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FROM MULTI-INSTANTONS TO EXACT RESULTS

by Jean ZINN-JUSTIN

1. Introduction.

We recall here a set of conjectures about the complete form of the perturbative expansion of the spectrum of a quantum hamiltonian H, in cases where the potential has degenerate minima [1], [2], [3], [4].

Perturbative expansions are obtained by first approximating the potential by a harmonic potential near its minimum. They are expansions for $\hbar \to 0$ valid for energy eigenvalues of order \hbar , in contrast with the WKB expansion where energies are non-vanishing in this limit (WKB approximation applies to large quantum numbers). When the potential has degenerate minima, perturbation series can be shown to be non-Borel summable. Moreover, quantum tunneling generates additional contributions to eigenvalues of order exp (-const. $/\hbar$), which have to be added to the perturbative expansion (for a review see for example [3]).

Therefore, the determination of eigenvalues starting from their expansion for \hbar small is a non-trivial problem. The conjectures we describe here, give a systematic procedure to calculate eigenvalues, for \hbar finite, from expansions which are shown to contain powers of \hbar , $\ln\hbar$ and $\exp(-\text{const. }/\hbar)$. Moreover, generalized Bohr–Sommerfeld formulae allow to derive the many series which appear in such formal expansions from only two of them, which can be extracted by suitable transformations from the corresponding WKB

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expansions. Note that the relation with the WKB expansion is not completely trivial. Indeed, the perturbative expansion corresponds, from the point of view of a semi-classical approximation, to a situation with confluent singularities and thus, for example, the WKB expressions for barrier penetration are not uniform when the energy goes to zero.

Finally, several of these conjectures have found a natural explanation in the framework of Ecalle's theory of resurgent functions and have now been proven by Pham and his collaborators [5], [6].

In Section 2.2, we describe the conjectures. In Section 3.3, we discuss the connection with the WKB expansion. In Sections 4.4, 5.5, we explain how these conjectures were suggested by a systematic calculation of socalled *multi-instanton* contributions to the path integral representation of the partition function tr $e^{-\beta H}$, in the example of the double-well potential.

Although several conjectures have now been proven, we believe that heuristic arguments based on instanton considerations are still useful because, suitably generalized, they could lead to new conjectures in more complicated situations.

More detail and explanations can be found in ref. [3].

Note that, in what follows, the symbol g plays the role of \hbar and the energy eigenvalues are measured in units of \hbar , a normalization adapted to perturbative expansions.

2. Generalized Bohr–Sommerfeld quantization formulae.

In what follows, we always assume that the potential is an analytic entire function. Moreover, all proven results apply to potentials belonging to this class.

We first explain the conjecture in the case of the so-called *double-well* potential.

2.1. The example of the quartic double-well potential.

The hamiltonian corresponding to the double-well potential can be written as

(2.1)
$$H = -\frac{g}{2} \left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{1}{g} V(q), \quad V(q) = \frac{1}{2} q^2 (1-q)^2.$$

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The hamiltonian is symmetric in the exchange $q \leftrightarrow 1 - q$ and thus commutes with the corresponding parity operator P, whose action on wave functions is

(2.2)
$$P\psi(q) = \psi(1-q) \Rightarrow [H,P] = 0.$$

The eigenfunctions of H satisfy

$$H\psi_{\varepsilon,N}(q) = E_{\epsilon,N}(g)\psi_{\varepsilon,N}(q), \quad P\psi_{\varepsilon,N}(q) = \epsilon\psi_{\varepsilon,N}(q),$$

where $\epsilon = \pm 1$ and $E_{\epsilon,N}(g) = N + 1/2 + O(g)$. We have conjectured [1] that the eigenvalues $E_{\epsilon,N}(g)$ of the hamiltonian (2.1) have a complete semiclassical expansion of the form

(2.3)
$$E_{\varepsilon,N}(g) = \sum_{0}^{\infty} E_{N,l}^{(0)} g^{l} + \sum_{n=1}^{\infty} \left(\frac{2}{g}\right)^{Nn} \left(-\epsilon \frac{e^{-1/6g}}{\sqrt{\pi g}}\right)^{n} \sum_{k=0}^{n-1} \left(\ln(-2/g)\right)^{k} \sum_{l=0}^{\infty} e_{N,nkl} g^{l}.$$

All the series in powers of g are non-Borel summable and have to be summed for g negative first and the value for g positive is then obtained by analytic continuation, consistently with the determination of $\ln(-g)$. In the analytic continuation from g negative to g positive, the Borel sums become complex and get imaginary parts exponentially smaller by about a factor $e^{-1/3g}$ than the real parts, which are cancelled by the imaginary parts coming from the function $\ln(-2/g)$.

Moreover, we have conjectured [7] that all these series can be obtained by expanding for g small a generalized Bohr–Sommerfeld quantization formula, which in the case of the double-well potential reads

(2.4)
$$\frac{1}{\Gamma(\frac{1}{2}-B)} + \frac{\epsilon i}{\sqrt{2\pi}} \left(-\frac{2}{g}\right)^{B(E,g)} e^{-A(E,g)/2} = 0,$$

 $(\epsilon = \pm 1)$ with

(2.5)
$$B(E,g) = -B(-E,-g) = E + \sum_{k=1}^{\infty} g^k b_{k+1}(E),$$

(2.6)
$$A(E,g) = -A(-E,-g) = \frac{1}{3g} + \sum_{k=1}^{\infty} g^k a_{k+1}(E).$$

The coefficients $b_k(E)$ and $a_k(E)$ are odd or even polynomials in E of degree k. The three first orders, for example, are

(2.7)
$$B(E,g) = E + g\left(3E^2 + \frac{1}{4}\right) + g^2\left(35E^3 + \frac{25}{4}E\right) + O\left(g^3\right),$$

(2.8) $A(E,g) = \frac{1}{3}g^{-1} + g\left(17E^2 + \frac{19}{12}\right) + g^2\left(227E^3 + \frac{187}{4}E\right) + O\left(g^3\right).$

The function B(E,g) has been inferred from the perturbative expansion. The function A(E,g) had initially been determined at this order by a combination of analytic and numerical calculations.

Asymmetric wells. — In the case of a potential with asymmetric wells, the generalized Bohr-Sommerfeld quantization formula takes the form

(2.9)
$$\frac{1}{\Gamma(\frac{1}{2} - B_1)\Gamma(\frac{1}{2} - B_2)} + \frac{1}{2\pi} \left(-\frac{2C_1}{g}\right)^{B_1(E,g)} \left(-\frac{2C_2}{g}\right)^{B_2(E,g)} e^{-A(g,E)} = 0,$$

where C_1 and C_2 are numerical constants, adjusted in such a way that A(E,g) has no term of order g^0 . The functions $B_1(E,g)$ and $B_2(E,g)$ are determined by the perturbative expansions around each of the two minima of the potential.

2.2. Exponentially small contributions at leading order.

At leading order the exponential contributions, that is the terms corresponding to l = 0 in the expansion (2.3), are obtained by expanding the equation

(2.10)
$$\frac{e^{-1/6g}}{\sqrt{2\pi}} \left(-\frac{2}{g}\right)^E = -\frac{\epsilon i}{\Gamma(\frac{1}{2} - E)} \iff \frac{\cos \pi E}{\pi}$$
$$= \epsilon i \frac{e^{-1/6g}}{\sqrt{2\pi}} \left(-\frac{2}{g}\right)^E \frac{1}{\Gamma(\frac{1}{2} + E)}.$$

We show in Section 4 that, from the point of view of the path integral representation, the successive terms correspond to *multi-instanton* contributions.

For example, the term n = 1 in (2.3), that is the one-instanton contribution, is

(2.11)
$$E_N^{(1)}(g) = -\frac{\epsilon}{N!} \left(\frac{2}{g}\right)^{N+1/2} \frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}} \left(1 + O(g)\right).$$

The term n = 2, that is the two-instanton contribution, is

$$E_N^{(2)}(g) = \frac{1}{\left(N!\right)^2} \left(\frac{2}{g}\right)^{2N+1} \frac{\mathrm{e}^{-1/3g}}{2\pi} \left[\ln(-2/g) - \psi(N+1) + O\left(g\ln g\right)\right],$$

where ψ is the logarithmic derivative of the Γ -function.

More generally, it can be verified that the n-instanton contribution has at leading order the form

$$E_N^{(n)}(g) = -\left(\frac{2}{g}\right)^{n(N+1/2)} \left(\frac{e^{-1/6g}}{\sqrt{2\pi}}\right)^n \left[P_n^N(\ln(-g/2)) + O\left(g(\ln g)^{n-1}\right)\right],$$

in which $P_n^N(\sigma)$ is a polynomial of degree n-1. For example, for N=0 one finds

(2.14)
$$P_2(\sigma) = \sigma + \gamma, \qquad P_3(\sigma) = \frac{3}{2} (\sigma + \gamma)^2 + \frac{\pi^2}{12},$$

in which γ is Euler's constant: $\gamma = -\psi(1) = 0.577215...$

Large order behaviour of perturbation series. — After an analytic continuation from g negative to g positive, two things happen: the Borel sums become complex and get an imaginary part exponentially smaller by about a factor $e^{-1/3g}$ than the real part. Simultaneously, the function $\ln(-2/g)$ also becomes complex and gets an imaginary part $\pm i\pi$. Since the sum of all contributions is real, the imaginary parts must cancel. This property leads to an evaluation of the imaginary part of the Borel sum of the perturbation series, or of the two-instanton contribution, for example.

From the imaginary part of P_2 , one derives

(2.15)
$$\operatorname{Im} E^{(0)}(g) \sim \frac{1}{\pi g} \mathrm{e}^{-1/3g} \mathrm{Im} \left[P_2 \left(\ln(-g/2) \right) \right]$$

and, therefore,

(2.16)
$$\operatorname{Im} E^{(0)}(g) \sim -\frac{1}{g} \mathrm{e}^{-1/3g}.$$

The coefficients of the perturbative expansion

(2.17)
$$E^{(0)}(g) = \sum_{k} E_{k}^{(0)} g^{k},$$

are related to the imaginary part by a Cauchy integral (k > 1)

$$E_k^{(0)} = \frac{1}{\pi} \int_0^\infty \mathrm{Im} \left[E^{(0)}(g) \right] \frac{\mathrm{d}g}{g^{k+1}}$$

For k large, the integral is dominated by small g values. From the asymptotic estimate of $\text{Im } E^{(0)}$ for $g \to 0$, on then derives the large order behaviour of the perturbative expansion [8]:

(2.18)
$$E_k^{(0)} \sim -\frac{1}{\pi} 3^{k+1} k!$$

From the imaginary part of P_3 , one derives the large order behaviour of the expansion

(2.19)
$$E^{(1)}(g) = -\frac{1}{\sqrt{\pi g}} e^{-1/6g} \left(1 + \sum_{k=1}^{\infty} E_{k}^{(1)} g^{k} \right)$$

by expressing that the imaginary part of $E^{(1)}(g)$ and $E^{(3)}(g)$ must cancel at leading order:

(2.20)
$$\operatorname{Im} E^{(1)}(g) \sim -\left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^3 \operatorname{Im} \left[P_3(\ln(-g/2))\right].$$

The coefficients $E_k^{(1)}$ again are given by a dispersion integral:

(2.21)
$$E_k^{(1)} = \frac{1}{\pi} \int_0^\infty \left\{ \operatorname{Im} \left[E^{(1)}(g) \right] \sqrt{\pi g} \, \mathrm{e}^{1/6g} \right\} \frac{\mathrm{d}g}{g^{k+1}}.$$

Using then equations (2.14) and (2.20), one finds

(2.22)
$$E_k^{(1)} \sim -\frac{1}{\pi} \int_0^\infty 3\left(\ln\frac{2}{g} + \gamma\right) e^{-1/3g} \frac{\mathrm{d}g}{g^{k+2}}.$$

At leading order for k large, g can be replaced by its saddle point value 1/3k in $\ln g$ and, finally, one obtains

(2.23)
$$E_k^{(1)} = -\frac{3^{k+2}}{\pi}k! \left[\ln 6k + \gamma + O\left(\frac{\ln k}{k}\right) \right].$$

Both results (2.18) and (2.23) have been checked against the numerical behaviour of the corresponding series for which 100 terms can easily be calculated.

The real part of the two-instanton contribution. — To check the real part of P_2 , the following quantity has been evaluated numerically:

(2.24)
$$\Delta(g) = 4 \frac{\left\{\frac{1}{2} \left(E_{+} + E_{-}\right) - \operatorname{Re}\left[\operatorname{Borel} \operatorname{sum} E^{(0)}(g)\right]\right\}}{\left(E_{+} - E_{-}\right)^{2} \left(\ln 2g^{-1} + \gamma\right)}$$

In this expression E_+ and E_- are the two lowest eigenvalues. In the sum $(E_+ + E_-)$ the contributions corresponding to an odd number of instantons cancel. Therefore, the numerator is dominated for g small by the real part of the two-instanton contribution. The difference $(E_+ - E_-)$, as we know, is dominated by the one-instanton contribution. Using the various expressions given above, it is easy to verify that $\Delta(g)$ should go to 1 when g goes to zero. If one performs an expansion in powers of g and inverse powers of $\ln(2/g)$ and keep only the first few terms in $\{1/\ln(2/g)\}$ in each term in the g-expansion, one finds [9]

$$\begin{split} \Delta(g) &\sim 1 + 3g - \frac{23}{2} \frac{g}{\ln(2/g)} \left[1 - \frac{\gamma}{\ln(2/g)} + \frac{\gamma^2}{\ln^2(2/g)} + O\left(\frac{1}{\ln^3(2/g)}\right) \right] \\ &+ \frac{53}{2} g^2 - 135 \frac{g^2}{\ln(2/g)} \left[1 - \frac{\gamma}{\ln(2/g)} + \frac{\gamma^2}{\ln^2(2/g)} + O\left(\frac{1}{\ln^3(2/g)}\right) \right] + O\left(g^3\right). \end{split}$$

The higher-order corrections, which are only logarithmically suppressed with respect to the leading terms 1 + 3g, change the numerical values quite significantly, even at small g. Of course, for larger values, significant deviations from the leading asymptotics must be expected due to higherorder effects; these are indeed observed. For example, at g = 0.1 the numerically determined value reads $\Delta(0.1) = 0.87684(1)$ whereas the first asymptotic terms sum up to a numerical value of 0.86029.

Table 1 displays results for g small in a range of values of g for which the numerical calculation is still reasonably precise; these are in agreement with the first few asymptotic terms up to numerical precision [9].

TABLE 1. – The ratio $\Delta(g)$ as a function of g.

g	0.005	0.006	0.007	0.008	0.009	0.010
$\Delta(g)$ num.	1.0063(5)	1.0075(5)	1.00832(5)	1.00919(5)	1.00998(5)	1.01078(5)
$\Delta(g)$ asymp.	1.00640	1.00739	1.00832	1.00919	1.01001	1.01078

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2.3. Other potentials.

The cosine potential. — This example differs from the preceding ones because the potential is still an entire function but no longer a polynomial. On the other hand, the periodicity of the potential simplifies the analysis because it allows to classify eigenvalues by the behaviour of the corresponding eigenfunction under a translation of one period:

$$\psi_{\varphi}(q+T) = \mathrm{e}^{\imath\varphi}\psi_{\varphi}(q),$$

where T is the period. For the cosine potential $\frac{1}{16}(1-\cos 4q)$, the conjecture then takes the form

(2.25)
$$\left(\frac{2}{g}\right)^{-B} \frac{e^{A(E,g)/2}}{\Gamma(\frac{1}{2}-B)} + \left(\frac{-2}{g}\right)^{B} \frac{e^{-A(g,E)/2}}{\Gamma(\frac{1}{2}+B)} = \frac{2\cos\varphi}{\sqrt{2\pi}}.$$

A similar analysis can be made. The respective expansions for the functions ${\cal A}$ and ${\cal B}$ are

(2.26)
$$B = E + g\left(E^2 + \frac{1}{4}\right) + g^2\left(3E^3 + \frac{5}{4}E\right) + O\left(g^3\right),$$

$$(2.27) A = g^{-1} + g\left(3E^2 + \frac{3}{4}\right) + g^2\left(11E^3 + \frac{23}{4}E\right) + O\left(g^3\right),$$

where again A has been obtained partially by numerical techniques. There exist numerical checks up to four-instanton order. The theory of resurgence has allowed to also investigate this potential, as well as general trigonometric potentials, [6].

The $O(\nu)$ -symmetric anharmonic oscillator. — The hamiltonian now is

(2.28)
$$H = -\frac{1}{2}\Delta + \frac{1}{2}\mathbf{q}^2 + g\left(\mathbf{q}^2\right)^2,$$

where \mathbf{q} belongs to \mathbb{R}^{ν} . For the $O(\nu)$ -symmetric quartic anharmonic oscillator, the conjecture corresponds to the expansion of the eigenvalues for g < 0 and, in particular, yields the instanton contributions to the large order behaviour of the perturbative expansion. The eigenfunctions of H can be classified according to the irreducible representations of the orthogonal $O(\nu)$ group. At fixed angular momentum l, the hamiltonian (2.28), expressed in terms of the radial variable $r = |\mathbf{q}|$, takes the form

$$H_{l} = -\frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}r} \right)^{2} - \frac{1}{2} \frac{\nu - 1}{r} \frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{2} \frac{l(l + \nu - 2)}{r^{2}} + \frac{1}{2} r^{2} + gr^{4}.$$

This is a one-dimensional hamiltonian but with a non-polynomial potential. The eigenvalues depend only on the combination $\mu = \nu + 2l$. The secular equation is then conjectured to be

(2.29)
$$i e^{-A(E,g)} \left(-\frac{2}{g}\right)^B e^{i\pi(B/2+\mu/4)} \frac{\Gamma\left(\frac{1}{4}\mu - \frac{1}{2}B\right)}{\Gamma\left(\frac{1}{4}\mu + \frac{1}{2}B\right)} = 1$$

where the first few terms of the expansions of the functions A and B are:

$$B = E - \frac{1}{2}g\left(3E^2 - \frac{1}{4}\mu^2 + \mu\right) + O\left(g^2\right),$$

$$A = -\frac{1}{3}g^{-1} + g\left(\frac{3}{16}\mu^2 - \frac{3}{4}\mu - \frac{5}{6} - \frac{17}{4}E^2\right) + O\left(g^2\right),$$

but these numbers lack numerical confirmation for $\mu > 2$. For $\mu = 2$, one can use the remarkable relation with the double-well potential:

$$B(E, g, \mu = 2) = 2B_{dw}(E/2, -g), \qquad A(E, g, \mu = 2) = A_{dw}(E/2, -g),$$

in the sense of formal series [10]. Note that the first proofs of this relation use path integral manipulations [11] or recursion relations [12]. A more rigorous analysis, generalized to the $O(\nu)$ hamiltonian and relying on differential equation techniques, can be found in reference [13].

3. Generalized WKB expansion and Riccati equation.

We write the Schrödinger equation as

(3.1)
$$[H\psi](q) \equiv -\frac{g}{2}\psi''(q) + \frac{1}{g}V(q)\psi(q) = E\psi(q).$$

We consider only potentials V that are entire functions. Moreover, for convenience we assume that q = 0 is the absolute minimum of V(q) and $V(q) \sim \frac{1}{2}q^2$ for q small. We first assume that the minimum is unique.

3.1. The spectrum: Standard WKB expansion.

The WKB expansion [14] is an expansion for $g \to 0$ at Eg fixed, in contrast with the perturbative expansion where E is fixed. A convenient way to generate it, is to use the Riccati equation obtained from Schrödinger equation (3.1) by setting

$$(3.2) S(q) = -g\psi'/\psi.$$

The function S then satisfies

(3.3)
$$gS'(q) - S^2(q) + S_0^2(q) = 0, \quad S_0^2(q) = 2V(q) - 2gE.$$

For E > 0, the function S_0 has two branch points $q_- < q_+$ on the real axis. We put the cut between q_- and q_+ and choose the determination of S_0 to be positive for $q > q_+$. This ensures the decrease of wave functions on the real axis for $|q| \to \infty$, at least in the WKB approximation.

Equation (3.3) is then expanded systematically in powers of g, at Eg fixed, starting from $S(q) = S_0(q)$. It is useful to introduce the decomposition

$$(3.4) \ S(q,g,E) = S_{+}(q,g,E) + S_{-}(q,g,E), \ S_{\pm}(q,-g,-E) = \pm S_{\pm}(q,g,E).$$

Then, equation (3.3) translates into

(3.5a)
$$gS'_{-} - S^2_{+} - S^2_{-} + S^2_{0} = 0,$$

(3.5b)
$$gS'_+ - 2S_+S_- = 0.$$

The second equation allows to express the wave function in terms of S_+ only:

(3.6)
$$\psi = (S_+)^{-1/2} \exp\left[-\frac{1}{g}\int^q \mathrm{d}q' S_+(q')\right].$$

For analytic potentials, the spectrum can then be determined by the condition

(3.7)
$$\frac{1}{2i\pi} \oint_C \mathrm{d}z \frac{\psi'(z)}{\psi(z)} = N_z$$

where N is the number of nodes of the eigenfunction and C a contour that encloses them. This elegant formulation, restricted however to one dimension and analytic potentials, bypasses the difficulties generally associated with turning points. In the semi-classical limit, C encloses the cut of $S_0(q)$ which joins the two turning points solutions of $S_0(q) = 0$. Then,

$$\frac{1}{2i\pi}\oint_C \mathrm{d} z \frac{S'_+(z)}{S_+(z)} = \frac{1}{2i\pi}\oint_C \mathrm{d} z \frac{S'_0(z)}{S_0(z)} = 1,$$

because the value of the integral depends only on the existence of the two branch points and higher order corrections for $g \to 0$ modify only their location.

In terms of S_+ , equation (3.7) thus becomes

(3.8)
$$-\frac{1}{2i\pi g} \oint_C \mathrm{d}z \, S_+(z) = N + \frac{1}{2}$$

Replacing S_+ by its WKB expansion and expanding each term in a power series of Eg, one obtains the function B(E,g) (the perturbative expansion):

(3.9)
$$-\frac{1}{2i\pi g} \oint_C \mathrm{d}z \, S_+(z) = B(E,g) = -B(-E,-g).$$

The quantization condition can also be written as

$$\exp\left[-\frac{1}{g}\oint_C \mathrm{d} z\,S_+(z)\right] + 1 = 0 \quad \text{with} \quad E(g=0) > 0.$$

Note, finally, that an alternative formalism is based on studying the differential equation satisfied by the diagonal matrix elements of the resolvent $(H - E)^{-1}$.

3.2. Potentials with degenerate minima.

Symmetric potentials with degenerate minima. — We now consider a potential of double-well type, symmetric with two degenerate minima. The function B(E,g) and the perturbative expansion are obtained by integrating S_+ around the cut corresponding to one well, for example $[q_1, q_2]$ in figure 1. However, this no longer yields the full answer. To obtain it, we start from a situation with one minimum only and proceed by analytic continuation. For example, we begin with E large enough. However, to avoid a singularity when E reaches the local maximum of the potential at the symmetry point, we take g, and thus Eg slightly complex. Eventually, the initial contour is deformed into a sum of three contours around the cuts

$$\pm([q_1,q_2]-[q_3,q_4])-[q_2,q_3]$$

of figure 1, respectively. The sign of the contribution $[q_1, q_2] - [q_3, q_4]$ depends on the analytic continuation and can be absorbed into a redefinition of the integral $[q_2, q_3]$ by changing Riemann sheet.

One can, thus, write the quantization condition as

$$\exp\left[-\frac{1}{g}\oint_{C'} \mathrm{d}z\,S_+(z)\right] + 1 = 0,$$

where C', the contour around the cut $[q_2, q_3]$, corresponds to barrier penetration. The expansion for Eg small of its WKB expansion yields the function A(g, E). Comparing with the instanton result, one infers the decomposition (in the notation of equation (2.9))

(3.10)
$$\frac{1}{g} \oint_{C'} \mathrm{d}z \, S_+(z) = A(E,g) - 2\ln\Gamma\left(\frac{1}{2} - B(E,g)\right) + 2B(E,g)\ln(-g/2C) + \ln(2\pi).$$

In the WKB expansion, the function $\Gamma(\frac{1}{2} - B)$ has to be replaced by its asymptotic expansion for *B* large. A calculation of A(E,g) at a finite order in *g* requires the WKB expansion and the asymptotic expansion of the Γ -function only to a finite order. For example, the expansions up to order g^2 (2.8) and (2.27) of the function A(E,g) for the double-well and cosine potentials, which have initially been determined in part by numerical calculations, are reproduced by the expansion of the two first WKB orders.

General potentials with degenerate minima. — In the case of general potentials with degenerate minima, each minimum is the starting point of a perturbative expansion and two functions B_1 and B_2 are generated (corresponding to contours enclosing $[q_1, q_2]$ and $[q_3, q_4]$ in figure 1, respectively). Moreover, an additional contour integral (the contour C' encloses $[q_2, q_3]$) arises, which is related to barrier penetration. The expansion for Eg small of its WKB expansion yields the function A(g, E). In the notation of equation (2.9), we conjectured

$$\frac{1}{g} \oint_{C'} \mathrm{d}z \, S_+(z) = A(E,g) + \ln(2\pi) - \sum_{i=1}^2 \ln\Gamma\left(\frac{1}{2} - B_i(E,g)\right) + B_i(E,g) \ln(-g/2C_i),$$

where the constants C_i are adjusted in such a way A(E,g) has no term of order g^0 and the coefficient of g^0 is proportional to the expansion of $B_1 + B_2$.

Finally, one verifies that the discontinuity as a function of E, for E small, of the integral in the l.h.s. of equation (3.11) yields the sums of the integrals around $[q_1, q_2]$ and $[q_3, q_4]$, that is the sum of integrals of the form (3.9) associated with each minimum, which is $-2i\pi(B_1 + B_2)$. In the r.h.s. the discontinuity comes from the asymptotic expansion of the Γ functions:

$$-\ln\Gamma\left(\frac{1}{2}-B_i\right) \underset{E\to\infty}{\sim} B_i \ln(-B_i) \underset{g\to0}{\sim} B_i \ln(-E),$$



Figure 1. The four turning points.

which also has a discontinuity $-2i\pi B_i$, showing the consistency of the equation. Moreover, the contribution proportional to $\ln(-g)$ yields the combination $\ln(gB_i)$ which is globally even in E, g as it should.

4. Instantons.

Let us now explain, in the case of the double-well potential, how path integrals and instanton calculus naturally lead to the conjectured form of the secular equations.

4.1. Partition function and resolvent.

The quantum partition function

$$\mathcal{Z}(eta) \equiv \mathrm{tr} \; \mathrm{e}^{-eta H} = \sum_{N \geqslant 0} \mathrm{e}^{-eta E_N}$$

contains the information about the spectrum of the hamiltonian H (which we assume discrete) and has a path integral representation. From the partition function, we can derive the trace G(E) of the resolvent of Hby Laplace transformation:

(4.1)
$$G(E) = \operatorname{tr} \frac{1}{H - E},$$

(4.2)
$$= \int_0^\infty \mathrm{d}\beta \,\mathrm{e}^{\beta E} \mathcal{Z}(\beta)$$

where, initially, Re E is chosen at the right of the spectrum of H, to ensure the convergence of the integral for $\beta \to \infty$.

The poles of G(E) then yield the spectrum of the hamiltonian H.

Note that the integral (4.2) does not necessarily converge at $\beta = 0$ because the eigenvalues may not increase fast enough. For example, the trace of the resolvent for the harmonic oscillator formally reads

$$G_{\rm osc.}(E) = \sum_{k=0}^{\infty} \frac{1}{n+1/2 - E} = \int_0^\infty \mathrm{d}\beta \, \frac{\mathrm{e}^{\beta E}}{2\sinh(\beta/2)},$$

which diverges. However, the first derivative of G(E) is defined. Integrating we can choose

$$G_{\text{osc.}}(E) = -\psi(1/2 - E),$$

where $\psi(z)$ is the logarithmic derivative of the Γ -function. In general different definitions of G(E) differ by a polynomial in E, but this does not affect the location of poles and their residues.

Finally, from G(E) one can derive the Fredholm determinant $\mathcal{D}(E) = \det(H - E)$ since

(4.3)
$$\ln \mathcal{D}(E) - \ln \mathcal{D}(E') = \operatorname{tr} \ln (H - E) - \operatorname{tr} \ln (H - E')$$

= $-\int_{E'}^{E} G(E'') dE''$,

where E' does not belong to the spectrum of H. Again, all expressions may have to be replaced by some regularized form. For the harmonic oscillator one definition leads to

(4.4)
$$\mathcal{D}_{\text{osc.}}(E) = 1/\Gamma(1/2 - E),$$

but irrespective of the precise definition, the spectrum is then solution of $\Delta(E) = 0$.

Note that for the double-well potential, we can separate eigenvalues corresponding to symmetric and antisymmetric eigenfunctions by considering the two functions

(4.5)
$$\mathcal{Z}_{\pm}(\beta) = \operatorname{tr} \left[\frac{1}{2}(1\pm P)\mathrm{e}^{-\beta H}\right] = \sum_{N=0} \mathrm{e}^{-\beta E_{\pm,N}}$$

where P is the parity operator (2.2). The eigenvalues are then poles of the Laplace transforms

(4.6)
$$G_{\epsilon}(E) = \int_{0}^{\infty} \mathrm{d}\beta \,\mathrm{e}^{\beta E} \mathcal{Z}_{\epsilon}(\beta),$$

 $(\epsilon = \pm 1).$

For the periodic cosine potential, one uses a generalized partition function with twisted boundary conditions depending on a rotation angle.

4.2. Path integrals and spectra of hamiltonians.

In the path integral formulation of quantum mechanics, the partition function is given by

(4.7)
$$Z(\beta) \propto \int_{q(-\beta/2)=q(\beta/2)} [\mathrm{d}q(t)] \exp\left[-\frac{1}{g}S(q(t))\right],$$

where the symbol $\int [dq(t)]$ means summation over all paths satisfying the boundary conditions, that is closed paths, and S(q) is the *euclidean* action:

(4.8)
$$\mathcal{S}(q) = \int_{-\beta/2}^{\beta/2} \left[\frac{1}{2} \dot{q}^2(t) + V(q(t)) \right] \mathrm{d}t.$$

In the case of the symmetric potential (2.1) which has degenerate minima, it is convenient to also consider the quantity

(4.9)
$$\mathcal{Z}_{\mathbf{a}}(\beta) \equiv \operatorname{tr}\left(P \mathrm{e}^{-\beta H}\right) \propto \int_{q(-\beta/2)+q(\beta/2)=1} \left[\mathrm{d}q(t)\right] \exp\left[-\frac{1}{g}\mathcal{S}\left(q(t)\right)\right]$$

where P is the parity operator (2.2). Then, eigenvalues corresponding to symmetric and antisymmetric eigenfunctions can be derived from the combinations (4.5).

Perturbation theory. — Perturbative expansions, that is expansions in powers of g for $g \to 0$, can be obtained by applying the steepest descent method to the path integral. Saddle points are solutions $q_c(t)$ to the euclidean equations of motion (euclidean equations are distinguished from the normal equations of classical mechanics by the sign in front of the potential). When the potential has a unique minimum, located for example at q = 0, the leading saddle point is $q_c(t) = 0$. A systematic expansion

around the saddle point then leads to an expansion of the eigenvalues of the hamiltonian of the form

$$E(g) = \sum_{n=0}^{\infty} E_n g^n.$$

In the case of a potential with degenerate minima, one must sum over several saddle points: to each saddle point corresponds an eigenvalue and thus several eigenvalues are degenerate at leading order. Because the potential (2.1) is symmetric, the lowest eigenvalue, and more generally all eigenvalues which remain finite when g goes to zero, are twice degenerate to all orders in perturbation theory:

(4.10)
$$E_{\pm,N}(g) = E_N^{(0)}(g) \equiv \sum_{n=0}^{\infty} E_{N,n}^{(0)} g^n.$$

4.3. Instantons.

Eigenvalues which remain finite when g goes to zero can be extracted from the large β expansion. In the infinite β limit, paths that dominate the path integral are solutions of the euclidean equations of motion that have a finite action. In the case of the path integral representation of $\mathcal{Z}_{\mathbf{a}}(\beta)$, constant solutions of the equation of motion do not satisfy the boundary conditions. Finite action solutions necessarily correspond to paths which connect the two minima of the potential (see figure 1).

In the example of the double-well potential (2.1), such solutions are

(4.11)
$$q_{\rm c}(t) = \left(1 + e^{\pm (t-t_0)}\right)^{-1}, \qquad S(q_{\rm c}) = 1/6.$$

Since the two solutions depend on an integration constant t_0 , one finds two one-parameter families of degenerated saddle points.

Non-constant solutions with finite action are called *instanton* solutions. Since the main contribution to the action comes from the region around $t = t_0$, one calls t_0 the position of the instanton.

The corresponding contribution to the path integral is proportional, at leading order in g and for $\beta \to \infty$, to $e^{-1/(6g)}$ and thus is nonperturbative. It is also proportional to β because one has to sum over all degenerate saddle points and the integration constant t_0 varies in $[0, \beta]$



Figure 2. The instanton configuration.

for β large but finite. It follows that the two lowest eigenvalues are now given by $(\epsilon = \pm 1)$:

(4.12)
$$E_{\epsilon,0}(g) = \lim_{\beta \to \infty} -\frac{1}{\beta} \ln \mathcal{Z}_{\epsilon}(\beta) \underset{g \to 0, \beta \to \infty}{=} E_0^{(0)}(g) - \epsilon E_0^{(1)}(g),$$

(4.13)
$$E_0^{(1)}(g) = \frac{1}{\sqrt{\pi g}} e^{-1/6g} (1 + O(g))$$

5. Multi-instantons.

Taking into account $E^{(0)}(g)$ and $E^{(1)}(g)$, one obtains for the functions (4.5) an expansion of the form

(5.1)
$$\mathcal{Z}_{\epsilon}(\beta) \underset{\beta \to \infty}{\sim} e^{-\beta (E_0^{(0)} - \epsilon E_0^{(1)})} \sim e^{-\beta E_0^{(0)}} \sum_{n=0}^{\infty} \frac{(\epsilon \beta)^n}{n!} \left(E_0^{(1)} \right)^n$$

Thus, the existence of a one-instanton contribution to eigenvalues implies the existence of *n*-instanton contributions to the functions (4.5), proportional to β^n .

For β finite, the path integrals indeed have other saddle points which correspond to oscillations in the well of the potential -V(q). In the infinite β limit, the solutions with n oscillations have an action $n \times 1/6$ and thus give contributions to the path integral of the expected form.

However, there are some subtleties: naively one would expect these configurations to give a contribution of order β (for β large) because a given classical trajectory depends only on one time integration constant. This has to contrasted with the expansion (5.1) where the n^{th} term is of order β^n .

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Moreover, one discovers that the gaussian integration near the saddle point involves the determinant of an operator which has eigenvalues which vanish exponentially in the large β limit. This divergence has the following origin: in the large β limit, the classical solution decomposes into a succession of infinitely separated instantons and fluctuations which tend to change the distance between instantons induce an infinitesimal variation of the action. It follows that, to properly study the limit, one has to introduce additional *collective coordinates*, which parametrize these configurations that are close to solutions of the euclidean equation of motion, even though they have a sightly different action. It can then also be understood where in the expansion (5.1) the factor β^n comes from. Although a given classical trajectory can only generate a factor β , these new configurations depend on *n* independent collective coordinates over which one has to integrate.

To summarize: we know that n-instanton contributions do exist. However, these contributions do not correspond, in general, to solutions of the classical equation of motion. They correspond to configurations of largely separated instantons connected in a way which we shall discuss, which become solutions of the equation of motion only asymptotically, in the limit of infinite separation. These configurations depend on n times more collective coordinates than the one-instanton configuration.

5.1. Multi-instanton configurations.

We now briefly explain, still with the example of the double-well potential, how multi-instanton contributions to the path integral can be evaluated at leading order for $g \to 0$.

In the infinite β limit, the instanton solutions can be written as

(5.2)
$$q_{\pm}(t) = f(\mp(t-t_0)),$$

(5.3)
$$f(t) = 1/(1 + e^t) = 1 - f(-t),$$

where the integration constant t_0 characterizes the instanton position.

The two-instanton configuration. — We first construct the twoinstanton configuration. We look for a configuration depending on an additional time parameter, the separation between instantons, which in the limit of infinite separation decomposes into two instantons, and which for large separation minimizes the variation of the action [15]. For this purpose, we could introduce a constraint in the path integral fixing the separation



Figure 3. The two-instanton configuration.

between instantons, and solve the equation of motion with a Lagrange multiplier for the constraint. Instead, we use a method which, at least at leading order, is simpler and more intuitive.

We consider a configuration $q_c(t)$ that is the sum of instantons separated by a distance θ , up to an additive constant adjusted in such a way as to satisfy the boundary conditions (figure 3):

(5.4)
$$q_{\rm c}(t) = f(t - \theta/2) + f(-t - \theta/2) - 1 = f(t - \theta/2) - f(t + \theta/2),$$

where f(t) is the function (5.3) (and of course all configurations deduced by time translation). This path has the following properties: it is continuous and differentiable and when θ is large it differs, near each instanton, from the instanton solution only by exponentially small terms of order $e^{-\theta}$. Although the calculation of the corresponding action is straightforward, we perform it stepwise to show that the ansatz (5.4) applies to more general symmetric potentials.

It is convenient to introduce some additional notation:

(5.5)
$$u(t) = f(t - \theta/2),$$

(5.6)
$$v(t) = u(t+\theta),$$

and thus $q_c = u - v$. The action corresponding to the path (5.4) can be written as

$$S(q_c) = \int dt \left[\frac{1}{2} \dot{q}_c^2 + V(q_c) \right]$$

(5.7)
$$= 2 \times \frac{1}{6} + \int dt \left[-\dot{u}\dot{v} + V(u-v) - V(u) - V(v) \right].$$

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The parity of q_c allows to restrict the integration to the region t > 0, where v is at least of order $e^{-\theta/2}$. After an integration by parts of the term $\dot{v}\dot{u}$, one finds

(5.8)
$$S(q_c) = \frac{1}{3} + 2\left\{v(0)\dot{u}(0) + \int_0^{+\infty} dt \left[v\ddot{u} + V(u-v) - V(u) - V(v)\right]\right\}.$$

One then expands the integrand in powers of v. Since the leading correction to S is of order $e^{-\theta}$, one needs only the expansion up to order v^2 . The term linear in v cancels as a consequence of the *u*-equation of motion. One obtains

(5.9)
$$S(q_c) - \frac{1}{3} \sim 2v(0)\dot{u}(0) + 2\left\{\int_0^{+\infty} \mathrm{d}t \left[\frac{1}{2}v^2 V''(u) - \frac{1}{2}V''(0)v^2\right]\right\}.$$

The function v decreases exponentially away from the origin so the main contributions to the integral come from the neighbourhood of t = 0, where $u = 1 + O(e^{-\theta/2})$ and thus $V''(u) \sim V''(1) = V''(0)$. Therefore, at leading order, the two terms in the integral cancel. At leading order,

$$v(0)\dot{u}(0) \sim -\mathrm{e}^{-\theta}$$

and thus

(5.10)
$$\mathcal{S}(q_c) = \frac{1}{3} - 2 e^{-\theta} + O\left(e^{-2\theta}\right).$$

It will become clearer later why the classical action is needed only up to order $e^{-\theta}$. In analogy with the partition function of a classical gas (instantons being identified with particles), one calls the quantity $-2e^{-\theta}$ interaction potential between instantons.

Actually, it is useful to perform the calculation for β large but finite. Symmetry between θ and $\beta - \theta$ then implies

(5.11)
$$S(q_c) = \frac{1}{3} - 2e^{-\theta} - 2e^{-(\beta-\theta)} + \text{ negligible contributions.}$$

The variation of the action. — We now show that if we modify infinitesimally (for θ large) the configuration to further decrease the variation of the action, the change r(t) of the path is of order $e^{-\theta}$ and the variation of the action of order $e^{-2\theta}$ at least. Setting

(5.12)
$$q(t) = q_c(t) + r(t)$$

and expanding the action up to second order in r(t), one finds

$$S(q_c + r) = S(q_c) + \int \left[\dot{q}_c(t) \dot{r}(t) + V'(q_c(t)) r(t) \right] dt$$

(5.13)
$$+ \frac{1}{2} \int dt \left[\dot{r}^2(t) + V''(q_c) r^2(t) \right] + O\left(\left[r(t) \right]^3 \right)$$

In the term linear in r(t), one integrates by parts $\dot{r}(t)$, in order to use the property that $q_c(t)$ approximately satisfies the equation of motion. In the term quadratic in r(t), one replaces V'' by 1, since r(t) is expected to be large only far from the instantons. One then verifies that the term linear in r is of order $e^{-\theta}$ while the quadratic term is of order 1. A shift of r to eliminate the linear term would then give a negligible contribution, of order $e^{-2\theta}$.

The n-instanton configuration. — We now consider a succession of n instantons separated by times θ_i with

(5.14)
$$\sum_{i=1}^{n} \theta_i = \beta.$$

At leading order, we need only consider "interactions" between nearest neighbour instantons. Other interactions are negligible because they are of higher order in $e^{-\theta}$.

The classical action $S_{c}(\theta_{i})$ can then be directly inferred from expression (5.11):

(5.15)
$$\mathcal{S}_{c}(\theta_{i}) = \frac{n}{6} - 2\sum_{i=1}^{n} e^{-\theta_{i}} + O\left(e^{-(\theta_{i}+\theta_{j})}\right).$$

Note that for *n* even, the *n*-instanton configurations contribute to tr $e^{-\beta H}$, while for *n* odd they contribute to tr $(Pe^{-\beta H})$ (*P* is the parity operator). But all contribute to the combination (4.5).

Remark. — Since we keep in the action all terms of order $e^{-\beta}$, we expect to find the contributions not only to the two lowest energies but also to all energies which remain finite when g goes to zero.

5.2. The *n*-instanton contribution.

We have calculated the n-instanton action. Let us now evaluate, at leading order, the contribution to the path integral of the neighbourhood

of the *n*-instanton configuration [1], [3]. We expand the action up to second order in the deviation from the classical path. Although the path is not a solution of the equation of motion, it has been chosen it in such a way that the linear terms in the expansion can be neglected. The gaussian integration involves then the determinant of the second derivative of the action at the classical path

(5.16)
$$M(t',t) = \left[-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + V''(q_{\mathrm{c}}(t))\right]\delta(t-t').$$

The operator M has the form of a hamiltonian with a potential that consists of n wells asymptotically identical to the well arising in the one-instanton problem, and which are largely separated. Therefore, at leading order, the corresponding spectrum is the spectrum arising in the one-instanton problem n-times degenerate. Corrections are exponentially small in the separation. Simultaneously, by introducing n collective time variables, we have suppressed n times the zero eigenvalue and generated the jacobian of the one-instanton case to the power n. Therefore, the n-instanton contribution to the combination (4.5):

$$\mathcal{Z}_{\epsilon}(\beta) = \frac{1}{2} \operatorname{tr} \left[(1 + \epsilon P) e^{-\beta H} \right],$$

 $(\epsilon = \pm 1)$, can be written as

(5.17)
$$\mathcal{Z}_{\epsilon}^{(n)}(\beta) = e^{-\beta/2} \frac{\beta}{n} \left(\epsilon \frac{e^{-1/6g}}{\sqrt{\pi g}} \right)^n$$

$$\int_{\theta_i \ge 0} \delta \left(\sum \theta_i - \beta \right) \prod_i d\theta_i \exp\left[\frac{2}{g} \sum_{i=1}^n e^{-\theta_i} \right].$$

All factors have already been explained, except the factor β , which comes from the integration over a global time translation, and the factor 1/nwhich arises because the configuration is invariant under a cyclic permutation of the θ_i . Finally, the normalization factor $e^{-\beta/2}$ corresponds to the partition function of the harmonic oscillator.

Let us define the "fugacity" of the instanton gas

(5.18)
$$\lambda = \frac{1}{\sqrt{\pi g}} \mathrm{e}^{-1/6g},$$

which is half the one-instanton contribution at leading order.

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If the instanton interactions are neglected, the integration over the θ_i 's is straightforward and the sum of the leading order *n*-instanton contributions

(5.19)
$$\Sigma_{\epsilon}(\beta, g) = e^{-\beta/2} + \sum_{n=1}^{\infty} \mathcal{Z}_{\epsilon}^{(n)}(\beta, g)$$

can be calculated:

(5.20)
$$\Sigma_{\epsilon}(\beta,g) = e^{-\beta/2} \left[1 + \beta \sum_{n=1}^{\infty} \frac{(\epsilon\lambda)^n}{n} \frac{\beta^{n-1}}{(n-1)!} \right] = e^{-\beta \left(\frac{1}{2} - \epsilon\lambda\right)}$$

We recognize the perturbative and one-instanton contribution, at leading order, to $E_{\epsilon,0}(g)$, the ground state and the first excited state energies:

(5.21)
$$E_{\epsilon,0}(g) = \frac{1}{2} + O(g) - \frac{\epsilon}{\sqrt{\pi g}} e^{-1/6g} (1 + O(g)).$$

Discussion. — To go beyond the one-instanton approximation, it is necessary to take into account the interaction between instantons. Unfortunately, if one examines expression (5.17), one discovers that the interaction between instantons is *attractive*. Therefore, for g small, the dominant contributions to the integral come from configurations in which the instantons are close. For such configurations, the concept of instanton is no longer meaningful, since they cannot be distinguished from fluctuations around the constant or the one-instanton solution.

We should have expected such a difficulty. Indeed, the large order behaviour analysis has shown that the perturbative expansion in the case of potentials with degenerate minima is not Borel summable. An ambiguity is expected at the two-instanton order. But if the perturbative expansion is ambiguous at the two-instanton order, contributions of the same order or even smaller are ill-defined. To proceed any further, we must first give a meaning to the sum of the perturbative expansion.

In the example of the double-well potential, it is possible to show that the perturbation series is Borel summable for g negative, by relating it to the perturbative expansion of the O(2) anharmonic oscillator. Therefore, we define the sum of the perturbation series as the analytic continuation of this Borel sum from g negative to $g = |g| \pm i0$. From the point of view of the Borel transformation, this corresponds to integrate above or below the real positive axis, respectively. We then note that for g negative the interaction between instantons is repulsive and, simultaneously, the expression (5.17) becomes meaningful. Therefore, we first calculate, for g small and negative, both the sum of the perturbation series and the instanton contributions, and perform an analytic continuation to g positive of all quantities consistently. In the same way, the perturbative expansion around each multi-instanton configuration is also non-Borel summable and the sum is defined by the same procedure.

5.3. The sum of leading order instanton contributions.

The Laplace transform

$$G_{\epsilon}^{(n)}(E) = \int_{0}^{\infty} \mathrm{d}\beta \,\mathrm{e}^{\beta E} \mathcal{Z}_{\epsilon}^{(n)}(\beta)$$

of the *n*-instanton contribution (5.17):

(5.22)
$$\mathcal{Z}_{\epsilon}^{(n)}(\beta) \sim \frac{\beta}{n} \mathrm{e}^{-\beta/2} (\epsilon \lambda)^n \int_{\theta_i \ge 0} \delta \left(\sum \theta_i - \beta \right) \prod_{i=1}^n \mathrm{d}\theta_i \exp \left[\frac{2}{g} \sum_{i=1}^n \mathrm{e}^{-\theta_i} \right],$$

yields the leading contribution $G_{\epsilon}^{(n)}(E)$ to the trace $G_{\epsilon}(E)$ of the resolvent (equation (4.6)). The integral over β is immediate and the integrals over the θ_i then factorize. One obtains

$$G_{\epsilon}^{(n)}(E) \sim \frac{(\epsilon\lambda)^n}{n} \frac{\partial}{\partial E} I^n(E-1/2)$$

with

(5.23)
$$I(s) = \int_0^{+\infty} e^{s\theta - \mu e^{-\theta}} d\theta,$$

where we have set $\mu = -2/g$.

To evaluate the integral (5.23), we change variables setting

(5.24)
$$\mu e^{-\theta} = t$$

and the integral becomes

(5.25)
$$I(s) = \int_0^{\mu} \frac{dt}{t} \left(\frac{\mu}{t}\right)^s e^{-t} = \int_0^{+\infty} \frac{dt}{t} \left(\frac{\mu}{t}\right)^s e^{-t} + O\left(e^{-\mu}/\mu\right)$$

for $g \to 0_-$ and thus μ positive and large. Up to an exponentially small correction, we thus obtain

(5.26)
$$I(s) \sim \mu^s \Gamma(-s).$$

The generating function $\mathcal{G}_{\epsilon}(E,g)$ of the leading order multi-instanton contributions (5.19) then is given by

$$\mathcal{G}_{\epsilon}(E,g) = \sum_{n=0} G_{\epsilon}^{(n)}(E) = -\frac{\partial}{\partial E} \ln \Delta_{\epsilon}(E)$$

with

$$\Delta_{\epsilon}(E) = 1 - \epsilon \lambda \mu^{E - 1/2} \Gamma(1/2 - E).$$

The functions $\Delta_{\epsilon}(E)$ give directly the sum of instanton contributions to the Fredholm determinant

$$\mathcal{D}_{\epsilon}(E) \propto \prod_{N} (1 - E/E_{\epsilon,N}) \quad \Rightarrow \det(H - E) \propto \mathcal{D}_{+}(E)\mathcal{D}_{-}(E),$$

but, unlike $\mathcal{D}_{\epsilon}(E)$, it is a meromorphic function of E because we have approximated the zero instanton contribution by its leading term for $\beta \to \infty$. Correcting for this effect amounts adding to $\mathcal{G}_{\epsilon}(E,g)$ the trace of the resolvent of the harmonic oscillator, and thus dividing $\Delta_{\epsilon}(E)$ by $\Gamma(1/2 - E)$. This cancels the poles and $\Delta_{\epsilon}(E)$ becomes an entire function

(5.27)
$$\Delta_{\epsilon}(E) = \frac{1}{\Gamma(1/2 - E)} - \epsilon \lambda \mu^{E - 1/2}.$$

The first term is simply the Fredholm determinant (4.4) corresponding to the harmonic oscillator. It is remarkable that the sum of all instanton contributions simply yield a kind of one-instanton correction (but *E*dependent) to the spectral equation.

Since λ is small, zeros of the equation $\Delta_{\epsilon}(E) = 0$ are close to eigenvalues of the harmonic oscillator

(5.28)
$$E_{\epsilon,N} = N + \frac{1}{2} + O(\lambda), \quad N \ge 0.$$

The zeros of the function (5.27) can then be expanded in a power series in λ :

(5.29)
$$E_{\epsilon,N}(g) = \sum_{n} E_N^{(n)}(g) (-\epsilon\lambda)^n.$$

One obtains from a unique equation the multi-instanton contributions to all energy eigenvalues $E_{\epsilon,N}(g)$ of the double-well potential at leading order.

The appearance of a factor $\ln g$ in equation (2.13) can now be simply understood by noting that the interaction terms are only relevant for $g^{-1}e^{-\theta}$ of order 1, that is θ of order $-\ln g$.

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Beyond leading order. — Assuming that an equation of the form (5.27) holds beyond leading order, we conclude that the argument of the Γ -function must be such that the equation reproduces at zero instanton order the perturbative expansion: E has to be replaced by the function B(E,g). If we further assume that, for B large, there is some connection with the WKB expansion, we infer that $(-2/g)^E$ must also be replaced by $(-2/g)^B$ to reconstruct a term of the form $B \ln(Eg)$. This is the origin of the conjecture.

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