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A GENERAL HILBERT-MUMFORD CRITERION

by Jürgen HAUSEN

1. Statement of the results.

Let a reductive group G act on a normal complex algebraic variety X. It is a central problem in Geometric Invariant Theory to construct all G-invariant open subsets $V \subset X$ admitting a *good quotient*, i.e. an affine G-invariant morphism $V \to V/\!\!/G$ onto a complex algebraic space such that locally $V/\!\!/G$ is the spectrum of the invariant functions. Let us call these $V \subset X$ for the moment the *good G-sets.*

In principle, it suffices to know all good T-sets $U \subset X$ for some fixed maximal torus $T \subset G$, because the good G-sets are precisely the G-invariant good T-sets, see [3]. The construction of "maximal" good T-sets is less hard, and in order to gain good G-sets one studies the following question: *Let $U \subset X$ be a good T-set. When is the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, a good G-set?*

The classical Hilbert-Mumford Criterion answers this question in the affirmative for sets of T-semistable points of G-linearized ample line bundles. Moreover, A. Białynicki-Birula and J. Święcicka settled in [2] the case of good T-sets defined by generalized moment functions, and in [3]

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the case $U = X$, as mentioned before. For $G = \mathrm{SL}_2$, several results can be found in [4], [5], and [12].

As indicated, one imposes maximality conditions on the good T-set U, e.g. projectivity or completeness of $U/\!\!/T$. The most general concept is *T-maximality*: U is not T-saturated in some properly larger good T-set U', where *T-saturated* means saturated with respect to the quotient map. For complete X and T-maximal $U \subset X$ which are invariant under the normalizer $N(T)$, A. Białynicki-Birula conjectures that $W(U)$ is a good G-set [1, Conj. 12.1].

We shall settle the case of *$(T,2)$-maximal* subsets. These are good T-sets $U \subset X$ such that $U/\!\!/T$ is embeddable into a toric variety, and U is not a T-saturated subset of some properly larger U' having the same properties, compare [14]. We shall assume that X is $\mathbb{Q}$-factorial, i.e. for every Weil divisor on X some multiple is Cartier. In Section 4, we prove:

THEOREM 1.1. — *Let a connected reductive group G act on a $\mathbb{Q}$-factorial complex variety X. Let $T \subset G$ be a maximal torus and $U \subset X$ a $(T,2)$-maximal open subset. Then the intersection $W(U)$ of all translates $g{\cdot}U$, $g \in G$, is open in X, there is a good quotient $W(U) \to W(U)/\!\!/G$, and $W(U)$ is T-saturated in U.*

This generalizes results by A. Białynicki-Birula and J. Święcicka for $X = \mathbb{P}^n$, see [6, Thm. C], and by J. Święcicka for smooth complete varieties X with $\mathrm{Pic}(X) = \mathbb{Z}$, see [14, Cor. 6.3]. As an application of Theorem 1.1, we obtain:

COROLLARY 1.2. — *Let a connected reductive group G act on a complete $\mathbb{Q}$-factorial toric variety X, and let $T \subset G$ be a maximal torus. Then we have*

(i) *For every T-maximal open subset $U \subset X$ the set $W(U)$ is open and admits a good quotient $W(U) \to W(U)/\!\!/G$.*

(ii) *Every G-invariant open subset $V \subset X$ admitting a good quotient $V \to V/\!\!/G$ is a G-saturated subset of some set $W(U)$ as in (i).*

Together with well-known fan-theoretical descriptions of the T-maximal open subsets, see e.g. [13], this corollary explicitly solves the quotient problem for actions of connected reductive groups G on $\mathbb{Q}$-factorial toric varieties. In [1, Problem 12.9] our corollary was conjectured (in fact for arbitrary toric varieties).

2. Background on good quotients.

We recall basic definitions and facts on good quotients, see also [1, Chap. 7], [3, Sec. 1] and [6, Sec. 2]. Let a reductive group G act morphically on a complex algebraic variety X. The concept of a good quotient is locally, with respect to the étale topology, modelled on the classical invariant theory quotient:

DEFINITION 2.1. — *A G-invariant morphism $p: X \to Y$ onto a separated complex algebraic space Y is called a good quotient for the G-action on X if Y is covered by étale neighbourhoods $V \to Y$ such that*

(i) *V and its inverse image $U := p^{-1}(V) = X \times_Y V$ are affine varieties,*

(ii) *$p^*: \mathcal{O}(V) \to \mathcal{O}(U)$ defines an isomorphism onto the algebra of G-invariants.*

A good quotient $p: X \to Y$ for the G-action on X is called geometric, if its fibres are precisely the G-orbits.

A good quotient $X \to Y$ for the G-action on X is categorical, i.e. any G-invariant morphism $X \to Z$ of algebraic spaces factors uniquely through $X \to Y$. In particular, good quotient spaces are unique up to isomorphism. This justifies the notation $X \to X/\!\!/G$ for good and $X \to X/G$ for geometric quotients.

In the sequel we say that an open subset $U \subset X$ of a G-variety X with good quotient is *G-saturated*, if U is saturated with respect to the quotient map $X \to X/\!\!/G$. The following well-known properties of good quotients are direct consequences of the corresponding statements in the affine case:

Remark 2.2. — Assume that the G-action on X has a good quotient $p: X \to X/\!\!/G$.

(i) If $A \subset X$ is G-invariant and closed, then $p(A)$ is closed in $X/\!\!/G$, and the restriction $p: A \to p(A)$ is a good quotient for the action of G on A.

(ii) If A and A' are disjoint G-invariant closed subsets of X, then $p(A)$ and $p(A')$ are disjoint.

(iii) If $U \subset X$ is G-saturated and open, then $p(U)$ is open in $X/\!\!/G$, and the restriction $p: U \to p(U)$ is a good quotient for the action of G on U.

(iv) If $A \subset X$ and $U \subset X$ are as in (i) and (iii), then $A \cap U$ is G-saturated in A.

Let X be normal (in particular irreducible) with a good quotient $X \to X/\!\!/G$. Then any reductive subgroup $H \subset G$ admits a good quotient $X \to X/\!\!/H$, see [7, Cor. 10]. If H is normal in G, then universality of good quotients [1, Thm. 7.1.4] allows to push down the G-action to $X/\!\!/H$. Moreover, we have

Proposition 2.3. — *Let $H \subset G$ be a reductive normal subgroup such that $X/\!\!/H$ is an algebraic variety. Then the canonical map $X/\!\!/H \to X/\!\!/G$ is a good quotient for the induced action of G/H on $X/\!\!/H$.*

We turn to the special case of an action of an algebraic torus T on a normal variety X. Good quotients for such torus actions are always affine morphisms of normal algebraic varieties, see [3, Cor. 1.3]. We work with the following maximality concepts for good quotients, compare [14, Def. 4.3]:

Definition 2.4. — *A T-invariant open subset $U \subset X$ with a good quotient $U \to U/\!\!/T$ is called a (T,k)-maximal subset of X if*

(i) *the quotient space $U/\!\!/T$ is an A_k-variety, i.e. any collection $y_1, \ldots, y_k \in U/\!\!/T$ admits a common affine neighbourhood in $U/\!\!/T$,*

(ii) *U does not occur as proper T-saturated subset of some T-invariant open $U' \subset X$ admitting a good quotient $U' \to U'/\!\!/T$ with an A_k-variety $U'/\!\!/T$.*

As usual, *T-maximal* stands for $(T,1)$-maximal. The collection of all (T,k)-maximal subsets is always finite, see [14, Thm. 4.4]. The case $k=2$ can also be characterized via embeddability of the quotient spaces: By [15, Thm. A], a normal variety has the A_2-property if and only if it embeds into a toric variety.

Proposition 2.5. — *Let X be a toric variety, and let the algebraic torus T act on X via a homomorphism $T \to T_X$ to the big torus $T_X \subset X$. Then the T-maximal subsets of X are precisely the $(T,2)$-maximal subsets of X.*

Proof. — First observe that every $(T,2)$-maximal subset is T-saturated in some T-maximal subset. Hence we only have to show that for any T-maximal $U \subset X$ the quotient space $U/\!\!/T$ is an A_2-variety. But this is known: By [13, Cor. 2.4 and 2.5], the set U is T_X-invariant, and $U/\!\!/T$ inherits the structure of a toric variety from U. In particular, $U/\!\!/T$ is an A_2-variety, see [15, p. 709]. □

3. Globally defined $(T, 2)$-maximal subsets.

Let G be a connected reductive group, $T \subset G$ a maximal torus, and X a normal G-variety. In this section, we reduce the construction of $(T, 2)$-maximal subsets to a purely toric problem in $\mathbb{C}^n$. The following notion is central:

DEFINITION 3.1. — *We say that a $(T, 2)$-maximal subset $U \subset X$ is globally defined in X, if there are T-homogeneous $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that each X_{f_i} is an affine open subset of U and any pair $x, x' \in U$ is contained in some X_{f_i}.*

Here, as usual, $f \in \mathcal{O}(X)$ is called *T-homogeneous*, if $f(t \cdot x) = \chi(t)f(x)$ holds with a character $\chi: T \to \mathbb{C}^*$, and X_f denotes the set of all $x \in X$ with $f(x) \neq 0$. Our reduction is split into two lemmas. The proofs are based on ideas of [11].

LEMMA 3.2. — *Let X be $\mathbb{Q}$-factorial, and let $U \subset X$ be $(T, 2)$-maximal. Then there are an algebraic torus H and a $\mathbb{Q}$-factorial quasi-affine $(G \times H)$-variety $\widehat{X}$ such that*

(i) *H acts freely on $\widehat{X}$ with a G-equivariant geometric quotient $q: \widehat{X} \to X$,*

(ii) *$\widehat{U} := q^{-1}(U)$ is a globally defined $(T \times H, 2)$-maximal subset of $\widehat{X}$.*

Proof. — Let $p: U \to U /\!\!/ T$ be the quotient. By assumption, we can cover $U /\!\!/ T$ by affine open subsets $Y_1, \ldots, Y_r$ such that any pair $y, y' \in U /\!\!/ T$ is contained in a common Y_i. Since p is affine, each $p^{-1}(Y_i)$ is affine. Hence each $X \setminus p^{-1}(Y_i)$ is of pure codimension one and, by $\mathbb{Q}$-factoriality, equals the support $\mathrm{Supp}(D_i)$ of an effective Cartier divisor D_i on X.

The Cartier divisors $D_1, \ldots, D_r$ generate a free abelian subgroup Λ of the group of all Cartier divisors of X. Enlarging Λ by adding finitely many generators, we achieve that every $x \in X$ admits an affine neighbourhood $X \setminus \mathrm{Supp}(D)$ for some effective member $D \in \Lambda$. The group Λ gives rise to a graded $\mathcal{O}_X$-algebra

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D := \bigoplus_{D \in \Lambda} \mathcal{O}_X(D).$$

After eventually replacing Λ with a subgroup of finite index, we can endow $\mathcal{A}$ with a G-sheaf structure, see [11, Prop. 3.5]: for any $g \in G$

and any open $V \subset X$, we then have a Λ-graded homomorphism $\mathcal{A}(V) \to \mathcal{A}(g \cdot V)$, these homomorphisms are compatible with restriction of $\mathcal{A}$ and multiplication of G, and the resulting G-representation on $\mathcal{A}(X)$ is rational.

We define the desired data; for details see [10, Sec. 2]. Let $\widehat{X} := \operatorname{Spec}(\mathcal{A})$. The inclusion $\mathcal{O}_X \to \mathcal{A}$ defines an affine morphism $q\colon \widehat{X} \to X$ with $q_*(\mathcal{O}_{\widehat{X}}) = \mathcal{A}$. For the canonical section of an effective $D \in \Lambda$, its zero set in $\widehat{X}$ is just $q^{-1}(\operatorname{Supp}(D))$. In particular, $\widehat{X}$ is covered by affine sets $\widehat{X}_f$ and hence is quasi-affine.

The Λ-grading of $\mathcal{A}$ corresponds to a free action of the torus $H := \operatorname{Spec}(\mathbb{C}[\Lambda])$ on $\widehat{X}$. This makes $q\colon \widehat{X} \to X$ to an H-principal bundle. In particular, q is a geometric quotient for the H-action, and $\widehat{X}$ is $\mathbb{Q}$-factorial. The G-sheaf structure of $\mathcal{A}$ induces a G-action on $\widehat{X}$ commuting with the H-action and making q equivariant.

We show that $\widehat{U} = q^{-1}(U)$ is $(\widehat{T}, 2)$-maximal, where we set $\widehat{T} := T \times H$. First note that the restriction $p \circ q\colon \widehat{U} \to U /\!\!/ T$ is a good quotient for the $\widehat{T}$-action. For $(\widehat{T}, 2)$-maximality, let $\widehat{U}$ be $\widehat{T}$-saturated in some $(\widehat{T}, 2)$-maximal $\widehat{U}_1 \subset \widehat{X}$. Then Lemma 2.3 gives a commutative diagram

$$\begin{tikzcd} \widehat{U}_1 \arrow[rr, "/\!\!/ \widehat{T}"] \arrow[dr, "q", "/H"'] & & \widehat{U}_1 /\!\!/ \widehat{T} \\ & U_1 \arrow[ur, "/\!\!/ T"'] & \end{tikzcd}$$

where $U_1 := q(\widehat{U}_1)$ is open in X. Since $\widehat{U}$ is $\widehat{T}$-saturated in $\widehat{U}_1$ and $\widehat{U}_1 \to U_1$ is surjective, this diagram shows that U is a T-saturated subset of U_1. By $(T, 2)$-maximality of U in X, this implies $U = U_1$ and hence $\widehat{U} = \widehat{U}_1$.

Finally, let $f_i \in \mathcal{O}(\widehat{X})$ be the canonical sections of some large positive multiples of the D_i. The zero set of f_i in $\widehat{X}$ is just $q^{-1}(\operatorname{Supp}(D_i))$. In particular, these zero sets are $\widehat{T}$-invariant, and hence the f_i are $\widehat{T}$-homogeneous. By construction, the sets $\widehat{X}_{f_i}$ equal $q^{-1}(p^{-1}(Y_i))$, and thus form an affine cover of $\widehat{U}$ as required in 3.1. □

Lemma 3.3. — *Let X be quasi-affine, and let $U \subset X$ be a globally defined $(T, 2)$-maximal subset of X. Then there exist a linear G-action on some $\mathbb{C}^n$ and a G-equivariant locally closed embedding $X \to \mathbb{C}^n$ such that*

(i) *the maximal torus $T \subset G$ acts on $\mathbb{C}^n$ by means of a homomorphism $T \to \mathbb{T}^n$ to the big torus $\mathbb{T}^n := (\mathbb{C}^*)^n$,*

(ii) *there is a* $\mathbb{T}^n$*-invariant open* $V \subset \mathbb{C}^n$ *containing* U *as a closed subset and admitting a good quotient* $V \to V/\!\!/T$.

Proof. — Let $f_1, \dots, f_r \in \mathcal{O}(X)$ be as in 3.1, and set $X_i := X_{f_i}$. By [10, Lemma 2.4], we can realize X as a G-invariant open subset of an affine G-variety $\overline{X}$ such that the f_i extend regularly to $\overline{X}$ and satisfy $\overline{X}_{f_i} = X_i$. Complete the f_i to a system $f_1, \dots, f_s$ of T-homogeneous generators of the algebra $\mathcal{O}(\overline{X})$.

To proceed, we use the standard representation $g \cdot f(x) := f(g^{-1} \cdot x)$ of G on $\mathcal{O}(\overline{X})$. Let $M_i \subset \mathcal{O}(\overline{X})$ be the G-module generated by $G \cdot f_i$. Fix a basis $f_{i1}, \dots, f_{in_i}$ of M_i such that all f_{ij} are T-homogeneous and for the first one we have $f_{i1} = f_i$. Denoting by N_i the dual G-module of M_i, we obtain G-equivariant maps

$$\Phi_i \colon \overline{X} \to N_i, \qquad x \mapsto [h \mapsto h(x)].$$

We identify N_i with $\mathbb{C}^{n_i}$ by associating to a functional of N_i its coordinates $z_{i1}, \dots, z_{in_i}$ with respect to the dual basis $f_{i1}^*, \dots, f_{in_i}^*$. Then the pullback $\Phi_i^*(z_{ij})$ is just the function f_{ij}. Now, consider the direct sum of the G-modules $\mathbb{C}^{n_i}$; we write this direct sum as $\mathbb{C}^n$ but still use the coordinates z_{ij}. The maps Φ_i fit together to a G-equivariant closed embedding:

$$\Phi \colon \overline{X} \to \mathbb{C}^n, \qquad x \mapsto (f_{11}(x), \dots, f_{1n_1}(x), \dots, f_{s1}(x), \dots, f_{sn_s}(x)).$$

In the sequel, we shall regard $\overline{X}$ as a G-invariant closed subset of $\mathbb{C}^n$. Thus the functions f_{ij} are just the restrictions of the coordinate functions z_{ij}. By construction, the maximal torus T of G acts diagonally on $\mathbb{C}^n$, that means that T acts by a homomorphism $T \to \mathbb{T}^n$ to the big torus $\mathbb{T}^n = (\mathbb{C}^*)^n$.

We come to the construction of the desired set $V \subset \mathbb{C}^n$. Let $V_i \subset \mathbb{C}^n$ be the complement of the coordinate hyperplane defined by z_{i1}. Note that $\overline{X} \cap V_i$ equals X_i. In particular, X_i is closed in V_i. Consider the union $V_0 := V_1 \cup \ldots \cup V_r$. Then V_0 is invariant under the big torus $\mathbb{T}^n$. Moreover, we have

$$\overline{X} \cap V_0 = \bigcup_{i=1}^{r} \overline{X} \cap V_i = \bigcup_{i=1}^{r} \overline{X}_{f_i} = \bigcup_{i=1}^{r} X_i = U.$$

Let $V \subset V_0$ be the minimal $\mathbb{T}^n$-invariant open subset with $U = \overline{X} \cap V$. Then every closed $\mathbb{T}^n$-orbit of V has nontrivial intersection with U. We show that V admits a good quotient by the action of T. By [11, Prop. 1.2], it suffices to verify that any two points with closed $\mathbb{T}^n$-orbits in V have a common T-invariant affine open neighbourhood in V.

Let $z, z' \in V$ have closed $\mathbb{T}^n$-orbits in V. Since these $\mathbb{T}^n$-orbits meet U, there are $t, t' \in \mathbb{T}^n$ such that $t\cdot z$ and $t'\cdot z'$ lie in U. By the choice of $f_1, \ldots, f_r$, the points $t\cdot z$ and $t'\cdot z'$ even lie in some common X_i. Consider the corresponding V_i and the good quotient $p\colon V_i \to V_i/\!\!/T$. The latter is a toric morphism of affine toric varieties.

Let $Z_i := V_i \setminus V$. Then Z_i is T-invariant and closed in V_i. Moreover, Z_i does not meet the T-invariant closed subset $X_i \subset V_i$. Thus $p(Z_i)$ and $p(X_i)$ are closed in $V_i/\!\!/T$ and disjoint from each other. In particular, neither $p(t\cdot z)$ nor $p(t'\cdot z')$ lie in $p(Z_i)$. Since Z_i is even $\mathbb{T}^n$-invariant, also $p(z)$ and $p(z')$ do not lie in $p(Z_i)$.

Consequently, there exists a T-invariant regular function on V_i that vanishes along Z_i but not in the points z and z'. Removing the zero set of this function from V_i yields the desired common T-invariant affine open neighbourhood of the points z and z' in V. This proves existence of a good quotient $V \to V/\!\!/T$. □

4. Proof of the results.

Proof of Theorem 1.1. — First we reduce to the case of globally defined subsets of quasi-affine varieties. So, assume for the moment that Theorem 1.1 holds in this setting. Consider the quasi-affine variety $\widehat{X}$, the torus H and the geometric quotient $q\colon \widehat{X} \to X$ provided by Lemma 3.2.

Then $\widehat{G} := G \times H$ is reductive with maximal torus $\widehat{T} := T \times H$, and $\widehat{U} = q^{-1}(U)$ is a globally defined $(\widehat{T}, 2)$-maximal subset of $\widehat{X}$. By assumption, the intersection $W(\widehat{U})$ of all translates $\widehat{g}\cdot\widehat{U}$ is open, admits a good quotient by $\widehat{G}$, and is $\widehat{T}$-saturated in $\widehat{U}$. Since each $\widehat{g}\cdot\widehat{U}$ is H-invariant and $q\colon \widehat{X} \to X$ is G-equivariant, we obtain

$$W(\widehat{U}) = \bigcap_{\widehat{g}\in\widehat{G}} \widehat{g}\cdot\widehat{U} = \bigcap_{g\in G} g\cdot\widehat{U} = \bigcap_{g\in G} g\cdot q^{-1}(U) = q^{-1}(W(U)).$$

In particular, $W(U)$ is open in X. Moreover, restricting q gives a geometric quotient $W(\widehat{U}) \to W(U)$ for the H-action. Lemma 2.3 tells us that the induced map from $W(U)$ onto $W(\widehat{U})/\!\!/\widehat{G}$ is a good quotient for the

G-action on $W(U)$. Similarly, we infer T-saturatedness of $W(U)$ in U from the commutative diagram

$$\begin{tikzcd}
\widehat{U} \arrow[rr, "/\!/\widehat{T}"] \arrow[dr, "q", "/H"'] & & \widehat{U}/\!/\widehat{T} \\
& U \arrow[ur, "/\!/T"'] &
\end{tikzcd}$$

We are left with proving 1.1 for quasi-affine X and globally defined $(T,2)$-maximal $U \subset X$. By Lemma 3.3, we may view X as a G-invariant locally closed subset of a G-module $\mathbb{C}^n$, where T acts via a homomorphism $T \to \mathbb{T}^n$ and U is closed in some $\mathbb{T}^n$-invariant open $V \subset \mathbb{C}^n$ with good quotient $V \to V/\!/T$. We regard $\mathbb{C}^n$ as the G-invariant open subset of $\mathbb{P}^n$ obtained by removing the zero set of the homogeneous coordinate z_0.

Let $V' \subset \mathbb{P}^n$ be a T-maximal open subset containing V as a T-saturated subset. Let $\overline{X}$ be the closure of X in $\mathbb{P}^n$, and set $X' := \overline{X} \cap V'$. Then X' is closed in V', and we have $U = X' \cap V$. Using 2.2 (i), (iii) and (iv), we subsume the situation in a commutative cube

$$\begin{tikzcd}
& U \arrow[rr] \arrow[dl] \arrow[dd] & & V \arrow[dl] \arrow[dd] \\
X' \arrow[rr, crossing over] \arrow[dd] & & V' & \\
& U/\!/T \arrow[rr, dashed] \arrow[dl] & & V/\!/T \arrow[dl] \\
X'/\!/T \arrow[rr] & & V'/\!/T \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the downwards arrows are good quotients by the respective actions of T, the right arrows are closed inclusions, the upper diagonal arrows are T-saturated inclusions and the lower diagonal arrows are open inclusions.

According to [6, Thm. C], the intersection $W(V')$ of all translates $g{\cdot}V'$ is open in $\mathbb{P}^n$ and admits a good quotient by the action of G. Recall from [6, Lemma 8.4] that $W(V')$ is T-saturated in V'. We transfer the desired properties step by step from $W(V')$ to $W(U)$. First note that by G-invariance of $\overline{X}$ we have

$$W(X') = \bigcap_{g \in G} g{\cdot}X' = \bigcap_{g \in G} g{\cdot}(\overline{X} \cap V') = \overline{X} \cap W(V') = X' \cap W(V').$$

Thus $W(X')$ is open in X', and by 2.2 (iv) it is T-saturated in X'. In particular, the T-action on $W(X')$ has a good quotient. Moreover, $W(X')$

is G-invariant and closed in $W(V')$. Thus 2.2 (i) ensures the existence of a good quotient

$$u: W(X') \to W(X')/\!\!/G.$$

Consider $B := X' \setminus X$. Since X is open in $\overline{X}$ and B equals $(\overline{X} \setminus X) \cap X'$, the set B is closed in X'. The intersection $W(B)$ of the translates $g \cdot B$, where $g \in G$, is G-invariant and closed in $W(X')$. We claim that it suffices to verify

$$W(U) = W(X') \setminus u^{-1}(u(W(B))). \tag{1}$$

Indeed, suppose we have (1). Then $W(U)$ is open in X', hence in U, and thus in X. Property 2.2 (iii) provides a good quotient $W(U) \to W(U)/\!\!/G$. Moreover, $W(U)$ is T-saturated in $W(X')$, because it is G-saturated and we have the induced map from $W(X')/\!\!/T$ onto $W(X')/\!\!/G$. Since $W(X')$ and U are T-saturated in X', we obtain that $W(U)$ is T-saturated in U.

We verify (1). Let $v: X' \to X'/\!\!/T$ be the quotient map. As a subvariety, $X'/\!\!/T$ inherits the A_2-property from $V'/\!\!/T$, which in turn satisfies it by 2.5. Thus, since U is $(T, 2)$-maximal in X, it is necessarily the maximal T-saturated subset of X' which is contained in $X \cap X'$. In terms of $B = X' \setminus X$ this means

$$U = X' \setminus v^{-1}(v(B)). \tag{2}$$

We check the inclusion "$\subset$" of (1). Let $x \in u^{-1}(u(W(B)))$. Then, by 2.2 (ii), the closure of $G \cdot x$ meets $W(B)$. The classical Hilbert-Mumford Lemma [8, Thm. 4.2] says that for some maximal torus $T' \subset G$ the closure of $T' \cdot x$ meets $W(B)$. Let $g \in G$ with $T = gT'g^{-1}$. Then the closure of $T \cdot g \cdot x$ meets $W(B)$. Hence $g \cdot x$ lies in $v^{-1}(v(B))$. By (2), the point x cannot belong to $W(U)$.

We turn to the inclusion "$\supset$" of (1). For this, consider the set $A := (X \cap X') \setminus U$. Then X' is the disjoint union of U, A and B. Consequently, we have

$$W(U) = \bigcap_{g \in G} g \cdot (X' \setminus (A \cup B)) = W(X') \setminus \bigcup_{g \in G} g \cdot A \cup g \cdot B.$$

So we have to show that u maps a given $x \in W(X') \cap g \cdot (A \cup B)$ to $u(W(B))$. Since $g^{-1} \cdot x \notin U$ holds, we infer from (2) that $g^{-1} \cdot x$ lies in

$v^{-1}(v(B))$. According to 2.2 (ii), the closure of $T \cdot g^{-1} \cdot x$ in X' meets B. Since $W(X')$ is T-saturated in X', this implies that the closure of $T \cdot g^{-1} \cdot x$ meets $W(X') \cap B$. But we have

$$W(X') \cap B \;=\; W(X') \setminus X \;=\; \bigcap_{g \in G} g \cdot (X' \setminus X) \;=\; W(B).$$

Hence we obtained that the closure of the orbit $G \cdot x$ intersects $W(B)$. This in turn shows that the image $u(x)$ lies in $u(W(B))$. $\square$

Proof of Corollary 1.2. — Recall from [9, Sec. 4] that the automorphism group of X is a linear algebraic group having the big torus $T_X \subset X$ as a maximal torus. Thus, by conjugating T_X we achieve that $T \subset G$ acts on X via a homomorphism $T \to T_X$. Proposition 2.5 then ensures that each T-maximal subset of X is as well $(T, 2)$-maximal, and statement (i) follows from Theorem 1.1.

For statement (ii), let $V \subset X$ be open and G-invariant with good quotient $V \to V /\!\!/ G$. Then [7, Cor. 10] provides a good quotient $V \to V /\!\!/ T$. Let $U \subset X$ be a T-maximal subset containing V as T-saturated subset. Then we have $V \subset W(U)$. Again by 2.5, the set U is $(T, 2)$-maximal. Thus Theorem 1.1 says that $W(U)$ is open, has a good quotient $u \colon W(U) \to W(U) /\!\!/ G$, and is T-saturated in U.

For G-saturatedness of V in $W(U)$ we have to show that any $x \in u^{-1}(u(V))$ with closed G-orbit in $W(U)$ belongs to V. For this note that V is T-saturated in $W(U)$, because both sets are so in U. Now, let $y \in V$ with $u(y) = u(x)$. Then x lies in the closure of $G \cdot y$. Thus [8, Thm. 4.2] provides a $g \in G$ such that the closure of $T \cdot g \cdot y$ meets $G \cdot x$. Since $g \cdot y$ lies in V and V is T-saturated in $W(U)$, we obtain $G \cdot x \subset V$, and hence $x \in V$. $\square$

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