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GAUSS-MANIN CONNECTIONS OF SCHLÄFLI TYPE FOR HYPERSPHERE ARRANGEMENTS

by Kazuhiko AOMOTO

1. Introduction.

The theory of the hypergeometric integrals associated with hyperplane arrangements has been developed by many authors, like P. Orlik, H. Terao , A. Varchenko, M. Yoshida, etc (see [9], [15]). An arrangement of one hypersphere and many hyperplanes is also interesting from geometric and combinatorial point of view. If we restrict this to the hypersphere, we have a general hypersphere arrangement. The purpose of this note is to present the Gauss-Manin connection of the hypergeometric integrals associated with a "generic" hypersphere arrangement in the unit hypersphere in an invariant form. This expression can be regarded as a natural extension of the classical Schläfli formula for a geodesic spherical simplex. The author has given in [1] various formulae about the hypergeometric integrals involved in quadratic exponentials. This note heavily depends on it. Especially we cite often the basic variational formulae stated in Propositions 1.3 and 2.3, and their consequences derived from contiguity relations stated in [1] I.

We give here a further extension and an application of some results in [1] in the case of hypersphere arrangements (See Theorems 5-8) . Finally we raise some related questions (Questions 1 and 2).

Keywords: Hypersphere arrangements – Twisted cohomology – Gauss-Manin connection.
Math. classification: 32S22 – 32S40 – 33C70 – 39B32.

2. Rational twisted de Rham cohomology (terminologies and basic facts).

Consider the n dimensional complex hypersphere $Q : f_0(x) = 0$ in the complex affine space $\mathbb{C}^{n+1}$ for $f_0(x) = 1 + x_1^2 + \cdots + x_{n+1}^2$, $x = (x_1, \cdots, x_{n+1})$.

Let $\mathcal{A} = \{H_j\}_{1 \leqslant j \leqslant m}$ be an arrangement of hyperplanes H_j defined by $f_j(x) = 0$, for $f_j(x) = \sum_{\nu=1}^{n+1} u_{j\nu} x_\nu + u_{j0}$.

The intersection $\mathcal{A}_Q = \{Q \cap H_j\}_{1 \leqslant j \leqslant m}$ defines an arrangement of $(n-1)$ dimensional hyperspheres in Q.

We denote by a_{ij} $(1 \leqslant i, j \leqslant m)$, $a_{i0} = a_{0i}$ $(1 \leqslant i \leqslant m)$, the inner product of f_i and f_j: $a_{ij} = \sum_{\nu=0}^{n+1} u_{i\nu} u_{j\nu}$, and $a_{i0} = u_{i0}$ respectively. We put $a_{00} = 1$. These quantities are invariant under the action of $O(n+1, \mathbb{C})$, the complex orthogonal group, which acts linearly in $\mathbb{C}^{n+1}$. Normalize f_j such that $a_{jj} = 1$ for all j.

Let A be the $(m+1) \times (m+1)$ symmetric matrix whose (i,j) entries are a_{ij} $(0 \leqslant i, j \leqslant m)$. We call A the configuration matrix associated with $\mathcal{A}$.

For a pair of the p tuples of different ordered indices $I = \{i_1, \cdots, i_p\}$ and $J = \{j_1, \cdots, j_p\}$ $(1 \leqslant i_\mu, j_\nu \leqslant m)$ the subdeterminants $A\binom{I}{J}$, $A\binom{0,\ I}{0,\ J}$ are defined as

$$A\begin{pmatrix} I \\ J \end{pmatrix} = \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_p} \\ \vdots & & \vdots \\ a_{i_p j_1} & \cdots & a_{i_p j_p} \end{vmatrix}, \quad A\begin{pmatrix} 0, & I \\ 0, & J \end{pmatrix} = \begin{vmatrix} a_{00} & a_{0j_1} & \cdots & a_{0j_p} \\ a_{i_1 0} & a_{i_1 j_1} & \cdots & a_{i_1 j_p} \\ \vdots & \vdots & & \\ a_{i_p 0} & a_{i_p j_1} & \cdots & a_{i_p j_p} \end{vmatrix}.$$

In the same way, for $I = \{i_1, \cdots, i_{p-1}\}$ and $J = \{j_1, \cdots, j_p\}$, we denote by $A\binom{0,\ I}{J}$ the subdeterminant

$$\begin{vmatrix} a_{0j_1} & \cdots & a_{0j_p} \\ a_{i_1 j_1} & \cdots & a_{i_1 j_p} \\ \vdots & & \vdots \\ a_{i_{p-1} j_1} & \cdots & a_{i_{p-1} j_p} \end{vmatrix}.$$

The determinants

$$[I] = \begin{cases} \det\left((u_{i_\mu \nu})_{\substack{1 \leqslant \mu \leqslant n+1 \\ 1 \leqslant \nu \leqslant n+1}} \right) & \text{for } I = \{i_1, \cdots, i_{n+1}\} \\[2ex] \det\left((u_{i_\mu \nu})_{\substack{1 \leqslant \mu \leqslant n+2 \\ 0 \leqslant \nu \leqslant n+1}} \right) & \text{for } I = \{i_1, \cdots, i_{n+2}\} \end{cases}$$

are identified with $\pm\sqrt{A(0,I)}$ and $\pm\sqrt{A(I)}$ respectively.

For $I=\{i_1,\cdots,i_p\}$, $\partial_\nu I$ denotes the deletion of the $\nu-$th element i_ν from $I:\{i_1,\cdots,i_{\nu-1},i_{\nu+1},\cdots,i_p\}$ and $\{j,I\}$ denotes the addition of the new index j to $I:\{j,i_1,\cdots,i_p\}$. $|I|$ denotes p the size of I.

Let $X=M(\mathcal{A})$ and $Y=M_Q(\mathcal{A})$ be the complements $X=\mathbb{C}^{n+1}-\sum_{j=1}^m H_j$ and $Y=Q-\sum_{j=1}^m Q\cap H_j$ respectively.

We consider the twisted rational cohomology $H^*(X,\nabla)$ in X with respect to the covariant differentiation

$$\nabla\psi=d\psi+\omega\wedge\psi,\quad \omega=\lambda_0\frac{df_0}{f_0}+\sum_{j=1}^m\lambda_j\frac{df_j}{f_j}$$

and the twisted rational cohomologies $H^*(X,\nabla_0)$ in X , $H^*(Y,\nabla_0)$ in Y with respect to the covariant differentiation

$$\nabla_0\psi=d\psi+\omega_0\wedge\psi,\quad \omega_0=\sum_{j=1}^m\lambda_j\frac{df_j}{f_j}$$

respectively.

We assume

($\mathcal{H}1$) (i) $A(I),A(0,J)\neq 0$ for all $I,J\subset\{1,2,\cdots,m\}$ such that $|I|\leqslant n+2$, $|J|\leqslant n+1$ respectively. (Remark that $A(I)=0$ if $|I|\geqslant n+3$ and $A(0,I)=0$ if $|I|\geqslant n+2$.)

(ii) $\sum_{j\in I}\lambda_j\notin\mathbb{Z}$ for all $I\subset\{1,2,\cdots,m\}$ and $2\lambda_0+\lambda_\infty\notin\mathbb{Z}$ where λ_∞ denotes $\sum_{j=1}^m\lambda_j$.

Then the following facts have essentially proved in [1] I (see also [5], [12] for more general schemes).

THEOREM 1.

$$H^p(X-Y;\nabla)\cong 0\qquad 0\leqslant p\leqslant n \tag{2.1}$$

$$H^{n+1}(X-Y;\nabla)\cong\mathbb{C}^l\quad \textit{where } l=\sum_{\nu=0}^{n+1}\binom{m}{\nu}. \tag{2.2}$$

THEOREM 2.

(i)

$$H^p(X;\nabla_0)\cong 0\qquad 0\leqslant p\leqslant n \tag{2.3}$$

$$H^{n+1}(X;\nabla_0)\cong\mathbb{C}^{l_0}\quad l_0=\binom{m-1}{n+1}. \tag{2.4}$$

(ii)

$$H^p(Y;\nabla_0) \cong 0 \qquad 0 \leqslant p \leqslant n-1 \tag{2.5}$$

$$H^n(Y;\nabla_0) \cong \mathbb{C}^{l_1} \qquad l_1 = \sum_{\nu=0}^{n} \binom{m}{\nu} + \binom{m-1}{n}. \tag{2.6}$$

As for the relative cohomology, we have

THEOREM 3.

$$H^p(X,Y;\nabla_0) \cong 0 \qquad 0 \leqslant p \leqslant n \tag{2.7}$$

$$H^{n+1}(X,Y;\nabla_0) \cong H^{n+1}(X;\nabla_0) \oplus H^n(Y;\nabla_0) \tag{2.8}$$

so that

$$H^{n+1}(X,Y;\nabla_0) \cong \mathbb{C}^l \tag{2.9}$$

where $l = l_0 + l_1$.

A basis of $H^{n+1}(X-Y;\nabla)$ can be given as

$$\varphi(I) = \frac{\tau}{f_{i_1} f_{i_2} \cdots f_{i_p}}$$

for $I = \{i_1, \cdots, i_p\}$, $1 \leqslant i_1 < \cdots < i_p \leqslant m$, $0 \leqslant p \leqslant n+1$, where

$$\tau = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}.$$

We denote by $\varphi(\emptyset)$ the $\varphi(I)$ when I is empty.

On the other hand, a basis of $H^{n+1}(X;\nabla_0)$ can be given by $\varphi(I)$ for $I = \{i_1, \ldots, i_{n+1}\}$, $|I| = n+1$,

$$\varphi(I) = [I]^{-1} d\log f_{i_1} \wedge \cdots \wedge df_{i_{n+1}}$$

with the following fundamental relations:

$$\omega_0 \wedge d\log f_{j_1} \wedge \cdots \wedge \log f_{j_n} \sim 0 \ \text{ in } H^{n+1}(X;\nabla_0) \tag{2.10}$$

for arbitrary indices $J = \{j_1, \cdots, j_n\}$, $1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant m$.

THEOREM 4. — *A basis of $H^n(Y;\nabla_0)$ can be given by*

$$\varphi_Q(I) = \frac{\tau_Q}{f_{i_1} f_{i_2} \cdots f_{i_p}}$$

for $I = \{i_1, \cdots, i_p\}$, $1 \leqslant i_1 < \cdots < i_p \leqslant m$, $0 \leqslant p \leqslant n+1$, where τ_Q denotes the invariant n-form

$$\tau_Q = 2\left[\frac{\tau}{df_0}\right]_{f_0=0} = \sum_{\nu=1}^{n+1} (-1)^\nu x_\nu dx_1 \wedge \cdots \wedge dx_{\nu-1} \wedge dx_{\nu+1} \wedge \cdots \wedge dx_{n+1}.$$

The fundamental relations are written as follows:

For arbitray $(n+2)$ indices $J = \{j_1, \cdots, j_{n+2}\}$,

$$\begin{aligned}(2.11)\qquad &\frac{1}{2}\sum_{1\leqslant\mu\neq\nu\leqslant n+2}(-1)^{\mu+\nu}\varphi_Q(\partial_\mu\partial_\nu J)\frac{A(0,\partial_\mu\partial_\nu J)}{A\binom{0,\partial_\mu J}{0,\partial_\nu J}}\\ &+\sum_{\mu=1}^{n+2}(-1)^\mu\varphi_Q(\partial_\mu J)\frac{A(\partial_\mu J)}{A\binom{0,\partial_\mu J}{J}} = 0.\end{aligned}$$

Proof 1. — There exist uniquely the constants $c_{\mu,\nu}$ and c_μ such that

$$f_0(x) = \sum_{1\leqslant\mu,\nu\leqslant n+2,\mu\neq\nu} c_{\mu,\nu}f_{j_\mu}(x)f_{j_\nu}(x) + \sum_{\mu=1}^{n+2} 2c_\mu f_{j_\mu}(x).$$

This identity gives (2.11). See (4.9), (4.11) in [1] I.

The Jacobi identities show

$$A(0,\partial_\mu J)A(0,\partial_\nu J) = A\begin{pmatrix}0,\partial_\mu J\\ 0,\partial_\nu J\end{pmatrix}^2,\ A(0,\partial_\mu J)A(J) = A\begin{pmatrix}0,\partial_\mu J\\ J\end{pmatrix}^2$$

since $A(0,J) = 0$.

For general $I = \{i_1, \cdots, i_p\}$, we denote further

$$\varphi_*(I) = \varphi(I) + \sum_{\nu=1}^{p}(-1)^\nu A\begin{pmatrix}0,\partial_\nu I\\ I\end{pmatrix}\varphi(\partial_\nu I)$$

$$\varphi_{Q,*}(I) = \varphi_Q(I) + \sum_{\nu=1}^{p}(-1)^\nu A\begin{pmatrix}0,\partial_\nu I\\ I\end{pmatrix}\varphi_Q(\partial_\nu I)$$

whence

$$\left[\frac{\varphi(I)}{df_0}\right]_{f_0=0} = \frac{1}{2}\varphi_Q(I),\quad \left[\frac{\varphi_*(I)}{df_0}\right]_{f_0=0} = \frac{1}{2}\varphi_{Q,*}(I).$$

As a consequence of Theorem 4

COROLLARY 1. — *Another basis of $H^n(Y,\nabla_0)$ can be given by the n-forms:*

$\varphi_Q(I)$, $|I| \leqslant n$ *and the logarithmic forms* $d\log f_{i_1}\wedge\cdots\wedge d\log f_{i_n}$

with the fundamental relations

$$(2.12)\qquad 0 \sim \omega_0 \wedge d\log f_{j_1}\wedge\cdots\wedge d\log f_{j_{n-1}}$$

for arbitrary indices $J = \{j_1, \cdots, j_{n-1}\}$.

Proof 2. — See (4.12) [1] I.

(2.11) and (2.12) are equivalent to each other, since we have the relations in $H^n(Y, \nabla_0)$, for $I = \{i_1, \cdots, i_n\}$,

$$\lambda_\infty d \log f_{i_1} \wedge \cdots \wedge d \log f_{i_n} \sim - \sum_{j \notin I, 1 \leqslant j \leqslant m} \lambda_j [j, I] \varphi_{Q,*}(j, I) \tag{2.13}$$

and conversely for $I = \{i_1, \cdots, i_{n+1}\}$,

$$[I] \frac{A(I)}{A(0,I)} \varphi_{Q,*}(I) \sim \sum_{\nu=1}^{n+1} (-1)^\nu d \log f_{i_1} \wedge \cdots \langle i_\nu \rangle \cdots \wedge d \log f_{i_{n+1}}. \tag{2.14}$$

3. Twisted cycles.

We put $x_j = \sqrt{-1}\xi_j$ and $u_{j0} = \sqrt{-1}u'_{j0}$, $(u'_{j0}, u_{j\nu} \in \mathbb{R})$ so that

$$f_j(x) = \sqrt{-1} f'_j(\xi) \tag{3.1}$$

where $f'_j(\xi) = \sum_{\nu=1}^{n+1} u_{j\nu}\xi_\nu + u'_{j0}$ define real polynomials.

The hypersphere Q is expressed as

$$Q : f'_0(\xi) = 1 - \xi_1^2 - \cdots - \xi_{n+1}^2 = 0.$$

The real part $\Re Q$ is defined to be the subset of Q such that all the ξ_j are real.

We define the real symmetric configuration matrix $A' = (a'_{ij})_{0 \leqslant i,j \leqslant m}$ with the entries

$$a'_{ij} = \sum_{\nu=1}^{n+1} u_{i\nu} u_{j\nu} - u'_{i0} u'_{j0} \quad (1 \leqslant i, j \leqslant m)$$
$$a'_{i0} = u'_{i0}, \ a'_{00} = -1.$$

The subdeterminants $A'\binom{I}{J}$, $A'\binom{0,I}{0,J}$, $A'\binom{0,I}{J}$, $A'\binom{I}{0,J}$ are defined as before. Hence $A'\binom{I}{J} = A\binom{I}{J}$, $\sqrt{-1}A'\binom{0,I}{J} = A\binom{0,I}{J}$, $-A'\binom{0,I}{0,J} = A\binom{0,I}{0,J}$, $A'(I) = A(I)$, $-A'(0,I) = A(0,I)$.

Assume now

($\mathcal{H}2$) For all I $(1 \leqslant |I| \leqslant n+1)$, $A'(I) > 0$ (remark that $-A'(0,I) > A'(I)$).

Under the hypotheses ($\mathcal{H}1$), ($\mathcal{H}2$) we have

PROPOSITION 1. — *The real hyperplane arrangement* $\mathcal{A}' = \{H'_j\}_{1\leqslant j\leqslant m}$ *gives the real* $(n-1)$ *dimensional hypersphere arrangement* $\{H'_j \cap \Re Q\}_{1\leqslant j\leqslant m}$ *in* Q.

Let $\mathcal{L}_{-\omega}$ and $\mathcal{L}_{-\omega_0}$ be the dual of the local systems in $X-Y$ and Y which are defined by the functions U' and U'_0:

$$U'(\xi) = f'_0(\xi)^{\lambda_0} f'_1(\xi)^{\lambda_1} \cdots f'_m(\xi)^{\lambda_m}$$
$$U'_0(\xi) = f'_1(\xi)^{\lambda_1} \cdots f'_m(\xi)^{\lambda_m}$$

respectively. Then $H^{n+1}(X-Y;\nabla)$ and $H_{n+1}(X-Y;\mathcal{L}_{-\omega})$ gives the perfect pairing of each other by integration,

$$H^{n+1}(X-Y;\nabla) \times H_{n+1}(X-Y;\mathcal{L}_{-\omega}) \longrightarrow \mathbb{C}$$
$$(3.2) \qquad (\varphi\ ,\ \mathcal{C}) \longrightarrow \hat{\varphi} = \int_{\mathcal{C}} U'\varphi$$

where φ and $\mathcal{C}$ denote an $(n+1)$ dimensional form and an $(n+1)$ dimensional twisted cycle in $X-Y$.

Similarly $H^{n+1}(Y;\nabla_0)$ and $H_n(Y;\mathcal{L}_{-\omega_0})$ gives the perfect pairing of each other

$$H^{n}(Y;\nabla_0) \times H_{n}(Y;\mathcal{L}_{-\omega_0}) \longrightarrow \mathbb{C}$$
$$(3.3) \qquad (\varphi\bar{Q}\ ,\ \mathcal{C}) \longrightarrow \hat{\varphi}\bar{Q} = \int_{\mathcal{C}} U'_0\varphi_Q.$$

The common parts of non-compact connected components of $\mathbb{R}^{n+1} - \cup_{j=1}^m H'_j$ with Q gives a basis of twisted cycles in $H_n(Y;\mathcal{L}_{-\omega_0})$.

The intersection of the closure of a non-compact connected component with Q corresponds exactly to a connected component of $\Re Q - \cup_{j=1}^m \Re Q \cap H'_j$. The number of such connected components is equal to l_1 which is also equal to $|\chi(Q - \cup_{j=1}^m Q \cap H'_j)|$ the absolute value of the Euler number of $Q - \cup_{j=1}^m Q \cap H'_j$.

Let Δ be a chamber in $X-Y$ which is a common part of the inside of $\Re Q$ and a non-compact chamber of $\mathcal{A}'$.

We consider the hypergeometric pairings

$$(3.4) \qquad \widehat{\varphi'}(I) = \int_{\Delta} U'(\xi)\varphi'(I)$$

$$(3.5) \qquad \widehat{\varphi'_Q}(I) = \int_{\partial\Delta\cap Q} U'_0(\xi)\varphi'_Q(I)$$

for

$$\varphi'(I) = \frac{\tau'}{f'_{i_1}\cdots f'_{i_p}} \quad (I = \{i_1, \cdots, i_p\},\ 0 \leqslant p \leqslant n+1)$$

$$\varphi'_Q(I) = \frac{\tau'_Q}{f'_{i_1}\cdots f'_{i_p}} \quad (I = \{i_1, \cdots, i_p\},\ 0 \leqslant p \leqslant n+1)$$

where

$$\tau' = d\xi_1 \wedge \cdots \wedge d\xi_{n+1}$$

$$\tau'_Q = \sum_{\nu=1}^{n+1} (-1)^\nu \xi_\nu d\xi_1 \wedge \cdots \langle \nu \rangle \cdots \wedge d\xi_{n+1}$$

respectively.

Remark that

$$\lim_{\lambda_0 \to -1} -2(\lambda_0+1) \int_\Delta U'(\xi)\varphi'(I) = \int_{\partial\Delta\cap Q} U'_0(\xi)\varphi'_Q(I). \tag{3.6}$$

We have similar relations to (2.13) and (2.14) between

$$\varphi_Q(I) = \frac{\tau'_Q}{f'_{i_1}\cdots f'_{i_n}}, \quad I = \{i_1, \cdots, i_n\}$$

and the logarithmic forms

$$d\log f'_{j_1} \wedge \cdots \wedge d\log f'_{j_n}.$$

4. Basic invariant 1-forms and Gauss-Manin connections (main results).

The arrangement $\mathcal{A}'$ has a singualrity if and only if

$$A'(I) = 0 \quad \text{or} \quad A'(0, I) = 0 \tag{4.1}$$

for a certain $I = \{i_1, \cdots, i_p\} \subset \{1, 2, \cdots, m\}$ $p \leqslant n+2$ or $p \leqslant n+1$. These correspond to the singular loci

$$df'_0 \wedge df'_{i_1} \wedge \cdots \wedge df'_{i_p} = 0, \quad f'_0 = f'_{i_1} = \cdots = f'_{i_p} = 0 \tag{4.2}$$

or

$$df'_{i_1} \wedge \cdots \wedge df'_{i_p} = 0, \quad f'_{i_1} = \cdots = f'_{i_p} = 0. \tag{4.3}$$

This observation seems to come as early as from F. Pham's paper [11].

$\{\widehat{\varphi}'(I)\}_{I, 0 \leqslant |I| \leqslant n+1}$ give a basis of linearly independent integrals for $\widehat{\varphi}'$ of (3.4).

On the other hand, the integrals

$$\widehat{\widehat{\varphi}}'(I) = \sqrt{A(I)} \int U' f_0'^q \varphi'(I) \qquad \text{for } |I| = 2q \leqslant n+1 \tag{4.4}$$

$$\widehat{\widehat{\varphi}}'(I) = \sqrt{A(0,I)} \int U' f_0'^q \varphi'(I) \quad \text{for } |I| = 2q-1 \leqslant n+1 \tag{4.5}$$

give another equivalent system of (3.4). Since $\{\varphi'(I)\}_I$ and $\{f_0'^q \varphi'(I)\}_I$ give two bases in the same cohomology $H^{n+1}(X-Y;\nabla)$, the corresponding integrals $\{\widehat{\widehat{\varphi}}'(I)\}_I$ and $\{\widehat{\varphi}'(I)\}_I$ are connected with each other by contiguity relations.

The Gauss-Manin connection concerning $\widehat{\widehat{\varphi}}'(I)$ can be expressed by "closed 1-forms" as follows.

DEFINITION 1. — *We define the closed 1-forms*

(i)

$$\omega'\begin{pmatrix} I \\ I,j,k \end{pmatrix} = \frac{1}{2\sqrt{-1}}\, d\log\left(\frac{-b+\sqrt{b^2-1}}{-b-\sqrt{b^2-1}}\right)$$

or

$$\omega'\begin{pmatrix} I \\ I,j,0 \end{pmatrix} = \frac{1}{2}\, d\log\left(\frac{-b+\sqrt{b^2-1}}{-b-\sqrt{b^2-1}}\right)$$

for $b = \frac{A'\binom{I,j}{I,k}}{\sqrt{A'(I,j)A'(I,k)}}$, $I \subset \{0,1,2,\cdots,m\}$ *and* $j,k \notin I$, $j \neq k$ *according as* $j,k \geqslant 1$ *or* $j \geqslant 1$, $k=0$.

The closed 1-form

$$db = \left\{\frac{A'(I)A'(I,j,k)}{A'(I,j)A'(I,k)}\right\}^{\frac{1}{2}} \omega'\begin{pmatrix} I \\ I,j,k \end{pmatrix}$$

will be denoted by $\omega'\binom{I,j}{I,k}$ *or* $\omega'\binom{I,k}{I,j}$.

(ii)

$$\omega'\begin{pmatrix} I \\ I,j \end{pmatrix} = d\log\frac{A'(I)}{A'(I,j)}.$$

The fundamental hierarchical identities among them hold as a result of Jacobi identities:

(i) For $I \subset J$ and $|J| = |I| + 4$,

$$\sum_{I\subset K\subset J, |K|=|I|+2} \omega'\begin{pmatrix} I \\ K \end{pmatrix} \wedge \omega'\begin{pmatrix} K \\ J \end{pmatrix} = 0. \tag{4.6}$$

(ii)

$$\omega'\binom{I}{I,j,k}\wedge\left\{\omega'\binom{I,k}{I,j,k}-\omega'\binom{I}{I,j}\right\}=0. \tag{4.7}$$

Proof 3. — See $(R.I_1)$ and $(R.I_2)$ in [1] I.

Under these notations, we have

THEOREM 5. — *Regarded as functions of the configuration matrix A', the Gauss-Manin connections for $\widehat{\widehat{\varphi}}'(I)$ can be expressed as*

$$d\widehat{\widehat{\varphi}}'(I)=F_1+F_2+F_3+F_4+F_5+F_6 \tag{4.8}$$

where F_1,F_2,F_3,F_4,F_5,F_6 are given as follows:

Case $|I|=2q$

$$F_1=\frac{1}{4(\lambda_0+q+1)}\left\{\sum_{j,k\geqslant 1;j,k\notin I,j\neq k}\lambda_j\lambda_k\omega'\binom{I}{I,j,k}\widehat{\widehat{\varphi}}'(I,j,k)\right\}$$

$$F_2=-\frac{(2\lambda_0+\lambda_\infty+n+2)}{2(\lambda_0+q+1)}\left\{\sum_{j\geqslant 1;j\notin I}\lambda_j\omega'\binom{I}{I,j,0}\widehat{\widehat{\varphi}}'(I,j)\right\}$$

$$F_3=\frac{1}{2}\left\{-\sum_{\nu=1}^{2q}\lambda_{i_\nu}\omega'\binom{\partial_\nu I}{I}-\sum_{k\geqslant 1;k\notin I}\lambda_k\omega'\binom{I}{I,k}\right.$$
$$\left.+(2\lambda_0+\lambda_\infty+n+2)\omega'\binom{I}{I,0}\right\}\widehat{\widehat{\varphi}}'(I)$$

$$F_4=\frac{1}{2}(\lambda_0+q)\sum_{1\leqslant\mu,\nu\leqslant 2q,\mu\neq\nu}\omega'\binom{\partial_\mu\partial_\nu I}{I}\widehat{\widehat{\varphi}}'(\partial_\mu\partial_\nu I)$$

$$F_5=\sum_{k\geqslant 1;k\notin I}\sum_{1\leqslant\nu\leqslant 2q}\lambda_k\omega\binom{I}{\partial_\nu I,k}\widehat{\widehat{\varphi}}'(\partial_\nu I,k)$$

$$F_6=-(2\lambda_0+\lambda_\infty+n+2)\sum_{1\leqslant\nu\leqslant 2q}\omega\binom{I}{\partial_\nu I,0}\widehat{\widehat{\varphi}}'(\partial_\nu I).$$

Case $|I| = 2q-1$

$$F_1 = \frac{1}{4(\lambda_0+q)}\Bigg\{\sum_{j,k\geqslant 1; j,k\notin I, j\neq k}\lambda_j\lambda_k\omega'\begin{pmatrix}0,I\\0,I,j,k\end{pmatrix}\widehat{\widehat{\varphi}}'(I,j,k)\Bigg\}$$

$$F_2 = \frac{1}{2}\Bigg\{-\sum_{\nu=1}^{2q-1}\lambda_{i_\nu}\omega'\begin{pmatrix}0,\partial_\nu I\\0,I\end{pmatrix}-\sum_{k\geqslant 1;k\notin I}\lambda_k\omega'\begin{pmatrix}0,I\\0,I,k\end{pmatrix}$$

$$+(2\lambda_0+\lambda_\infty+n+2)\omega'\begin{pmatrix}I\\0,I\end{pmatrix}\Bigg\}\widehat{\widehat{\varphi}}'(I)$$

$$F_3 = (\lambda_0+q)\sum_{1\leqslant\mu,\nu\leqslant 2q-1,\mu\neq\nu}\omega'\begin{pmatrix}0,\partial_\mu\partial_\nu I\\0,I\end{pmatrix}\widehat{\widehat{\varphi}}'(\partial_\mu\partial_\nu I)$$

$$F_4 = -2(\lambda_0+q)\sum_{1\leqslant\nu,\nu\leqslant 2q-1}\omega'\begin{pmatrix}\partial_\nu I\\0,I\end{pmatrix}\widehat{\widehat{\varphi}}'(\partial_\nu I)$$

$$F_5 = \sum_{k\geqslant 1;k\notin I}\sum_{1\leqslant\nu\leqslant 2q-1}\lambda_k\omega'\begin{pmatrix}0,I\\0,\partial_\nu I,k\end{pmatrix}\widehat{\widehat{\varphi}}'(\partial_\nu I,k)$$

$$F_6 = -\sum_{k\geqslant 1}\lambda_k\omega'\begin{pmatrix}0,I\\\partial_\nu I,k\end{pmatrix}\widehat{\widehat{\varphi}}'(I,k).$$

Similarly we can obtain an explicit expression of Gauss-Manin connection for $\widehat{\varphi}'(I)$*. But in this case it is not written in closed 1-forms.*

Proof 4. — This theorem has been essentially proved in Proposition 2.4_p, Propositon $2.4'_p$, Proposition $2.4''_p$ in [1] by the homogenization of the integrals (3.4) and (3.5) , also of (4.4) and (4.5). In fact, we put $t_j = t_0 x_j \ \ 1\leqslant j\leqslant n+1$ and

$$\tilde f_j(t) = t_0 f_j\Big(\frac{t_1}{t_0},\cdots,\frac{t_{n+1}}{t_0}\Big)$$

and consider the homogeneous integrals

$$\tilde\varphi_+(I) = \int_{\tilde C}\exp\left(-\frac{1}{2}(t_0^2+t_1^2+\cdots+t_{n+1}^2)\right)$$

$$t_0^{\mu_0}\prod_{j=1}^m\tilde f_j(t)^{\lambda_j}\frac{dt_0\wedge dt_1\wedge\cdots\wedge dt_{n+1}}{\tilde f_{i_1}\cdots\tilde f_{i_p}}$$

and

$$\tilde\varphi_-(I) = \int_{\tilde C}\exp\left(-\frac{1}{2}(t_0^2+t_1^2+\cdots+t_{n+1}^2)\right)$$

$$t_0^{\mu_0-1}\prod_{j=1}^m\tilde f_j(t)^{\lambda_j}\frac{dt_0\wedge dt_1\wedge\cdots\wedge dt_{n+1}}{\tilde f_{i_1}\cdots\tilde f_{i_p}}$$

respectively for $I=\{i_1,\cdots,i_p\},\ i_\nu \geqslant 1$. Then by the identification (3.1) $\tilde{\varphi}(I),\ \tilde{\varphi}(0,I)$ equal the integrals

$$\Gamma\Big(-\lambda_0-\frac{p}{2}\Big)2^{-\lambda_0-\frac{p+2}{2}}(\sqrt{-1})^{\lambda_\infty+n+1-p}\int_{\mathcal{C}} U'(\xi)f_0'^{\frac{p}{2}}(\xi)\varphi'(I)$$

$$\Gamma\Big(-\lambda_0-\frac{p+1}{2}\Big)2^{-\lambda_0-\frac{p+3}{2}}(\sqrt{-1})^{\lambda_\infty+n+1-p}\int_{\mathcal{C}} U'(\xi)f_0'^{\frac{p+1}{2}}(\xi)\varphi'(I)$$

integrated over suitable cycles $\mathcal{C}$ and $\tilde{\mathcal{C}}$ respectively. Here we put $\mu_0 = -2\lambda_0-\lambda_\infty-n-2$. We apply Propositions 2.4_p, $2.4'_p$, $2.4''_p$ [1] I to $\tilde{\varphi}(I),\ \tilde{\varphi}(0,I)$ and obtain (4.8).

To get an explicit form of Gauss-Manin connection for $\widehat{\varphi'_{Q,*}}(I)$ we first define the following 1-forms $\theta'\binom{\emptyset}{I}$ in an inductive way.

DEFINITION 2.

$$\theta'\begin{pmatrix}\emptyset\\ i\end{pmatrix}=da'_{i0}$$

$$\theta'\begin{pmatrix}\emptyset\\ i,j\end{pmatrix}=da'_{ij}-\frac{A'\binom{0,j}{i,j}}{A'(0,j)}da'_{0j}+\frac{A'\binom{0,i}{i,j}}{A'(0,i)}da'_{0i}$$

$$\cdots$$

$$\theta'\begin{pmatrix}\emptyset\\ I\end{pmatrix}=\sum_{\nu=1}^{p}(-1)^\nu\frac{A'\binom{0,\partial_\nu I}{I}}{A'(0,\partial_\nu I)}\theta'\begin{pmatrix}\emptyset\\ \partial_\nu I\end{pmatrix}$$

for $I=\{i_1,\cdots,i_p\}\subset\{1,2,\cdots,m\}$, $(p\geqslant 3)$.

One may point out an important property for $\theta'\binom{\emptyset}{I}$.

LEMMA 1. — *If* $p\geqslant n+2$, *then*

$$\theta'\begin{pmatrix}\emptyset\\ I\end{pmatrix}=0. \tag{4.9}$$

This fact has been proved in [2] in an analytic way.

Furthermore one can make the following conjecture. For an arbitrary I, we have

$$\theta'\begin{pmatrix}\emptyset\\ I\end{pmatrix}\cong 0 \bmod(A'(0,I),dA'(0,I)) \tag{4.10}$$

as module of differential forms, i.e., if $A'(0,I)=0$, then $\theta'\binom{\emptyset}{I}=0$.

In fact, for $p=2$ the following identity holds (see [2]):

$$\frac{A'\binom{0,i}{0,j}}{A'(0,i,j)}\theta'\binom{\emptyset}{i,j} = \frac{1}{2}\{-dA'(0,i,j) + A'(0,i,j)d\log\left(A'(0,i)A'(0,j)\right)\}.$$

We can write explicitly the Gauss-Manin connection for $\widehat{\varphi'_Q}(\emptyset)$ by using the 1-forms $\theta'\binom{\emptyset}{I}$.

THEOREM 6. — *In $X-Y$, the integral $\widehat{\varphi}'(\emptyset)$ satisfies the variational formula*

$$(4.11)\quad \begin{aligned} d\widehat{\varphi}'(\emptyset) = \sum_{p=1}^{n+1}\frac{1}{p!}\sum_{I=(i_1,\cdots,i_p)} &\frac{\lambda_{i_1}\cdots\lambda_{i_p}}{(\mu_0+1)\cdots(\mu_0+p-1)} \\ &\cdot\theta'\binom{\emptyset}{I}\frac{A'(I)}{A'(0,I)}\widehat{\varphi'_*}(I)\end{aligned}$$

for $\mu_0 = -2\lambda_0 - \lambda_\infty - n - 2$, where I moves over the set of different indices $I \subset \{1,2,\cdots,m\}$.

Proof 5. — This follows from $(E.III_0)$ in [1] I and Lemma 1.

By taking the limit $\lambda_0 \to -1$ and taking into consideration (3.6), we have

THEOREM 7.

$$(4.12)\quad \begin{aligned} d\widehat{\varphi'_Q}(\emptyset) = \sum_{p=1}^{n+1}\frac{1}{p!}\sum_{1\leqslant i_1<\cdots<i_p\leqslant m} &\frac{\lambda_{i_1}\cdots\lambda_{i_p}}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty-n+p-1)} \\ &\cdot\theta'\binom{\emptyset}{I}\frac{A'(I)}{A'(0,I)}\widehat{\varphi'_{Q,*}}(I).\end{aligned}$$

Here φ'_* and $\varphi'_{Q,*}$ are defined by

$$\varphi'_*(I) = \varphi'(I) - \sum_{\nu=1}^{p}(-1)^\nu A'\binom{0,\partial_\nu I}{I}\varphi'(\partial_\nu I)$$

$$\varphi'_{Q,*}(I) = \varphi'_Q(I) - \sum_{\nu=1}^{p}(-1)^\nu A'\binom{0,\partial_\nu I}{I}\varphi'_Q(\partial_\nu I).$$

One can get similar formulae for $\widehat{\varphi'_*}(I)$ although they are more complicated.

Suppose now

$$a'_{0i} = 0 \quad \text{for all } i,$$

then

$$\theta'\binom{\emptyset}{i} = 0, \quad \theta'\binom{\emptyset}{i,j} = da'_{ij}, \quad \theta'\binom{\emptyset}{I} = 0 \quad \text{for } |I| \geqslant 3$$

and $\varphi'_*(I) = \varphi'(I)$, $\varphi'_{Q,*}(I) = \varphi'_Q(I)$ whence (4.11), (4.12) reduce

$$d\widehat{\varphi'}(\emptyset) = -\frac{1}{2(\mu_0+1)} \sum_{1\leqslant i,j\leqslant m, i\neq j} \lambda_i\lambda_j da'_{ij}\widehat{\varphi'}(i,j) \tag{4.13}$$

$$d\widehat{\varphi'_Q}(\emptyset) = -\frac{1}{2(\lambda_\infty+n-1)} \sum_{1\leqslant i,j\leqslant m, i\neq j} \lambda_i\lambda_j da'_{ij}\widehat{\varphi'_Q}(i,j) \tag{4.14}$$

respectively.

Furthermore if $\lambda_1, \cdots, \lambda_m$ tend to 0, (4.13), (4.14) reduce to the Schälfli formula for the volume of a geodesic simplex.

5. Variational formulae for volumes.

The angle $< i,j >$ between the hyperplanes H'_i and H'_j subtended by Δ is defined uniquely by the equation

$$-\cos < i,j > = a'_{ij} \tag{5.1}$$

$< i,j >$ is also equal to the angle between the hyperspheres $H'_i \cap \Re Q$ and $H'_j \cap \Re Q$ subtended by the domain $f'_i \geqslant 0$, $f'_j \geqslant 0$ in $\Re Q$.

On the other hand, the normalized distance of H'_i and the origin is equal to $\frac{a'_{0i}}{-\sqrt{-A'(0i)}}$. These quantities are invariant under the action of $O(n+1,\mathbb{R})$.

We now consider the case where $m = n+1$. By taking the limit $\lambda_1, \cdots, \lambda_{n+1} \to 0$, $\widehat{\varphi'_Q}(\emptyset)$ becomes the volume for the simplex $\Delta \cap Q$ in Q,

$$V_\emptyset = \int_{\Delta\cap Q} \tau'_Q \tag{5.2}$$

where

$$\Delta \cap Q : f'_1 \geqslant 0, \cdots, f'_{n+1} \geqslant 0.$$

We can apply the formulae (4.13), (4.14) to $V_\emptyset$. The intersection $f'_1 = \cdots f'_{n+1} = 0$ with Q is isomorphic up to similarity to the $(n-p)$ dimensional unit sphere:

$$Q_p : 1 - \sum_{j=1}^{n-p+1} \eta_j^2 = 0.$$

The volume form τ'_{Q_p} is given by

$$\tau'_{Q_p} = \sum_{\nu=1}^{n-p+1} (-1)^\nu \eta_\nu d\eta_1 \wedge \cdots \langle \nu \rangle \cdots \wedge \eta_{n-p+1}. \tag{5.3}$$

Then we have

$$\left[\frac{\tau'_Q}{df'_{i_1} \wedge \cdots \wedge df'_{i_p}} \right]_{f'_{i_1} = \cdots = f'_{i_p} = 0} = \frac{(-1)^p}{\sqrt{A'(I)}} \left\{ - \frac{A'(I)}{A'(0,I)} \right\}^{\frac{n-p}{2}} \tau'_{Q_p}. \tag{5.4}$$

Let V_I be the $(n-p)$ dimensional volume

$$V_I = \int_{\Delta \cap Q \cap H'_{i_1} \cap \cdots \cap H'_{i_p}} \tau'_{Q_p}. \tag{5.5}$$

Let A'_I denote the $(m+1-p) \times (m+p-1)$ configuration matrix whose entries b_{jk} are:

$$b_{00} = -1$$

$$b_{0j} = b_{j0} = \frac{A'\binom{I,0}{I,j}}{\sqrt{-A'(I,0)A'(I,j)}} \quad j \notin I$$

$$b_{jk} = \frac{A'\binom{I,j}{I,k}}{\sqrt{A'(I,j)A'(I,k)}} \quad j,k \notin I.$$

We can define the admissible 1-forms $\theta'\binom{\emptyset}{J}$ for A'_I. We denote these forms by $\theta'_I\binom{\emptyset}{J}$ for $J \subset \{1,2,\cdots,m\} - I$.

The hierarchical variational system for the volumes V_I can be stated in the following theorem which follows from Theorem 7, by taking $\lambda_j \to 0$.

THEOREM 8.

$$dV_\emptyset = \sum_{p=1}^{n} \frac{1}{p!} \sum_{|I|=p} \frac{1}{(n-1)\cdots(n-p+1)}$$

$$\theta'\binom{\emptyset}{I} \frac{1}{\sqrt{A'(I)}} \left\{ - \frac{A'(I)}{A'(0,I)} \right\}^{\frac{n-p+2}{2}} V_I. \tag{5.6}$$

Each V_I has the lower dimensional variational formula

$$dV_I = \sum_{q=1}^{n-p} \frac{1}{q!} \sum_{J,|J|=q} \frac{1}{(n-p-1)\cdots(n-p-q+1)}$$

$$\theta'_I\binom{\emptyset}{J} \frac{1}{\sqrt{A'_I(J)}} \left\{ -\frac{A'_I(J)}{A'_I(0,J)} \right\}^{\frac{n-p-q+2}{2}} V_J. \tag{5.7}$$

COROLLARY 2. — *$V_\emptyset$ can be expressed as iterated integrals of the 1-forms $\theta'_I\binom{\emptyset}{J}\frac{1}{\sqrt{A'_I(J)}}\left\{-\frac{A'_I(J)}{A'_I(0,J)}\right\}^{\frac{n-p-q+2}{2}}$ in the sense of K.T. Chen.*

6. Degenerate cases and some questions.

Suppose $A'(I) = 0$, for some I, $|I| = p \leqslant n+1$, then the chamber

$$f'_{i_1} \geqslant 0, \cdots, f'_{i_p} \geqslant 0 \text{ in } \Re Q$$

reduces to a point. This is a vanishing cycle in $H^n(Y, \nabla_0)$ corresponding to the sigularity

$$\begin{aligned} df'_0 \wedge df'_{i_1} \wedge \cdots \wedge df'_{i_p} &= 0 \\ f'_0 = f'_{i_1} = \cdots = f'_{i_p} &= 0. \end{aligned} \tag{6.1}$$

The rank of $H^n(Y, \nabla_0)$ decreases by one. There arises a new linear relation among the cohomology classes of $\varphi'_Q(I)$. This relation can be described as follows, as a result of a series of contiguity relations (see (D.III$^*_p$),(D.IV$^*_p$) [1] I).

There exists a system of rational functions $g\binom{I}{J}$ of a'_{ij} and a'_{i0} such that

$$\begin{aligned} &\sum_j \sum_{\mu_1,\cdots,\mu_q} g\begin{pmatrix} I \\ j, \partial_{\mu_1} \cdots \partial_{\mu_q} I \end{pmatrix} \varphi'_Q(j, \partial_{\mu_1} \cdots \partial_{\mu_q} I) \\ &+ \sum_{\mu_1,\cdots,\mu_q} g\begin{pmatrix} I \\ \partial_{\mu_1} \cdots \partial_{\mu_q} I \end{pmatrix} \varphi'_Q(\partial_{\mu_1} \cdots \partial_{\mu_q} I) \sim 0. \end{aligned} \tag{6.2}$$

Notice that $g\binom{I}{j,I} = \lambda_j A'\binom{0,I}{j,I}$.

The special case where $m = n+2$, and all $A'(I) = 0$ $(|I| = n+1)$ seems interesting, for example if $n = 2$, $m = 4$ then

$$A'(2,3,4) = A'(1,3,4) = A'(1,2,4) = A'(1,2,3) = 0$$

and $l_1 = 10$.

Finally one may ask the following general questions.

Question 1. Let $\Omega^\cdot$ be a subspace of closed forms on Y such that $\omega_0 \wedge \Omega^\cdot \subset \Omega^\cdot$. Then ∇_0 can be defined on $\Omega^\cdot$ too so that we have the subcomplex $(\nabla_0,\ \Omega^\cdot)$,

$$\nabla_0 \quad : \varphi \longrightarrow \omega_0 \wedge \varphi \quad \varphi \in \Omega^\cdot.$$

In this situation one may ask "Does it exist $\Omega^\cdot$ such that the isomorphism

$$H^*(Y, \nabla_0) \cong H^*(\Omega, \nabla_0) \tag{6.3}$$

hold for a general hypersphere arrangement $\mathcal{A}_Q$?"

One can prove

PROPOSITION 2. — *Under* $(\mathcal{H}1)$ *and* $(\mathcal{H}2)$,

$$H^p(Y,\mathbb{C}) \cong \sum_{I,|I|=p} \mathbb{C}\{d\log f_{i_1} \wedge \cdots \wedge d\log f_{i_p}\} \quad 0 \leqslant p \leqslant n-1 \tag{6.4}$$

$$H^n(Y,\mathbb{C}) \cong \sum_{I,|I|=n} \mathbb{C}\{d\log f_{i_1} \wedge \cdots \wedge d\log f_{i_n}\} \oplus \sum_{p=0}^{n} \sum_{I,|I|=p} \frac{\tau_Q}{f_{i_1}\cdots f_{i_p}} \tag{6.5}$$

where I denotes the set of indices $\{i_1, \cdots, i_p\}$ and $\{i_1, \cdots, i_n\}$ respectively. Hence

$$rkH^p(Y,\mathbb{C}) = \binom{m}{p} \quad 0 \leqslant p \leqslant n-1 \tag{6.6}$$

$$rkH^n(Y,\mathbb{C}) = \sum_{\nu=0}^{n} \binom{m}{\nu} + \binom{m}{n}. \tag{6.7}$$

As a corollary the Euler number $\chi(Y)$ equals:

COROLLARY 3.

$$\chi(Y) = \sum_{\nu=0}^{n-1} (-1)^\nu \binom{m}{\nu} + (-1)^n \left\{ \sum_{\nu=0}^{n-1} \binom{m}{\nu} + 2\binom{m}{n} \right\} = (-1)^n l_1$$

owing to the identity

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{m}{\nu} = (-1)^n \binom{m-1}{n}$$

(l_1 was defined in (2.6)).

We take as $\Omega^{\cdot}$:

$$(6.8)\quad \Omega^{\cdot} = \sum_{p=0}^{n} \sum_{I,|I|=p} \mathbb{C}\{d\log f_{i_1} \wedge \cdots \wedge d\log f_{i_p}\} \oplus \sum_{p=0}^{n} \sum_{I,|I|=p} \mathbb{C}\Big\{\frac{\tau_Q}{f_{i_1}\cdots f_{i_p}}\Big\}$$

then for a generic $\lambda = (\lambda_1, \cdots, \lambda_m)$ we have (6.3) in view of Theorem 4 and its corollary.

Remark that (6.3) is valid in case of general hyperplane arrangements, when we take as $\Omega^{\cdot}$ the space of logarithmic forms (the Orlik-Solomon algebra)(see [9].) In case of general hypersphere arrangements, the first summand of the RHS of (6.8) appears in the Orlik-Solomom algebra, while the second recently appears in [7], [10], [14]. It seems interesting to discuss them in relation to hypersphere arrangements.

Question 2. Let Θ be a rational flat connection, such that

$$(6.9)\qquad d\Theta + \Theta \wedge \Theta = 0.$$

Does there exist a rational matrix function Φ such that

$$\Theta' = \Phi^{-1}\Theta\Phi + \Phi^{-1}d\Phi$$

is closed? In this situation, the integrability condition (6.9) becomes algebraic:

$$d\Theta' = \Theta' \wedge \Theta' = 0.$$

Question 2 was answered affirmatively in case of hyperplane arrangements of general position and their degeneration of codimension 1 and more generally in non-resonance cases (see [13], [6]).

BIBLIOGRAPHY

[1] K. Aomoto, Configurations and invariant Gauss-Manin connections of integrals I, II, Tokyo J. of Math., 5 (1982), 249–287; ibid, 6 (1983), 1–24.

[2] K. Aomoto, Vanishing of certain 1-form attached to a configuration, Tokyo J. of Math., 9 (1986), 453–455.

[3] K. Aomoto, Errata. Tokyo J. of Math., 22 (1999), 511–512.

[4] K. Aomoto, Hyperlogarithmic expansion and the volume of a hyperbolic Simplex, Partial Differential Equations, Banach Center Publications, 27 (1992), 9–21.

[5] K. Aomoto, M. Kita, P. Orlik and H. Terao, Twisted de Rham cohomology groups of logarithmic forms, Adv. in Math., 128 (1997), 119–152.

[6] D.C. Cohen and P. Orlik, Gauss-Manin connections for arrangements, II non-resonant weights, preprint, 2002.

[7] H. HORIUCHI and H. TERAO, The Poincaré series of the algebra of rational functions which are regular outside hyperplanes, preprint, 2002, March.

[8] Yi HU, On the homology of complements of arrangements of subspaces and spheres, Proc. Amer. Math. Soc., 122 (1994), 285–290.

[9] P. ORLIK and H. TERAO, Arrangements and Hypergeometric Integrals, MSJ memoirs, n° 9 (2001).

[10] P. ORLIK and H. TERAO, Commutative algebras for arrangements, Nagoya Math. J., 134 (1994), 65–73.

[11] F. PHAM, Introduction à l'Etude Topologique des Singularités de Landau, Gauthiers Villars, 1967.

[12] C. SABBAH, On the comparison theorem for elementary irregular $\mathcal{D}$-modules, Nagoya Math. J., 141 (1996), 107–124.

[13] H. TERAO, Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss-Manin connections, Topology and its Appl., 118 (2002), 255–274.

[14] H. TERAO, Algebras generated by reciprocals of linear forms, J. of Algebra, 250 (2002), 549–558.

[15] K. YOSHIDA, Hypergeometric Functions, My Love, Vieweg Verlag, 1997.

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