



# ANNALES

DE

# L'INSTITUT FOURIER

Marek GOLASIŃSKI

**On  $G$ -disconnected injective models**

Tome 53, n° 2 (2003), p. 625-664.

[http://aif.cedram.org/item?id=AIF\\_2003\\_\\_53\\_2\\_625\\_0](http://aif.cedram.org/item?id=AIF_2003__53_2_625_0)

© Association des Annales de l'institut Fourier, 2003, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>

## ON $G$ -DISCONNECTED INJECTIVE MODELS

by Marek GOLASIŃSKI

---

### Introduction.

The purpose of this paper is to redefine the equivariant minimality and redevelop some results on the rational homotopy theory of  $G$ -disconnected spaces. This was originally studied by the author in [7] but some of the constructions used must be corrected.

The central results of Sullivan's theory of minimal models [18] has been generalized by Triantafillou [19], [20] to equivariant simply connected and nilpotent spaces when  $G$  is finite, but only to  $G$ -connected and finite type spaces. Since a space  $X$  is  $G$ -connected only if all the fixed point subspaces  $X^H$  are non-empty and connected for all subgroups  $H \subseteq G$ , this is a very restrictive condition. To model these equivariant spaces it is necessary to use not only the de Rham algebra  $A_X$  of  $\mathbb{Q}$ -differential forms on  $X$  but also a functor  $\mathcal{A}_X$  on the orbit category  $\mathcal{O}(G)$  which contains all of the algebras  $A_{X^H}$  for all subgroups  $H \subseteq G$ . The key to producing equivariant minimal models is to study the injectivity of such functors.

---

*Keywords:* Differential graded algebra – de Rham algebra –  $EI$ -category –  $i$ -elementary extension –  $i$ -minimal model – Linearly compact (complete)  $k$ -module – Postnikov tower – Quasi-isomorphism – Rationalization –  $G$ -simplicial set.

*Math. classification:* 55P62 – 55P91 – 16W80 – 18G30.

This paper addresses two issues in moving to more general  $G$ -disconnected spaces. First, it was observed by Scull that the original definition of the equivariant minimality used in [19] is incorrect because of an error concerning algebraic properties. In [17] there is an explanation of this error, with counterexamples, and a new definition of the equivariant minimality is developed for simply connected spaces. Consequently, methods presented in [7] for the  $G$ -disconnected case and based on [19], [20] must also be rebuilt and main results redeveloped. On the other hand, from the context of [19], [20], it follows that some constructions presented there and then used in [7] really lead to the correct one.

The second issue is that in the  $G$ -disconnected case the category  $\mathcal{O}(G)$  was originally replaced by the category  $\mathcal{O}(G, X)$  with one object for each component of each fixed point simplicial subsets  $X^H$  of a  $G$ -simplicial set  $X$ , for all subgroups  $H \subseteq G$  (see the unpublished Ph. D. thesis by Fine [5]). Unfortunately, this does not lead to a construction of equivariant minimal models. The problem is that in the category of functors from  $\mathcal{O}(G, X)$  to the category of (finitely generated)  $\mathbb{Q}$ -modules, there is not sufficiently many injectives, and injectivity in that category is not preserved by the object-wise tensor product. These properties are crucial to make further steps in [5] for the rational homotopy type studies of disconnected  $G$ -spaces (even of finite  $G$ -type). Furthermore, under an action of a finite group on a simplicial set we can never find a base point unless a group action has a fixed point. This paper defines another category which overcomes this problem.

In Section 1 we study the category  $k\mathbb{I}\text{-Mod}$  of covariant functors on an  $EI$ -category  $\mathbb{I}$  to the category of  $k$ -modules over a field  $k$ . To ensure the existence problem of sufficiently many injectives and preserving injectivity by the object-wise tensor product, we adopt the category of functors from an  $EI$ -category  $\mathbb{I}$  to the category  $k\text{-Mod}^c$  of linearly compact  $k$ -modules considered by Lefschetz in [13]. Then we investigate the category of functors from an  $EI$ -category  $\mathbb{I}$  to the category  $DGA_k$  of differential graded algebras over a field  $k$ . We generalize the approach presented in [17] and redefine a minimal model of a complete injective  $k\mathbb{I}$ -algebra. Proposition 1.7, a replacement for [7, Proposition 2.4] shows that minimal  $k\mathbb{I}$ -algebras behave (up to homotopy) as cofibrant ones. Then Theorem 1.12 gives an existence of minimal models.

Section 2 applies results assembled in Section 1 to the category of  $G$ -simplicial sets, where  $G$  is a finite group. To show an existence of the

minimal model  $\mathcal{M}_X$  for a disconnected  $G$ -simplicial set  $X$  we replace  $\mathcal{O}(G, X)$  by the more subtle category  $\tilde{\mathcal{O}}(G, X)$  with one object for each 0-simplex of fixed point simplicial subsets  $X^H$ , for all subgroups  $H \subseteq G$ . To any  $G$ -simplicial set  $X$ , we associate the complete and injective de Rham  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}_X$ , where  $\mathbb{Q}$  is the field of rationals. The main result stated in [7, Theorem 3.11], relating the rational homotopy type of a nilpotent Kan  $G$ -simplicial set  $X$  by means of the minimal model of the de Rham algebra  $\mathcal{A}_X$  and others, have been redeveloped.

In a forthcoming paper we plan to apply these new minimal models to showing equivariant formality of  $G$ -Kähler manifolds, the result aimed at in [5], [6].

The author would like to express his gratitude to Laura S. Scull for sending her Ph. D. thesis and helpful discussions, and to the referee for carefully reading the paper and many suggestions to settle its final version.

## 1. Injectivity and minimal models of functors to algebras.

For a field  $k$ , let  $k\text{-Mod}$  denote the category of (left)  $k$ -modules. The dual category  $k\text{-Mod}^{op}$  is isomorphic to the category  $k\text{-Mod}^c$  of linearly compact  $k$ -modules considered in [13]. Given a  $k$ -module  $M$ , let  $M^* = \text{Hom}_k^t(M, k)$  be its topological dual. Write  $M \hat{\otimes} N$  for the completion  $(M \otimes N)^\wedge$ , called the *complete tensor product* of  $M$  and  $N$  (see e.g., [7] for its properties).

Let  $DGA_k$  denote the category of homologically connected commutative differential graded  $k$ -algebras with unit (or simply  *$k$ -algebras*). Given a  $k$ -algebra  $A$  with a differential  $d$ , write  $Z^{n+1}A$  for the  $k$ -module of its cocycles of degree  $n+1$ . For a  $k$ -module  $M$ , denote by  $SM$  the free  $k$ -algebra generated by  $M$  in degree  $n$  with the trivial differential. Then any  $k$ -map  $\tau : M \rightarrow Z^{n+1}A$  gives a rise to the differential  $d_\tau$  on the tensor product  $A \otimes S(M)$  such that

$$d_\tau(a \otimes 1) = d(a) \otimes 1 \quad \text{and} \quad d_\tau(1 \otimes m) = \tau(m) \otimes 1$$

for  $a \in A$  and  $m \in M$ . The  $k$ -algebra  $A(M) = (A \otimes SM, d_\tau)$  is called in [11] the *elementary extension* of  $A$  with respect to  $\tau$ . There are two ways of defining simply connected minimal  $k$ -algebras. One is to use the intrinsic algebraic condition that the differential is decomposable; this is the original approach taken by Sullivan [18]. The other is to define a minimal algebra to be a union of a sequence of elementary extensions of  $k$ -algebras; this

approach was developed by Bousfield-Gugenheim [1] and Halperin [11]. Non-equivariantly these two definitions are equivalent.

However, it was observed in [17] that “equivariantly these two approaches are not the same, as it was erroneously claimed in [19], [20]”. The original work of Triantafyllou defined the equivariant minimality using an analogue of the decomposable differential. In using minimal models to encode geometric information, the stages in the Postnikov tower of a space correspond to elementary extensions of the algebra. Therefore, in order to obtain a connection to geometry, it is necessary to redefine the equivariant minimality following the elementary extension approach and restate its algebraic properties using the new definition. Consequently, methods for the  $G$ -disconnected case presented in [7] and based on [19], [20] must also be rebuilt and main results redeveloped.

Now we recall some notions and results presented in [7], [8], [10]. For a small category  $\mathbb{I}$ , a covariant functor  $\mathbb{I} \rightarrow k\text{-Mod}$  is called a *left  $k\mathbb{I}$ -module*. We also have contravariant functors  $\mathbb{I} \rightarrow k\text{-Mod}$ , alias *right  $k\mathbb{I}$ -modules*. *Projective* (resp. *injective*)  $R\mathbb{I}$ -modules are defined by usual lifting properties. Observe that the category of projective right  $k\mathbb{I}$ -modules is isomorphic to the category of injectives in the category of covariant functors from  $\mathbb{I}$  to the category  $k\text{-Mod}^c$ . A covariant functor from  $\mathbb{I}$  to  $k\text{-Mod}^c$  is called a *linearly compact left  $k\mathbb{I}$ -module*. For two linearly compact left  $k\mathbb{I}$ -modules  $M, N$  we define the object-wise *complete tensor product*  $M \widehat{\otimes} N$  as the linearly compact left  $k\mathbb{I}$ -module such that  $(M \widehat{\otimes} N)(I) = M(I) \widehat{\otimes} N(I)$  for all  $I \in \text{Ob}(\mathbb{I})$ . We assume in this section that the category  $\mathbb{I}$  is such that the constant left  $k\mathbb{I}$ -module  $\underline{k}$  determined by the field  $k$  (of characteristic zero) is injective and that the above object-wise complete tensor product of two injectives in this category is again injective. For each object  $I \in \text{Ob}(\mathbb{I})$  we have the right  $k\mathbb{I}$ -module

$$k\mathbb{I}(-, I) : \mathbb{I} \longrightarrow k\text{-Mod}$$

determined by the Yoneda functor  $\mathbb{I}(-, I)$  and similarly, the left  $k\mathbb{I}$ -module  $k\mathbb{I}(I, -)$ . Given a category  $\mathbb{C}$ , any covariant (resp. contravariant) functor  $\mathbb{I} \rightarrow \mathbb{C}$  we call a covariant (resp. contravariant)  $\mathbb{I}$ -*system* in the category  $\mathbb{C}$ .

Recall that an *EI-category* is a small category such that any endomorphism is an isomorphism. Define the partial order on the set  $\text{Iso}(\mathbb{I})$  of isomorphism classes  $\bar{I}$  of objects  $I \in \text{Ob}(\mathbb{I})$  by

$$\bar{I} \leq \bar{J} \quad \text{if} \quad \mathbb{I}(I, J) \neq \emptyset.$$

Write  $\bar{I} < \bar{J}$  if  $\bar{I} \leq \bar{J}$  and  $\bar{I} \neq \bar{J}$ .

Henceforward, we assume that  $\mathbb{I}$  is a *cofinite EI*-category (each isomorphism class  $\bar{I}$  has only finitely many predecessors) with a filtration  $\emptyset = T_0 \subseteq T_1 \cdots \subseteq T_m = \text{Iso}(\mathbb{I})$  such that

$$(\star) \quad \bar{I} \in T_k, \bar{J} \in T_l, \bar{I} < \bar{J} \text{ implies } l < k$$

provided that  $\bar{I} \notin T_{k'}$  and  $\bar{J} \notin T_{l'}$  for  $k' < k$  and  $l' < l$ . We point out that this condition always holds for a finite *EI*-category.

We make use of [8], [10] to show briefly how injective left  $k\mathbb{I}$ -modules can be constructed from injective modules over some group rings. If  $I \in \text{Ob}(\mathbb{I})$  with the automorphism group  $\text{Aut}(I)$ , we write  $k[I]$  for the group ring of  $\text{Aut}(I)$  over  $k$ ,  $k[I]\text{-Mod}$  for the category of left  $k[I]$ -modules and  $k\mathbb{I}\text{-Mod}$  for the category of left  $k\mathbb{I}$ -modules.

For a fixed object  $I \in \text{Ob}(\mathbb{I})$ , we introduce the following covariant functors (cf. [15] for the dual notions).

*Cosplitting* functor  $S_I : k\mathbb{I}\text{-Mod} \rightarrow k[I]\text{-Mod}$ .

If  $\mathbf{M}$  is a  $k\mathbb{I}$ -module, let  $S_I(\mathbf{M})$  be the  $k[I]$ -submodule of  $\mathbf{M}(I)$  equal to the intersection of kernels of all  $k$ -maps  $\mathbf{M}(f) : \mathbf{M}(I) \rightarrow \mathbf{M}(J)$  induced by all non-isomorphisms  $f : I \rightarrow J$  with  $I$  as the source. Each automorphism  $g \in \text{Aut}(I)$  induces a map  $\mathbf{M}(g) : \mathbf{M}(I) \rightarrow \mathbf{M}(I)$  which maps  $S_I(\mathbf{M})$  into itself. Thus  $S_I(\mathbf{M})$  becomes a left  $k[I]$ -module. It is clear how  $S_I$  is defined on morphisms.

*Corestriction* functor  $\text{Res}_I : k\mathbb{I}\text{-Mod} \rightarrow k[I]\text{-Mod}$  sends  $\mathbf{M}$  to  $\mathbf{M}(I)$ .

*Coextension* functor  $E_I : k[I]\text{-Mod} \rightarrow k\mathbb{I}\text{-Mod}$  sends  $N$  to  $\text{Hom}_{k[I]}(k\mathbb{I}(-, I), N)$ .

*Coinclusion* functor  $\text{In}_I : k[I]\text{-Mod} \rightarrow k\mathbb{I}\text{-Mod}$  assigns to a  $k[I]$ -module  $N$  the  $k\mathbb{I}$ -module  $\text{In}_I(N)$  defined by

$$\text{In}_I(N)(J) = \begin{cases} \text{Hom}_{k[I]}(k\mathbb{I}(J, I), N) & \text{if } \bar{J} = \bar{I}, \\ 0 & \text{if } \bar{J} \neq \bar{I}. \end{cases}$$

Given a subset  $T \subseteq \text{Iso}(\mathbb{I})$ , a  $k\mathbb{I}$ -module  $\mathbf{M}$  is called of *type*  $T$  if the set  $\{\bar{I} \in \text{Iso}(\mathbb{I}); \mathbf{M}(I) \neq 0\}$  is contained in  $T$ . For any  $\bar{I} \in T$  choose a representative  $I \in \bar{I}$  and fix a  $k[I]$ -monomorphism

$$0 \rightarrow \mathbf{M}(I) \rightarrow Q_I,$$

where  $Q_I$  is an injective  $k[I]$ -module. If  $\mathbf{M}$  is of type  $T$  then we get a monomorphism of  $k\mathbb{I}$ -modules

$$0 \rightarrow \mathbf{M} \rightarrow \prod_{\bar{I} \in T} E_I Q_I.$$

In particular, it follows that any injective  $k\mathbb{I}$ -module of type  $T$  is a direct summand of a  $k\mathbb{I}$ -module  $\prod_{\bar{I} \in T} E_I Q_I$ , where  $Q_I$  are injective  $k[I]$ -modules for  $\bar{I} \in T$ .

Then the next result follows easily from the above definitions.

LEMMA 1.1. — (1) *The functors  $E_I$  and  $\text{Res}_I$  and the functors  $S_I$  and  $\text{In}_I$  are adjoint, i.e., there are natural isomorphisms of  $k$ -modules*

$$\text{Hom}_{k\mathbb{I}}(\mathbf{M}, E_I N) \xrightarrow{\cong} \text{Hom}_{k[I]}(\text{Res}_I \mathbf{M}, N)$$

and

$$\text{Hom}_{k[I]}(N, S_I \mathbf{M}) \xrightarrow{\cong} \text{Hom}_{k\mathbb{I}}(\text{In}_I N, \mathbf{M}).$$

(2) *The functor  $S_I E_I : k[I]\text{-Mod} \rightarrow k[I]\text{-Mod}$  is naturally equivalent to the identity functor. The composition  $S_J E_I$  is zero for  $\bar{I} \neq \bar{J}$ .*

(3)  *$S_I$  and  $E_I$  preserve products, monomorphisms and injective modules.*

Let now  $\mathbf{M}$  be a  $k\mathbb{I}$ -module and  $T_0 \subseteq \text{Iso}(\mathbb{I})$ . Then the adjunction induces a natural  $k\mathbb{I}$ -map

$$\mathcal{J}_I \mathbf{M} : \mathbf{M} \longrightarrow (E_I \text{Res}_I) \mathbf{M}$$

for any  $I \in \text{Ob}(\mathbb{I})$ . The product of the maps  $\mathcal{J}_I \mathbf{M}$  running over  $\bar{I} \in T_0$  yields

$$\mathcal{J}_{T_0} \mathbf{M} : \mathbf{M} \longrightarrow \prod_{\bar{I} \in T_0} (E_I \text{Res}_I) \mathbf{M}.$$

Write  $\ker_{T_0} \mathbf{M}$  for the kernel of  $\mathcal{J}_{T_0} \mathbf{M}$ . If  $\mathbf{M}$  is of type  $T$  then  $\ker_{T_0} \mathbf{M}$  is of type  $T \setminus (T \cap T_0)$ . The following full description of injective left  $k\mathbb{I}$ -modules has been developed in [8].

THEOREM 1.2 (Filtration Theorem for injective  $k\mathbb{I}$ -modules). — *Let  $\mathbb{I}$  be an  $EI$ -category and  $\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_m = T$  a filtration satisfying*

( $\star$ ) for a subset  $T \subseteq \text{Iso}(\mathbb{I})$ . If  $\mathbf{Q}$  is an injective left  $k\mathbb{I}$ -module of type  $T$  then there is a natural filtration by  $k\mathbb{I}$ -modules

$$0 = \mathbf{Q}_m \xrightarrow{\text{in}_m} \mathbf{Q}_{m-1} \xrightarrow{\text{in}_{m-1}} \dots \hookrightarrow \mathbf{Q}_1 \xrightarrow{\text{in}_1} \mathbf{Q}_0 = \mathbf{Q}$$

satisfying:

(1)  $\mathbf{Q}_l = \ker_{T_l} \mathbf{Q}_{l-1}$  and  $\text{in}_l : \mathbf{Q}_l = \ker_{T_l} \mathbf{Q}_{l-1} \rightarrow \mathbf{Q}_{l-1}$  is the inclusion map;

(2)  $\mathbf{Q}_l$  is injective of type  $T \setminus T_l$ ;

(3) let  $\text{in}^l : \mathbf{Q}_l \rightarrow \mathbf{Q}$  be the composition  $\text{in}_1 \text{in}_2 \dots \text{in}_l$ . Then  $S_I \text{in}^l : S_I \mathbf{Q}_l \rightarrow S_I \mathbf{Q}$  is an isomorphism for  $\bar{I} \in T \setminus T_l$ ;

(4) there is a natural exact sequence which splits (not naturally with respect to  $\mathbf{Q}$ )

$$0 \longrightarrow \mathbf{Q}_l \xrightarrow{\text{in}_l} \mathbf{Q}_{l-1} \longrightarrow \prod_{\bar{I} \in T_k \setminus T_{l-1}} (E_I S_I) \mathbf{Q} \longrightarrow 0$$

for  $l = 1, \dots, m$

Conversely, if a  $k\mathbb{I}$ -module  $\mathbf{Q}$  of type  $T$  satisfies (1), (3) and (4), and  $S_I \mathbf{Q}_l$  are injective  $k[I]$ -modules for  $\bar{I} \in T$  then the  $\mathbb{I}$ -modules  $\mathbf{Q}_l$  are injective for  $l = 0, \dots, m$ .

A covariant functor  $\mathbb{I} \rightarrow \text{DGA}_k$  is called a  $k\mathbb{I}$ -algebra (or simply an  $\mathbb{I}$ -system of  $k$ -algebras). We say that a  $k\mathbb{I}$ -algebra  $\mathcal{A}$  is linearly compact (resp. complete) if the algebras  $\mathcal{A}(I)$  are linearly compact (resp. complete) for all  $I \in \text{Ob}(\mathbb{I})$ . Define the  $n$ th graded piece  $\mathcal{A}^n$  of  $\mathcal{A}$  as the left  $k\mathbb{I}$ -module  $\mathcal{A}^n(I) = (\mathcal{A}(I))^n$  for all  $I \in \text{Ob}(\mathbb{I})$ . A  $k\mathbb{I}$ -algebra  $\mathcal{A}$  is called injective if the left  $k\mathbb{I}$ -modules  $\mathcal{A}^n$  are all injective for  $n \geq 0$ .

Given a graded linearly compact left  $k\mathbb{I}$ -module  $\mathbf{Q} = \{\mathbf{Q}_i\}_{i \geq 0}$ , we get graded linearly compact  $k$ -modules  $\mathbf{Q}(I) = \{\mathbf{Q}_i(I)\}_{i \geq 0}$  for all  $I \in \text{Ob}(\mathbb{I})$  and let  $|q| = i$  for  $q \in \mathbf{Q}_i(I)$ . In the light of the associativity of the complete tensor product  $\widehat{\otimes}$ , we can define graded linearly compact left  $k\mathbb{I}$ -modules  $\widehat{T}^n \mathbf{Q}$  and  $\widehat{S}^n \mathbf{Q}$  studied in [7] (called the  $n$ th tensor and symmetric power, respectively) as follows:

$$(\widehat{T}^n \mathbf{Q})_i(I) = \bigoplus_{i_1 + \dots + i_n = i} \mathbf{Q}_{i_1}(I) \widehat{\otimes} \dots \widehat{\otimes} \mathbf{Q}_{i_n}(I)$$

and

$$(\widehat{S}^n \mathbf{Q})_i(I) = ((S^n \mathbf{Q})_i(I))^\wedge$$

for  $i, n \geq 0$ , where  $((S^n \mathbf{Q})_i(I))^\wedge = (T^n \mathbf{Q}_i(I)/(R^n \mathbf{Q})_i(I))^\wedge$  and  $(R^n \mathbf{Q})_i(I)$  is the homogeneous  $k$ -submodule of  $(T^n \mathbf{Q})_i(I) = (\mathbf{Q}^{\otimes n})_i(I)$  generated by elements

$$q_1 \otimes \cdots \otimes q_n - (-1)^{|q_k||q_{k+1}|} q_1 \otimes \cdots \otimes q_{k+1} \otimes q_k \otimes \cdots \otimes q_n$$

for  $q_k \in \mathbf{Q}_{|q_k|}(I)$  and  $k = 1, \dots, n$ .

Furthermore, we define  $\widehat{T}\mathbf{Q}$  and  $\widehat{S}\mathbf{Q}$ , the graded linearly compact tensor and symmetric left  $k\mathbb{L}$ -algebra, respectively with  $(\widehat{T}\mathbf{Q})_i = \bigoplus_{n \geq 0} (\widehat{T}^n \mathbf{Q})_i$  and  $(\widehat{S}\mathbf{Q})_i = \bigoplus_{n \geq 0} (\widehat{S}^n \mathbf{Q})_i$  for  $i \geq 0$ . Observe that  $\widehat{S}\mathbf{Q} = \widehat{T}\mathbf{Q}/\widehat{R}\mathbf{Q}$ , where  $\widehat{R}\mathbf{Q}$  is the closed homogeneous ideal of  $\widehat{T}\mathbf{Q}$  generated by elements  $x \widehat{\otimes} y - (-1)^{|x||y|} y \widehat{\otimes} x$  for  $x, y \in \widehat{T}\mathbf{Q}$ . The assumptions on the category  $\mathbb{I}$  yield

PROPOSITION 1.3. — *If  $\mathbf{Q} = \{\mathbf{Q}_i\}_{i \geq 0}$  is a graded degree-wise injective linearly compact left  $k\mathbb{L}$ -module then the graded linearly compact  $k\mathbb{L}$ -algebras  $\widehat{S}\mathbf{Q} = \{(\widehat{S}\mathbf{Q})_i\}_{i \geq 0}$  and  $\widehat{T}\mathbf{Q} = \{(\widehat{T}\mathbf{Q})_i\}_{i \geq 0}$  are injective.*

Proof. — The natural map  $\pi_I : T^n \mathbf{Q}(I) \rightarrow T^n \mathbf{Q}(I)/R^n \mathbf{Q}(I)$  determines

$$\widehat{\pi}_I : \widehat{T}^n \mathbf{Q}(I) \rightarrow \widehat{S}^n \mathbf{Q}(I)$$

for  $I \in \text{Ob}(\mathbb{I})$ . Furthermore, given the  $n$ th symmetric group  $S_n$ , there is the natural action

$$S_n \times T^n \mathbf{Q}(I) \rightarrow T^n \mathbf{Q}(I)$$

such that  $\tau(q_1 \otimes \cdots \otimes q_n) = \varepsilon(\tau) q_{\tau(1)} \otimes \cdots \otimes q_{\tau(n)}$  for  $\tau \in S_n$ ,  $q_k \in \mathbf{Q}_{|q_k|}(I)$ ,  $I \in \text{Ob}(\mathbb{I})$  and  $k = 1, \dots, n$ , where  $\varepsilon : S_n \rightarrow \{+1, -1\}$  is the sign determined by the obvious interchange map. Because characteristic of  $k$  is zero and this action of  $S_n$  on  $T^n \mathbf{Q}(I)$  vanishes on  $R^n \mathbf{Q}(I)$ , so there is a natural map

$$\sigma_I : S^n \mathbf{Q}(I) \rightarrow T^n \mathbf{Q}(I)$$

such that

$$\sigma_I(q_1 \otimes \cdots \otimes q_n + R^n \mathbf{Q}(I)) = \frac{1}{n!} \sum_{\tau \in S_n} \tau(q_1 \otimes \cdots \otimes q_n)$$

for  $q_k \in \mathbf{Q}_{|q_k|}(I)$ ,  $I \in \text{Ob}(\mathbb{I})$  and  $k = 1, \dots, n$ . Then we get the induced natural map  $\widehat{\sigma}_I : \widehat{S}^n \mathbf{Q}(I) \rightarrow \widehat{T}^n \mathbf{Q}(I)$  such that  $\widehat{\pi}_I \widehat{\sigma}_I = \text{id}_{\widehat{S}^n \mathbf{Q}(I)}$  for  $I \in \text{Ob}(\mathbb{I})$  and consequently  $\widehat{S}^n \mathbf{Q}$  is a direct summand of  $\widehat{T}^n \mathbf{Q}$ .

But the constant left  $k\mathbb{L}$ -module  $\underline{k}$  is injective, the iterated complete tensor products  $\widehat{T}^n\mathbf{Q}(I)$  of linearly compact injectives are injective and  $\widehat{S}^n\mathbf{Q}(I)$  are direct summands of  $\widehat{T}^n\mathbf{Q}(I)$ , and hence the proof is complete.  $\square$

To state the notion of the minimality for a  $k\mathbb{L}$ -algebra we first make the following crucial construction. Let  $\mathcal{A}$  be a complete  $k\mathbb{L}$ -algebra,  $\mathbf{M}$  a linearly compact left  $k\mathbb{L}$ -module and  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  a  $k\mathbb{L}$ -map, where  $Z^{n+1}\mathcal{A}$  is the left  $k\mathbb{L}$ -module of cocycles of degree  $n + 1$  of the  $k\mathbb{L}$ -algebra  $\mathcal{A}$  for some  $n \geq 0$ . We construct a linearly topological  $k\mathbb{L}$ -algebra  $\mathcal{A}_\tau(\mathbf{M})$  called the *i-elementary extension* of  $\mathcal{A}$  with respect to  $\tau$ .

Given a linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}$ , the property  $(\star)$  leads in [8, Corollary 2.2] to an existence of the minimal injective resolution

$$0 \longrightarrow \mathbf{M} \xrightarrow{\omega} \mathbf{M}_0 \xrightarrow{\omega_0} \mathbf{M}_1 \xrightarrow{\omega_1} \dots,$$

i.e.,  $\mathbf{M}_l$  is the injective envelope of  $\text{Im } \omega_{l-1}$  for  $l = 0, 1, \dots$ . Then, the map  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  induces  $k\mathbb{L}$ -maps  $\tau_l : \mathbf{M}_l \rightarrow \mathcal{A}^{n+l+1}$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{M} & \xrightarrow{\omega} & \mathbf{M}_0 & \xrightarrow{\omega_0} & \mathbf{M}_1 & \xrightarrow{\omega_1} & \dots \\ & & \tau \downarrow & & \tau_0 \downarrow & & \tau_1 \downarrow & & \\ & & Z^{n+1}\mathcal{A} & \longrightarrow & \mathcal{A}^{n+1} & \xrightarrow{d} & \mathcal{A}^{n+2} & \xrightarrow{d} & \dots \end{array}$$

commutes. Let  $\mathbf{M}_* = \{\mathbf{M}_l\}_{l \geq 0}$  be a graded left  $k\mathbb{L}$ -module, where  $\mathbf{M}_l$  has degree  $n + l$ . Define the complete  $k\mathbb{L}$ -algebra  $\mathcal{A}_\tau(\mathbf{M}) = \mathcal{A} \widehat{\otimes} \widehat{\mathbf{S}}\mathbf{M}_*$ , where the differential on  $\mathcal{A}_\tau(\mathbf{M})$  restricts on  $\mathcal{A}$  to the one given on  $\mathcal{A}$  and on  $\mathbf{M}_l$  to  $\omega_l + (-1)^l \tau_l$  for  $l = 0, 1, \dots$ . Note that by Proposition 1.3 the  $k\mathbb{L}$ -algebra  $\mathcal{A}_\tau(\mathbf{M})$  is injective and linearly compact provided that  $\mathcal{A}$  is so.

For a complete  $k\mathbb{L}$ -algebra  $\mathcal{A}$ , we define the object-wise cohomology functor by  $\mathbf{H}^n(\mathcal{A})(I) = H^n(\mathcal{A}(I))$  for  $I \in \text{Ob}(\mathbb{I})$ . We say that a map  $f : \mathcal{A} \rightarrow \mathcal{B}$  of complete  $k\mathbb{L}$ -algebras is a *quasi-isomorphism* (resp. an *n-quasi-isomorphism*) if the induced maps  $\mathbf{H}^m(f) : H^m(\mathcal{A}) \rightarrow H^m(\mathcal{B})$  are isomorphisms for  $m \geq 0$  (resp. isomorphisms for  $m \leq n$  and a monomorphism for  $m = n + 1$ ). For a linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}$ , we also define cohomology by looking at the natural transformations  $\text{Hom}(\mathbf{M}, \mathcal{A})$ . This is a cochain complex with a differential induced by the differential on  $\mathcal{A}$  and write  $H^n(\mathcal{A}, \mathbf{M})$  for its cohomology. If  $\mathcal{A}$  is injective, then the standard homological algebra arguments yield a cohomology spectral sequence

$$E_2^{p,q} = \text{Ext}^p(\mathbf{M}, \mathbf{H}^q(\mathcal{A})) \implies H^{p+q}(\mathcal{A}, \mathbf{M})$$

being an essential tool in investigations presented in [17], [19]. Observe that a map  $f : \mathcal{A} \rightarrow \mathcal{B}$  of injective  $k\mathbb{L}$ -algebras is a quasi-isomorphism (resp. an  $n$ -quasi-isomorphism) if and only if the induced maps  $H^m(f, \mathbf{M}) : H^m(\mathcal{A}, \mathbf{M}) \rightarrow H^m(\mathcal{B}, \mathbf{M})$  are isomorphisms for  $m \geq 0$  (resp. isomorphisms for  $m \leq n$  and a monomorphism for  $m = n + 1$ ) and any linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}$ . Given a map  $f : \mathcal{A} \rightarrow \mathcal{B}$  of  $k\mathbb{L}$ -algebras with differentials  $d$  and  $d'$ , respectively consider its cocone  $\mathcal{C}_f$  as the graded left  $k\mathbb{L}$ -module  $\mathcal{C}_f^n = \mathcal{A}^n \times \mathcal{B}^{n-1}$  with the differential  $\delta$  such that  $\delta(a, b) = (da, fa - d'b)$  for  $(a, b) \in \mathcal{C}_f^n$  and  $n \geq 0$ . Then we form relative cohomology groups

$$\mathbf{H}^n(f : \mathcal{A} \rightarrow \mathcal{B}) = \mathbf{H}^n(\mathcal{C}_f), \quad H^n(f : \mathcal{A} \rightarrow \mathcal{B}, \mathbf{M}) = H^n(\text{Hom}(\mathbf{M}, \mathcal{C}_f))$$

to get long exact sequences

$$\dots \rightarrow \mathbf{H}^n(\mathcal{A}) \rightarrow \mathbf{H}^n(\mathcal{B}) \rightarrow \mathbf{H}^{n+1}(f : \mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathbf{H}^{n+1}(\mathcal{A}) \rightarrow \dots$$

and

$$\dots \rightarrow H^n(\mathcal{A}, \mathbf{M}) \rightarrow H^n(\mathcal{B}, \mathbf{M}) \rightarrow H^{n+1}(f : \mathcal{A} \rightarrow \mathcal{B}, \mathbf{M}) \rightarrow H^{n+1}(\mathcal{A}, \mathbf{M}) \rightarrow \dots$$

Moreover, any commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A}' & \xrightarrow{f'} & \mathcal{B}' \end{array}$$

yields maps

$$\mathbf{H}^n(f : \mathcal{A} \rightarrow \mathcal{B}) \longrightarrow \mathbf{H}^n(f' : \mathcal{A}' \rightarrow \mathcal{B}') \\ \text{and } H^n(f : \mathcal{A} \rightarrow \mathcal{B}, \mathbf{M}) \rightarrow H^n(f' : \mathcal{A}' \rightarrow \mathcal{B}', \mathbf{M})$$

for all  $n \geq 0$  and any linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}$ . Furthermore, a sequence of maps

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

produces two long exact sequences

$$\dots \rightarrow \mathbf{H}^n(\mathcal{A} \xrightarrow{gf} \mathcal{C}) \rightarrow \mathbf{H}^n(\mathcal{B} \xrightarrow{g} \mathcal{C}) \rightarrow \mathbf{H}^{n+1}(\mathcal{A} \xrightarrow{f} \mathcal{B}) \rightarrow \dots$$

and

$$\dots \rightarrow H^n(\mathcal{A} \xrightarrow{gf} \mathcal{C}, \mathbf{M}) \rightarrow H^n(\mathcal{B} \xrightarrow{g} \mathcal{C}, \mathbf{M}) \rightarrow H^{n+1}(\mathcal{A} \xrightarrow{f} \mathcal{B}, \mathbf{M}) \rightarrow \dots$$

Given a map  $f : \mathcal{A} \rightarrow \mathcal{B}$  of injective  $k\mathbb{L}$ -algebras and a linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}$ , homological algebra also produces a cohomology spectral sequence

$$E_2^{p,q} = \text{Ext}^p(\mathbf{M}, \mathbf{H}^q(f : \mathcal{A} \rightarrow \mathcal{B})) \implies H^{p+q}(f : \mathcal{A} \rightarrow \mathcal{B}, \mathbf{M}).$$

The proofs of [17, Lemma 11.51, Lemma 11.53] lead to the following.

LEMMA 1.4. — *Let  $\mathcal{A}$  be a complete  $k\mathbb{L}$ -algebra and  $\mathbf{M}$  a complete left  $k\mathbb{L}$ -module.*

- (1) *For a  $k\mathbb{L}$ -map  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  there is an isomorphism*

$$\mathbf{H}^{n+1}(\mathcal{A} \hookrightarrow \mathcal{A}_\tau(\mathbf{M})) \xrightarrow{\cong} \mathbf{M}$$

*of left  $k\mathbb{L}$ -modules.*

- (2) *Given  $k\mathbb{L}$ -maps  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  and  $k\mathbb{L}$ -map  $\tau' : \mathbf{M}' \rightarrow Z^{n+1}\mathcal{A}$ , let  $f : \mathcal{A}_\tau(\mathbf{M}) \rightarrow \mathcal{A}_{\tau'}(\mathbf{M}')$  be a  $k\mathbb{L}$ -map of  $i$ -elementary extensions with the properties:*

- (i) *the map restricts to an isomorphism on  $\mathcal{A}$ ;*
- (ii)  *$f(m) = g(m) + \alpha(m)$  for  $m \in \mathbf{M}$ , where  $g : \mathbf{M} \rightarrow \mathbf{M}'$  is an isomorphism and  $\alpha(m) \in \mathcal{A}$ .*

*Then  $f$  is an isomorphism.*

A  $k\mathbb{L}$ -map  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  gives rise to an element  $[\tau] \in H^{n+1}(\mathcal{A}, \mathbf{M})$ . Thus, making use of [17, Proposition 11.52] it might be shown

PROPOSITION 1.5. — *Given  $k\mathbb{L}$ -maps  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  and  $\tau' : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$ , the  $i$ -elementary extensions  $\mathcal{A}_\tau(\mathbf{M})$  and  $\mathcal{A}_{\tau'}(\mathbf{M})$  are isomorphic if and only if  $[\tau] = [\tau']$  in  $H^{n+1}(\mathcal{A}, \mathbf{M})$ .*

The notion of the minimality for systems of simple  $k$ -algebras stated in [17] as a union of an increasing sequence of  $i$ -elementary extensions is exactly the statement of [19, Corollary 5.11]. But, as it was observed by Scull [17], this is not isomorphic to that one stated in [19, Definition 5.1]. Now, we extend Scull's notion [17] of the minimality on  $k\mathbb{L}$ -algebras via a series of  $i$ -elementary extensions with respect to two indices and correct [20, Definition 2.3] as well. A complete  $k\mathbb{L}$ -algebra  $\mathcal{M}$  is said to be  *$i$ -minimal* if

(1)

$$\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}(n),$$

where  $\mathcal{M}(0) = \underline{k}$  and  $\mathcal{M}(n) = \bigcup_{l \geq 0} \mathcal{M}(n, l)$  with  $\mathcal{M}(n, 0) = \mathcal{M}(n - 1)$ ,  $\mathcal{M}(n, l + 1) = \mathcal{M}(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  is an  $i$ -elementary extension of  $\mathcal{M}(n, l)$  with respect to a map  $\tau_{n,l} : \mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}(n, l)$  with a linearly compact left  $k\mathbb{L}$ -module  $\mathbf{M}_{n,l}$  for all  $l \geq 0$  and  $n \geq 1$ ;

(2) the canonical map  $\mathbf{M}_{n,l} \simeq \mathbf{H}^{n+1}(\mathcal{M}(n, l) \hookrightarrow \mathcal{M}(n, l + 1)) \rightarrow \mathbf{H}^{n+1}(\mathcal{M}(n, l) \hookrightarrow \mathcal{M})$  is an isomorphism for all  $l \geq 0$  and  $n \geq 1$ .

Note that an  $i$ -minimal algebra  $\mathcal{M}$  is injective and linearly compact, and the natural inclusion map  $\mathcal{M}(n) \hookrightarrow \mathcal{M}$  is an  $n$ -quasi-isomorphism for all  $n \geq 0$ .

Suppose  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a map of complete  $k\mathbb{L}$ -algebras,  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$  and  $\varphi : \mathbf{M} \rightarrow \mathcal{B}^n$  maps of left  $k\mathbb{L}$ -modules. If  $d\varphi = f\tau$ , where  $d$  is the differential on  $\mathcal{B}$  then by [7] there is a map  $\tilde{f} : \mathcal{A}_\tau(\mathbf{M}) \rightarrow \mathcal{B}$  extending  $f$  and  $\varphi$ . To state a generalization of that let  $k(t, dt)$  be the free  $k$ -algebra generated by  $t$  in degree 0 and  $dt$  in degree 1, with  $d(t) = dt$ . A homotopy between maps  $f, g : \mathcal{A} \rightarrow \mathcal{B}$  of  $k\mathbb{L}$ -algebras is a map  $H : \mathcal{A} \rightarrow \mathcal{B} \widehat{\otimes} k(t, dt)$  such that  $p_0H = f$  and  $p_1H = g$ , where  $p_0, p_1 : \mathcal{B} \widehat{\otimes} k(t, dt) \rightarrow \mathcal{B}$  are the natural projections. We write  $f \simeq g$  for homotopic maps  $f$  and  $g$ . Define the map

$$\int_0^1 : \mathcal{B} \widehat{\otimes} k(t, dt) \longrightarrow \mathcal{B}$$

by  $\int_0^1 b \otimes t^i = 0$  and  $\int_0^1 b \otimes t^i dt = (-1)^{(|b|} b \otimes \frac{t^{i+1}}{i+1}$  for  $b \in \mathcal{B}$  and  $i \geq 0$ .

Then the obstruction and its relative version developed in [5, Lemma 6.2.2, Lemma 6.2.4] and fixed in [17, Lemma 12.55, Lemma 12.56] can be reproved for complete  $k\mathbb{L}$ -algebras as well.

LEMMA 1.6. — *Let  $\mathcal{A}_\tau(\mathbf{M})$  be an  $i$ -elementary extension of a  $k\mathbb{L}$ -algebra  $\mathcal{A}$  with respect to  $\tau : \mathbf{M} \rightarrow Z^{n+1}\mathcal{A}$ .*

(1) *Suppose we have a diagram of complete  $k\mathbb{L}$ -algebras*

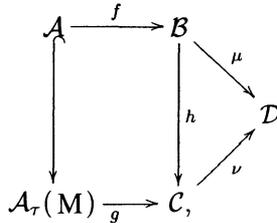
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow h \\ \mathcal{A}_\tau(\mathbf{M}) & \xrightarrow{g} & \mathcal{C} \end{array}$$

with  $g|_{\mathcal{A}} \simeq hf$  by a homotopy  $H : \mathcal{A} \rightarrow \mathcal{B} \widehat{\otimes} k(t, dt)$ . Then the map

$$O : \mathbf{M} \longrightarrow \mathcal{B}^{n+1} \times \mathcal{C}^n$$

given by  $O(m) = (fd(m), g(m) + \int_0^1 Hd(m))$  for  $m \in \mathbf{M}$  defines  $[O] \in H^{n+1}(h : \mathcal{B} \rightarrow \mathcal{C}, \mathbf{M})$  and there is an extension  $\tilde{f} : \mathcal{A}_\tau(\mathbf{M}) \rightarrow \mathcal{B}$  of the map  $f : \mathcal{A} \rightarrow \mathcal{B}$  with  $h\tilde{g} \simeq g$  if and only if the obstruction class  $[O] \in H^{n+1}(h : \mathcal{B} \rightarrow \mathcal{C}, \mathbf{M})$  vanishes.

(2) Suppose we have a diagram of complete  $k\mathbb{L}$ -algebras



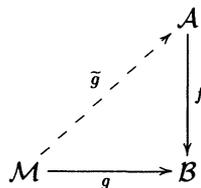
where we assume the following:

- (i)  $\mu$  is onto;
- (ii)  $\mu f = \nu g|_{\mathcal{A}}$ ;
- (iii)  $\ker \mu$  and  $\ker \nu$  are injective;
- (iv) there exists a homotopy  $H$  from  $hf$  to  $g|_{\mathcal{A}}$  such that  $(\nu \widehat{\otimes} \text{id})H$  is constant;
- (v) there exists a  $k\mathbb{L}$ -map  $\theta : \mathbf{M} \rightarrow \mathcal{A}$  such that  $\nu g|_{\mathbf{M}} = \mu\theta$ .

Then the obstruction class  $[O] \in H^{n+1}(h : \mathcal{B} \rightarrow \mathcal{C}, \mathbf{M})$  vanishes if and only if there are an extension  $\tilde{H} : \mathcal{A}_\tau(\mathbf{M}) \rightarrow \mathcal{C} \widehat{\otimes} (t, dt)$  of  $H$  and an extension  $\tilde{f} : \mathcal{A}_\tau(\mathbf{M}) \rightarrow \mathcal{B}$  of  $f$  such that  $\mu\tilde{f} = \nu g$  and  $(\nu \widehat{\otimes} \text{id})\tilde{H}$  is constant.

Now we are in a position to show that  $i$ -minimal  $k\mathbb{L}$ -algebras defined above play the same role in the category of complete and injective  $k\mathbb{L}$ -algebras as minimal algebras in the category of  $k$ -algebras. First, we redevelop the result presented in [7].

PROPOSITION 1.7. — Let  $f : \mathcal{A} \rightarrow \mathcal{B}$  be a quasi-isomorphism of  $k\mathbb{L}$ -algebras and  $g : \mathcal{M} \rightarrow \mathcal{B}$  a map of  $k\mathbb{L}$ -algebras, where  $\mathcal{M}$  is  $i$ -minimal. Then there is a map  $\tilde{g} : \mathcal{M} \rightarrow \mathcal{A}$  (unique up to homotopy) such that the diagram



commutes up to homotopy.

*Proof.* — The  $k\mathbb{L}$ -algebra  $\mathcal{M}$  is  $i$ -minimal, so  $\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}(n)$ , where  $\mathcal{M}(0) = \underline{k}$ ,  $\mathcal{M}(n) = \bigcup_{l \geq 0} \mathcal{M}(n, l)$  and  $\mathcal{M}(n, l + 1) = \mathcal{M}(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  is an  $i$ -elementary extension of  $\mathcal{M}(n, l)$  with respect to a map  $\mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}(n, l)$  for  $l \geq 0$  and  $n \geq 1$ .

We proceed inductively to construct the  $k\mathbb{L}$ -map  $\tilde{g} : \mathcal{M} \rightarrow \mathcal{A}$ . Let  $\tilde{g}_0 = \eta : \mathcal{M}(0) = \underline{k} \rightarrow \mathcal{A}$  be the unit map. Assume we have defined  $\tilde{g}_{n-1} = \tilde{g}_{n,0} : \mathcal{M}(n-1) = \mathcal{M}(n, 0) \rightarrow \mathcal{A}$  and  $\tilde{g}_{n,l} : \mathcal{M}(n, l) \rightarrow \mathcal{A}$  with  $f\tilde{g}_{n-1} \simeq g|_{\mathcal{M}(n-1)}$  and  $f\tilde{g}_{n,l} \simeq g|_{\mathcal{M}(n,l)}$ . Then consider the commutative (up to homotopy) diagram

$$\begin{array}{ccc} \mathcal{M}(n, l) & \xrightarrow{\tilde{g}_{n,l}} & \mathcal{A} \\ \downarrow & & \downarrow f \\ \mathcal{M}(n, l + 1) & \xrightarrow{g|_{\mathcal{M}(n,l+1)}} & \mathcal{B}. \end{array}$$

By Lemma 1.6 (1) the obstruction to extend  $\tilde{g}_{n,l}$  to  $\mathcal{M}(n, l + 1) = \mathcal{M}(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  lies in  $H^{n+1}(f : \mathcal{A} \rightarrow \mathcal{B}, \mathbf{M}_{n,l})$  which vanishes because the map  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a quasi-isomorphism.

The relative obstruction theory arguments of Lemma 1.6 (2) yield uniqueness (up to homotopy) of the map  $\tilde{g}$  and the proof is complete.  $\square$

Therefore,  $i$ -minimal  $k\mathbb{L}$ -algebras are cofibrant up to homotopy. Then, by means of the obstruction theory as in [17, Proposition 3.5] and closely following the non-equivariant proof (e.g., [1, Proposition 6.3] and [4]), one can show that the homotopy  $\simeq$  is an equivalence relation between maps  $\mathcal{M} \rightarrow \mathcal{A}$  whenever  $\mathcal{M}$  is  $i$ -minimal. Writing  $[\mathcal{M}, \mathcal{A}]$  for the set of homotopy classes of maps from  $\mathcal{M}$  to  $\mathcal{A}$  we obtain

**COROLLARY 1.8.** — *If  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a quasi-isomorphism of complete  $k\mathbb{L}$ -algebras and  $\mathcal{M}$  is an  $i$ -minimal  $k\mathbb{L}$ -algebra then the induced map  $f_* : [\mathcal{M}, \mathcal{A}] \rightarrow [\mathcal{M}, \mathcal{B}]$  of the sets of homotopy classes is a bijection.*

To state the next result we need to reprove [17, Lemma 13.57] in our context. Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $i$ -minimal  $k\mathbb{L}$ -algebras. Then

$$\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}(n) \quad \text{and} \quad \mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}(n),$$

where  $\mathcal{M}(0) = \underline{k}$ ,  $\mathcal{M}(n) = \bigcup_{l \geq 0} \mathcal{M}(n, l)$ ,  $\mathcal{N}(0) = \underline{k}$ ,  $\mathcal{N}(n) = \bigcup_{l \geq 0} \mathcal{N}(n, l)$  and  $\mathcal{M}(n, l + 1) = \mathcal{M}(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$ ,  $\mathcal{N}(n, l + 1) = \mathcal{N}(n, l)_{\tau'_{n,l}}(\mathbf{N}_{n,l})$  are  $i$ -elementary extensions of  $\mathcal{M}(n, l)$  and  $\mathcal{N}(n, l)$  with respect to maps

$\tau_{n,l} : \mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}(n, l)$  and  $\tau'_{n,l} : \mathbf{N}_{n,l} \rightarrow Z^{n+1}\mathcal{N}(n, l)$ , respectively for  $l \geq 0$  and  $n \geq 1$ .

LEMMA 1.9. — A map  $f : \mathcal{M} \rightarrow \mathcal{N}$  of  $i$ -minimal  $k\mathbb{I}$ -algebras is homotopic to a map  $g : \mathcal{M} \rightarrow \mathcal{N}$  which takes  $\mathcal{M}(n, l)$  to  $\mathcal{N}(n, l)$  for all  $l \geq 0$  and  $n \geq 1$ .

Proof. — We proceed again inductively taking  $g_0 = f_0 = \text{id}_k$ . Let  $\iota_{\mathcal{M}(n,l)} : \mathcal{M}(n, l) \hookrightarrow \mathcal{M}$  and  $\iota_{\mathcal{N}(n,l)} : \mathcal{N}(n, l) \hookrightarrow \mathcal{N}$  be the inclusion maps and assume we have  $g_{n-1} = g_{n,0} : \mathcal{M}(n-1) = \mathcal{M}(n, 0) \rightarrow \mathcal{N}(n-1) = \mathcal{N}(n, 0)$  with  $\iota_{\mathcal{M}(n,0)}g_{n,0} \simeq f|_{\mathcal{M}(n,0)}$ .

Given inductively  $g_{n,l} : \mathcal{M}(n, l) \rightarrow \mathcal{N}(n, l)$  with  $\iota_{\mathcal{M}(n,l)}g_{n,l} \simeq f|_{\mathcal{M}(n,l)}$  consider the diagram

$$\begin{array}{ccccc}
 \mathcal{M}(n, l) & \xrightarrow{g_{n,l}} & \mathcal{N}(n, l) & \hookrightarrow & \mathcal{N}(n, l+1) \\
 \downarrow & & & & \downarrow \\
 \mathcal{M}(n, l+1) & \xrightarrow{f|_{\mathcal{M}(n,l+1)}} & \mathcal{N} & & 
 \end{array}$$

which commutes up to homotopy. Now  $\mathcal{M}(n, l+1) = \mathcal{M}(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  for  $\tau_{n,l} : \mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}(n, l)$  and the obstruction  $O : \mathbf{M}_{n,l} \rightarrow \mathcal{N}(n, l+1)^{n+1} \times \mathcal{N}^n$  from Lemma 1.6 (1) lies in  $H^{n+1}(\mathcal{N}(n, l+1) \hookrightarrow \mathcal{N}, \mathbf{M}_{n,l})$ . But the sequence of the inclusion maps  $\mathcal{N}(n, l) \hookrightarrow \mathcal{N}(n, l+1) \hookrightarrow \mathcal{N}$  leads to the long exact sequence

$$\begin{aligned}
 0 \rightarrow H^{n+1}(\mathcal{N}(n, l+1) \hookrightarrow \mathcal{N}, \mathbf{M}_{n,l}) \xrightarrow{\delta^{n+1}} H^{n+2}(\mathcal{N}(n, l) \hookrightarrow \mathcal{N}(n, l+1), \mathbf{M}_{n,l}) \\
 \rightarrow H^{n+2}(\mathcal{N}(n, l) \hookrightarrow \mathcal{N}, \mathbf{M}_{n,l}) \rightarrow \dots \text{ with } \delta^{n+1}([O]) = 0.
 \end{aligned}$$

Consequently the obstruction class  $[O] \in H^{n+1}(\mathcal{N}(n, l+1) \hookrightarrow \mathcal{N}, \mathbf{M}_{n,l})$  vanishes and in the light of Lemma 1.6 (1) there exist a map

$$g_{n,l+1} : \mathcal{M}(n, l+1) \rightarrow \mathcal{N}(n, l+1)$$

extending  $\mathcal{M}(n, l) \xrightarrow{g_{n,l}} \mathcal{N}(n, l) \hookrightarrow \mathcal{N}(n, l+1)$  and a homotopy

$$\iota_{\mathcal{N}(n,l+1)}g_{n,l+1} \simeq f|_{\mathcal{M}(n,l+1)}.$$

Then, we produce the map

$$g(n) = \bigcup_{l \geq 0} g_{n,l} : \mathcal{M}(n) = \bigcup_{l \geq 0} \mathcal{M}(n, l) \rightarrow \mathcal{N}(n) = \bigcup_{l \geq 0} \mathcal{N}(n, l)$$

and the inductive step is complete. □

In general, any quasi-isomorphism of  $i$ -minimal  $k\mathbb{I}$ -algebras is not an isomorphism. Nevertheless we are able to show

PROPOSITION 1.10. — *If a map  $f : \mathcal{M} \rightarrow \mathcal{N}$  of  $i$ -minimal  $k\mathbb{I}$ -algebras is a quasi-isomorphism then there exists an isomorphism  $g : \mathcal{M} \rightarrow \mathcal{N}$  with  $f \simeq g$ .*

*Proof.* — By Lemma 1.9 there is a map  $g : \mathcal{M} \rightarrow \mathcal{N}$  which is homotopic to  $f$  and takes  $\mathcal{M}(n, l)$  to  $\mathcal{N}(n, l)$  for all  $l \geq 0$  and  $n \geq 1$ . We proceed inductively to show that  $g$  is an isomorphism.

Given the isomorphism  $g_{n-1} = g_{n,0} : \mathcal{M}(n-1) = \mathcal{M}(n,0) \rightarrow \mathcal{N}(n-1) = \mathcal{N}(n,0)$  assume inductively that  $g_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{N}(n,l)$  is also an isomorphism. But  $g : \mathcal{M} \rightarrow \mathcal{N}$  is a quasi-isomorphism, hence the 5-lemma applied to the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{H}^n(\mathcal{M}) & \longrightarrow & \mathbf{H}^{n+1}(\mathcal{M}(n,l) \hookrightarrow \mathcal{M}) & \longrightarrow & \mathbf{H}^{n+1}(\mathcal{M}(n,l)) \longrightarrow \dots \\ & & \downarrow & & (g_{n,l}, g)^* \downarrow & & \downarrow \\ \dots & \longrightarrow & \mathbf{H}^n(\mathcal{N}) & \longrightarrow & \mathbf{H}^{n+1}(\mathcal{N}(n,l) \hookrightarrow \mathcal{N}) & \longrightarrow & \mathbf{H}^{n+1}(\mathcal{N}(n,l)) \longrightarrow \dots \end{array}$$

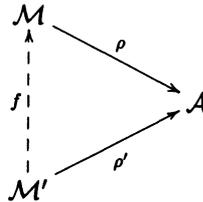
leads to the isomorphism

$$(g_{n,l}, g)^* : \mathbf{M}_{n,l} = \mathbf{H}^{n+1}(\mathcal{M}(n,l) \hookrightarrow \mathcal{M}) \xrightarrow{\cong} \mathbf{N}_{n,l} = \mathbf{H}^{n+1}(\mathcal{N}(n,l) \hookrightarrow \mathcal{N}).$$

Thus there is an isomorphism  $(\mathbf{M}_{n,l})_* \xrightarrow{\cong} (\mathbf{N}_{n,l})_*$  of graded left  $k\mathbb{I}$ -modules associated to minimal resolutions of  $\mathbf{M}_{n,l}$  and  $\mathbf{N}_{n,l}$ , respectively. Consequently,  $\mathcal{M}(n, l+1)$  and  $\mathcal{N}(n, l+1)$  are isomorphic as graded left  $k\mathbb{I}$ -modules. Moreover, as in [Theorem 3.8] we can see that for any  $m \in \mathbf{M}_{n,l+1}$  there is  $\alpha(m) \in \mathcal{N}(n, l)$  with  $g_{n,l+1}(m) = (g_{n,l}, g)^*(m) + \alpha(m)$ . Finally, in virtue of Lemma 1.4 (2), the map  $g_{n,l+1} : \mathcal{M}(n, l+1) \rightarrow \mathcal{N}(n, l+1)$  is an isomorphism of  $k\mathbb{I}$ -algebras. □

Then Proposition 1.7 and Proposition 1.10 lead to the conclusion.

COROLLARY 1.11 ([7]). — *If  $\rho : \mathcal{M} \rightarrow \mathcal{A}$  and  $\rho' : \mathcal{M}' \rightarrow \mathcal{A}$  are quasi-isomorphisms, where  $\mathcal{M}$  and  $\mathcal{M}'$  are  $i$ -minimal  $k\mathbb{I}$ -algebras then there is an isomorphism  $f : \mathcal{M} \rightarrow \mathcal{M}'$  such that the diagram*



*commutes up to homotopy.*

Now let  $\mathcal{A}$  be a complete and homologically connected  $k\mathbb{L}$ -algebra (i.e., the unit map  $\eta : \underline{k} \rightarrow \mathcal{A}$  induces an isomorphism  $\eta_* : \underline{k} \xrightarrow{\cong} \mathbf{H}^0(\mathcal{A})$ ). For an  $i$ -minimal  $k\mathbb{L}$ -algebra  $\mathcal{M}$  a quasi-isomorphism  $\rho : \mathcal{M} \rightarrow \mathcal{A}$  is called the  $i$ -minimal model of  $\mathcal{A}$ . Observe that the above uniqueness results make this definition meaningful. The methods similar to the proof of [17, Theorem 3.11] give rise to the existence (cf. [7, Theorem 2.8]) of  $i$ -minimal models. Nevertheless, we present a sketch of its proof.

**THEOREM 1.12.** — *If  $\mathcal{A}$  is an injective, complete and homologically connected  $k\mathbb{L}$ -algebra with linearly compact cohomology  $\mathbf{H}^n(\mathcal{A})$  for all  $n \geq 0$  then it has an  $i$ -minimal model  $\rho : \mathcal{M} \rightarrow \mathcal{A}$ .*

*Proof.* — We construct inductively maps  $\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}$  for  $l \geq 0$  and  $n \geq 1$ . Let  $\rho_{1,0} = \eta : \underline{k} \rightarrow \mathcal{A}$  and assume inductively that  $\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}$  is an  $(n - 1)$ -quasi-isomorphism for some  $l \geq 0$  and  $n \geq 1$ .

Writing  $\mathbf{M}_{n,l} = \mathbf{H}^{n+1}(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A})$ , we make use of the spectral sequence

$$E_2^{p,q} = \text{Ext}^p(\mathbf{M}_{n,l}, \mathbf{H}^q(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A})) \implies H^{p+q}(f : \mathcal{M}(n,l) \rightarrow \mathcal{A}, \mathbf{M}_{n,l}).$$

Because  $\mathbf{H}^q(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}) = 0$  for  $q \leq n$  and  $E_2^{p,q} = \text{Ext}^p(\mathbf{M}_{n,l}, \mathbf{H}^q(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A})) = 0$  for  $p + q = n + 1$  and  $p > 0$ , the identity map  $\text{id}_{\mathbf{M}_{n,l}} \in \text{Hom}(\mathbf{M}_{n,l}, \mathbf{H}^{n+1}(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A})) = E_2^{0,n+1}$ . Hence  $E_2^{0,n+1} = E_\infty^{0,n+1}$  and  $\text{id}_{\mathbf{M}_{n,l}}$  represents a class  $[\text{id}_{\mathbf{M}_{n,l}}] \in H^{n+1}(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}, \mathbf{M}_{n,l})$ . In virtue of the exact sequence

$$\begin{aligned} \dots &\rightarrow H^{n+1}(\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}, \mathbf{M}_{n,l}) \\ &\rightarrow H^{n+1}(\mathcal{M}(n,l), \mathbf{M}_{n,l}) \xrightarrow{\rho_{n,l}^*} H^{n+1}(\mathcal{A}, \mathbf{M}_{n,l}) \rightarrow \dots \end{aligned}$$

$[\text{id}_{\mathbf{M}_{n,l}}]$  yields an element  $[\tau_{n,l}] \in H^{n+1}(\mathcal{M}(n,l), \mathbf{M}_{n,l})$  with  $\rho_{n,l}^*([\tau_{n,l}]) = 0$ . So  $\rho_{n,l}\tau_{n,l}$  is a coboundary and there is a map  $\varphi : \mathbf{M}_{n,l} \rightarrow \mathcal{A}^n$  such that  $d\varphi = \rho_{n,l}\tau_{n,l}$ , where  $d$  is the differential on  $\mathcal{A}$ .

Defining  $\mathcal{M}(n,l + 1) = \mathcal{M}(n,l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  to be the  $i$ -elementary extension of  $\mathcal{M}(n,l)$  with respect to  $\tau_{n,l} : \mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}(n,l)$ , we get a map  $\rho_{n,l+1} : \mathcal{M}(n,l + 1) \rightarrow \mathcal{A}$  extending  $\rho_{n,l}$  and  $\varphi$ . Now given  $\rho_{n,l} : \mathcal{M}(n,l) \rightarrow \mathcal{A}$  for all  $l \geq 0$ , let  $\rho_n : \mathcal{M}(n) \rightarrow \mathcal{A}$  be the extension of all the maps  $\rho_{n,l}$  on  $\mathcal{M}(n) = \bigcup_{l \geq 0} \mathcal{M}(n,l)$ .

Finally, given  $\rho_n : \mathcal{M}(n) \rightarrow \mathcal{A}$  for all  $n \geq 0$ , let  $\rho : \mathcal{M} \rightarrow \mathcal{A}$  be the extension of all the maps  $\rho_n$  on  $\mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}(n)$ . The desired properties are not hard to verify.  $\square$

To extend the notion of the  $i$ -minimal model to any (not necessary injective)  $k\mathbb{I}$ -algebra we recall the result on injective envelopes developed in [10, Theorem 2.1] and generalizing constructions presented in [8]. By induction over the filtration of  $\text{Iso}(\mathbb{I})$  we have proved

**THEOREM 1.13.** — *If  $\mathbb{I}$  is an EI-category such that the group ring  $k[I]$  is semisimple for all  $I \in \text{Ob}(\mathbb{I})$  then for any complete  $k\mathbb{I}$ -algebra  $\mathcal{A}$  there is a complete and injective  $k\mathbb{I}$ -algebra  $\Omega(\mathcal{A})$  and a natural inclusion  $\mathcal{A} \hookrightarrow \Omega(\mathcal{A})$  which is a quasi-isomorphism.*

**Remark 1.14.** — This theorem allows us to produce  $i$ -minimal models for a wider class of  $k\mathbb{I}$ -algebras. Let  $\mathcal{A}$  be any complete and homologically connected  $k\mathbb{I}$ -algebra with linearly compact cohomology and  $\Omega(\mathcal{A})$  the associated injective  $k\mathbb{I}$ -algebra determined by Theorem 1.13. Then by Theorem 1.12 there is an  $i$ -minimal  $k\mathbb{I}$ -algebra  $\mathcal{M}$  and a quasi-isomorphism  $\rho : \mathcal{M} \rightarrow \Omega(\mathcal{A})$  called the  $i$ -minimal model of  $\mathcal{A}$ .

## 2. Equivariant disconnected rational homotopy theory.

Let now  $G$  be a finite group. Recall that a simplicial set  $X$  with a simplicial action of  $G$  is called a  $G$ -simplicial set. Of course, fixed point subsets  $X^H$  are simplicial for any subgroup  $H \subseteq G$ . Much of topological information about a  $G$ -simplicial set  $X$  is encoded in the form of functors from the category  $\mathcal{O}(G, X)$  objects of which consist of pairs  $(G/H, \alpha)$ , for a subgroup  $H \subseteq G$  and  $\alpha$  in the set  $\pi_0(X^H)$  of connected components of the fixed point simplicial subset  $X^H$ . Morphisms  $(G/K, \beta) \rightarrow (G/H, \alpha)$  in  $\mathcal{O}(G, X)$  are  $G$ -maps  $\phi : G/K \rightarrow G/H$  such that  $\pi_0(\bar{\phi})(\alpha) = \beta$  with the induced map  $\bar{\phi} : X^H \rightarrow X^K$  of fixed point simplicial subsets.

The geometric information we are modeling depends on Postnikov towers and homotopy groups, which are based constructions. Therefore we need to replace in [7] the category  $\mathcal{O}(G, X)$  by a closely related indexing category  $\tilde{\mathcal{O}}(G, X)$  which allows us to keep track of base points in various components. This category  $\tilde{\mathcal{O}}(G, X)$  is defined as follows:

objects  $\text{Ob}(\tilde{\mathcal{O}}(G, X))$  are pairs  $(G/H, x)$  with a subgroup  $\subseteq G$  and  $x \in X_0^H$ , where  $X_0^H$  is the set of 0-simplexes of the fixed point simplicial subset  $X^H$ ;

morphisms  $(G/H, x) \rightarrow (G/K, y)$  are given by  $G$ -maps  $\phi : G/H \rightarrow G/K$  such that  $\bar{\phi}_0(y) = x$  with the induced map  $\bar{\phi}_0 : X_0^K \rightarrow X_0^H$ .

*Remark 2.1.* — If  $\mathbb{I}$  is a small category and  $F : \mathbb{I} \rightarrow \mathcal{C}at$  is a contravariant functor into the category  $\mathcal{C}at$  of small categories, then the Grothendieck construction on  $F$  is the category  $\mathbb{I} \int F$  defined as follows: the objects of  $\mathbb{I} \int F$  are pairs  $(i, x)$  where  $i$  is an object of  $\mathbb{I}$  and  $x$  one of  $F(i)$ ; a morphism  $(i, x) \rightarrow (i, x')$  in  $\mathbb{I} \int F$  is a pair  $(\varphi, \psi)$  with  $\varphi : i \rightarrow i'$  in  $\mathbb{I}$  and  $\psi : x \rightarrow F(\varphi)(x')$  in  $F(i)$ .

For the category  $\mathcal{O}(G)$  of canonical orbits of the group  $G$ , any  $G$ -set  $X$  leads to the contravariant functor  $X_0^{(-)} : \mathcal{O}(G) \rightarrow \mathcal{C}at$  given by  $X_0^{(-)}(G/H) = X^H$  for  $G/H \in \mathcal{O}(G)$  provided that sets are considered as discrete small categories. Then

$$\tilde{\mathcal{O}}(G, X) = \mathcal{O}(G) \int X_0^{(-)}.$$

In order to apply the results of Section 1 in the sequel, we need to demonstrate that the category  $\tilde{\mathcal{O}}(G, X)$  has appropriate properties. First, we show that it admits the required filtration. We identify an object  $(G/H, x)$  in  $\tilde{\mathcal{O}}(G, X)$  with the 0-simplex  $x$ . For the isomorphism class  $\bar{x}$  of any object  $x$ , let  $\varrho(x) = n$  be the largest number such that there is a sequence  $\bar{x} = \bar{x}_1 < \dots < \bar{x}_n$ . The group  $G$  is finite, so  $\tilde{\mathcal{O}}(G, X)$  is a cofinite  $EI$ -category. Furthermore, we can define the filtration

$$\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_m = \text{Iso}(\tilde{\mathcal{O}}(G, X))$$

on the set of isomorphism classes  $\text{Iso}(\tilde{\mathcal{O}}(G, X))$  satisfying  $(\star)$ , where  $T_l = \{\bar{x} ; \varrho(x) \leq l\}$  for  $l = 0, 1, \dots, m$ .

Next, we need to correct [Lemma 3.6] which is essential for an existence of  $i$ -minimal models of  $G$ -simplicial sets and show that the category  $\tilde{\mathcal{O}}(G, X)$  satisfies the assumptions about injective functors. Recall that the dual left  $k\tilde{\mathcal{O}}(G, X)$ -module  $(k\tilde{\mathcal{O}}(G, X)(-, (G/H, x)))^*$  for some object  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  is called a *co-Yoneda* module, where  $k$  is a field. Of course, any such a  $k\tilde{\mathcal{O}}(G, X)$ -module is injective.

LEMMA 2.2. — *Let  $k$  be a field and  $X$  a  $G$ -simplicial set. Then*

- (1) *the object-wise complete tensor product  $\hat{\otimes}$  of two injective linearly compact left  $k\tilde{\mathcal{O}}(G, X)$ -modules is again injective;*
- (2) *for any  $k$ -module  $M$ , the constant left  $k\tilde{\mathcal{O}}(G, X)$ -module  $\underline{M}$  determined by  $M$  is injective.*

*Proof.* — We prove part (1) by means of methods used in [7, Lemma 3.6] and [9]. First, observe that any injective linearly compact left  $k\tilde{\mathcal{O}}(G, X)$ -module is a direct summand of a product of co-Yoneda  $k\tilde{\mathcal{O}}(G, X)$ -modules  $(k\tilde{\mathcal{O}}(G, X)(-, (G/H, x)))^*$  for some  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$ .

Therefore, it is sufficient to show that the complete tensor product of two co-Yoneda  $k\tilde{\mathcal{O}}(G, X)$ -modules is a product of co-Yoneda  $k\tilde{\mathcal{O}}(G, X)$ -modules. Observe that, for an object  $(G/K, y)$  in the category  $\tilde{\mathcal{O}}(G, X)$ , the free  $k$ -module

$$k\tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_1, x_1)) \otimes k\tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_2, x_2))$$

is freely generated by the set

$$\tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_1, x_1)) \times \tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_2, x_2)).$$

The  $G$ -set  $G/H_1 \times G/H_2$  is in one-to-one correspondence with a disjoint union  $\bigcup_{i=1}^m G/K_i$ , where  $K_i$  is the isotropy group of a point

$$(g'_i H_1, g''_i H_2) \in G/H_1 \times G/H_2 \text{ for } i = 1, \dots, m$$

and the set

$$\tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_1, x_1)) \times \tilde{\mathcal{O}}(G, X)((G/K, y), (G/H_2, x_2))$$

is in one-to-one correspondence with a disjoint union

$$\bigcup_{\substack{i=1 \\ g'_i x_1 = g''_i x_2}}^m \tilde{\mathcal{O}}(G, X)((G/K, y), (G/K_i, g'_i x_1)).$$

Thus we obtain a topological isomorphism of left  $k\tilde{\mathcal{O}}(G, X)$ -modules

$$\begin{aligned} (k\tilde{\mathcal{O}}(G, X)(-, (G/H_1, x_1)))^* \widehat{\otimes} (k\tilde{\mathcal{O}}(G, X)(-, (G/H_2, x_2)))^* \\ \cong \prod_{\substack{i=1 \\ g'_i x_1 = g''_i x_2}}^m (k\tilde{\mathcal{O}}(G, X)(-, (G/K_i, g'_i x_1)))^* \end{aligned}$$

and this demonstrates part (1).

To prove (2) we make the following observations:

(a) if  $G_x$  is the isotropy subgroup of a point  $x \in X_0$  then for any object  $(G/H, x)$  there is a unique morphism  $(G/H, x) \rightarrow (G/G_x, x)$  in  $\tilde{\mathcal{O}}(G, X)$ ;

(b) any connected component of the category  $\tilde{\mathcal{O}}(G, X)$  is the full subcategory given by all objects  $(G/H, gx)$  with  $g \in G, H \subseteq G$  for a fixed point  $x \in X_0$  in the set of 0-simplices.

Let  $\underline{M}$  be the constant left  $k\tilde{\mathcal{O}}(G, X)$ -module determined by a  $k$ -module  $M$  and consider the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathbf{M} & \xrightarrow{\beta} & \mathbf{N} \\
 & & \downarrow \alpha & & \\
 & & \underline{M} & & 
 \end{array}$$

in the category of left  $k\tilde{\mathcal{O}}(G, X)$ -modules. Given a decomposition  $X_0 = \bigcup_{\lambda \in \Lambda} Gx_\lambda$  of the set  $X_0$  of 0-simplices into disjoint orbits, fix maps  $\gamma_\lambda : \mathbf{N}(G/G_{x_\lambda}, x_\lambda) \rightarrow M$  and such that the following diagrams:

$$\begin{array}{ccc}
 0 \longrightarrow & \mathbf{M}(G/G_{x_\lambda}, x_\lambda) & \xrightarrow{\beta_\lambda} & \mathbf{N}(G/G_{x_\lambda}, x_\lambda) \\
 & \downarrow \alpha_\lambda & \nearrow \gamma_\lambda & \\
 & M & & 
 \end{array}$$

commute, where  $\alpha_\lambda = \alpha(G/G_{x_\lambda}, x_\lambda)$  and  $\beta_\lambda = \beta(G/G_{x_\lambda}, x_\lambda)$  for  $\lambda \in \Lambda$ . Then, the unique morphism  $(G/G_{gx_\lambda}, gx_\lambda) \rightarrow (G/G_{x_\lambda}, x_\lambda)$  and  $\gamma_\lambda$  give rise to a map

$$\gamma(G/G_{gx_\lambda}, gx_\lambda) : \mathbf{N}(G/G_{gx_\lambda}, gx_\lambda) \rightarrow \mathbf{N}(G/G_{x_\lambda}, x_\lambda) \rightarrow M$$

for any  $g \in G$ .

Now, for any object  $(G/H, x)$  in  $\tilde{\mathcal{O}}(G, X)$ , note that  $x = gx_\lambda$  for some  $\lambda \in \Lambda$  and let  $(G/H, x) \rightarrow (G/G_{gx_\lambda}, gx_\lambda)$  be the unique morphism from observation (a). Then  $(G/H, x) \rightarrow (G/G_{gx_\lambda}, gx_\lambda)$  and  $\gamma(G/G_{gx_\lambda}, gx_\lambda)$  determine a map

$$\gamma(G/H, x) : \mathbf{N}(G/H, x) \rightarrow \mathbf{N}(G/G_{gx_\lambda}, gx_\lambda) \rightarrow M.$$

Consequently, in the light of observation (b), we get a map  $\gamma : \mathbf{N} \rightarrow \underline{M}$  in the category of left  $\tilde{\mathcal{O}}(G, X)$ -modules with the commutative diagram

$$\begin{array}{ccc}
 0 & \longrightarrow & M & \xrightarrow{\beta} & N \\
 & & \downarrow \alpha & \searrow \gamma & \\
 & & \underline{M} & & 
 \end{array}$$

□

In particular, the constant left  $k\tilde{\mathcal{O}}(G, X)$ -module  $\underline{k}$  determined by the field  $k$  is injective.

Given a simplicial set  $X$ , one can form the  $\mathbb{Q}$ -algebra  $A_X$  over the field  $\mathbb{Q}$  of rationals by taking collections of  $\mathbb{Q}$ -polynomial forms on each simplex (sums of terms of type  $\omega(t_0, \dots, t_n) dt_{i_0} \wedge \dots \wedge dt_{i_l}$  for  $l \geq 0$ , where  $\omega$  is a  $\mathbb{Q}$ -polynomial in indeterminates  $t_0, \dots, t_n$ ) that agree when restricted to common faces (see [1] for more details). The algebra  $A_X$  is the key link between geometry and algebra, and  $A_X$  admits the following natural linearly complete topology.

For a fixed  $n \geq 0$  consider all simplicial maps  $\tilde{x} : \Delta(l) \rightarrow X$ , where  $\Delta(l)$  is the standard  $l$ th simplex. We have proved in [7] the  $\mathbb{Q}$ -submodules  $\ker(A_X^n(\tilde{x}) : A_X^n \rightarrow A^n(\Delta(l)))$ , form a fundamental system of neighborhoods of zero for a linear topology on  $A_X^n$ ,  $n \geq 0$ . Furthermore,  $A_X$  is linearly complete with respect to that linear topology and the induced topology on cohomology of  $A_X$  is linearly compact as well.

Let now  $C_n(X, \mathbb{Q})$  be the discrete  $\mathbb{Q}$ -module of  $n$ -chains on a simplicial set  $X$  with coefficients in  $\mathbb{Q}$  for  $n \geq 0$ . Then on the  $\mathbb{Q}$ -module  $C^n(X, \mathbb{Q})$  of  $n$ -cochains there is linearly compact topology for  $n \geq 0$ . In particular, it follows that the induced topology on the cohomology groups  $H^n(X, \mathbb{Q})$  is linearly compact for  $n \geq 0$  and the topological dual  $\mathbb{Q}$ -module  $(H^n(X, \mathbb{Q}))^*$  is isomorphic to the homology group  $H_n(X, \mathbb{Q})$  for  $n \geq 0$ . On the other hand, the map

$$i_X : A_X \longrightarrow C^*(X, \mathbb{Q})$$

given by the integration of forms, in virtue of [7, Corollary 3.2], is natural and continuous. Consequently, by the de Rham Theorem [1, Theorem 2.2] the induced map on cohomology  $i_X^n : H^n(A_X) \longrightarrow H^n(X, \mathbb{Q})$  is a continuous isomorphism for all  $n \geq 0$ .

Equivariantly, given a  $G$ -simplicial set  $X$ , let  $\mathcal{A}_X$  be the  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra defined by  $\mathcal{A}_X(G/H, x) = A_{X_x^H}$ , where  $X_x^H$  denotes the connected component of  $X^H$  corresponding to  $x \in X_0^H$  in the set  $X_0^H$ . Maps are induced by the action of  $G$  on the connected components of the fixed point simplicial subsets. By [7], the  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}_X$  is complete for any  $G$ -simplicial set  $X$ .

HOMOLOGY AND COHOMOLOGY OF  $G$ -SIMPLICIAL SETS. Let now  $R$  be a commutative ring and  $\mathcal{C}_*(X, R)$  be the differential graded right  $R\tilde{\mathcal{O}}(G, X)$ -module given by

$$\mathcal{C}_n(X, R)(G/H, x) = C_n(X_x^H, R)$$

for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 0$ . We note that  $\mathcal{C}_n(X, R)$  is a projective  $R\tilde{\mathcal{O}}(G, X)$ -module for all  $n \geq 0$ . Given a left  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$ , we write  $\mathcal{C}_*^G(X, \mathbf{M}) = \mathcal{C}_*(X, R) \otimes_{\tilde{\mathcal{O}}(G, X)} \mathbf{M}$  for the chain complex induced by the tensor product of  $R\tilde{\mathcal{O}}(G, X)$ -modules and then consider two types of homology for  $X$ :

- (1) the right  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{H}_n(X, R)$  such that

$$\mathbf{H}_n(X, R)(G/H, x) = H_n(X_x^H, R)$$

for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 0$ ;

- (2) the homology  $H_n^G(X, \mathbf{M}) = H_n(\mathcal{C}_*^G(X, \mathbf{M}))$  with coefficients in a left  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$  for  $n \geq 0$ .

Then standard homological algebra arguments yield a homology spectral sequence

$$E_{p,q}^2 = \text{Tor}_p(\mathbf{H}_q(X, R), \mathbf{M}) \implies H_{p+q}^G(X, \mathbf{M}).$$

Dually, let  $\mathcal{C}^*(X, R)$  be the differential graded left  $R\tilde{\mathcal{O}}(G, X)$ -module given by

$$\mathcal{C}^n(X, R)(G/H, x) = C^n(X_x^H, R)$$

for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 0$ . Given a right  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$ , we write  $\mathcal{C}_G^*(X, \mathbf{M}) = \text{Hom}(\mathcal{C}_*(X, R), \mathbf{M})$  for the cochain complex given by natural transformations and then consider two types of cohomology for  $X$ :

- (1) the left  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{H}^n(X, R)$  such that

$$\mathbf{H}^n(X, R)(G/H, x) = H^n(X_x^H, R)$$

for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 0$ ;

(2) the cohomology  $H_G^n(X, \mathbf{M}) = H^n(\mathcal{C}_G^*(X, \mathbf{M}))$  with coefficients in a right  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$  for  $n \geq 0$ .

Standard homological algebra arguments also yield a cohomology spectral sequence

$$E_2^{p,q} = \text{Ext}^p(\mathbf{H}_q(X, R), \mathbf{M}) \implies H_G^{p+q}(X, \mathbf{M}).$$

Any map  $f : X \rightarrow Y$  of  $G$ -simplicial sets gives rise to the right  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{H}_n(f : X \rightarrow Y, R)$  and the left  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{H}^n(f : X \rightarrow Y, R)$  such that

$$\mathbf{H}_n(f : X \rightarrow Y, R)(G/H, x) = H_n(f^H : X_x^H \rightarrow Y_{f(x)}^H, R)$$

and

$$\mathbf{H}^n(f : X \rightarrow Y, R)(G/H, x) = H^n(f^H : X_x^H \rightarrow Y_{f(x)}^H, R)$$

are given by the ordinary relative homology and cohomology groups, respectively for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 0$ .

Furthermore, we can also consider relative homology  $H_n^G(f : X \rightarrow Y, \mathbf{M})$  and cohomology  $H_G^n(f : X \rightarrow Y, \mathbf{N})$  groups with coefficients in a left  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$  and a right  $R\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{N}$ , respectively for which there are long exact sequences

$$\begin{aligned} \cdots \rightarrow H_{n+1}^G(f : X \rightarrow Y, \mathbf{M}) \rightarrow H_n^G(X, \mathbf{M}) \\ \rightarrow H_n^G(Y, \mathbf{M}) \rightarrow H_n^G(f : X \rightarrow Y, \mathbf{M}) \rightarrow \cdots \end{aligned}$$

and

$$\begin{aligned} \cdots \rightarrow H_G^n(f : X \rightarrow Y, \mathbf{N}) \rightarrow H_G^n(Y, \mathbf{N}) \\ \rightarrow H_G^n(X, \mathbf{N}) \rightarrow H_G^{n+1}(f : X \rightarrow Y, \mathbf{N}) \rightarrow \cdots \end{aligned}$$

Homological algebra yields appropriate relative homology and cohomology spectral sequences as well.

For a discrete right  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$ , let  $\mathbf{M}^*$  denote its topological dual left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module defined by  $\mathbf{M}^*(G/H, x) = \text{Hom}(\mathbf{M}(G/H, x), \mathbb{Q})$  with the induced linearly compact topology on each  $\mathbf{M}^*(G/H, x)$  for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$ .

The natural map

$$\mathbf{i}_X : \mathcal{A}_X \longrightarrow \mathcal{C}^*(X, \mathbb{Q})$$

leads to continuous isomorphisms

$$i_X^n : \mathbf{H}^n(\mathcal{A}_X) \xrightarrow{\cong} \mathbf{H}^n(X, \mathbb{Q})$$

for all  $n \geq 0$ . Consequently, cohomology groups  $\mathbf{H}^n(\mathcal{A}_X)$  are linearly compact left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -modules for all  $n \geq 0$ . Taking these facts into account we can get as in [7, Theorem 3.5], by means of the cohomology spectral sequence above, an isomorphism

$$H_G^n(X, \mathbf{M}) \xrightarrow{\cong} H^n(\mathcal{A}_X, \mathbf{M}^*)$$

for all  $n \geq 0$ . Now we follow the proof of [8, Theorem 2.3] to show that the  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}_X$  is also injective.

PROPOSITION 2.3. — *If  $X$  is a  $G$ -simplicial set then the  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}_X$  is injective.*

*Proof.* — Given a  $G$ -simplicial set  $X$ , we identify an object  $(G/H, x)$  in  $\tilde{\mathcal{O}}(G, X)$  with the 0-simplex  $x$ . We do not also distinguish between the  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}_X$  and the graded  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module  $\mathcal{A}_X$ . Throughout we keep notations from the proof of Theorem 1.2.

The group  $G$  is finite, hence the group rings  $\mathbb{Q}[x]$  are semisimple and  $S_x(\mathcal{A}_X)$  are injective  $\mathbb{Q}[x]$ -modules for all  $x \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$ . Given the filtration

$$\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_m = \text{Iso}(\tilde{\mathcal{O}}(G, X))$$

defined above let  $\mathcal{A}_X^{(0)} = \mathcal{A}_X$  and  $\mathcal{A}_X^{(l)} = \ker_{T_l} \mathcal{A}_X^{(l-1)}$  for all  $l = 1, \dots, m$ . Then  $\mathcal{A}_X^{(l)}$  are of type  $T \setminus T_l$  and there is a natural filtration

$$0 = \mathcal{A}_X^{(m)} \hookrightarrow \mathcal{A}_X^{(m-1)} \hookrightarrow \dots \hookrightarrow \mathcal{A}_X^{(1)} \hookrightarrow \mathcal{A}_X^{(0)} = \mathcal{A}_X$$

of  $\mathcal{A}_X$  determined by the exact sequences

$$0 \longrightarrow \mathcal{A}_X^{(l)} \longrightarrow \mathcal{A}_X^{(l-1)} \xrightarrow{\mathcal{J}_l} \prod_{\bar{x} \in T_l \setminus T_{l-1}} (E_x S_x) \mathcal{A}_X$$

for  $l = 1, \dots, m$ . In the light of Theorem 1.2, to prove the injectivity of  $\mathcal{A}_X$ , we need to show that the maps

$$\mathcal{J}_l(y) : \mathcal{A}_X^{(l-1)}(y) \longrightarrow \prod_{\bar{x} \in T_l \setminus T_{l-1}} ((E_x S_x) \mathcal{A}_X)(y)$$

are surjective for all objects  $y \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  for  $l = 1, \dots, m$ .

Observe that any element in  $\prod_{\bar{x} \in T_i \setminus T_{i-1}} ((E_x S_x) \mathcal{A}_X)(y)$  determines a collection of polynomial forms  $\omega_x^H$  on  $X_x^H$  for  $\bar{x} \in T_i \setminus T_{i-1}$ . Given a simplex  $z \in X_x^H \cap X_{x'}^{H'}$  with  $H \neq H'$  let  $G_z$  denote its isotropy group. Then  $G_z$  contains properly  $H$  or  $H'$  and there is a 0-simplex  $x''$  such that  $X_{x''}^{G_z} \subseteq X_x^H \cap X_{x'}^{H'}$ . Therefore we get a polynomial form  $\omega$  on the union  $\bigcup_{\bar{x} \in T_i \setminus T_{i-1}} X_x^H$ .

Let the object  $y \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  be determined by the pair  $(G/K, y)$ . Then the set

$$\tilde{\mathcal{O}}(G, X)((G/K, y), (G/H, x))$$

is one-to-one correspondence with the disjoint union

$$\bigcup_{\substack{i=1 \\ K \subseteq H_i}}^n \tilde{\mathcal{O}}(G, X)((G/H_i, y), (G/H, x)),$$

where  $H_1, \dots, H_n$  are some distinct subgroups of  $G$  conjugate to  $H$ . Now the polynomial form  $\omega$  on  $\bigcup_{\bar{x} \in T_i \setminus T_{i-1}} X_x^H$  gives rise to such a form  $\tilde{\omega}$  on the union  $\bigcup_{i=1}^n X_y^{H_i} \subseteq X_y^K$  and by [1, Lemma 2.7] there is a polynomial form  $\bar{\omega}$  on  $X_y^K$  extending  $\tilde{\omega}$ . Then  $\mathcal{J}_l(y)(\bar{\omega}) = \omega$  for all  $l = 1, \dots, m$  and the proof is complete.  $\square$

Of course, the unit map  $\eta : \mathbb{Q} \rightarrow \mathcal{A}_X$  induces an isomorphism  $\mathbb{Q} \xrightarrow{\cong} \mathbf{H}^0(\mathcal{A}_X)$  and finally we are in a position to apply Theorem 1.12 and state the result.

**THEOREM 2.4.** — *If  $X$  is a  $G$ -simplicial set then there is an  $i$ -minimal  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{M}_X$  and a quasi-isomorphism  $\rho_X : \mathcal{M}_X \rightarrow \mathcal{A}_X$ .*

The map  $\rho_X : \mathcal{M}_X \rightarrow \mathcal{A}_X$  is called the  *$i$ -minimal model* of the  $G$ -simplicial set  $X$ .

**HOMOTOPY OF  $G$ -SIMPLICIAL SETS AND OBSTRUCTION.** To make a geometric use of  $i$ -minimal models we briefly review basic facts on  $G$ -simplicial sets. A map  $f : X \rightarrow Y$  of  $G$ -simplicial sets (or simply a  $G$ -map) is said to be a *Kan  $G$ -fibration* if the induced maps  $f^H : X^H \rightarrow Y^H$  of fixed point simplicial subsets are Kan fibrations in the sense of [16, Definition 7.1] for all subgroups  $H \subseteq G$ . In particular, a  $G$ -simplicial set  $X$  is called a *Kan set* if the map  $X \rightarrow *$  is a Kan fibration, where  $*$  denotes the  $G$ -simplicial set generated by a single point. The preimage  $f^{-1}(Y^{(0)})$  of the 0-skeleton  $Y^{(0)}$  of  $Y$  is called the *fibre* of the Kan fibration  $f : X \rightarrow Y$ . Note that for any  $G$ -simplicial set  $X$  there is a canonical  $G$ -map  $\delta : X \rightarrow X_0$  (given

by the iterations of the 0-face operation) and its retraction  $\sigma : X_0 \rightarrow X$  (given the iteration of the 0-degeneracy operation), where the  $G$ -set  $X_0$  of 0-simplexes is identified with the corresponding constant  $G$ -simplicial set. We say that a map  $f : X \rightarrow Y$  of  $G$ -simplicial sets is *pointed* if  $X$  and  $Y$  have the same sets of 0-simplexes and  $f$  restricts to the identity map on  $X_0$ .

For a Kan  $G$ -simplicial set  $X$ , let

$$\pi_n(X) : \tilde{\mathcal{O}}(G, X) \longrightarrow \mathbb{G}p$$

be the contravariant functor to the category  $\mathbb{G}p$  of groups such that

$$\pi_n(X)(G/H, x) = \pi_n(X_x^H, x)$$

is the  $n$ th homotopy group of the based simplicial set  $(X_x^H, x)$  for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 1$ .

Given a map  $f : X \rightarrow Y$  of Kan  $G$ -simplicial sets, we consider also the contravariant functor

$$\pi_n(f : X \rightarrow Y) : \tilde{\mathcal{O}}(G, X) \longrightarrow \mathbb{G}p$$

such that

$$\pi_n(f : X \rightarrow Y)(G/H, x) = \pi_n(f^H : X_x^H \rightarrow Y_{f(x)}^H)$$

is given by relative homotopy groups for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  and  $n \geq 1$ . In the same way we define a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system  $\pi_0(f : X \rightarrow Y)$  of sets. Then the non-equivariant natural relative Hurewicz map leads to a map

$$\mathbf{h}_n : \pi_n(f : X \rightarrow Y) \longrightarrow \mathbf{H}_n(f : X \rightarrow Y, \mathbb{Z})$$

for  $n \geq 2$ , where  $\mathbb{Z}$  denotes the ring of integers. It can be shown (cf. [2]) that a map  $f : X \rightarrow Y$  of  $G$ -simplicial sets is a  $G$ -homotopy equivalence if and only if the induced maps of fixed point simplicial subsets  $f^H : X^H \rightarrow Y^H$  are homotopy equivalence for all subgroups  $H \subseteq G$ . Therefore, by the non-equivariant simplicial Whitehead Theorem [16, Theorem 12.5] a map  $f : X \rightarrow Y$  of Kan  $G$ -simplicial sets is a  $G$ -homotopy equivalence if and only if  $\pi_n(f : X \rightarrow Y) = 0$  for all  $n \geq 0$ . Furthermore, a sequence  $X \xrightarrow{g} Y \xrightarrow{f} Z$  of pointed maps of Kan  $G$ -simplicial sets gives rise to the long exact sequence

$$\dots \rightarrow \pi_n(fg : X \rightarrow Z) \rightarrow \pi_n(g : Y \rightarrow Z) \rightarrow \pi_{n-1}(f : X \rightarrow Y) \rightarrow \dots$$

If  $\pi_n(X) = \pi$  and  $\pi_m(X) = 0$  for  $m \neq n$  with  $n \geq 1$  then  $X$  is said to be an *Eilenberg-MacLane  $G$ -simplicial set* of type  $(\pi, n)$ . We denote such a  $G$ -simplicial set by  $K(\pi, n)$ . We indicate two ways an Eilenberg-MacLane  $G$ -simplicial set can be constructed.

For a  $G$ -set  $X$  and  $n \geq 1$ , let  $\pi$  be a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system of groups (abelian, provided  $n \geq 2$ ). Then consider the contravariant system of  $\tilde{\mathcal{O}}(G, X)$ -simplicial sets determined by the non-equivariant Eilenberg-MacLane simplicial sets  $K(\pi(G/H, x), n)$  for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$  (studied e.g., in [16]). Defining

$$K(\pi, n)^H = \prod_{x \in X^H} K(\pi(G/H, x), n)$$

for any subgroup  $H \subseteq G$ , we get the  $\mathcal{O}(G)$ -system  $\mathcal{K}(\pi, n) = \{K(\pi, n)^H\}_{H \subseteq G}$  of simplicial sets. Then the coalescence functor  $c$  (developed e.g., in [3]) from the category of systems of  $\mathcal{O}(G)$ -simplicial sets to  $G$ -simplicial sets gives rise the required Eilenberg-MacLane  $G$ -simplicial set

$$K(\pi, n) = c(\mathcal{K}(\pi, n)).$$

On the other hand, considering the group  $G$  as a category with a single object  $*$ , any  $G$ -set determines a contravariant functor  $X : G \rightarrow \text{Cat}$  and the Grothendieck construction leads to the category  $G \int X$  such that  $\text{Ob}(G \int X) = X$  and morphisms  $x \rightarrow y$  are given by the set  $\{g \in G; gy = x\}$  for  $x, y \in X$ .

Given a covariant  $G \int X$ -system  $\mathcal{X}$  of  $G$ -sets, let  $\mathbf{F}\mathcal{X}$  be the contravariant projective  $\tilde{\mathcal{O}}(G, H)$ -system of abelian groups defined as in [19] by  $\mathbf{F}\mathcal{X}(G/H, x) = F(\mathcal{X}(x)^H)$ , where  $F(\mathcal{X}(x)^H)$  is the free abelian group generated by the set  $\mathcal{X}(x)^H$  for any  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$ . Because in the category of contravariant  $\tilde{\mathcal{O}}(G, X)$ -systems of abelian groups there are enough projectives, so the  $\tilde{\mathcal{O}}(G, X)$ -system  $\pi$  of abelian groups leads to a projective resolution

$$\cdots \longrightarrow \mathbf{F}\mathcal{X}_n \longrightarrow \cdots \longrightarrow \mathbf{F}\mathcal{X}_1 \longrightarrow \mathbf{F}\mathcal{X}_0 \longrightarrow \pi.$$

Then, to construct the Eilenberg-MacLane  $K(\pi, n)$ , we follow [19] and replace cells and spheres by standard simplexes and their boundaries, respectively. We stress the analogy between such a construction of an

Eilenberg-MacLane  $G$ -simplicial set and the algebraic construction in the previous section to define an  $i$ -elementary extension.

Let  $X$  and  $Y$  be  $G$ -simplicial sets with the same sets of 0-simplexes,  $X' \subseteq X$  a  $G$ -simplicial subset,  $X^{(n)}$  the  $n$ -skeleton of  $X$  for  $n \geq 1$  and let  $Y$  be a Kan set as well. Suppose  $\varphi : X^{(n)} \cup X' \rightarrow Y$  is a  $G$ -map,  $x \in X$  is an  $(n + 1)$ -simplex and  $G_x \subseteq G$  its isotropy group, and let  $x \in X_{x_0}^{G_x}$  for some 0-simplex  $x_0$ . If  $Y_{\varphi(x_0)}^{G_x}$  is  $n$ -simple and  $\dot{x}$  denote the boundary of  $x$  then  $\varphi(\dot{x})$  represents an element in  $\pi_n(Y_{x_0}^{G_x}, x_0)$ . Consequently,  $\varphi$  defines  $c_\varphi \in \text{Hom}(\mathcal{C}_{n+1}(X' \hookrightarrow X), \pi_n(Y))$ . As in [19] we can prove that this is a cocycle defining a cohomology class

$$[c_\varphi] \in H_G^{n+1}(X' \hookrightarrow X, \pi_n(Y)),$$

which is the *obstruction* to extending the restriction  $\varphi|_{X^{(n-1)} \cup X'}$  to a  $G$ -map  $X^{(n+1)} \cup X' \rightarrow Y$ .

A pointed map  $f : X \rightarrow Y$  of  $G$ -simplicial sets is called *constant* if it factors through the constant  $G$ -simplicial set  $X_0$ . Given an Eilenberg-MacLane  $G$ -simplicial set  $K(\pi, n)$  with a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system  $\pi$  of abelian groups and  $n \geq 1$ , consider the constant  $G$ -map  $\theta : X \xrightarrow{\delta} X_0 \xrightarrow{\sigma} K(\pi, n)$ . Then, we are ready to apply the obstruction theory above and derive an isomorphism

$$H^n(X, \pi) \xrightarrow{\cong} [X, K(\pi, n)]_G^{X_0}$$

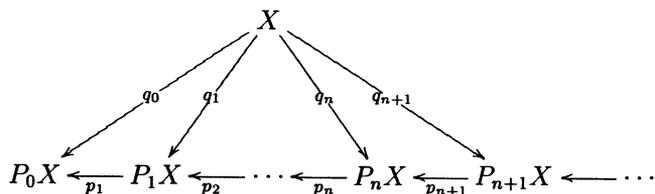
for  $n \geq 1$  the proof of which offers no difficulties, where  $[X, K(\pi, n)]_G^{X_0}$  denotes the set of  $G$ -homotopy classes of pointed maps from  $X$  to  $K(\pi, n)$ .

POSTNIKOV TOWER AND RATIONALIZATION OF  $G$ -SIMPLICIAL SETS. Let  $X_k$  be the  $G$ -set of  $k$ -simplexes of a  $G$ -simplicial set  $X$  for all  $k \geq 0$ . For each  $n \geq 0$  define an equivalence relation  $R_n$  on  $X_k$  by  $xR_n y$  if each face of  $x$  with dimension  $\leq n$  agrees with the corresponding face of  $y$ . Then we can define a simplicial set  $P_n X$  by

$$(P_n X)_k = X_k / R_n$$

for all  $k \geq 0$ , where the face and degeneracy operations are induced from those of  $X$ . Note that the  $G$ -structure on  $X$  determines such a structure

on each  $P_n X$  for  $n \geq 0$ . Write  $q_n : X \rightarrow P_n X$  and  $p_n : P_n X \rightarrow P_{n-1} X$  for the pointed natural  $G$ -maps. Then the diagram



is called the Postnikov tower of a  $G$ -simplicial set  $X$  and  $P_n X$  the  $n$ th stage of this tower for  $n \geq 0$ . We can follow *mutatis mutandis* the non-equivariant results [16, Proposition 8.2 and Theorem 8.4] to show

PROPOSITION 2.5. — *Let  $X$  be a Kan  $G$ -simplicial set. Then*

- (1) each  $q_n : X \rightarrow P_n X$  is a Kan  $G$ -fibration;
- (2) for each  $n \geq m$  the natural map  $P_n X \rightarrow P_m X$  is a Kan  $G$ -fibration;
- (3) each  $P_n X$  is a Kan  $G$ -simplicial set;
- (4)  $\pi_l(P_n X) = 0$  for  $l \geq n$ ;
- (5) each  $q_n : X \rightarrow P_n X$  yields an isomorphism  $\pi_l(q_n) : \pi_l(X) \xrightarrow{\cong} \pi_l(P_n X)$  for  $l \leq n$ .

The action  $\pi_1(X_x^H, x) \times \pi_n(X_x^H, x) \rightarrow \pi_n(X_x^H, x)$  defines a natural transformation

$$\pi_1(X) \times \pi_n(X) \rightarrow \pi_n(X).$$

The functoriality of the lower central series of a group allows to define inductively functors

$$\Gamma_l \pi_n(X) : \tilde{\mathcal{O}}(G, X) \rightarrow \mathbb{G}p$$

for all  $l \geq 0$  as follows:

$$\Gamma_0 \pi_n(X) = \pi_n(X)$$

and

$$\Gamma_{l+1} \pi_n(X)(G/H, x) = \{\gamma - \alpha\gamma ; \gamma \in \Gamma_l \pi_n(X)(G/H, x), \alpha \in \pi_1(X_x^H, x)\}.$$

This yields a decreasing filtration

$$\pi_n(X) = \Gamma_0\pi_n(X) \supseteq \Gamma_1\pi_n(X) \supseteq \Gamma_2\pi_n(X) \supseteq \dots$$

of  $\pi_n(X)$  for all  $n \geq 1$  and short exact sequences

$$0 \rightarrow \Gamma_l\pi_n(X)/\Gamma_{l+1}\pi_n(X) \rightarrow \pi_n(X)/\Gamma_{l+1}\pi_n(X) \rightarrow \pi_n(X)/\Gamma_l\pi_n(X) \rightarrow 0$$

with the trivial action of  $\pi_1(X)$  on  $\Gamma_l\pi_n(X)/\Gamma_{l+1}\pi_n(X)$  for  $l \geq 0$  and  $n \geq 1$ . We say that  $G$ -simplicial set is *nilpotent* if for any  $n \geq 1$  there is a sufficiently large integer  $l_n$  with  $\Gamma_{l_n}\pi_n(X) = 0$ .

To state the next result we need to recall the notion of a *simplicial set of paths*. Let  $X$  be a Kan simplicial set, with the base point  $*$  and the face operations  $d_i$  and the degeneracy operations  $s_i$ , then the simplicial set of paths  $\mathcal{P}_*X$  is to be

$$(\mathcal{P}_*X)_n = \{x \in X_{n+1}; d_1 \cdots d_{n+1}x = *\},$$

and for each  $0 \leq i \leq n$ ,  $d_i$  on  $(\mathcal{P}_*X)_n$  is to be the restriction of  $d_{i+1}$  on  $X_{n+1}$ , and similarly for each  $s_i$ . Then the map  $p_* : \mathcal{P}_*X \rightarrow X$  such that  $p_*(x) = d_0x$  for  $x \in (\mathcal{P}_*X)_n$  is a Kan fibration.

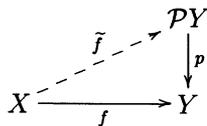
Let now  $X$  be a  $G$ -simplicial set and  $x_0 \in X_0$ . Then any element  $g \in G$  induces a simplicial map  $\mathcal{P}_{x_0}X \rightarrow \mathcal{P}_{gx_0}X$ . Hence on the simplicial set

$$\mathcal{P}X = \coprod_{x_0 \in X_0} \mathcal{P}_{x_0}X$$

a  $G$ -structure is defined and the Kan fibrations  $p_{x_0} : \mathcal{P}_{x_0}X \rightarrow X$  give rise to a Kan  $G$ -fibration

$$p : \mathcal{P}X \rightarrow X.$$

Then a map  $f : X \rightarrow Y$  of pointed  $G$ -simplicial sets is  $G$ -homotopic to the constant map  $\theta : X \xrightarrow{\delta} X_0 \xrightarrow{\sigma} Y$  if and only if there is a map  $\tilde{f} : X \rightarrow \mathcal{P}Y$  such that the diagram



is commutative.

A pointed Kan  $G$ -fibration  $f : X \rightarrow Y$  is called *principal* of type  $(\pi, n)$  if it is induced by a classifying map  $k : Y \rightarrow K(\pi, n + 1)$ , i.e., there is a pullback

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{P}K(\pi, n + 1) \\ f \downarrow & & \downarrow p \\ Y & \xrightarrow{k} & K(\pi, n + 1), \end{array}$$

where  $\pi$  is a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system of groups and  $n \geq 1$ . So a pointed Kan  $G$ -fibration  $f : X \rightarrow Y$  of type  $(\pi, n)$  is characterized by a cohomology class  $[k] \in H^{n+1}(Y, \pi)$ .

PROPOSITION 2.6. — *If  $X$  is a Kan nilpotent  $G$ -simplicial set and*

$$\begin{array}{ccccccc} & & X & & & & \\ & \swarrow & & \searrow & & & \\ & q_0 & & q_1 & & q_n & q_{n+1} \\ & & & & & & \\ P_0 X & \longleftarrow & P_1 X & \longleftarrow & \dots & \longleftarrow & P_n X & \longleftarrow & P_{n+1} X & \longleftarrow & \dots \end{array}$$

is its Postnikov tower then the Kan  $G$ -fibration  $p_n : P_n X \rightarrow P_{n-1} X$  admits a refinement

$$P_n X = P_{n, l_n} X \rightarrow P_{n, l_n - 1} X \rightarrow \dots \rightarrow P_{n, 1} X \rightarrow P_{n, 0} X = P_{n-1} X,$$

where  $P_{n, l} X \rightarrow P_{n, l-1} X$  is a principal  $G$ -fibration with the fibre

$$K(\Gamma_{l-1} \pi_n(X) / \Gamma_l \pi_n(X), n + 1)$$

for  $l = 1, \dots, l_n$  and  $n \geq 1$ .

*Proof.* — By Proposition 2.5 the map  $p_n : P_n X \rightarrow P_{n-1} X$  is a Kan  $G$ -fibration with the fibre  $K(\pi_n(X), n)$  for  $n \geq 0$ . Suppose that  $\Gamma_l \pi_n(X) = 0$  for  $l > l_n$ . From the non-equivariant relative Hurewicz Theorem we deduce that  $p_n : P_n X \rightarrow P_{n-1} X$  leads to a natural isomorphism

$$\mathbf{h}_{n+1} : \pi_n(X) / \Gamma_1 \pi_n(X) \xrightarrow{\cong} \mathbf{H}_{n+1}(p_n : P_n X \rightarrow P_{n-1} X, \mathbb{Z}).$$

Then, in virtue of the relative cohomology spectral sequence

$$\begin{aligned} E_2^{p, q} &= \text{Ext}^p(\mathbf{H}_q(p_n : P_n X \rightarrow P_{n-1} X, \mathbb{Z}), \pi_n(X) / \Gamma_1 \pi_n(X)) \\ &\implies H_G^{p+q}(p_n : P_n X \rightarrow P_{n-1} X, \pi_n(X) / \Gamma_1 \pi_n(X)), \end{aligned}$$

we see that  $\mathbf{h}_{n+1}^{-1}$  may be regarded as an element of the relative cohomology group  $H_G^{n+1}(p_n : P_n X \rightarrow P_{n-1} X, \pi_n(X)/\Gamma_1 \pi_n(X))$ . From the cohomology long exact sequence

$$\begin{aligned} \cdots &\rightarrow H_G^{n+1}(p_n : P_n X \rightarrow P_{n-1} X, \pi_n(X)/\Gamma_1 \pi_n(X)) \\ &\rightarrow H_G^{n+1}(P_{n-1} X, \pi_n(X)/\Gamma_1 \pi_n(X)) = [P_{n-1} X, K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1)]_G^{X_0} \\ &\xrightarrow{p_n^*} H_G^{n+1}(P_n X, \pi_n(X)/\Gamma_1 \pi_n(X)) \rightarrow \cdots \end{aligned}$$

it follows that  $\mathbf{h}_{n+1}^{-1}$  gives rise to a pointed map

$$k_{n,0} : P_{n-1} X \rightarrow K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1)$$

such that  $k_{n,1} p_n : P_{n-1} X \rightarrow K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1)$  is  $G$ -homotopic to the constant map  $\theta : P_{n-1} X \rightarrow K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1)$ . Then the pullback

$$\begin{array}{ccc} P_{n,1} X & \longrightarrow & \mathcal{P}K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1) \\ p_{n,1} \downarrow & & \downarrow \\ P_{n-1} X & \xrightarrow{k_{n,0}} & K(\pi_n(X)/\Gamma_1 \pi_n(X), n+1) \end{array}$$

determines a factorization  $p_n : P_n X \xrightarrow{r_{n,1}} P_{n,1} X \xrightarrow{p_{n,1}} P_{n-1} X$ , the long homotopy sequence of which reduces to

$$0 \rightarrow \Gamma_2 \pi_n(X) \rightarrow \pi_n(X) \rightarrow \pi_n(X)/\Gamma_2 \pi_n(X) \rightarrow 0.$$

Hence we may repeat the procedure above, with  $r_{n,1}$  replacing  $p_n$ , and, continuing in this way, we reach

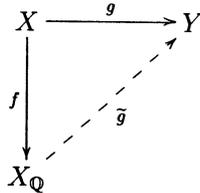
$$P_n X \xrightarrow{r_{n,l_n}} P_{n,l_n} X \xrightarrow{p_{n,l_n}} P_{n,l_n-1} X \rightarrow \cdots \xrightarrow{p_{n,2}} P_{n,1} X \xrightarrow{p_{n,1}} P_{n,0} X = P_{n-1} X$$

each  $p_{n,l} : P_{n,l} X \rightarrow P_{n,l-1} X$  being a principal  $G$ -fibration induced by  $k_{n,l-1} : P_{n,l-1} X \rightarrow K(\Gamma_{l-1} \pi_n(X)/\Gamma_l \pi_n(X), n+1)$ . However, the homotopy groups  $\pi_m(r_{n,l_n})$  vanish for all  $m \geq 0$ , so that  $r_{n,l_n} : P_n X \rightarrow P_{n,l_n} X$  is a  $G$ -homotopy equivalence and we have proved the existence of the required refinement at the stage  $n$ . □

*Remark 2.7.* — It can be also shown that any Kan  $G$ -simplicial set the Postnikov tower of which admits a principal refinement is nilpotent.

**THE MAIN RESULT.** We say that a Kan nilpotent  $G$ -simplicial set  $X$  is *rational* if the homotopy groups  $\pi_n(X^H, x)$  are uniquely divisible for any

subgroup  $H \subseteq G$ ,  $x \in X_0^H$  and  $n \geq 1$ . Note that for  $n \geq 2$  this means that  $\pi_n(X^H, x)$  are  $\mathbb{Q}$ -modules. We say that a pointed map  $f : X \rightarrow X_{\mathbb{Q}}$  of nilpotent  $G$ -simplicial sets is a  $G$ -rationalization of  $X$  if  $X_{\mathbb{Q}}$  is rational and for every map  $g : X \rightarrow Y$  of nilpotent Kan  $G$ -simplicial sets with  $Y$  rational there is a (unique up to  $G$ -homotopy) map  $\tilde{g} : X_{\mathbb{Q}} \rightarrow Y$  such that the diagram

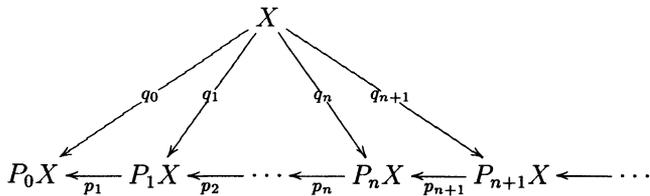


commutes (up to  $G$ -homotopy). Given a nilpotent Kan  $G$ -simplicial set  $X$  we follow [19], [20] to sketch how the principal refinement of the Postnikov tower of  $X$  leads to its  $G$ -rationalization.

PROPOSITION 2.8. — *Let  $X$  be a nilpotent Kan  $G$ -simplicial set  $X$ . Then there is a rationalization  $f : X \rightarrow X_{\mathbb{Q}}$  and, for a pointed map  $f : X \rightarrow Y$  of nilpotent Kan  $G$ -simplicial sets with  $Y$  rational, the following statements are equivalent:*

- (1) *the map  $f : X \rightarrow Y$  is a  $G$ -rationalization of  $X$ ;*
- (2)  *$H_G^n(f, \mathbf{M}) : H_G^n(Y, \mathbf{M}) \rightarrow H_G^n(X, \mathbf{M})$  is an isomorphism for all left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -modules  $\mathbf{M}$  and  $n \geq 0$ ;*
- (3)  *$H_n^G(f, \mathbf{M}) : H_n^G(X, \mathbf{M}) \rightarrow H_n^G(Y, \mathbf{M})$  is an isomorphism for all right  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -modules  $\mathbf{M}$  and  $n \geq 0$ ;*
- (4)  *$\mathbf{H}_n(f, \mathbb{Q}) : \mathbf{H}_n(X, \mathbb{Q}) \rightarrow \mathbf{H}_n(Y, \mathbb{Z})$  is a isomorphism for  $n \geq 0$ ;*
- (5)  *$\pi_n(f) : \pi_n(X) \otimes \mathbb{Q} \rightarrow \pi_n(Y)$  is an isomorphism for  $n \geq 0$ ;*

*Proof.* — Given a nilpotent Kan  $G$ -simplicial set, consider its Postnikov tower



and the principal refinement

$$P_n X = P_{n,l_n} X \rightarrow P_{n,l_n-1} X \rightarrow \cdots \rightarrow P_{n,1} X \rightarrow P_{n,0} X = P_{n-1} X$$

of the Kan  $G$ -fibration  $p_n : P_n X \rightarrow P_{n-1} X$ .

We construct inductively  $G$ -rationalizations  $f_{n,l} : P_{n,l} X \rightarrow (P_{n,l} X)_{\mathbb{Q}}$  for  $n \geq 1$  and  $0 \leq l \leq l_n$ . Because  $P_{1,1} X = K(\pi_1(X)/\Gamma_1 \pi_1(X), 1)$ , we define  $(P_{1,1} X)_{\mathbb{Q}} = K((\pi_1(X)/\Gamma_1 \pi_1(X)) \otimes \mathbb{Q}, 1)$ . Then the canonical map

$$\pi_1(X)/\Gamma_1 \pi_1(X) \rightarrow (\pi_1(X)/\Gamma_1 \pi_1(X)) \otimes \mathbb{Q}$$

of  $\tilde{\mathcal{O}}(G, X)$ -systems of abelian groups induces the rationalization  $f_{1,1} : P_{1,1} X \rightarrow (P_{1,1} X)_{\mathbb{Q}}$ . Given a  $G$ -rationalization  $f_{n,l} : P_{n,l} X \rightarrow (P_{n,l} X)_{\mathbb{Q}}$  for some  $n \geq 1$  and  $l \geq 0$ , we first consider the  $G$ -map

$$K(\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X), n+1) \longrightarrow K((\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X)) \otimes \mathbb{Q}, n+1)$$

induced by the canonical map

$$\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X) \longrightarrow (\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X)) \otimes \mathbb{Q}$$

of  $\tilde{\mathcal{O}}(G, X)$ -systems of abelian groups. Then the classifying map  $k_{n,l} : P_{n,l} X \rightarrow K(\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X), n+1)$  for the principal Kan  $G$ -fibration  $p_{n,l+1} : P_{n,l+1} X \rightarrow P_{n,l} X$  yields a commutative (up to  $G$ -homotopy) diagram

$$\begin{array}{ccc} P_{n,l} X & \xrightarrow{f_{n,l}} & (P_{n,l} X)_{\mathbb{Q}} \\ \downarrow k_{n,l} & & \downarrow (k_{n,l})_{\mathbb{Q}} \\ K(\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X), n+1) & \longrightarrow & K((\Gamma_l \pi_n(X)/\Gamma_{l+1} \pi_n(X)) \otimes \mathbb{Q}, n+1). \end{array}$$

Let  $(p_{n,l+1})_{\mathbb{Q}} : (P_{n,l+1} X)_{\mathbb{Q}} \rightarrow (P_{n,l} X)_{\mathbb{Q}}$  be the principal Kan  $G$ -fibration determined by the map  $(k_{n,l})_{\mathbb{Q}}$ . Because the composed  $G$ -map

$$(k_{n,l})_{\mathbb{Q}} f_{n,l} p_{n,l+1} : X_{n,l+1} \rightarrow K(\Gamma_l \pi_n/\Gamma_{l+1} \pi_n(X), n+1)$$

is  $G$ -homotopic to the constant  $G$ -map, so there is a rationalization

$$f_{n,l+1} : P_{n,l+1} X \rightarrow (P_{n,l+1} X)_{\mathbb{Q}}$$

and such that  $(p_{n,l+1})_{\mathbb{Q}} f_{n,l+1} = f_{n,l} p_{n,l+1}$ .

Finally, passing to the inverse limit, the maps  $f_{n,l} : P_{n,l}X \rightarrow (P_{n,l}X)_{\mathbb{Q}}$  give rise to the required  $G$ -rationalization

$$f : X \longrightarrow \varprojlim_{n,l} P_{n,l}X \longrightarrow X_{\mathbb{Q}} = \varprojlim_{n,l} (P_{n,l}X)_{\mathbb{Q}}.$$

For the proof of its universal property we mimic the non-equivariant case (cf. e.g., [12, Theorem 2A] or [14, Theorem 1.4]).

Let now consider (1)  $\implies$  (2). Given a left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}$ , by means of the universal property of the  $G$ -rationalization  $f : X \rightarrow Y$ , the induced map  $H_G^n(f, \mathbf{M}) : H_G^n(Y, \mathbf{M}) = [Y, K(\mathbf{M}, n)]_{X_0}^G \rightarrow H_G^n(X, \mathbf{M}) = [X, K(\mathbf{M}, n)]_{X_0}^G$  is an isomorphism for all  $n \geq 0$ . To show (2)  $\implies$  (1) we can make use of the principal refinement of the Postnikov tower of  $X$ , the representability of cohomology by Eilenberg-MacLane  $G$ -simplicial sets and then follow the non-equivariant procedure (see e.g., [14, Theorem 1.3]).

The implications (4)  $\implies$  (2) and (4)  $\implies$  (3) follow from cohomology and homology spectral sequences. For the proof of (2)  $\implies$  (4) and (3)  $\implies$  (4) we take for  $\mathbf{M}$  the Yoneda  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -modules  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)(-, (G/H, x))$  and its dual  $(\tilde{\mathcal{O}}(G, X)((G/H, x), -), \mathbb{Q})$ , respectively for  $(G/H, x) \in \text{Ob}(\tilde{\mathcal{O}}(G, X))$ . The equivalence (4)  $\iff$  (5) is a direct consequence of results (cf. e.g., [12, Theorem 3B]) on the non-equivariant rationalization of nilpotent spaces. □

To formulate the main result containing the heart of the matter and relating the rational homotopy with  $i$ -minimal models, we show as in [17, Theorem 4.15 and Theorem 4.16] how the algebraic structure of the  $i$ -minimal model of a nilpotent Kan  $G$ -simplicial set corresponds to the structure of its Postnikov tower. The first step is to model the basic pieces which compose the tower, the Eilenberg-MacLane  $G$ -simplicial set and then pointed principal Kan  $G$ -fibrations as well.

For a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system  $\pi$  of  $\mathbb{Q}$ -modules, let  $\pi^*$  be its topological dual determined by the linearly compact  $\mathbb{Q}$ -modules given by  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)(\pi(G/H, x), \mathbb{Q})$  for  $G/H, x \in \tilde{\mathcal{O}}(G, X)$ . Write  $\underline{\mathbb{Q}}_{\tau}(\pi^*)$  for the  $i$ -elementary extension of the constant  $\mathbb{Q}\hat{\mathcal{O}}(G, X)$ -algebra  $\underline{\mathbb{Q}}$  with respect to the trivial map  $\tau : \pi^* \rightarrow Z^{n+1}\underline{\mathbb{Q}} = 0$  for  $n \geq 1$ . Given a nilpotent  $G$ -simplicial set  $X$  we say that a  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}$  is *geometric* for  $X$  if there is a quasi-isomorphism  $\mathcal{A} \rightarrow \mathcal{A}_X$ .

LEMMA 2.9 (The Hirsch Lemma). — *Let  $X$  be a nilpotent  $G$ -simplicial set, let a  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebra  $\mathcal{A}$  be geometric for  $X$  and  $\pi$  a contravariant  $\tilde{\mathcal{O}}(G, X)$ -system of  $\mathbb{Q}$ -modules. Then,*

(1) the set  $[X, K(\boldsymbol{\pi}, n)]_{X_0}^G$  of homotopy classes of  $G$ -maps is in the one-to-one correspondence with the set of homotopy classes  $[\underline{\mathbb{Q}}_\tau(\boldsymbol{\pi}^*), \mathcal{A}]$  of  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -maps;

(2) the set of isomorphism classes of pointed principal Kan  $G$ -fibrations  $Y \rightarrow X$  of type  $(\boldsymbol{\pi}, n)$  is in one-to-one correspondence with the set of isomorphism classes of  $i$ -elementary extensions  $\mathcal{A}_\tau(\boldsymbol{\pi}^*)$  and the map  $X \rightarrow K(\boldsymbol{\pi}, n + 1)$  classifying  $X \rightarrow Y$  corresponds to  $\tau : \boldsymbol{\pi}^* \rightarrow Z^{n+1}\mathcal{A}$ . Moreover,  $\mathcal{A}_\tau(\boldsymbol{\pi}^*)$  is geometric for  $Y$ .

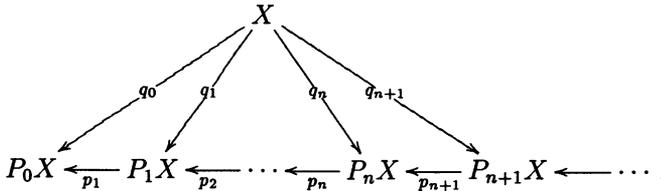
*Proof.* — For (1) consider the minimal injective resolution

$$0 \longrightarrow \boldsymbol{\pi}^* \longrightarrow \mathbf{M}_0 \longrightarrow \mathbf{M}_1 \longrightarrow \dots$$

We know that the set  $[X, K(\boldsymbol{\pi}, n)]_{X_0}^G$  corresponds to  $H^n(X, \boldsymbol{\pi}) \xrightarrow{\cong} H^n(\mathcal{A}, \boldsymbol{\pi}^*) \xrightarrow{\cong} H^{n+1}(\underline{\mathbb{Q}} \rightarrow \mathcal{A}, \boldsymbol{\pi}^*)$  and by Lemma 1.6 the relative cohomology group  $H^{n+1}(\underline{\mathbb{Q}} \rightarrow \mathcal{A}, \boldsymbol{\pi}^*)$  corresponds to the set of homotopy classes  $[\underline{\mathbb{Q}}_\tau(\boldsymbol{\pi}^*), \mathcal{A}]$ .

Part (2) is a direct consequence of (1). □

Now we may show how the  $i$ -minimal model of a nilpotent Kan  $G$ -simplicial set  $X$  captures all of the rational homotopy information contained in its Postnikov tower



Let

$$P_n X = P_{n,l_n} X \rightarrow P_{n,l_n-1} X \rightarrow \dots \rightarrow P_{n,1} X \rightarrow P_{n,0} X = P_{n-1} X$$

be the principal refinement of the pointed Kan  $G$ -fibration  $p_n : P_n X \rightarrow P_{n-1} X$ , where  $p_{n,l} : P_{n,l} X \rightarrow P_{n,l-1} X$  is a principal  $G$ -fibration of type  $(\Gamma_{l-1}\boldsymbol{\pi}_n(X)/\Gamma_l\boldsymbol{\pi}_n(X), n + 1)$  for  $l = 1, \dots, l_n$  and  $n \geq 1$ . The  $G$ -simplicial set  $P_{n,l} X$  is called the  $(n, l)$ -stage of this tower.

Write  $\rho_X : \mathcal{M}_X \rightarrow \mathcal{A}_X$  for the  $i$ -minimal model of  $X$  given by Theorem 2.4, where  $\mathcal{M}_X = \bigcup_{n \geq 0} \mathcal{M}_X(n)$ ,  $\mathcal{M}_X(0) = \mathbb{Q}$ ,  $\mathcal{M}_X(n) = \bigcup_{l \geq 0} \mathcal{M}_X(n, l)$  and  $\mathcal{M}_X(n, l + 1) = \mathcal{M}_X(n, l)_{\tau_{n,l}}(\mathbf{M}_{n,l})$  is an  $i$ -elementary extension of  $\mathcal{M}_X(n, l)$  with respect to a map  $\mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}_X(n, l)$  of left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -modules for  $l \geq 0$  and  $n \geq 1$ .

**THEOREM 2.10.** — *Let  $X$  be a nilpotent Kan  $G$ -simplicial set and  $\rho_X : \mathcal{M}_X \rightarrow \mathcal{A}_X$  its  $i$ -minimal model. Then the correspondence  $X \mapsto \mathcal{M}_X$  is a bijection from equivariant rational homotopy types of nilpotent Kan  $G$ -simplicial sets to isomorphisms classes of  $i$ -minimal  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -algebras, and  $\mathcal{M}_X$  encodes the rational homotopy information via the following natural isomorphisms:*

(1)  $H_G^n(X, \mathbf{M}) \cong H^n(\mathcal{A}_X, \mathbf{M}^*) \cong H^n(\mathcal{M}_X, \mathbf{M}^*)$  for all  $n \geq 0$ , where  $\mathbf{M}$  is a right  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module and  $\mathbf{M}^*$  its dual linearly compact left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module;

(2)  $\mathbf{H}^n(X, \mathbb{Q}) \cong \mathbf{H}^n(\mathcal{A}_X) \cong \mathbf{H}^n(\mathcal{M}_X)$  for all  $n \geq 0$ ;

(3) the map  $k_{n,l} : P_{n,l}X \rightarrow K(\Gamma_l\pi_n(X)/\Gamma_{l+1}\pi_n(X), n+1)$  classifying the pointed Kan  $G$ -fibration  $p_{n,l+1} : P_{n,l+1}X \rightarrow P_{n,l}X$  corresponds to  $\tau_{n,l} : \mathbf{M}_{n,l} \rightarrow Z^{n+1}\mathcal{M}_X(n, l)$  determining the  $i$ -elementary extension  $\mathcal{M}_X(n, l+1) = \mathcal{M}_X(n, l)(\mathbf{M}_{n,l})$  and there is an isomorphism

$$(\Gamma_l\pi_n(X)/\Gamma_{l+1}\pi_n(X)) \otimes \mathbb{Q} \xrightarrow{\cong} (\mathbf{M}_{n,l})^*,$$

where  $(\mathbf{M}_{n,l})^*$  is the topological dual of the linearly compact left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module  $\mathbf{M}_{n,l}$  for all  $l, n \geq 1$ ;

(4)  $\mathcal{M}_X(n, l)$  is the  $i$ -minimal model for the  $(n, l)$ -stage  $P_{n,l}X$  of the Postnikov tower of  $X$ .

*Proof.* — The proof that the correspondence  $X \mapsto \mathcal{M}_X$  is a bijection, is based on Lemma 2.9 (2).

To show (1) consider the quasi-isomorphism  $\rho_X : \mathcal{M}_X \rightarrow \mathcal{A}_X$ . By means of cohomology spectral sequences, this leads to isomorphisms

$$H_G^n(X, \mathbf{M}) \cong H^n(\mathcal{A}_X, \mathbf{M}^*) \cong H^n(\mathcal{M}_X, \mathbf{M}^*)$$

for all  $n \geq 0$ , where  $\mathbf{M}$  is a right  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module and  $\mathbf{M}^*$  its dual linearly compact left  $\mathbb{Q}\tilde{\mathcal{O}}(G, X)$ -module.

The maps  $\mathbf{H}^n(X, \mathbb{Q}) \xrightarrow{\cong} \mathbf{H}^n(\mathcal{A}_X) \xrightarrow{\cong} \mathbf{H}^n(\mathcal{M}_X)$  are isomorphisms for all  $n \geq 0$  by the de Rham Theorem [1, Theorem 2.2] and (2) follows.

Given  $\mathcal{M}_X(n, l)$  geometric for  $P_{n,l}X$  we consider the pointed principal Kan  $G$ -fibration  $p_{n,l+1} : P_{n,l+1}X \rightarrow P_{n,l}X$  to get, in the light of Lemma 2.9, the  $i$ -elementary extension  $\mathcal{M}_X(n, l+1) = \mathcal{M}_X(n, l)(\mathbf{M}_{n,l})$  with  $\mathbf{M}_{n,l} = (\Gamma_l\pi_n(X)/\Gamma_{l+1}\pi_n(X)) \otimes \mathbb{Q}$ , so (3)-(4) are shown and the proof is complete. □

(1) We plan to apply our new  $i$ -minimal models to show equivariant formality (over the rationals  $\mathbb{Q}$ ) of disconnected compact Kähler  $G$ -manifolds and their holomorphic maps, results aimed at in [5], [6].

(2) We say that a Kan  $G$ -simplicial set  $X$  is *complete* if the canonical map

$$\pi_n(X) \longrightarrow \varprojlim_l \pi_n(X)/\Gamma_l \pi_n(X)$$

is an isomorphism for all  $n \geq 1$ . Of course, any nilpotent Kan  $G$ -simplicial set is complete. The results above can be generalized on complete  $G$ -simplicial sets as well. The rationalization of Kan  $G$ -simplicial sets must be then replaced by their  $\mathbb{Q}$ -completion.

(3) Let  $\mathbb{T}$  be the circle group. We point out that a smooth version of our methods might be extend the algebraization [17] of rational  $\mathbb{T}$ -equivariant homotopy theory not only to nilpotent but even to complete disconnected  $\mathbb{T}$ -spaces.

## BIBLIOGRAPHY

- [1] A.K. BOUSFIELD & V.K.A.M. GUGENHEIM, On PL de Rham theory and rational homotopy type, *Memories of the Amer. Math. Soc.*, 179 (1976).
- [2] G. BREDON, *Equivariant Cohomology Theories*, *Lecture Notes in Math.*, vol. 34 (1967).
- [3] J.-M. CORDIER & T. PORTER, Homotopy coherent category theory, *Trans. Amer. Math. Soc.*, 349 (1) (1997), 1-54.
- [4] P. DELIGNE, P. GRIFFITHS, J. MORGAN & D. SULLIVAN, Real homotopy theory of Kähler manifolds, *Invent. Math.*, vol. 29 (1975), 245-274.
- [5] B.L. FINE, *Disconnected equivariant rational homotopy theory and formality of compact  $G$ -Kähler manifolds*, Ph.D. thesis, Chicago (1992).
- [6] B.L. FINE & G.V. TRIANTAFILLOU, On the equivariant formality of Kähler manifolds with finite group action, *Can. J. Math.*, 45 (1993), 1200-1210.
- [7] M. GOLASIŃSKI, Injective models of  $G$ -disconnected simplicial sets, *Ann. Inst. Fourier, Grenoble*, 47-5 (1997), 1491-1522.
- [8] M. GOLASIŃSKI, Injectivity of functors to modules and  $DGA'$ s, *Comm. Algebra*, 27 (1999), 4027-4038.
- [9] M. GOLASIŃSKI, On the object-wise tensor product of functors to modules, *Theory Appl. Categ.*, vol. 7, n° 1 (2000), 227-235.
- [10] M. GOLASIŃSKI, Component-wise injective models of functors to  $DGAs$ , *Colloq. Math.*, 73 (1997), 83-92.
- [11] S. HALPERIN, *Lectures on minimal models*, *Mémoires S.M.F., Nouvelle série* 9-10 (1983).
- [12] P. HILTON, G. MISLIN & J. ROITBERG, *Localization of Nilpotent Groups and Spaces*, *Noth-Holland Mathematics Studies* 15, Amsterdam (1975).
- [13] S. LEFSCHETZ, *Algebraic Topology*, *Amer. Math. Soc. Colloq. Publ.*, vol. XXVII (1942).

- [14] D. LEHMANN, Théorie homotopique des formes différentielles, Astérisque 45, S.M.F. (1977).
- [15] W. LÜCK, Transformation groups and Algebraic K-Theory, Lect. Notes in Math. 1408, Springer-Verlag, (1989).
- [16] J.P. MAY, Simplicial Objects in Algebraic Topology, Van Nostrand Mathematical Studies no. 11, Princeton-Toronto-London-Melbourne (1967).
- [17] L.S. SCULL, Rational  $S^1$ -equivariant homotopy theory, Trans. Amer. Math. Soc., 354 (2002), 1-45.
- [18] D. SULLIVAN, Infinitesimal Computations in Topology, Publ. Math. I.H.E.S., 47 (1977), 269-331.
- [19] G.V. TRIANTAFILLOU, Equivariant minimal models, Trans. Amer. Math. Soc., 274 (1982), 509-532.
- [20] G.V. TRIANTAFILLOU, An algebraic model for  $G$ -homotopy types, Astérisque, 113-114 (1984), 312-337.

Manuscrit reçu le 27 septembre 2001,  
révisé le 22 avril 2002,  
accepté le 10 septembre 2002.

Marek GOLASIŃSKI,  
Nicholas Copernicus University  
Faculty of Mathematics and Computer Science  
Chopina 12/18  
87-100 Toruń (Poland).  
marek@mat.uni.torun.pl