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A general Hilbert-Mumford criterion

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A GENERAL HILBERT-MUMFORD CRITERION

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1. Statement of the results.

Let a reductive group $G$ act on a normal complex algebraic variety $X$. It is a central problem in Geometric Invariant Theory to construct all $G$-invariant open subsets $V \subset X$ admitting a good quotient, i.e. an affine $G$-invariant morphism $V \to V//G$ onto a complex algebraic space such that locally $V//G$ is the spectrum of the invariant functions. Let us call these $V \subset X$ for the moment the good $G$-sets.

In principle, it suffices to know all good $T$-sets $U \subset X$ for some fixed maximal torus $T \subset G$, because the good $G$-sets are precisely the $G$-invariant good $T$-sets, see [3]. The construction of “maximal” good $T$-sets is less hard, and in order to gain good $G$-sets one studies the following question: Let $U \subset X$ be a good $T$-set. When is the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, a good $G$-set?

The classical Hilbert-Mumford Criterion answers this question in the affirmative for sets of $T$-semistable points of $G$-linearized ample line bundles. Moreover, A. Białynicki-Birula and J. Święcicka settled in [2] the case of good $T$-sets defined by generalized moment functions, and in [3]

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the case $U = X$, as mentioned before. For $G = \text{SL}_2$, several results can be found in [4], [5], and [12].

As indicated, one imposes maximality conditions on the good $T$-set $U$, e.g. projectivity or completeness of $U/T$. The most general concept is $T$-maximality: $U$ is not $T$-saturated in some properly larger good $T$-set $U'$, where $T$-saturated means saturated with respect to the quotient map. For complete $X$ and $T$-maximal $U \subset X$ which are invariant under the normalizer $N(T)$, A. Bialynicki-Birula conjectures that $W(U)$ is a good $G$-set [1, Conj. 12.1].

We shall settle the case of $(T, 2)$-maximal subsets. These are good $T$-sets $U \subset X$ such that $U/T$ is embeddable into a toric variety, and $U$ is not a $T$-saturated subset of some properly larger $U'$ having the same properties, compare [14]. We shall assume that $X$ is $\mathbb{Q}$-factorial, i.e. for every Weil divisor on $X$ some multiple is Cartier. In Section 4, we prove:

**Theorem 1.1.** — Let a connected reductive group $G$ act on a $\mathbb{Q}$-factorial complex variety $X$. Let $T \subset G$ be a maximal torus and $U \subset X$ a $(T, 2)$-maximal open subset. Then the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, is open in $X$, there is a good quotient $W(U) \to W(U)/\!/G$, and $W(U)$ is $T$-saturated in $U$.

This generalizes results by A. Białyńcki-Birula and J. Święcicka for $X = \mathbb{P}^n$, see [6, Thm. C], and by J. Święcicka for smooth complete varieties $X$ with Pic$(X) = \mathbb{Z}$, see [14, Cor. 6.3]. As an application of Theorem 1.1, we obtain:

**Corollary 1.2.** — Let a connected reductive group $G$ act on a complete $\mathbb{Q}$-factorial toric variety $X$, and let $T \subset G$ be a maximal torus. Then we have

(i) For every $T$-maximal open subset $U \subset X$ the set $W(U)$ is open and admits a good quotient $W(U) \to W(U)/\!/G$.

(ii) Every $G$-invariant open subset $V \subset X$ admitting a good quotient $V \to V/\!/G$ is a $G$-saturated subset of some set $W(U)$ as in (i).

Together with well-known fan-theoretical descriptions of the $T$-maximal open subsets, see e.g. [13], this corollary explicitly solves the quotient problem for actions of connected reductive groups $G$ on $\mathbb{Q}$-factorial toric varieties. In [1, Problem 12.9] our corollary was conjectured (in fact for arbitrary toric varieties).
2. Background on good quotients.

We recall basic definitions and facts on good quotients, see also [1, Chap. 7], [3, Sec. 1] and [6, Sec. 2]. Let a reductive group $G$ act morphically on a complex algebraic variety $X$. The concept of a good quotient is locally, with respect to the étale topology, modelled on the classical invariant theory quotient:

**Definition 2.1.** A $G$-invariant morphism $p: X \to Y$ onto a separated complex algebraic space $Y$ is called a good quotient for the $G$-action on $X$ if $Y$ is covered by étale neighbourhoods $V \to Y$ such that

(i) $V$ and its inverse image $U := p^{-1}(V) = X \times_Y V$ are affine varieties,

(ii) $p^*: \mathcal{O}(V) \to \mathcal{O}(U)$ defines an isomorphism onto the algebra of $G$-invariants.

A good quotient $p: X \to Y$ for the $G$-action on $X$ is called geometric, if its fibres are precisely the $G$-orbits.

A good quotient $X \to Y$ for the $G$-action on $X$ is categorical, i.e. any $G$-invariant morphism $X \to Z$ of algebraic spaces factors uniquely through $X \to Y$. In particular, good quotient spaces are unique up to isomorphism. This justifies the notation $X \to X//G$ for good and $X \to X/G$ for geometric quotients.

In the sequel we say that an open subset $U \subset X$ of a $G$-variety $X$ with good quotient is $G$-saturated, if $U$ is saturated with respect to the quotient map $X \to X//G$. The following well-known properties of good quotients are direct consequences of the corresponding statements in the affine case:

**Remark 2.2.** — Assume that the $G$-action on $X$ has a good quotient $p: X \to X//G$.

(i) If $A \subset X$ is $G$-invariant and closed, then $p(A)$ is closed in $X//G$, and the restriction $p: A \to p(A)$ is a good quotient for the action of $G$ on $A$.

(ii) If $A$ and $A'$ are disjoint $G$-invariant closed subsets of $X$, then $p(A)$ and $p(A')$ are disjoint.

(iii) If $U \subset X$ is $G$-saturated and open, then $p(U)$ is open in $X//G$, and the restriction $p: U \to p(U)$ is a good quotient for the action of $G$ on $U$.

(iv) If $A \subset X$ and $U \subset X$ are as in (i) and (iii), then $A \cap U$ is $G$-saturated in $A$. 

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Let $X$ be normal (in particular irreducible) with a good quotient $X \to X//G$. Then any reductive subgroup $H \subset G$ admits a good quotient $X \to X//H$, see [7, Cor. 10]. If $H$ is normal in $G$, then universality of good quotients [1, Thm. 7.1.4] allows to push down the $G$-action to $X//H$. Moreover, we have

**Proposition 2.3.** — Let $H \subset G$ be a reductive normal subgroup such that $X//H$ is an algebraic variety. Then the canonical map $X//H \to X//G$ is a good quotient for the induced action of $G/H$ on $X//H$.

We turn to the special case of an action of an algebraic torus $T$ on a normal variety $X$. Good quotients for such torus actions are always affine morphisms of normal algebraic varieties, see [3, Cor. 1.3]. We work with the following maximality concepts for good quotients, compare [14, Def. 4.3]:

**Definition 2.4.** — A $T$-invariant open subset $U \subset X$ with a good quotient $U \to U//T$ is called a $(T,k)$-maximal subset of $X$ if

(i) the quotient space $U//T$ is an $A_k$-variety, i.e. any collection $y_1, \ldots, y_k \in U//T$ admits a common affine neighbourhood in $U//T$,

(ii) $U$ does not occur as proper $T$-saturated subset of some $T$-invariant open $U' \subset X$ admitting a good quotient $U' \to U'//T$ with an $A_k$-variety $U'//T$.

As usual, $T$-maximal stands for $(T,1)$-maximal. The collection of all $(T,k)$-maximal subsets is always finite, see [14, Thm. 4.4]. The case $k=2$ can also be characterized via embeddability of the quotient spaces: By [15, Thm. A], a normal variety has the $A_2$-property if and only if it embeds into a toric variety.

**Proposition 2.5.** — Let $X$ be a toric variety, and let the algebraic torus $T$ act on $X$ via a homomorphism $T \to T_X$ to the big torus $T_X \subset X$. Then the $T$-maximal subsets of $X$ are precisely the $(T,2)$-maximal subsets of $X$.

**Proof.** — First observe that every $(T,2)$-maximal subset is $T$-saturated in some $T$-maximal subset. Hence we only have to show that for any $T$-maximal $U \subset X$ the quotient space $U//T$ is an $A_2$-variety. But this is known: By [13, Cor. 2.4 and 2.5], the set $U$ is $T_X$-invariant, and $U//T$ inherits the structure of a toric variety from $U$. In particular, $U//T$ is an $A_2$-variety, see [15, p. 709].
3. Globally defined \((T,2)\)-maximal subsets.

Let \(G\) be a connected reductive group, \(T \subset G\) a maximal torus, and \(X\) a normal \(G\)-variety. In this section, we reduce the construction of \((T,2)\)-maximal subsets to a purely toric problem in \(\mathbb{C}^n\). The following notion is central:

**Definition 3.1.** We say that a \((T,2)\)-maximal subset \(U \subset X\) is globally defined in \(X\), if there are \(T\)-homogeneous \(f_1, \ldots, f_r \in \mathcal{O}(X)\) such that each \(X_{f_i}\) is an affine open subset of \(U\) and any pair \(x, x' \in U\) is contained in some \(X_{f_i}\).

Here, as usual, \(f \in \mathcal{O}(X)\) is called \(T\)-homogeneous, if \(f(t \cdot x) = \chi(t)f(x)\) holds with a character \(\chi: T \to \mathbb{C}^*\), and \(X_f\) denotes the set of all \(x \in X\) with \(f(x) \neq 0\). Our reduction is split into two lemmas. The proofs are based on ideas of [11].

**Lemma 3.2.** Let \(X\) be \(\mathbb{Q}\)-factorial, and let \(U \subset X\) be \((T,2)\)-maximal. Then there are an algebraic torus \(H\) and a \(\mathbb{Q}\)-factorial quasi-affine \((G \times H)\)-variety \(\hat{X}\) such that

(i) \(H\) acts freely on \(\hat{X}\) with a \(G\)-equivariant geometric quotient \(q: \hat{X} \to X\),

(ii) \(\hat{U} := q^{-1}(U)\) is a globally defined \((T \times H, 2)\)-maximal subset of \(\hat{X}\).

**Proof.** Let \(p: U \to U/\!/T\) be the quotient. By assumption, we can cover \(U/\!/T\) by affine open subsets \(Y_1, \ldots, Y_r\) such that any pair \(y, y' \in U/\!/T\) is contained in a common \(Y_i\). Since \(p\) is affine, each \(p^{-1}(Y_i)\) is affine. Hence each \(X \setminus p^{-1}(Y_i)\) is of pure codimension one and, by \(\mathbb{Q}\)-factoriality, equals the support \(\text{Supp}(D_i)\) of an effective Cartier divisor \(D_i\) on \(X\).

The Cartier divisors \(D_1, \ldots, D_r\) generate a free abelian subgroup \(\Lambda\) of the group of all Cartier divisors of \(X\). Enlarging \(\Lambda\) by adding finitely many generators, we achieve that every \(x \in X\) admits an affine neighbourhood \(X \setminus \text{Supp}(D)\) for some effective member \(D \in \Lambda\). The group \(\Lambda\) gives rise to a graded \(\mathcal{O}_X\)-algebra

\[ A := \bigoplus_{D \in \Lambda} A_D := \bigoplus_{D \in \Lambda} \mathcal{O}_X(D). \]

After eventually replacing \(\Lambda\) with a subgroup of finite index, we can endow \(A\) with a \(G\)-sheaf structure, see [11, Prop. 3.5]: for any \(g \in G\)
and any open $V \subset X$, we then have a $\Lambda$-graded homomorphism $A(V) \to A(g \cdot V)$, these homomorphisms are compatible with restriction of $A$ and multiplication of $G$, and the resulting $G$-representation on $A(X)$ is rational.

We define the desired data; for details see [10, Sec. 2]. Let $\hat{X} := \text{Spec}(A)$. The inclusion $\mathcal{O}_X \to A$ defines an affine morphism $q: \hat{X} \to X$ with $q_*(\mathcal{O}_{\hat{X}}) = A$. For the canonical section of an effective $D \in \Lambda$, its zero set in $\hat{X}$ is just $q^{-1}(\text{Supp}(D))$. In particular, $\hat{X}$ is covered by affine sets $\hat{X}_f$ and hence is quasi-affine.

The $\Lambda$-grading of $A$ corresponds to a free action of the torus $H := \text{Spec}(\mathbb{C}[\Lambda])$ on $\hat{X}$. This makes $q: \hat{X} \to X$ to an $H$-principal bundle. In particular, $q$ is a geometric quotient for the $H$-action, and $\hat{X}$ is $\mathbb{Q}$-factorial. The $G$-sheaf structure of $A$ induces a $G$-action on $\hat{X}$ commuting with the $H$-action and making $q$ equivariant.

We show that $\hat{U} = q^{-1}(U)$ is $(\hat{T}, 2)$-maximal, where we set $\hat{T} := T \times H$. First note that the restriction $p \circ q: \hat{U} \to U//T$ is a good quotient for the $\hat{T}$-action. For $(\hat{T}, 2)$-maximality, let $U$ be $\hat{T}$-saturated in some $(\hat{T}, 2)$-maximal $U_1 \subset \hat{X}$. Then Lemma 2.3 gives a commutative diagram

$$
\begin{array}{ccc}
\hat{U}_1 & \overset{q}{\longrightarrow} & \hat{U}_1//\hat{T} \\
\downarrow \scriptstyle{\hat{T}} & & \downarrow \scriptstyle{\hat{T}} \\
U_1 & \underset{H}{\longrightarrow} & \hat{U}_1//T
\end{array}
$$

where $U_1 := q(\hat{U}_1)$ is open in $X$. Since $\hat{U}$ is $\hat{T}$-saturated in $\hat{U}_1$ and $\hat{U}_1 \to U_1$ is surjective, this diagram shows that $U$ is a $T$-saturated subset of $U_1$. By $(T, 2)$-maximality of $U$ in $X$, this implies $U = U_1$ and hence $\hat{U} = \hat{U}_1$.

Finally, let $f_i \in \mathcal{O}(\hat{X})$ be the canonical sections of some large positive multiples of the $D_i$. The zero set of $f_i$ in $\hat{X}$ is just $q^{-1}(\text{Supp}(D_i))$. In particular, these zero sets are $\hat{T}$-invariant, and hence the $f_i$ are $\hat{T}$-homogeneous. By construction, the sets $\hat{X}_{f_i}$ equal $q^{-1}(p^{-1}(Y_i))$, and thus form an affine cover of $\hat{U}$ as required in 3.1.

**Lemma 3.3.** — Let $X$ be quasi-affine, and let $U \subset X$ be a globally defined $(T, 2)$-maximal subset of $X$. Then there exist a linear $G$-action on some $\mathbb{C}^n$ and a $G$-equivariant locally closed embedding $X \to \mathbb{C}^n$ such that

(i) the maximal torus $T \subset G$ acts on $\mathbb{C}^n$ by means of a homomorphism $T \to \mathbb{T}^n$ to the big torus $\mathbb{T}^n := (\mathbb{C}^*)^n$, 

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(ii) there is a $T^n$-invariant open $V \subset \mathbb{C}^n$ containing $U$ as a closed subset and admitting a good quotient $V \rightarrow V/\!/T$.

\textbf{Proof.} — Let $f_1, \ldots, f_r \in \mathcal{O}(X)$ be as in 3.1, and set $X_i := X_{f_i}$. By [10, Lemma 2.4], we can realize $X$ as a $G$-invariant open subset of an affine $G$-variety $\overline{X}$ such that the $f_i$ extend regularly to $\overline{X}$ and satisfy $\overline{X}_{f_i} = X_i$. Complete the $f_i$ to a system $f_1, \ldots, f_s$ of $T$-homogeneous generators of the algebra $\mathcal{O}(\overline{X})$.

To proceed, we use the standard representation $g \cdot f(x) := f(g^{-1} \cdot x)$ of $G$ on $\mathcal{O}(\overline{X})$. Let $M_i \subset \mathcal{O}(\overline{X})$ be the $G$-module generated by $G \cdot f_i$. Fix a basis $f_{i1}, \ldots, f_{in_i}$ of $M_i$ such that all $f_{ij}$ are $T$-homogeneous and for the first one we have $f_{i1} = f_i$. Denoting by $N_i$ the dual $G$-module of $M_i$, we obtain $G$-equivariant maps

$$
\Phi_i : \overline{X} \rightarrow N_i, \quad x \mapsto [h \mapsto h(x)].
$$

We identify $N_i$ with $\mathbb{C}^{n_i}$ by associating to a functional of $N_i$ its coordinates $z_{i1}, \ldots, z_{in_i}$ with respect to the dual basis $f^*_1, \ldots, f^*_i$. Then the pullback $\Phi_i^*(z_{ij})$ is just the function $f_{ij}$. Now, consider the direct sum of the $G$-modules $\mathbb{C}^{n_i}$; we write this direct sum as $\mathbb{C}^n$ but still use the coordinates $z_{ij}$. The maps $\Phi_i$ fit together to a $G$-equivariant closed embedding:

$$
\Phi : \overline{X} \rightarrow \mathbb{C}^n, \quad x \mapsto (f_{i1}(x), \ldots, f_{in_1}(x), \ldots, f_{s1}(x), \ldots, f_{sn_s}(x)).
$$

In the sequel, we shall regard $\overline{X}$ as a $G$-invariant closed subset of $\mathbb{C}^n$. Thus the functions $f_{ij}$ are just the restrictions of the coordinate functions $z_{ij}$. By construction, the maximal torus $T$ of $G$ acts diagonally on $\mathbb{C}^n$, that means that $T$ acts by a homomorphism $T \rightarrow \mathbb{T}^n$ to the big torus $\mathbb{T}^n = (\mathbb{C}^\ast)^n$.

We come to the construction of the desired set $V \subset \mathbb{C}^n$. Let $V_i \subset \mathbb{C}^n$ be the complement of the coordinate hyperplane defined by $z_{i1}$. Note that $\overline{X} \cap V_i$ equals $X_i$. In particular, $X_i$ is closed in $V_i$. Consider the union $V_0 := V_1 \cup \ldots \cup V_r$. Then $V_0$ is invariant under the big torus $\mathbb{T}^n$. Moreover, we have

$$
\overline{X} \cap V_0 = \bigcup_{i=1}^r \overline{X} \cap V_i = \bigcup_{i=1}^r \overline{X}_{f_i} = \bigcup_{i=1}^r X_i = U.
$$

Let $V \subset V_0$ be the minimal $\mathbb{T}^n$-invariant open subset with $U = \overline{X} \cap V$. Then every closed $\mathbb{T}^n$-orbit of $V$ has nontrivial intersection with $U$. We
show that $V$ admits a good quotient by the action of $T$. By [11, Prop. 1.2], it suffices to verify that any two points with closed $\mathbb{T}^n$-orbits in $V$ have a common $T$-invariant affine open neighbourhood in $V$.

Let $z, z' \in V$ have closed $\mathbb{T}^n$-orbits in $V$. Since these $\mathbb{T}^n$-orbits meet $U$, there are $t, t' \in \mathbb{T}^n$ such that $t \cdot z$ and $t' \cdot z'$ lie in $U$. By the choice of $f_1, \ldots, f_r$, the points $t \cdot z$ and $t' \cdot z'$ even lie in some common $X_i$. Consider the corresponding $V_i$ and the good quotient $p: V_i \to V_i/T$. The latter is a toric morphism of affine toric varieties.

Let $Z_i := V_i \setminus V$. Then $Z_i$ is $T$-invariant and closed in $V_i$. Moreover, $Z_i$ does not meet the $T$-invariant closed subset $X_i \subset V_i$. Thus $p(Z_i)$ and $p(X_i)$ are closed in $V_i/T$ and disjoint from each other. In particular, neither $p(t \cdot z)$ nor $p(t' \cdot z')$ lie in $p(Z_i)$. Since $Z_i$ is even $\mathbb{T}^n$-invariant, also $p(z)$ and $p(z')$ do not lie in $p(Z_i)$.

Consequently, there exists a $T$-invariant regular function on $V_i$ that vanishes along $Z_i$ but not in the points $z$ and $z'$. Removing the zero set of this function from $V_i$ yields the desired common $T$-invariant affine open neighbourhood of the points $z$ and $z'$ in $V$. This proves existence of a good quotient $V \to V/T$. \hfill \Box

4. Proof of the results.

Proof of Theorem 1.1.— First we reduce to the case of globally defined subsets of quasi-affine varieties. So, assume for the moment that Theorem 1.1 holds in this setting. Consider the quasi-affine variety $\hat{X}$, the torus $H$ and the geometric quotient $q: \hat{X} \to X$ provided by Lemma 3.2.

Then $\hat{G} := G \times H$ is reductive with maximal torus $\hat{T} := T \times H$, and $\hat{U} = q^{-1}(U)$ is a globally defined $(\hat{T}, 2)$-maximal subset of $\hat{X}$. By assumption, the intersection $W(\hat{U})$ of all translates $g \cdot \hat{U}$ is open, admits a good quotient by $\hat{G}$, and is $\hat{T}$-saturated in $\hat{U}$. Since each $g \cdot \hat{U}$ is $H$-invariant and $q: \hat{X} \to X$ is $G$-equivariant, we obtain

$$W(\hat{U}) = \bigcap_{\hat{g} \in \hat{G}} \hat{g} \cdot \hat{U} = \bigcap_{g \in G} g \cdot \hat{U} = \bigcap_{g \in G} g \cdot q^{-1}(U) = q^{-1}(W(U)).$$

In particular, $W(\hat{U})$ is open in $X$. Moreover, restricting $q$ gives a geometric quotient $W(\hat{U}) \to W(U)$ for the $H$-action. Lemma 2.3 tells us that the induced map from $W(U)$ onto $W(\hat{U})/\hat{G}$ is a good quotient for the
Similarly, we infer $T$-saturatedness of $W(U)$ in $U$ from the commutative diagram

\[
\begin{array}{ccc}
\hat{U} & \xrightarrow{\mathbb{T}} & \hat{U} \sslash \hat{T} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\mathbb{T}} & U \sslash \mathbb{T}
\end{array}
\]

We are left with proving 1.1 for quasi-affine $X$ and globally defined $(T, 2)$-maximal $U \subset X$. By Lemma 3.3, we may view $X$ as a $G$-invariant locally closed subset of a $G$-module $\mathbb{C}^n$, where $T$ acts via a homomorphism $T \to \mathbb{T}^n$ and $U$ is closed in some $\mathbb{T}^n$-invariant open $V \subset \mathbb{C}^n$ with good quotient $V \to V \sslash T$. We regard $\mathbb{C}^n$ as the $G$-invariant open subset of $\mathbb{P}^n$ obtained by removing the zero set of the homogeneous coordinate $z_0$.

Let $V' \subset \mathbb{P}^n$ be a $T$-maximal open subset containing $V$ as a $T$-saturated subset. Let $\overline{X}$ be the closure of $X$ in $\mathbb{P}^n$, and set $X' := \overline{X} \cap V'$. Then $X'$ is closed in $V'$, and we have $U = X' \cap V$. Using 2.2 (i), (iii) and (iv), we subsume the situation in a commutative cube

\[
\begin{array}{ccc}
U & \xrightarrow{\mathbb{T}} & V \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\mathbb{T}} & V' \\
\downarrow & & \downarrow \\
X' \sslash \mathbb{T} & \xrightarrow{\mathbb{T}} & V' \sslash \mathbb{T}
\end{array}
\]

where the downwards arrows are good quotients by the respective actions of $T$, the right arrows are closed inclusions, the upper diagonal arrows are $T$-saturated inclusions and the lower diagonal arrows are open inclusions.

According to [6, Thm. C], the intersection $W(V')$ of all translates $g \cdot V'$ is open in $\mathbb{P}^n$ and admits a good quotient by the action of $G$. Recall from [6, Lemma 8.4] that $W(V')$ is $T$-saturated in $V'$. We transfer the desired properties step by step from $W(V')$ to $W(U)$. First note that by $G$-invariance of $\overline{X}$ we have

\[
W(X') = \bigcap_{g \in G} g \cdot X' = \bigcap_{g \in G} g \cdot (\overline{X} \cap V') = \overline{X} \cap W(V') = X' \cap W(V').
\]

Thus $W(X')$ is open in $X'$, and by 2.2 (iv) it is $T$-saturated in $X'$. In particular, the $T$-action on $W(X')$ has a good quotient. Moreover, $W(X')$
is $G$-invariant and closed in $W(V')$. Thus 2.2 (i) ensures the existence of a good quotient

$$u: W(X') \to W(X')/G.$$  

Consider $B := X' \setminus X$. Since $X$ is open in $X$ and $B$ equals $(\overline{X} \setminus X) \cap X'$, the set $B$ is closed in $X'$. The intersection $W(B)$ of the translates $g \cdot B$, where $g \in G$, is $G$-invariant and closed in $W(X')$. We claim that it suffices to verify

\[(1) \quad W(U) = W(X') \setminus u^{-1}(u(W(B))).\]

Indeed, suppose we have (1). Then $W(U)$ is open in $X'$, hence in $U$, and thus in $X$. Property 2.2 (iii) provides a good quotient $W(U) \to W(U)/G$. Moreover, $W(U)$ is $T$-saturated in $W(X')$, because it is $G$-saturated and we have the induced map from $W(X')/T$ onto $W(X')/G$. Since $W(X')$ and $U$ are $T$-saturated in $X'$, we obtain that $W(U)$ is $T$-saturated in $U$.

We verify (1). Let $v: X' \to X'/T$ be the quotient map. As a subvariety, $X'/T$ inherits the $A_2$-property from $V'/T$, which in turn satisfies it by 2.5. Thus, since $U$ is $(T, 2)$-maximal in $X$, it is necessarily the maximal $T$-saturated subset of $X'$ which is contained in $X \cap X'$. In terms of $B = X' \setminus X$ this means

\[(2) \quad U = X' \setminus v^{-1}(v(B)).\]

We check the inclusion $\subset$ of (1). Let $x \in u^{-1}(u(W(B)))$. Then, by 2.2 (ii), the closure of $G \cdot x$ meets $W(B)$. The classical Hilbert-Mumford Lemma [8, Thm. 4.2] says that for some maximal torus $T' \subset G$ the closure of $T' \cdot x$ meets $W(B)$. Let $g \in G$ with $T = gT'g^{-1}$. Then the closure of $T \cdot g \cdot x$ meets $W(B)$. Hence $g \cdot x$ lies in $v^{-1}(v(B))$. By (2), the point $x$ cannot belong to $W(U)$.

We turn to the inclusion $\supset$ of (1). For this, consider the set $A := (X \cap X') \setminus U$. Then $X'$ is the disjoint union of $U$, $A$ and $B$. Consequently, we have

$$W(U) = \bigcap_{g \in G} g \cdot (X' \setminus (A \cup B)) = W(X') \setminus \bigcup_{g \in G} g \cdot A \cup g \cdot B.$$  

So we have to show that $u$ maps a given $x \in W(X') \cap g \cdot (A \cup B)$ to $u(W(B))$. Since $g^{-1} \cdot x \notin U$ holds, we infer from (2) that $g^{-1} \cdot x$ lies in
According to 2.2 (ii), the closure of $T \cdot g^{-1} \cdot x$ in $X'$ meets $B$. Since $W(X')$ is $T$-saturated in $X'$, this implies that the closure of $T \cdot g^{-1} \cdot x$ meets $W(X') \cap B$. But we have

$$W(X') \cap B = W(X') \setminus X = \bigcap_{g \in G} g \cdot (X' \setminus X) = W(B).$$

Hence we obtained that the closure of the orbit $G \cdot x$ intersects $W(B)$. This in turn shows that the image $u(x)$ lies in $u(W(B))$. □

**Proof of Corollary 1.2.** — Recall from [9, Sec. 4] that the automorphism group of $X$ is a linear algebraic group having the big torus $T_X \subset X$ as a maximal torus. Thus, by conjugating $T_X$ we achieve that $T \subset G$ acts on $X$ via a homomorphism $T \to T_X$. Proposition 2.5 then ensures that each $T$-maximal subset of $X$ is as well $(T, 2)$-maximal, and statement (i) follows from Theorem 1.1.

For statement (ii), let $V \subset X$ be open and $G$-invariant with good quotient $V \to V/G$. Then [7, Cor. 10] provides a good quotient $V \to V/T$. Let $U \subset X$ be a $T$-maximal subset containing $V$ as $T$-saturated subset. Then we have $V \subset W(U)$. Again by 2.5, the set $U$ is $(T, 2)$-maximal. Thus Theorem 1.1 says that $W(U)$ is open, has a good quotient $u: W(U) \to W(U)/G$, and is $T$-saturated in $U$.

For $G$-saturatedness of $V$ in $W(U)$ we have to show that any $x \in u^{-1}(u(V))$ with closed $G$-orbit in $W(U)$ belongs to $V$. For this note that $V$ is $T$-saturated in $W(U)$, because both sets are so in $U$. Now, let $y \in V$ with $u(y) = u(x)$. Then $x$ lies in the closure of $G \cdot y$. Thus [8, Thm. 4.2] provides a $g \in G$ such that the closure of $T \cdot g \cdot y$ meets $G \cdot x$. Since $g \cdot y$ lies in $V$ and $V$ is $T$-saturated in $W(U)$, we obtain $G \cdot x \subset V$, and hence $x \in V$. □

**BIBLIOGRAPHY**


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