [K1] or [D1] for a detailed exposition.

Let u be a psh function in the unit ball $B \subset \mathbb{C}^2$ endowed with coordinates (z, w) . (Kiselman numbers depend on a choice of coordinates whereas the Lelong number does not.) Fix a weight $(\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2$.

The Kiselman number of u at the point $p = (0, 0)$ with weight (α_1, α_2) is defined as

$$\nu(p, u, (\alpha_1, \alpha_2)) := \lim_{r \rightarrow 0} \frac{\alpha_1 \alpha_2}{\log r} \sup_{\Delta(r^{1/\alpha_1}) \times \Delta(r^{1/\alpha_2})} u.$$

For $(\alpha_1, \alpha_2) = (1, 1)$ we recover the usual Lelong number

$$\nu(p, u, (1, 1)) = \nu(p, u).$$

We have the following homogeneity property:

$$(2.2) \quad \nu(p, u, (\lambda\alpha_1, \lambda\alpha_2)) = \lambda \nu(p, u, (\alpha_1, \alpha_2)),$$

for any $\lambda > 0$. The inequality

$$(2.3) \quad \nu(p, u, (\alpha_1, \alpha_2)) \geq \min(\alpha_1, \alpha_2) \nu(p, u)$$

always holds (see [D1]).

We can now state the following

PROPOSITION 2.3. — *Let $u \in PSH(B)$, $u \leq 0$ and assume that the positive closed current $S = dd^c u$ does not charge the curve $\{z = 0\}$. Then*

$$\lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} \nu(p, S, (\alpha, 1)) = 0.$$

Remark. — The same types of considerations were made by Souad Khemeri Mimouni in [M]. In fact, she studies more generally the transformation of Lelong numbers under any sequence of blowing-ups. In our case, we blow-up only at the intersection point of the exceptional divisor and the strict transform of the curve $\{z = 0\}$.

Proof of Proposition 2.3. — After having proved this result we learned that S. K. Mimouni had obtained a more general version of it [M], so we only outline a proof here. It relies on the approximation process by Demailly [D2]. Introduce the following Hilbert space:

$$\mathcal{H}(su) := \{h \in (B) \mid |h|_{su}^2 := \int_B |h|^2 e^{-2su} < +\infty\}.$$

Set

$$u_s := s^{-1} \sup_{|h|_{su} \leq 1} \log |h| \in PSH(B).$$

One checks that $u_s = (2s)^{-1} \log \sum |\sigma_i|^2$ for any orthonormal basis $\{\sigma_i\}$ of $\mathcal{H}(su)$, and that u_s is the logarithm of a real analytic function. The following result connects the singularities of u and u_s . Its proof is an adaptation of the argument in [D2].

LEMMA 2.4. — For any point $p \in B$ and all $(\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2$, one has

- (1) $\nu(p, u_s, (\alpha_1, \alpha_2)) \geq \nu(p, u, (\alpha_1, \alpha_2)) - s^{-1}(\alpha_1 + \alpha_2)$;
- (2) $\nu(p, u_s) \leq \nu(p, u)$.

We then conclude the proof of Proposition 2.3 as follows. As $dd^c u$ does not charge $\{z = 0\}$ there exists a point $p = (0, w)$ such that $\nu(p, u) = 0$ (by Siu's theorem). In particular by Lemma 2.4 (2), u_s does not charge $\{z = 0\}$ either, hence there exists a holomorphic function $h \in \mathcal{H}(su)$ which does not vanish identically on $\{z = 0\}$. For such a function, we apply Łojasiewicz' inequality [Lo, p. 243] and get

$$|h(z, w)| + |z| \geq C \operatorname{dis}((z, w), I)^\theta,$$

with $I = h^{-1}(0) \cap \{z = 0\}$ and for some constants $C, \theta > 0$. We infer for any w and any $\alpha < \theta^{-1}$,

$$\sup_{\Delta(r^{1/\alpha}) \times w + \Delta(r)} \log |h| \geq C'\theta \log r.$$

From this, it is easy to see that $\lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} \nu(p, \log |h|, (\alpha, 1)) = 0$; hence

$$\lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} \nu(p, u_s, (\alpha, 1)) \leq \lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} s^{-1} \nu(p, \log |h|, (\alpha, 1)) = 0.$$

In particular, we get for any $s \geq 0$,

$$\lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} \nu(p, u, (\alpha, 1)) \leq s^{-1} + \lim_{\alpha \rightarrow 0} \sup_{p \in \{z=0\}} \nu(p, u_s, (\alpha, 1)) \leq s^{-1},$$

which implies the result. □

Proof of Lemma 2.4. — This lemma is standard for Lelong numbers (see [D2]). We emphasize that the inequality (2) is not obvious and

relies on the Ohsawa-Takegoshi extension theorem. The generalization is straightforward for Kiselman numbers. We nevertheless give the arguments for assertion (1) for completeness. Let $h \in \mathcal{H}(su)$ normalized by $|h|_{su} = 1$. The mean value property inequality for subharmonic functions implies

$$\begin{aligned} |h(z, w)|^2 &\leq \frac{C_1}{r^{2(1/\alpha_1+1/\alpha_2)}} \int_{|z-\xi|<r^{1/\alpha_1}, |w-\zeta|<r^{1/\alpha_2}} |h(\xi, \zeta)|^2 d\xi d\zeta \\ &\leq \frac{C_1^{1/2}}{r^{2(1/\alpha_1+1/\alpha_2)}} \sup_{|z-\xi|<r^{1/\alpha_1}, |w-\zeta|<r^{1/\alpha_2}} e^{2su}. \end{aligned}$$

Hence for all $(z, w) \in B$, we have

$$u_s(z, w) \leq s^{-1} \log \left(\frac{C_1}{r^{1/\alpha_1+1/\alpha_2}} \right) + \sup_{|z-\xi|<r^{1/\alpha_1}, |w-\zeta|<r^{1/\alpha_2}} u.$$

Write $p = (z_0, w_0)$. By definition of Kiselman number we have

$$\sup_{|z_0-\xi|<r^{1/\alpha_1}, |w_0-\zeta|<r^{1/\alpha_2}} u \leq (\alpha_1\alpha_2)^{-1} \nu(p, u, (\alpha_1, \alpha_2)) \log r + C_3$$

hence

$$\begin{aligned} \sup_{|z_0-\xi|<r^{1/\alpha_1}, |w_0-\zeta|<r^{1/\alpha_2}} u_s &\leq \sup_{|z_0-\xi|<2r^{1/\alpha_1}, |w_0-\zeta|<2r^{1/\alpha_2}} u \\ &\leq (\alpha_1\alpha_2)^{-1} \log(Cr) \nu(p, u, (\alpha_1, \alpha_2)) + C_3 - s^{-1} \log \left(\frac{C_1}{r^{1/\alpha_1+1/\alpha_2}} \right) \end{aligned}$$

with $C = 2^{-1/\max\{\alpha_1, \alpha_2\}}$. We hence get

$$\nu(p, u_s, (\alpha_1, \alpha_2)) \geq \nu(p, u, (\alpha_1, \alpha_2)) - s^{-1}(\alpha_1 + \alpha_2)$$

as desired. □

3. Asymptotic multiplicities.

As explained earlier, the main ingredient in the proof of Theorem A are estimates from below of $\text{Vol } f^n B$ for a ball B . This is equivalent to estimates of the Jacobian Jf^n , and more precisely to the asymptotic order of vanishing of Jf^n . In this section, we also consider two other asymptotic multiplicities for a holomorphic mapping of \mathbb{P}^2 .

We will make frequent use of the following *strong Birkhoff theorem* due to the first author (see [F3] or [F2] Theorem 2.5.14):

THEOREM 3.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$ and let $\kappa_n : \mathbb{P}^2 \rightarrow [1, +\infty[$ be a sequence of functions satisfying the following conditions:*

- (1) *for any $n \geq 0$, κ_n is upper semicontinuous (usc) with respect to the analytic Zariski topology;*
- (2) *for any $n, m \geq 0$ and any $x \in \mathbb{P}^2$,*

$$\kappa_{n+m}(x) \leq \kappa_n(x) \kappa_m(f^n x);$$

we say κ_n defines a submultiplicative cocycle;

- (3) *for any $n \geq 0$, $\min_{\mathbb{P}^2} \kappa_n = 1$.*

Then, for any $x \in \mathbb{P}^2$, the sequence $\kappa_n(x)^{1/n}$ converges. Let $\kappa_\infty(x)$ be its limit. We have $\kappa_\infty \circ f = \kappa_\infty$. Further, if $\kappa_\infty(x) > 1$, then

- *either x is preperiodic,*
- *or some iterate of x belongs to a (not necessarily irreducible) fixed curve V such that $\min_V \kappa_\infty = \kappa_\infty(x)$.*

Remark 3.2. — *Note that the curve V of the preceding theorem must contain an irreducible component of the proper analytic subset $\mathcal{C} := \{\kappa_1 > 1\}$.*

3.1. Asymptotic multiplicity of the Jacobian.

First we study the asymptotic order of vanishing of the Jacobian of f . Fix local charts $U \ni p, V \ni fp$, and denote by Jf the Jacobian determinant of $f : U \rightarrow V$.

DEFINITION 3.3. — *Let $\mu(p, Jf) \in \mathbb{N}$ be the order of vanishing of Jf at p .*

This number does not depend on the choice of chart. It can be interpreted analytically as the Lelong number

$$(3.1) \quad \mu(p, Jf) = \nu(p, dd^c \log |Jf|)$$

of the positive closed $(1, 1)$ current $dd^c \log |Jf|$. Note that $\mu(p, Jf) \geq 1$ if and only if p belongs to the critical set \mathcal{C}_f .

We are interested in studying the growth of $\mu(p, Jf^n)$ when n tends to infinity. It is straightforward to see that

$$(3.3) \quad \mu(p, Jf^{k+n}) = \mu(p, Jf^n) + \mu(p, Jf^k \circ f^n)$$

for any $n, k \geq 0$. The sequence $\{\mu(p, Jf^n)\}_{n \geq 0}$ is not submultiplicative, but we will see that we can treat it as such. First we have the following inequality.

PROPOSITION 3.4 (see [F1] Remark 3). — *For any $p \in \mathbb{P}^2$ and any $n, k \geq 0$, the following inequality:*

$$(3.2) \quad \mu(p, Jf^k \circ f^n) \leq (3 + 2\mu(p, Jf^n)) \cdot \mu(f^n p, Jf^k)$$

holds.

From this it follows that

$$\begin{aligned} 3 + 2\mu(p, Jf^{k+n}) &= 3 + 2\mu(p, Jf^n) + 2\mu(p, Jf^k \circ f^n) \\ &\leq 3 + 2\mu(p, Jf^n) + (3 + 2\mu(p, Jf^n)) (2\mu(f^n p, Jf^k)) \\ &\leq (3 + 2\mu(p, Jf^n)) \cdot (3 + 2\mu(f^n p, Jf^k)). \end{aligned}$$

Introduce

$$\widehat{\mu}(p, Jf^n) := 3 + 2\mu(p, Jf^n).$$

The last inequality can then be rewritten as

$$(3.4) \quad \widehat{\mu}(p, Jf^{k+n}) \leq \widehat{\mu}(p, Jf^n) \cdot \widehat{\mu}(f^n p, Jf^k).$$

The sequence $\widehat{\mu}(p, Jf^n)$ hence defines a submultiplicative cocycle. It is moreover usc with respect to the analytic Zariski topology on \mathbb{P}^2 (e.g. by Siu's theorem). Thus Theorem 3.1 applies and yields:

PROPOSITION 3.5. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. For any $p \in \mathbb{P}^2$, the sequence $\widehat{\mu}(p, Jf^n)^{1/n}$ converges to a real number $\mu_\infty(p) \geq 1$. We have $\mu_\infty \circ f = \mu_\infty$. Further, if $\mu_\infty(p) > 1$, then one of the following holds:*

- (i) $f^N p$ is a periodic critical point for some $N \geq 0$;

(ii) *there exists a fixed curve V such that $f^N p \in V$ for some $N \geq 0$, and $\min_V \mu_\infty = \mu_\infty(p)$.*

Remark 3.6. — The sequence $(1 + \mu(p, Jf^n))^{1/n}$ also converges to $\mu_\infty(p)$. Moreover, define $\mu_n(p) := \mu(p, Jf \circ f^n)$. This sequence is clearly increasing. From (3.2), it follows that

$$n^{-1} \mu(p, Jf^n) \leq \mu_{n+1}(p) \leq \mu(p, Jf^n),$$

and so $(1 + \mu_n(p))^{1/n}$ also converges towards $\mu_\infty(p)$.

PROPOSITION 3.7. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$. Then*

$$(3.5) \quad 0 \leq \mu(p, Jf) \leq 3(d - 1)$$

for all $p \in \mathbb{P}^2$. Hence

$$(3.6) \quad 1 \leq \mu_\infty(p) \leq d$$

for all $p \in \mathbb{P}^2$.

Proof. — The multiplicity $\mu(p, Jf)$ is always smaller than the degree of the critical set of f , which is $3(d - 1)$. Applying (3.5) to f^n and letting $n \rightarrow \infty$ we get (3.6). □

3.2. Asymptotic topological degree.

An important quantity in the study of totally invariant sets is the local topological degree $e(p, f)$ of f at a point p . By definition, this is the topological degree of the germ of an open map induced by f at p . Clearly $e(p, f) > 1$ if and only if $p \in \mathcal{C}_f$ and moreover $e(p, f)$ is an usc function with respect to the Zariski topology (see e.g. [F3]). Further e satisfies the composition formula

$$e(p, f^{k+n}) = e(p, f^n) \cdot e(f^n p, f^k).$$

Theorem 3.1 again applies and shows that

PROPOSITION 3.8. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. For any $p \in \mathbb{P}^2$, the sequence $e(p, f^n)^{1/n}$ converges to a real number*

$e_\infty(p) \geq 1$. We have $e_\infty \circ f = e_\infty$. Further, if $e_\infty(p) > 1$, then one of the following holds:

- (i) $f^N p$ is a periodic critical point for some $N \geq 0$;
- (ii) there exists a fixed curve V such that $f^N p \in V$ for some $N \geq 0$ and $\min_V e_\infty = e_\infty(p)$.

PROPOSITION 3.9. — Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$. Then

$$(3.7) \quad 1 \leq e(p, f) \leq d^2$$

for all $p \in \mathbb{P}^2$ and $\{e(p, f) > d\}$ is a finite set whose cardinality is bounded only in terms of the degree d . We have

$$(3.8) \quad 1 \leq e_\infty(p) \leq d^2$$

for all $p \in \mathbb{P}^2$ and the set $\{e_\infty(p) = d^2\}$ is finite, totally invariant and contained in $\{e(p, f) = d^2\}$.

Proof of Proposition 3.9. — Since the (global) topological degree of f is d^2 we have $1 \leq e(p, f) \leq d^2$ for all p . Let us show that $\{e > d\}$ is a finite set. We follow the proof of Theorem 4.7 in [FS3]. Take a point z in the regular locus of \mathcal{C}_f such that fz belongs to the regular locus of $f\mathcal{C}_f$, and z is a regular point for $f|_{\mathcal{C}_f}$. One can find local coordinates so that $f(z, w) = (z, w^e)$ with $e := e(p, f)$. On the other hand, we have in terms of currents $f_*[\mathcal{C}_f] = e[f\mathcal{C}_f]$ and $f^*f_*[\mathcal{C}_f] = ef^*[f\mathcal{C}_f] \geq e^2[\mathcal{C}_f]$. As $\deg(f_*[\mathcal{C}_f]) = d \deg(\mathcal{C}_f)$, we get $e^2 \leq d^2$. Hence outside the finite set $E := (\mathcal{C}_f) \cup f^{-1}(f\mathcal{C}_f) \cup (f|_{\mathcal{C}_f})$ we have $e \leq d$. We conclude the proof noting that the cardinality of E can be bounded only in terms of d .

If $e_\infty(p) = d^2$, then the orbit of p must visit the finite set $\{e = d^2\}$ infinitely many times and is therefore preperiodic to a periodic orbit in that set. So we may write $f^m p = q = f^n q$ with $e(q, f) = d^2$. But then $e_\infty(p) = e(q, f^n)^{1/n}$, so we must have $e(f^i q, f) = d^2$ for $0 \leq i < n$. Thus $f^{-1}(f^{i+1} q) = \{f^i q\}$ for all i , and that implies that $p = f^i q$ for some i . We conclude that p is periodic and that the orbit of p is totally invariant and contained in $\{e = d^2\}$. \square

3.3. Asymptotic diameter.

A third quantity that we will need is related to the asymptotic diameter of balls $f^n B$. To define this, fix local coordinates around p and fp so that $p = fp = 0$. Define $c(p, f)$ to be the largest integer c such that

$$|f(\zeta)| \leq A|\zeta|^c \quad \text{as } \zeta \rightarrow 0$$

for some constant $A > 0$. Alternatively, $c(p, f)$ is the order of the first nonvanishing term in the Taylor expansion of f . It can also be interpreted as the Lelong number $c(p, f) = \nu(p, \log |f|)$; hence the definition does not depend on the choice of local coordinates and c is used with respect to the Zariski topology. Notice that the set $\{c(p, f) > 1\}$ is finite. Moreover, it is clear from the definition that

$$(3.9) \quad c(p, f^{k+n}) \geq c(p, f^n) \cdot c(f^n p, f^k),$$

and so c defines a supermultiplicative cocycle. We have

LEMMA 3.10. — $\mu(p, f) \geq 2(c(p, f) - 1)$.

Proof. — This is a local result so we may assume $p = fp = 0$. Write $c = c(p, f)$ and $\mu = \mu(p, f)$. Let $f(\zeta) = f_c(\zeta) + O(|\zeta|^{c+1})$ where f_c is a homogeneous polynomial of degree c . The Jacobian determinant of f_c is a homogeneous polynomial of degree $2c - 2$ or vanishes identically. Thus $Jf(\zeta) = Jf_c(\zeta) + O(|\zeta|^{2c-1})$ and $\mu \geq 2c - 2$. □

This estimate and (3.9) allow us to deduce the following result from Proposition 3.5.

PROPOSITION 3.11. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. For any $p \in \mathbb{P}^2$, the sequence $c(p, f^n)^{1/n}$ converges to a real number $c_\infty(p) \geq 1$. We have $c_\infty \circ f = c_\infty$. Further, if $c_\infty(p) > 1$, then $f^N p$ is a periodic critical point for some $N \geq 0$. Further, if $c_\infty(p) > 1$, then one of the following holds:*

- (i) $f^N p$ is a periodic critical point for some $N \geq 0$;
- (ii) there exists a fixed curve V such that $f^N p \in V$ for some $N \geq 0$, and $\min_V c_\infty = c_\infty(p)$.

Notice that case (ii) in Proposition 3.5 does not appear as the set of p such that $c(p, f^n) > 1$ for some $n \geq 1$ is countable.

PROPOSITION 3.12. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$. Then*

$$(3.10) \quad 1 \leq c(p, f) \leq d,$$

$$(3.11) \quad 1 \leq c_\infty(p, f) \leq d,$$

for any $p \in \mathbb{P}^2$. Moreover $c(p, f) = d$ if and only if p is a homogeneous point for f , i.e., f maps the pencil of lines through p to the pencil of lines through fp .

Proof. — By pre- and post-composing by projective linear maps of \mathbb{P}^2 we may assume that $p = fp = 0$. Write $f(\zeta) = (P(\zeta)/R(\zeta), Q(\zeta)/R(\zeta))$ where P, Q and R are polynomials of degree d and $P(0) = Q(0) = 0, R(0) = 1$. Then clearly f can only vanish up to order d and so $1 \leq c(p, f) \leq d$. Further, $c(p, f) = d$ if and only if P and Q are homogeneous polynomials of degree d , which means precisely that p is a homogeneous point. \square

3.4. Properties of the multiplicities.

We summarize in the following proposition the inequalities relating the multiplicities μ, c , and e considered above.

PROPOSITION 3.13. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree $d \geq 2$. For all $p \in \mathbb{P}^2$, we have*

$$(3.12) \quad 2(c(p, f) - 1) \leq \mu(p, Jf) \leq 2(e(p, f) - 1),$$

$$(3.13) \quad c(p, f) \leq \sqrt{e(p, f)}.$$

Hence

$$(3.14) \quad c_\infty(p) \leq \mu_\infty(p) \leq e_\infty(p) \leq d,$$

$$(3.15) \quad c_\infty(p) \leq \sqrt{e_\infty(p)} \leq d.$$

The set $\{c_\infty = d\} \subset \{e_\infty = d^2\}$ is finite and totally invariant.

Example 3.14. — A totally invariant point, i.e., a point with $e(p, f) = d^2$, is not necessarily superattracting as the example $f(z, w) = (2z + w^d, z^d)$ from [FS3] shows. In this example the origin is a totally invariant fixed point with one expanding eigenvalue $2 > 1$. One can check that $c_\infty = \mu_\infty = 1$, whereas $e_\infty = d^2$ for this map.

Proof of Proposition 3.13. — Equations (3.14) and (3.15) are consequences of (3.12) and (3.13), and the last assertion follows from (3.15) and Proposition 3.9.

Equations (3.12) and (3.13) are local so we may assume $p = fp = 0$ and for sake of simplicity we write $\mu(p, f) = \mu$, $c(p, f) = c$, $e(p, f) = e$. Because of Lemma 3.10 we only have to show the inequalities $\mu \leq 2e - 2$ and $c \leq \sqrt{e}$.

To prove $\mu \leq 2(e - 1)$ we use the fact that $|f(\zeta)| \geq C|\zeta|^e$ (see e.g. [F1]) and (by definition) $|Jf(\zeta)| \leq D|\zeta|^\mu$ for some constants $C, D > 0$. For any ball $B(r)$ of radius $r > 0$, one gets

$$\begin{aligned} \text{Vol } fB(r) &\geq \text{Vol } B(Cr^e) = C'r^{4e} \\ \text{Vol } fB(r) &= e^{-1} \int_{B(r)} |Jf|^2 \leq e^{-1} \int_{B(r)} D|\zeta|^{2\mu} = D'r^{2\mu+4} \end{aligned}$$

for some constants $C', D' > 0$. By letting $r \rightarrow 0$ we get $\mu \leq 2(e - 1)$.

We now give an analytic proof of $c \leq \sqrt{e}$. We first note that $\delta_p = (dd^c \log |\zeta|)^2$. It follows from [D1] Corollary 6.8 that

$$\begin{aligned} e &= \nu(f^* \delta_p, p) \\ &= \nu(f^* dd^c \log |\zeta| \wedge f^* dd^c \log |\zeta|, p) \\ &\geq \nu(f^* dd^c \log |\zeta|, p)^2 = c^2. \end{aligned}$$

This concludes the proof. □

3.5. Exceptional sets.

We define two exceptional sets as follows.

DEFINITION 3.15. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be holomorphic of degree d .*

- *Define the first exceptional set \mathcal{E}_1 to be the union of irreducible curves V such that $\mu_\infty(p) = d$ for all $p \in V$.*
- *Define the second exceptional set \mathcal{E}_2 to be the set of points p with $c_\infty(p) = d$.*

Finally, we define the exceptional set \mathcal{E} by $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2$.

The exceptional set \mathcal{E} is where f is most volume contracting; we will spend the next few sections analyzing it. In particular we will show that \mathcal{E} is algebraic, totally invariant and superattracting.

4. The first exceptional set \mathcal{E}_1 .

The key to the description of the first and second exceptional sets \mathcal{E}_1 and \mathcal{E}_2 lies in understanding the loci $\{c_\infty < \mu_\infty = d\}$ and $\{c_\infty = d\}$. Being in the first locus means, roughly speaking, that volume is contracted much faster than diameter. This implies that the image of a ball is very close to being one-dimensional, and that can only happen if the critical set is very recurrent. Being in the second locus means that diameter is decreasing very fast, and this leads to totally invariant and superattracting points. In fact, we already know from Proposition 3.13 that the second exceptional set $\mathcal{E}_2 = \{c_\infty = d\}$ is finite and totally invariant. In this section, we show

THEOREM 4.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. Then the first exceptional set \mathcal{E}_1 is the union of all totally invariant curves for f and equals the (Zariski) closure of the locus $\{c_\infty < \mu_\infty = d\}$. Moreover, \mathcal{E}_1 is the union of at most three lines in general position.*

The proof relies essentially on the following local result. Notice that c_∞ and μ_∞ can be defined for a germ fixing a point.

THEOREM 4.2. — *Let $f : (\mathbb{C}^2, 0) \dashrightarrow (\mathbb{C}^2, 0)$ be a holomorphic germ. Let V_1, \dots, V_k be the irreducible components of the critical set \mathcal{C}_f . Assume that $c_\infty(0) < \mu_\infty(0)$. Then there exist $a_1, \dots, a_k \geq 0$ such that*

$$(4.1) \quad f^* \left(\sum_i a_i [V_i] \right) \geq \mu_\infty(0) \left(\sum_i a_i [V_i] \right).$$

Proof of Theorem 4.1. — First we show that any totally invariant curve V is contained in \mathcal{E}_1 . Given $n \geq 1$ and $p \in V$ outside a finite subset (depending on n) we may pick local coordinates at p and at $f^n p$ so that f^n is given by $(z, w) \mapsto (z^{d^n}, w)$. It follows that $\mu(p, Jf^n) = d^n - 1$ on V outside a finite set, and thus $\mu(p, Jf^n) \geq d^n - 1$ on all of V by upper semicontinuity. This implies $\mu_\infty = d$ on V hence $V \subset \mathcal{E}_1$.

Next let us consider the set $X := \{c_\infty < \mu_\infty = d\}$. We claim that X is contained in the union of totally invariant curves. In view of Proposition 1.1 this will complete the proof, since the set $\{c_\infty = d\}$ is finite.

Thus pick a point $p \in X$, $c_\infty(p) < \mu_\infty(p) = d$. By Proposition 3.5, either p is preperiodic or $f^m p \in W$ for some $m \geq 0$ and some fixed curve W with $\min_W \mu_\infty = \mu_\infty(p) = d$.

In the latter case, we have $f^*[W] \geq l[W]$ for some maximal $l \in [2, d]$. For a generic point $q \in W$, we have $d = \mu_\infty(q) = e_\infty(q)$, and for any $j \geq 0$, $e(q, f^j) = l^j$. Hence $l = d$, and we infer that W is totally invariant.

If p is preperiodic, then $f^m p = q = f^N q$ for some $q \in C_f$ and $m \geq 0$. Clearly $\mu_\infty(q, f^N) = \mu_\infty(p, f)^N = d^N$ and $c_\infty(q, f^N) = c_\infty(p, f)^N < d^N$. Apply Theorem 4.2 to f^N and find non-negative integers a_1, \dots, a_k such that $f^{N*} \sum_i a_i [V_i] \geq d^N \sum_i a_i [V_i]$. As f^N is of degree d^N , we have equality. In particular the union W of the critical components passing through p with $a_i > 0$ is a totally invariant set for f^N . But then the curve $W' = W \cup \dots \cup f^{N-1}W$ is totally invariant for f . Since $f^m p \in W'$ we have in fact $p \in W'$. This completes the proof. \square

Proof of Theorem 4.2. — Pick holomorphic maps ϕ_i so that $V_i = \phi_i^{-1}(0)$, and set $\phi := Jf$. Then ϕ_i are the irreducible factors of ϕ . There exist integers $t_{ij} \geq 0$ so that

$$(4.2) \quad \phi_i \circ f = \tilde{\phi}_i \times \prod_{j=1}^k \phi_j^{t_{ij}}$$

for some holomorphic $\tilde{\phi}_i$ with $\tilde{\phi}_i^{-1}(0) \cap C_f = \{0\}$. By Lojasiewicz' inequality [Lo, p. 243] there exist constants $C, \alpha > 0$ such that

$$(4.3) \quad |\tilde{\phi}_i(\zeta)| + |\phi(\zeta)| \geq C|\zeta|^\alpha$$

in a neighborhood of the origin. For fixed $n \geq 0$ write $f^n = f_{c_n} + O(|\zeta|^{c_n+1})$ where $f_{c_n} \not\equiv 0$ is a homogeneous polynomial of degree c_n . Similarly, set $\phi \circ f^n = \phi_{\mu_n} + O(|\zeta|^{\mu_n+1})$ for a non-degenerate homogeneous polynomial ϕ_{μ_n} of degree $\mu_n = \mu(0, Jf \circ f^n)$.

We know that c_n is an increasing supermultiplicative sequence such that $c_n^{1/n} \rightarrow c_\infty(0)$, and that μ_n is an increasing sequence such that $\mu_n^{1/n} \rightarrow \mu_\infty(0)$ (see Section 3). By hypothesis $c_\infty(0) < \mu_\infty(0)$, so for any fixed c, μ with $c_\infty(0) < c < \mu < \mu_\infty(0)$ we have

$$c_n \leq Ac^n$$

$$\mu_n \geq B\mu^n$$

for $n \geq 0$ and some constants $A, B > 0$. We infer that for a generic $|\zeta| \ll 1$ and for any $n \geq 0$

$$(4.4) \quad |\zeta_n| := |f^n(\zeta)| \geq C_n |\zeta|^{Ac^n}$$

for some $C_n > 0$. On the other hand for all $|\zeta| \ll 1$ and all $n \geq 0$ we have

$$(4.5) \quad |\phi \circ f^n(\zeta)| = |\phi(\zeta_n)| \leq D_n |\zeta|^{B\mu^n}.$$

Let M be the k by k matrix $[t_{ij}]$ and let $\rho > 0$ be the spectral radius of M . Denote $M^n := [t_{ij}^n]$ and fix $K > 0$ so that

$$(4.6) \quad 0 \leq t_{ij}^n \leq K\rho^n.$$

For n large enough, so that $Ac^n \leq B\mu^n$, and for generic ζ , we can apply (4.3), and get

$$|\tilde{\phi}_j(\zeta_n)| \geq C|\zeta_n|^\alpha - |\phi(\zeta_n)| \geq CC_n^\alpha |\zeta|^{Ac^n} - D_n |\zeta|^{B\mu^n} \geq C'_n |\zeta|^{Ac^n}.$$

Hence

$$\begin{bmatrix} |\phi_1| \\ \vdots \\ |\phi_k| \end{bmatrix} (\zeta_{n+1}) = \begin{bmatrix} |\tilde{\phi}_1| \prod_j |\phi_j|^{t_{1j}} \\ \vdots \\ |\tilde{\phi}_k| \prod_j |\phi_j|^{t_{kj}} \end{bmatrix} (\zeta_n) \geq C'_n |\zeta|^{Ac^n} \begin{bmatrix} |\phi_1| \\ \vdots \\ |\phi_k| \end{bmatrix}^M (\zeta_n),$$

where we let ${}^t[|\phi_1|, \dots, |\phi_k|]^M := {}^t[\prod_j |\phi_j|^{t_{1j}}, \dots, \prod_j |\phi_j|^{t_{kj}}]$.

By induction and using (4.6) for $1 \leq i \leq k$ we have

$$|\phi_i(\zeta_{n+1})| \geq C''_n |\zeta|^{AK \frac{\rho^{n+1} - c^{n+1}}{\rho - c}}.$$

On the other hand (4.5) shows that

$$D_{n+1} |\zeta|^{B\mu^{n+1}} \geq |\phi(\zeta_{n+1})| \geq C'''_n |\zeta|^{A'K \frac{\rho^{n+1} - c^{n+1}}{\rho - c}}.$$

As ζ was generic, we can let it tend to zero and let n tend to infinity. We infer $\rho \geq \mu$, and therefore $\rho \geq \mu_\infty(0)$ as $\mu < \mu_\infty(0)$ was chosen arbitrary.

The Perron-Frobenius theorem now implies the existence of a (not necessarily unique) eigenvector (a_1, \dots, a_k) for M with non-negative coefficients associated to the eigenvalue ρ . We have $f^* \sum_i a_i [V_i] = \rho(\sum_i a_i [V_i]) \geq \mu_\infty(0) \sum_i a_i [V_i]$, which completes the proof. \square

5. The second exceptional set \mathcal{E}_2 .

The second exceptional set \mathcal{E}_2 is both hard and interesting to analyze in detail. We will give some partial results that are enough for the purpose of Theorems A and A'.

PROPOSITION 5.1. — *The second exceptional set is given by the set of*

- *points $p \notin \mathcal{E}_1$ with $\mu_\infty(p) = d$;*
- *totally invariant periodic orbits f in \mathcal{E}_1 .*

Further, \mathcal{E}_2 is finite, totally invariant and superattracting.

Proof. — Proposition 3.13 shows that $\mathcal{E}_2 = \{c_\infty = d\}$ is finite and totally invariant. A point $p \in \mathcal{E}_2$, $f^n p = p$, is superattracting as the Taylor series of f^n at p vanishes to order $c(p, f^n) \geq 2$. Let us now prove the characterization of \mathcal{E}_2 . First consider a point $p \notin \mathcal{E}_1$. If $p \in \mathcal{E}_2$, then $c_\infty(p) = d$ and so $\mu_\infty(p) = d$ by Proposition 3.13. Conversely, if $\mu_\infty(p) = d$, then $c_\infty(p) = d$ by Theorem 4.1 so $p \in \mathcal{E}_2$.

Next consider $p \in \mathcal{E}_1$. We want to show that $c_\infty(p) = d$ if and only if p is periodic and the orbit of p is totally invariant. Both of these properties are preserved under replacing f by an iterate, so we may assume that the line ($w = 0$) is totally invariant for f and that p is on this line. Notice that the restriction $R := f|_{w=0}$ is a rational map of degree d and that $c(p, f^n) \leq c(p, R^n) = e(p, R^n)$.

Assume first $c_\infty(p) = d$. Then $e_\infty(p, R) = d$ hence p belongs to a totally invariant orbit.

Conversely, suppose that the orbit of p is totally invariant. After replacing f by f^2 we may assume that $p = (0, 0)$ is a totally invariant fixed point for R . Thus we may write

$$f(z, w) = ((z^d + wQ)(1 + \eta), w^d(1 + \eta))$$

in local coordinates $\zeta = (z, w)$, for holomorphic Q, η with $\eta(0, 0) = 0$. One checks that $c(p, f^n) \geq d^{n-1}$, thus $c_\infty(p) = d$. This completes the proof. \square

Let us give some examples of maps with nonempty second exceptional set.

Example 5.2. — We already know from Proposition 5.1 that a totally invariant point on a totally invariant line is in \mathcal{E}_2 . This will happen if

$f[z : w : t] = [z^d + wQ(z, w, t) : w^d : R(z, w, t)]$, where Q and R are homogeneous of degree $d - 1$ and d , respectively.

Example 5.3. — A homogeneous point, i.e., a point p with the property that the family of lines through p is invariant under f , is a point in \mathcal{E}_2 . In fact, $c(p, f^n) = d^n$ and so $c_\infty(p) = d$. In homogeneous coordinates where $p = [0 : 0 : 1]$ we have $f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)]$.

Example 5.4. — More generally, if f preserves a linear pencil of curves, then any base point p of the pencil belongs to \mathcal{E}_2 for this map. Holomorphic maps of \mathbb{P}^2 preserving a pencil of curves were studied in [DJ]: it turns out that the mappings and the base points are always of one of the types described in Examples 5.2 or 5.3.

It is a very interesting problem whether all points in \mathcal{E}_2 are of the types described in Examples 5.2 or 5.3. We postpone its discussion to a later paper. Assuming this is the case, we refer to the last section of the paper for a list of different possible configurations of the exceptional sets \mathcal{E}_1 and \mathcal{E}_2 .

6. Volume estimates outside the exceptional set.

In this section we give a lower bound on the volumes $f^n E$ for sets E that avoid a fixed neighborhood of the exceptional set \mathcal{E} under iterations.

THEOREM 6.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. Fix an open neighborhood $\Omega \supset \mathcal{E}$ of the exceptional set. Then there exist a constant $\lambda < d$ and constants $C_1, C_2 > 0$ such that*

$$(6.1) \quad \text{Vol } f^n E \geq (C_1 \text{Vol } E)^{C_2 \lambda^n}$$

for any Borel set $E \subset \mathbb{P}^2$ and any integer $n \geq 0$ with $E, \dots, f^n E \subset \mathbb{P}^2 \setminus \Omega$.

The key idea ingredient in the proof is the following upper bound on the multiplicities of the Jacobian.

PROPOSITION 6.2. — *In the setting of Theorem 6.1 there exist $\rho < d$ and $C > 0$ such that*

$$(6.2) \quad \mu(x, Jf^n) \leq C\rho^n$$

for any $x \notin \mathcal{E}$ and any integer $n \geq 0$.

We defer the proof of Proposition 6.2 until the end of this section and show how to deduce Theorem 6.1 from it. The key to doing so is the following result, which connects the multiplicity of the Jacobian to the volume of fE .

PROPOSITION 6.3 (see [F2] Chapitre 4). — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic mapping and $K \subset \mathbb{P}^2$ be a compact set. Define*

$$\tau_f(K) := \max\{\mu(p, Jf), p \in K\}.$$

Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(6.3) \quad \text{Vol } fE \geq C_\varepsilon (\text{Vol } E)^{1+\tau_f(K)+\varepsilon}$$

for any Borel set $E \subset K$.

Proof. — Write $\tau_\varepsilon := \tau_f(K) + \varepsilon$. Let $E \subset K$ be a Borel set. We are looking for a lower bound for $\text{Vol } fE$ in terms of $\text{Vol } E$. To this end we apply the Kiselman-Skoda estimate (Theorem 2.2) to the function $\log |Jf|$ in each chart of a given atlas of \mathbb{P}^2 . Notice that $\nu(z, \log |Jf|) = \mu(z, Jf)$. We conclude that there exists a constant $C_\varepsilon > 0$ such that

$$(6.4) \quad \text{Vol} (K \cap \{|Jf|^2 \leq t\}) \leq C_\varepsilon t^{1/\tau_\varepsilon}$$

for all $t \geq 0$.

If we define t by $C_\varepsilon t^{1/\tau_\varepsilon} = \frac{1}{2} \text{Vol } E$, then Chebyshev's inequality gives

$$\begin{aligned} \text{Vol } fE &\geq d^{-2} \int_E |Jf|^2 \geq d^{-2} t (\text{Vol } E - \text{Vol} \{|Jf|^2 \leq t\}) \\ &\geq \frac{t}{2d^2} \text{Vol } E \geq C'_\varepsilon (\text{Vol } E)^{1+\tau_\varepsilon}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 6.1. — Choose an integer N so that $C\rho^N < d^N$ for the constant $\rho < d$ given by Proposition 6.2. Fix $\lambda < d$ so that $C\rho^N < \lambda^N < d^N$. Proposition 6.3 applied to f^N with $K := \mathbb{P}^2 \setminus \Omega$ yields a constant $C > 0$ so that

$$(6.5) \quad \text{Vol } f^N E \geq C (\text{Vol } E)^{\lambda^N}$$

for any Borel set $E \subset \mathbb{P}^2 \setminus \Omega$. In a same way, one can find constants $D, D' > 0$ so that

$$(6.6) \quad \text{Vol } f^j E \geq D' (\text{Vol } E)^D$$

for any $0 \leq j \leq N - 1$ and any $E \subset \mathbb{P}^2$.

Take a Borel set $E \subset \mathbb{P}^2$ and $n \geq 0$ so that $E, \dots, f^n E \subset \mathbb{P}^2 \setminus \Omega$. Write $n = kN + l$ with $l \geq N - 1$. We have

$$\begin{aligned} \text{Vol } f^n E &= \text{Vol } f^l (f^{kN} E) \geq D' \text{Vol } f^{kN} E^D \\ &\geq D' \left((\text{Vol } E)^{\lambda^{kN}} C^{\sum_{i=1}^{k-1} \lambda^{iN}} \right)^D \geq (C'' \text{Vol } E)^{D'' \lambda^n}, \end{aligned}$$

which completes the proof of Theorem 6.1. □

Finally we prove the estimate for multiplicities in Proposition 6.2.

Proof of Proposition 6.2. — Denote by V_1, \dots, V_k the irreducible components of \mathcal{C}_f that are not in \mathcal{E}_1 . For each i , pick a point $x_i \in V_i$ so that $\mu_\infty(x_i) < d$, and fix $\lambda < d$ with $\max_i \mu_\infty(x_i) < \lambda$. One can find a constant $C > 0$ so that

$$(6.7) \quad \mu(x_i, Jf^n) \leq C \lambda^n$$

for all $n, i \geq 0$. Introduce the set $\mathcal{F}_N := \{x \in \mathbb{P}^2 \setminus \mathcal{E}, \mu(x, Jf^N) > C \lambda^N\}$ for a suitable $N \geq 0$ to be chosen later. Because of (6.7), it is a finite set.

Let $x \in \mathbb{P}^2$ and $n \geq 0$.

• First assume $\{x, fx, \dots, f^n x\} \cap \mathcal{F}_N = \emptyset$. Write $n = kN + l$ with $0 \leq l \leq N - 1$. We have

$$\begin{aligned} \mu(x, Jf^n) &= \mu(x, Jf^{kN+l}) = \mu(x, Jf^l) + \mu(x, Jf^{kN} \circ f^l) \\ &\leq \mu(x, Jf^l) + (3 + 2\mu(x, Jf^l))\mu(f^l x, Jf^{kN}). \end{aligned}$$

By (3.4), we infer

$$3 + 2\mu(f^l x, Jf^{kN}) \leq \prod_{j=0}^{k-1} (3 + 2\mu(f^{l+j} x, Jf^N)) \leq (3 + 2C \lambda^N)^k.$$

Set $C_N := \max_{\mathbb{P}^2} \mu(\cdot, Jf^N)$. Note that $C_N \geq C_{N-1}$. We get

$$\begin{aligned} \mu(x, Jf^n) &\leq C_N + (3C_N + 2)\mu(f^l x, Jf^{Nk}) \\ &\leq 3^{-1}(3C_N + 2)(3\mu(f^l x, Jf^{Nk}) + 2) + C_N - 2/3(3C_N + 2) \\ &\leq (C_N + 2/3)(3 + 2C \lambda^N)^k. \end{aligned}$$

Now for a fixed $\lambda < \rho < d$ we take $N \gg 1$ large enough to conclude

$$(6.8) \quad \mu(x, Jf^n) \leq C' \rho^n$$

for some constant $C' > 0$.

- Now assume $\{x, fx, \dots, f^n x\} \cap \mathcal{F}_N \neq \emptyset$.

The set \mathcal{F}_N is finite. By definition it does not intersect \mathcal{E} , hence one can find constants $\lambda' < d$, $C'' > 0$ such that

$$(6.9) \quad \mu(p, Jf^n) \leq C''(\lambda')^n$$

for all $p \in \mathcal{F}_N$ and $n \geq 0$.

Let l be the smallest integer such that $f^l x \in \mathcal{F}_N$. Applying (3.4) and (6.9), we get

$$\begin{aligned} \mu(x, Jf^n) &\leq \mu(x, Jf^l) + \mu(f^l x, Jf^{n-l})(3 + 2\mu(x, Jf^l)) \\ &\leq C' \rho^l + (3 + 2C' \rho^l)(C''(\lambda')^{n-l}) \\ &\leq D \max\{\rho, \lambda'\}^n, \end{aligned}$$

and the proof is complete. □

7. Volume estimates at the first exceptional set \mathcal{E}_1 .

We now analyze the dynamics near \mathcal{E}_1 . Recall that \mathcal{E}_1 is a totally invariant union of at most three lines (in general position).

PROPOSITION 7.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. Fix small open neighborhoods $\Omega_i \supset \mathcal{E}_i$ of the first and second exceptional sets, respectively. Then there exists a constant $C > 0$ and an integer $N \geq 1$ such that*

$$(7.1) \quad \text{Vol } f^n E \geq (C \text{Vol } E)^{d^n}$$

for any Borel set $E \subset \mathbb{P}^2$ and any integer $n \geq N$ with $E, \dots, f^n E \subset \Omega_1 \setminus \Omega_2$.

We will prove Proposition 7.1 using the structure of the Jacobian Jf near \mathcal{E}_1 . First we prove the following lemma.

LEMMA 7.2. — *There exists $C > 0$ such that for all $s > 0$ we have*

$$(7.2) \quad \text{Vol} \{p \in \Omega_1 : |Jf(p)| < s\} \leq C s^{\frac{2}{d-1} \log s}.$$

Further, there exists $N \geq 1$ and for any $n \geq N$ a constant $C_n > 0$ such that

$$(7.3) \quad \text{Vol} \{p \in \Omega_1 \setminus \Omega_2 : |Jf^n(p)| < s\} \leq C_n s^{\frac{2}{d^n-1}}$$

for all $s > 0$.

Proof of Lemma 7.2. — The result is local. Pick $p \in \mathcal{E}_1$. We may assume that $p = (0, 0)$ and that $\mathcal{E}_1 = (w = 0)$ or $\mathcal{E}_1 = (zw = 0)$ locally at p . Further, write

$$Jf(z, w) = w^{d-1} \prod_{i=1}^k (z - \alpha_i(w)),$$

where $1 \leq k \leq d-1$ is the multiplicity of $z = 0$ as a critical point of $f|_{(w=0)}$ and where α_i are multivalued functions with $\alpha_i(0) = 0$. Let X_s be the subset of the bidisk $\Delta^2 = \{|z|, |w| < 1\}$ where $|Jf| < s$. We will show that

$$\text{Vol} X_s \leq C s^{\frac{2}{d-1} \log s},$$

which will prove (7.2). For the rest of the proof we will let C denote various positive constants. Let $A = \sup_{\Delta^2} |\prod_{i=1}^k (z - \alpha_i(w))|$. For $\tau < A$ and fixed $w \in \Delta$ we may estimate

$$\begin{aligned} \text{Aera} \left\{ z \in \Delta : \left| \prod_{i=1}^k (z - \alpha_i(w)) \right| \leq \tau \right\} &\leq \sum_i \text{Aera} \left\{ |(z - \alpha_i(w))| \leq \tau^{1/k} \right\} \\ &\leq C \tau^{2/k}. \end{aligned}$$

We can then use Fubini's Theorem to estimate the volume of X_s :

$$\begin{aligned} \text{Vol} X_s &\leq 2\pi \int_0^{(s/A)^{\frac{1}{d-1}}} r \, dr \\ &\quad + \int_{(s/A)^{\frac{1}{d-1}}}^1 \text{Aera} \left\{ \left| \prod_{i=1}^k (z - \alpha_i(w)) \right| < \frac{s}{r^{d-1}} \right\} r \, dr \\ (7.4) \quad &\leq C s^{\frac{2}{d-1}} + C s^{\frac{2}{k}} \int_{(s/A)^{\frac{1}{d-1}}}^1 r^{1 - \frac{2(d-1)}{k}} \, dr. \end{aligned}$$

If $k < d - 1$, then the second term in (7.4) can be estimated by

$$Cs^{\frac{2}{k}} s^{\frac{1}{d-1} (2 - \frac{2(d-1)}{k})} = Cs^{\frac{2}{d-1}},$$

and so $\text{Vol } X_s \leq Cs^{2/(d-1)}$ in this case.

If $k = d - 1$ then the second term in (7.4) is instead bounded by

$$Cs^{\frac{2}{d-1}} \log(s^{\frac{1}{d-1}}) = Cs^{\frac{2}{d-1}} \log s,$$

and so $\text{Vol } X_s \leq Cs^{2/(d-1)} \log s$. This proves (7.2).

As for (7.3) we notice that for $n \geq N$, all the critical points for $f^n|_{(w=0)}$, except the ones at \mathcal{E}_2 , will have multiplicity $< d^n - 1$. Thus the above calculations imply (7.3). □

We now show how Lemma 7.2 implies Proposition 7.1.

Proof of Proposition 7.1. — The proof is similar to that of Proposition 6.3. First assume that $N \leq n \leq 2N$, with N from Lemma 7.2. We define t_n by

$$C_n t_n^{\frac{1}{d^n-1}} = \frac{1}{2} \text{Vol } E,$$

with C_n from Lemma 7.2 and get

$$\begin{aligned} \text{Vol } f^n E &\geq d^{-2n} \int_E |Jf^n|^2 \geq d^{-2n} t_n (\text{Vol } E - \text{Vol } \{|Jf^n|^2 \leq t_n\}) \\ &\geq \frac{t_n}{2d^{2n}} \text{Vol } E \geq C'_n (\text{Vol } E)^{d^n}. \end{aligned}$$

It is now easy to iterate this estimate and arrive at (7.1). □

8. Attenuation of Lelong numbers.

For the proof of Theorem A we need further information on the dynamics near \mathcal{E}_1 . To this end we prove the following result.

THEOREM 8.1. — *Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$ and let $S = \omega + dd^c u$ be a positive closed current on \mathbb{P}^2 such that*

- *S does not charge any component of \mathcal{E}_1 ;*
- *the Lelong number $\nu(p, S) = 0$ at any point $p \in \mathcal{E}_2 \cap \mathcal{E}_1$.*

Then

$$\sup_{p \in \mathcal{E}_1} \nu(p, d^{-n} f^{n*} S) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We first prove Theorem 8.1 under the weaker assumptions

(A) $u \not\equiv -\infty$ on any irreducible component of \mathcal{E}_1 ;

(B) u is bounded at each point $p \in \mathcal{E}_2 \cap \mathcal{E}_1$,

Proof of Theorem 8.1 under assumptions (A) and (B). — Let V be an irreducible component of \mathcal{E}_1 , i.e., a line. After replacing f by an iterate we may assume that V is fixed by f and hence $R := f|_V$ induces a rational map of V of degree d . Let S be a current as in the statement of the lemma. By (A) we may define the probability measure $m_S := S|_V$. We have

$$\nu(p, d^{-n} f^{n*} S) \leq \nu(p, d^{-n} R^{n*} m_S) = d^{-n} e(p, R^n) \nu(p, m_S).$$

By (B) the measure m_S does not charge totally invariant orbits of R . On the other hand, one can find $\lambda < d$ such that for large $n \geq 0$ and any $p \in V \setminus \mathcal{E}_2$ $e(p, R^n) \leq \lambda^n$. We conclude

$$\sup_{p \in \mathcal{E}_1} \nu(p, d^{-n} f^{n*} S) \leq (\lambda/d)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

□

Proof of Theorem 8.1 in the general case. — We will use of Proposition 2.3 on the behavior of Kiselman numbers when one weight tends to zero. Let $V \subset \mathcal{E}_1$ and $R := f|_V$ be as above. We cover V by a finite number of coordinates chart $U_i \ni (z_i, w_i)$ such that $V \cap U_i = \{z_i = 0\}$. In the open set $f^{-1}U_j \cap U_i$, the map f can be written in the form

$$f(z_i, w_i) = (z_i^d, R(w_i) + O(|z_i|)) (1 + O(|z_i, w_i|)).$$

For a point $p \in U_i$, we denote by $\nu(p, S, (\alpha_1, \alpha_2))$ the Kiselman number of S at p with weights $(\alpha_1, \alpha_2) \in (\mathbb{R}_+^*)^2$ associated to the coordinate systems (z_i, w_i) . (Recall that the Kiselman numbers, as opposed to the Lelong number, depend on a choice of coordinates.) Assume that we can prove the following result:

LEMMA 8.2. — For any point $p \in f^{-1}U_j \cap U_i$ and any $\alpha \leq 1$, we have

$$\nu(p, d^{-1} f^* S, (\alpha, d + 1)) \leq \nu(fp, S, (\alpha e(p, R)/d, d + 1)).$$

As before fix a constant $\lambda < d$ such that for large $n \geq 0$ and any point $p \in V \setminus \mathcal{E}_2$, we have $e(p, R^n) \leq \lambda^n$. By assumptions S does not charge V and $\nu(p, S) = 0$ for any point $p \in V \cap \mathcal{E}_2$; hence for any $n \geq 0$ large enough we get

$$\begin{aligned} \nu(p, d^{-n} f^{n*} S) &\leq C \nu(p, d^{-n} f^{n*} S, (1, d + 1)) \\ &\leq C \nu(f^n p, S, (e(p, R^n)/d^n, d + 1)) \\ &\leq C \sup_{p \in V} \nu(p, S, (e(p, R^n)/d^n, d + 1)) \\ &\leq C \sup_{p \in V} \nu(p, S, ((\lambda/d)^n, d + 1)) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the first line follows from (2.2) (with $C := d + 1$), the second from Lemma 8.2, and the last from Proposition 2.3. This concludes the proof. □

Proof of Lemma 8.2. — This is a local result so we may assume $p = fp = (0, 0)$ and

$$f(z, w) = (z^d(1 + O(|z|)), w^k(1 + O(|w|)) + O(|z|))(1 + O(|z, w|)),$$

with $k = e(p, R)$. We easily check that there exist constants $C, C' > 0$ such that for any $\alpha \leq 1$,

$$f\left(\Delta(r^{1/\alpha}) \times \Delta(r^{1/(d+1)})\right) \supset \Delta(Cr^{d/\alpha}) \times \Delta(C'r^{k/(d+1)}).$$

We remark that this property is easy to verify but nevertheless central to the proof. Write $S = dd^c u$ for some local psh potential u . We infer

$$\begin{aligned} \nu(p, d^{-1} f^* S, (\alpha, d + 1)) &= \lim_{r \rightarrow 0} \frac{\alpha(d + 1)}{d \log r} \sup_{\Delta(r^{1/\alpha}) \times \Delta(r^{1/(d+1)})} u \circ f \\ &\leq \lim_{r \rightarrow 0} \frac{\alpha(d + 1)}{d \log r} \sup_{\Delta(Cr^{d/\alpha}) \times \Delta(C'r^{k/(d+1)})} u \\ &= k \nu(fp, S, (\alpha/d, (d + 1)/k)) \\ &= \nu(fp, S, (\alpha k/d, d + 1)), \end{aligned}$$

which completes the proof. □

9. Proof of the main results.

This section is devoted to the proof of Theorem A and its two Corollaries B and C.

Proof of Theorem A. — We argue by contradiction. Suppose that S is a positive closed current on \mathbb{P}^2 for which the assumptions, but not the conclusions, of Theorem A hold. As in Section 1 we write $S = \omega + dd^c u$ with $u \leq 0$ qsh, and conclude that there exists a ball B , a positive number α and a sequence $n_j \rightarrow \infty$ such that

$$(9.1) \quad f^{n_j} B \subset \{u < -\alpha d^{n_j}\}.$$

We will get a contradiction from (9.1) by estimating the volumes of the two sides.

Fix small neighborhoods Ω_1, Ω_2 of the exceptional sets \mathcal{E}_1 and \mathcal{E}_2 , respectively. By the superattracting nature of \mathcal{E}_1 and \mathcal{E}_2 we may assume that $f\Omega_i \subset \subset \Omega_i$ for $i = 1, 2$. In order to reach a contradiction, it is sufficient to consider three different cases.

- Let us first assume that $f^n B$ avoids $\Omega_1 \cup \Omega_2$ for all $n \geq 0$. Then Theorem 6.1 applies and shows that

$$\text{Vol } f^n B \geq (C_1 \text{Vol } B)^{C_2 \lambda^n}$$

for some $\lambda < d$. On the other hand, the Kiselman-Skoda estimate (Theorem 2.2) shows that

$$\text{Vol } \{u \leq -\alpha d^n\} \leq C \exp(-\beta d^n)$$

for some $\beta > 0$ and for all $n \geq 0$. This yields a contradiction.

- The second case is when $f^n B \subset \Omega_1 \setminus \Omega_2$ for all $n \geq 0$. We then use the results from Sections 4, 7 and 8 on the dynamics near the first exceptional set \mathcal{E}_1 . First, by Proposition 7.1 there exists a constant $C > 0$ such that

$$(9.2) \quad \text{Vol } f^n B \geq (C \text{Vol } B)^{d^n}$$

for sufficiently large n . Second, by Proposition 8.1, for arbitrarily large $A > 0$, one can find an integer $m \geq 0$ so that $\sup_{\mathcal{E}_1} \nu(p, d^{-m} f^{m*} S) < 1/A$.

Hence by the Kiselman-Skoda estimate (Theorem 2.2) one has

$$(9.3) \quad \text{Vol} \{p \in \Omega_1 \setminus \Omega_2 \mid d^{-m}u \circ f^m \leq -t\} \leq \exp(-At)$$

for large enough t . For $n_j \gg m$, (9.1), (9.2) and (9.3) then imply

$$\begin{aligned} (C \text{Vol} B)^{d^{n_j-m}} &\leq \text{Vol} f^{n_j-m} B \\ &\leq \text{Vol} \{d^{-m}u \circ f^m < -\alpha d^{n_j-m}\} \\ &\leq \exp(-A\alpha d^{n_j-m}). \end{aligned}$$

We get a contradiction by choosing A so that $\exp(-\alpha A) < C \text{Vol} B$ and letting $n_j \rightarrow \infty$.

• The third and last case is when $f^n B \subset \Omega_2$ for all $n \geq 0$. But by our assumption u is bounded at \mathcal{E}_2 and so (9.1) clearly cannot hold. This completes the proof of Theorem A. □

Proof of Corollary B. — If $S = k^{-1}[C]$ is the current of integration on a curve C of degree $k \geq 1$, then S satisfies the assumptions of Theorem A unless

- C contains an irreducible component of \mathcal{E}_1 ; or
- $C \cap \mathcal{E}_2 \neq \emptyset$.

This concludes the proof as the set of curves C satisfying either of these conditions is a algebraic proper subset of \mathbb{P}^N . □

Proof of Corollary C. — Let $\mathcal{H} \subset \text{Hol}_d$ be the set of holomorphic maps f of degree d for which $\mathcal{E}_f \neq \emptyset$. By Theorem 4.1 and Proposition 5.1, \mathcal{E}_f consists of at most three totally invariant lines and a totally invariant set whose cardinality is bounded by some integer $N(d)$. It is easy to check from this that \mathcal{H} defines an algebraic set in Hol_d . To conclude the proof we only have to exhibit one holomorphic map $f \in \text{Hol}_d$ with $\mathcal{E}_f = \emptyset$. We follow a construction of Ueda.

Take a Lattès map in \mathbb{P}^1 of degree d for instance $R(z) := (z - 2/z)^d$. Consider the holomorphic map $g(z, w) := (R(z), R(w)) : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. It has topological degree d^2 . The quotient $\mathbb{P}^1 \times \mathbb{P}^1$ by the symmetry $(z, w) \rightarrow (w, z)$ is isomorphic to \mathbb{P}^2 and g induces a holomorphic map f on the quotient. The topological degree of f is d^2 hence $f \in \text{Hol}_d$. As R does not contain critical periodic points, the same is true for g and for f too. Hence $\mathcal{E}_f = \emptyset$ and we are done. □

10. The proof of Theorem A' and totally invariant currents.

In this section we will work under the assumption that *every point in $\mathcal{E}_2 \setminus \mathcal{E}_1$ is a homogeneous point* i.e., f preserves the pencil of lines through that point. It is possible that this assumption is valid for any holomorphic map of \mathbb{P}^2 . Our goal is to prove Theorem A' and to exhibit totally invariant currents associated with the sets \mathcal{E}_1 and \mathcal{E}_2 .

10.1. Local dynamics near \mathcal{E}_2 .

Near the points of \mathcal{E}_2 , the dynamics has a simple form and this will allow us to prove good volume estimates.

LEMMA 10.1. — *Assume p is a homogeneous point. Then f is locally conjugate at p to a map of the form*

$$(10.1) \quad (z, w) \mapsto (P(z, w), Q(z, w)),$$

where P, Q are homogeneous polynomial of degree d .

LEMMA 10.2. — *Assume $p \in \mathcal{E}_2 \cap \mathcal{E}_1$. Then f is locally conjugate at p to a map of the form*

$$(10.2) \quad (z, w) \mapsto (z^d + wh(z, w), w^d),$$

where h is holomorphic.

Proof of Lemma 10.1. — Assume that $p = [0 : 0 : 1]$. In homogeneous coordinates, f can be written $f[z : w : t] = [P(z, w) : Q(z, w) : R(z, w, t)]$ for homogeneous polynomials P, Q, R of degree d with $R(0, 0, 1) = 1$. Hence, locally, $f(z, w) = (P(z, w)(1 + \eta), Q(z, w)(1 + \eta))$ for some germ η with $\eta(0, 0) = 1$. As f is contracting, one can define the map

$$\phi := \prod_{j=0}^{\infty} (1 + \eta \circ f^j)^{1/d^{j+1}}$$

and one checks the map $(z, w) \rightarrow (z\phi(z, w), w\phi(z, w))$ conjugates f to (P, Q) . □

Proof of Lemma 10.2. — Again assume $p = [0 : 0 : 1]$. We may assume that the set \mathcal{E}_1 is given by $zw = 0$ or by $w = 0$. In the first of these cases, p is a homogeneous point and f is locally conjugate to (z^d, w^d) by Lemma 10.1. In the second case, we have

$$f[z : w : t] = [z^d + wQ(z, w, t) : w^d : R(z, w, t)]$$

in homogeneous coordinates, where $R(0, 0, 1) = 1$. Hence, locally, $f(z, w) = ((z^d + wQ(z, w))(1 + \eta), w^d(1 + \eta))$ for some germ η with $\eta(0, 0) = 1$. As in the proof of Lemma 10.1 we define

$$\phi := \prod_{j=0}^{\infty} (1 + \eta \circ f^j)^{1/d^{j+1}}$$

and conclude that the map $(z, w) \rightarrow (z\phi(z, w), w\phi(z, w))$ conjugates f to the desired form. □

COROLLARY 10.3. — *There exists $\alpha > 0$ such that $c_n(p) \geq \alpha d^n$ for any $p \in \mathcal{E}_2$.*

Proof. — This follows immediately from the normal forms in Lemma 10.1 and Lemma 10.2. □

PROPOSITION 10.4. — *Let $p \in \mathcal{E}_2$ and let Ω be a small neighborhood of p . Then for any Borel set $E \subset \Omega$ of positive volume $\text{Vol}(E) > 0$, there exists $\gamma(E) > 0$ such that*

$$(10.3) \quad \text{Vol } f^n E \geq \gamma(E)^{d^n} \quad \text{for all } n \geq 0.$$

Proof. — We first consider the case $p \in \mathcal{E}_2 \cap \mathcal{E}_1$ and write f in the skew product form (10.2), which we may rewrite as

$$(10.4) \quad (z, w) \mapsto (f_w(z), g(w)) = \left(\psi \prod_{i=1}^d (z - \alpha_i(w)), w^d \right),$$

where α_i are multi-valued with $\alpha_i(0) = 0$ and $\psi(0, 0) = 1$.

Fix $\varepsilon_0 > 0$ small. It follows from (10.4) that there exists a constant $c > 0$ such that for any $w \in \mathbb{D}(0, \varepsilon_0)$ and any Borel set $E'' \subset \mathbb{D}(0, \varepsilon_0)$ we have $\text{Aera } f_w E'' \geq c(\text{Aera } E'')^d$. Further, for any Borel set $E' \subset \mathbb{D}(0, \varepsilon_0)$ we have $\text{Aera } gE' \geq c(\text{Aera } E')^d$. Iterating these estimates yields

Aera $g^n E' \geq (c' \text{Aera } E')^{d^n}$ and Aera $f_w^n E'' \geq (c' \text{Aera } E'')^{d^n}$, where $f_w^n = f_{g^{n-1}(w)} \circ \dots \circ f_w$.

Now pick a Borel set $E \subset \Omega$ with $\text{Vol } E > 0$. After iterating forward we may assume that $E \subset \mathbb{D}^2(0, \varepsilon_0)$. For $w \in \mathbb{D}(0, \varepsilon_0)$ we write $E''_w = \{z \in \mathbb{D}(0, \varepsilon_0) \mid (z, w) \in E\}$. There exists $\delta > 0$ and a set $E' \in \mathbb{D}(0, \varepsilon_0)$ with $\text{Aera } E' > \delta$ such that $\text{Aera } E''_w > \delta$ for $w \in E'$. But then the previous estimates imply that $\text{Aera } g^n E' > (c' \delta)^{d^n}$ and $\text{Aera } f_w^n E_w > (c' \delta)^{d^n}$ for $w \in E'$, so by Fubini's Theorem we get $\text{Vol } f^n E \geq \gamma^{d^n}$ as desired.

The remaining case, when $p \in \mathcal{E}_2$ is a homogeneous point, is similar. We use the skew product structure (10.1). The only new observation that we need is that if $g : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is a rational map of degree $d \geq 2$, then there exists $c > 0$ such that $\text{Aera } g^n E \geq (c \text{Aera } E)^{d^n}$ for any Borel set E . \square

After these preliminaries we now prove Theorem A'.

Proof of Theorem A'. — The implication (1) \Rightarrow (2) is relatively easy. If S puts mass on a totally invariant curve $V \subset \mathcal{E}_1$, say $S \geq c[V]$, then for all $n \geq 0$ we have $d^{-n} f^{n*} S \geq d^{-n} f^{n*} c[V] = c[V]$. Since T has bounded potential we cannot have convergence towards T . Similarly, if $p \in \mathcal{E}_2$ with $\nu(p, S) > 0$, then one immediately checks that $\nu(p, d^{-n} f^{n*} S) \geq d^{-n} c_n \nu(S, p)$. Hence, by Corollary 10.3, $\nu(p, d^{-n} f^{n*} S) \geq \varepsilon$ for some $\varepsilon > 0$, which also prevents the sequence to converge towards T .

Conversely, suppose that the current S satisfies (2) of Theorem A'. To prove that $d^{-n} f^{n*} S \rightarrow T$ we follow the proof of Theorem A up to the third case, i.e. when $f^n B \subset \Omega_2$ for all $n \geq 0$. We pick a constant $\varepsilon \ll 1$ small enough. As $\nu(p, S) = 0$ for all $p \in \mathcal{E}_2$, by Theorem 2.2 one can find a constant $C_\varepsilon > 0$ such that

$$(10.5) \quad \text{Vol } \{u \leq -t\} \leq C_\varepsilon \exp(-t/\varepsilon),$$

for all $t \geq 0$. Combining (10.5) and the hypothesis $f^{n_j} B \subset \{u < -\alpha d^{n_j}\}$ with Proposition 10.4, we get

$$\gamma(B)^{d^{n_j}} \leq C_\varepsilon \exp(-\alpha d^{n_j} / \varepsilon),$$

hence $\gamma(B) \leq \exp(-\alpha/\varepsilon)$ by letting $n_j \rightarrow +\infty$. But $\gamma(B) > 0$ is fixed and ε is arbitrarily small, so this yields a contradiction. \square

10.2. Totally invariant currents.

Let us discuss the existence of totally invariant currents. Consider the cone \mathcal{S} of positive closed currents on \mathbb{P}^2 of unit mass such that $d^{-1}f^*T = T$. Let \mathcal{S}^e be the set of extremal points in \mathcal{S} . It is known [FS4] that the Green current T is in \mathcal{S}^e (this follows e.g. from Theorem A).

The following result is an immediate consequence of Theorem A.

COROLLARY 10.5. — *If $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$, then $\mathcal{S}^e = \mathcal{S} = \{T\}$.*

Conversely we want to show that if either \mathcal{E}_1 or \mathcal{E}_2 is nonempty, then \mathcal{S}^e contains currents other than T . Recall that the Green current T has zero Lelong number at every point and, in particular, does not put mass on any curve in \mathbb{P}^2 .

PROPOSITION 10.6. — *If the first exceptional set \mathcal{E}_1 is nonempty, then there exists a current $S \in \mathcal{S}^e$ supported on \mathcal{E}_1 .*

Proof. — Since \mathcal{E}_1 is totally invariant, the current $(\deg \mathcal{E}_1)^{-1}[\mathcal{E}_1]$ is in \mathcal{S} . This current need not be extremal, but can be decomposed into currents in \mathcal{S}^e supported on \mathcal{E}_1 . □

In the sequel, $\|S\| := \int S \wedge \omega$ denotes the projective mass of the positive closed current S .

PROPOSITION 10.7. — *If $p \in \mathcal{E}_2 \setminus \mathcal{E}_1$ is a homogeneous point, then there exists $S \in \mathcal{S}^e$ with positive Lelong number at p and with continuous potential outside p . More precisely, we have $\nu(p, S) = \|S\| = 1$.*

Proof. — Assume f preserves the pencil of lines through p . Then f induces a rational map g of \mathbb{P}^1 (the set of lines) of degree d . Let μ be the measure of maximal entropy for g . This satisfies $g^*\mu = d\mu$ and if we define

$$S = \int_{a \in \mathbb{P}^1} [L_a] d\mu(a),$$

where $[L_a]$ denotes the current of integration on the line through p corresponding to $a \in \mathbb{P}^1$, then $f^*S = dS$, so $S \in \mathcal{S}$. Assume $S_1 \leq S$ with $f^*S_1 = dS_1$. From the local structure of S , we infer the existence of a positive measure μ_1 such that $S_1 = \int_{a \in \mathbb{P}^1} [L_a] d\mu_1(a)$. The equation $f^*S_1 = dS_1$ is equivalent to $f^*\mu_1 = d\mu_1$. As μ_1 has no atoms, this forces

$\mu_1 = c\mu$ for some constant $c > 0$. Hence $S_1 = cS$, and $S \in \mathcal{S}^e$. Finally, a direct computation yields $\nu(p, S) = 1$. □

PROPOSITION 10.8. — *If $p \in \mathcal{E}_2 \cap \mathcal{E}_1$, then there exists $S \in \mathcal{S}^e$ with positive Lelong number at p and with continuous potential outside p .*

More precisely, for any positive real numbers $0 < \alpha \leq 1$, there exists a current $S_\alpha \in \mathcal{S}^e$ with

$$\nu(p, S_\alpha, (1, \alpha/d^{-1})) = \|S_\alpha\| = 1.$$

Proof. — Pick $\alpha \in]0, 1[$. Introduce the homogeneous real analytic function on $\mathbb{C}^3 \setminus \{0\}$

$$U(z, w, t) := |z|^d + |w|^d + |w|^\alpha |t|^{d-\alpha}.$$

It vanishes exactly on the ray $\mathbb{C} \cdot (0, 0, 1)$ and by homogeneity the positive closed current $d^{-1}dd^c \log U$ can be pushed down to \mathbb{P}^2 as a current ω_0 , smooth outside p , with a pole at p whose Lelong number is $\nu(\omega_0, p) = 1$.

Define

$$V(z, w, t) := \frac{U \circ F}{U^d}$$

where $F = (z^d + wQ, w^d, R)$ is a lift a f to \mathbb{C}^3 . Then V induces by projection a function on \mathbb{P}^2 which is real analytic outside p . We claim there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq V \leq C_2$$

for any point in \mathbb{P}^2 .

Indeed, as p is totally invariant, U and $U \circ F$ both vanishes exactly along $\mathbb{C} \cdot (0, 0, 1)$, hence the inequality has to be checked only in a neighborhood V of p in the chart $\{t = 1\}$. To do so, you may decompose V in two sets $\{|z|^d \leq A|w|^\alpha\}$ and $\{|z|^d \geq A|w|^\alpha\}$ for a well chosen A . In each of these sets, the estimates follow from a direct computation we leave to the reader. By normalizing U , we can assume $C_2 = 1$.

We now follow the standard construction of the Green current. We have $f^*\omega_0 = d\omega_0 + dd^c \log V$, hence

$$d^{-k} f^{k*} \omega_0 = \omega_0 + dd^c \left(\sum_{j=0}^k d^{-j-1} \circ f^j \right)$$

for all $k \geq 0$. The sequence of function $\sum_{j=0}^k d^{-j-1} \circ f^j$ is decreasing converging uniformly on compact sets. Hence the limit G_p is a L^1 function, continuous outside p , and bounded everywhere. The positive closed current $S := \omega_0 + dd^c G_p$ belongs to \mathcal{S} , has a continuous potential outside p , and a singularity at p with Lelong number $\nu(p, S) = 1/d$. More precisely, in the coordinates z, w , the Kiselman number of S with weight $(1, \alpha/d^{-1})$ is given by

$$\nu(p, S, (1, \alpha/d^{-1})) = 1.$$

To conclude we show S is extremal in the cone \mathcal{S} . Assume $S = S_1 + S_2$ with $f^* S_i = dS_i$ for $i = 1, 2$. Kiselman numbers behave additively hence

$$\nu(p, S_1, (1, \alpha/d^{-1})) + \nu(p, S_2, (1, \alpha/d^{-1})) = \nu(p, S, (1, \alpha/d^{-1})) = 1.$$

On the other hand, the following inequalities are standard (see [K1], [D1]):

$$\nu(p, S_i, (1, \alpha/d^{-1})) \leq \nu(p, S_i) \leq \|S_i\|.$$

Whence

$$1 = \nu(p, S_1, (1, \alpha/d^{-1})) + \nu(p, S_2, (1, \alpha/d^{-1})) \leq \|S_1\| + \|S_2\| = \|S\| = 1$$

and we infer

$$(10.6) \quad \nu(p, S_i, (1, \alpha/d^{-1})) = \|S_i\|$$

for $i = 1, 2$. Pick a global potential $S_i = dd^c u_i$ defined in \mathbb{C}^2 . By definition of Kiselman numbers we have in a neighborhood of the origin

$$u_i - \nu(p, S_i, (1, \alpha/d^{-1})) \times d^{-1} \log(|z^d| + |w^\alpha|) \leq C$$

for some constant $C > 0$. Together with equation (10.6), we deduce that $v_i := u_i - \|S_i\| d^{-1} \log(|z|^d + |w|^\alpha + |w|^d)$ is globally bounded from above. As $u_1 + u_2 = G_p + d^{-1} \log(|z|^d + |w|^\alpha + |w|^d)$, and G_p is bounded, we conclude that v_1, v_2 are also bounded everywhere. Hence

$$S_i = \lim_{n \rightarrow \infty} d^{-n} f^{n*} S_i = \lim_{n \rightarrow \infty} dd^c(d^{-n} v_i \circ f^n) + \|S_i\| d^{-n} f^{n*} \omega_0 = \|S_i\| \cdot S,$$

showing that $S \in \mathcal{S}^e$. □

Example 10.9. — For the map $f[z : w : t] = [z^d : w^d : t^d]$, the set \mathcal{S}^e is quite large. Given $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$ define

$$u_{\alpha, \beta, \gamma}(z, w, t) = \alpha \log |z| + \beta \log |w| + \gamma \log |t|.$$

Then \mathcal{S}^e contains all the currents $S = \omega + dd^c u$ with

$$u(z, w, t) = \max_{i=1,2} u_{\alpha_i, \beta_i, \gamma_i}(z, w, t) - \log |(z, w, t)|$$

such that

$$\min_{i=1,2} \alpha_i = \min_{i=1,2} \beta_i = \min_{i=1,2} \gamma_i = 0$$

or

$$u(z, w, t) = \max_{i=1,2,3} u_{\alpha_i, \beta_i, \gamma_i}(z, w, t) - \log |(z, w, t)|$$

such that

$$\min_{i=1,2,3} \alpha_i = \min_{i=1,2,3} \beta_i = \min_{i=1,2,3} \gamma_i = 0.$$

Notice that the Green current T is of the latter form.

10.3. Configurations of exceptional sets.

We conclude the paper by listing the different possible configurations of the exceptional sets \mathcal{E}_1 and \mathcal{E}_2 and the corresponding mappings f in case $\mathcal{E}_2 \setminus \mathcal{E}_1$ contains only homogeneous points. The case of totally invariant curves was treated in [FS3] (see Proposition 1.1). We summarize the results in Table 1.1 below.

- P, Q, R denotes homogeneous polynomials in three variables z, w, t except if we state it otherwise;

- p_z, p_w and p_t denote the points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$, respectively;

- Z, W and T denote the lines $(z = 0)$, $(w = 0)$ and $(t = 0)$, respectively;

- in the cases where $\#\mathcal{E}_1 \geq 2$ or $\#\mathcal{E}_2 \geq 2$, we mention only the maps preserving all irreducible components of \mathcal{E}_1 and each point in \mathcal{E}_2 . To be complete, one has to add maps which permute these sets.

The proof is essentially elementary. There are essentially only two points to check: any intersection point between two irreducible components of \mathcal{E}_1 is in \mathcal{E}_2 ; and if \mathcal{E}_2 contains two homogeneous points p, q , then \mathcal{E}_1

contains the line H passing through p and q . The first of these statements is easy; for the second note that $f^{-1}H$ is a union of lines passing through p as p is homogeneous, and also a union of lines passing through q . Hence $f^{-1}H = H$ is a totally invariant line. If we blow up \mathbb{P}^2 at the two points p, q , we can lift f to a holomorphic map for which the strict transform of H is totally invariant. We can hence contract it to a point, and the induced map becomes a holomorphic map on $\mathbb{P}^1 \times \mathbb{P}^1$. If $\{p, q\} = \{p_z, p_w\}$, this shows f can be written under the form $f = [P(z, t) : Q(w, t) : t^d]$. The other cases can be treated in a similar way.

Note that we in particular have (assuming $\mathcal{E}_2 \setminus \mathcal{E}_1$ contains only homogeneous points):

PROPOSITION 10.10. — *There are at most 3 distinct points in \mathcal{E}_2 .*

TABLE 1. — Configuration of exceptional sets.

$\#\mathcal{E}_1$	$\#\mathcal{E}_2$	\mathcal{E}_1	\mathcal{E}_2	f
1	0	T		$[P : Q : t^d]$
0	1		p_w	$[P(z, t) : Q : R(z, t)]$
1	1	T	p_z	$[P : w^d + tQ : t^d]$
		T	p_t	$[P(z, w) : Q(z, w) : t^d]$
1	2	T	p_z, p_w	$[P(z, t) : w^d + tQ : t^d]$
2	1	W, T	p_z	$[P : w^d : t^d]$
2	2	W, T	p_z, p_w	$[z^d + tP : w^d : t^d]$
2	3	W, T	p_z, p_w, p_t	$[z^d + wtP : w^d : t^d]$
3	3	Z, W, T	p_z, p_w, p_t	$[z^d : w^d : t^d]$

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