ANNALES

DE

L'INSTITUT FOURIER

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Tome 52, n^{o} 6 (2002), p. 1785-1792.

<http://aif.cedram.org/item?id=AIF_2002__52_6_1785_0>

Ann. Inst. Fourier, Grenoble
52, 6 (2002), 1785–1792

OBSTRUCTIONS TO GENERIC EMBEDDINGS

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In Grauert's paper [G] it is noted that finite dimensionality of cohomology groups sometimes implies vanishing of these cohomology groups. Later on Laufer formulated a zero or infinity law for the cohomology groups of domains in Stein manifolds. In this paper we generalize Laufer's Theorem in [L] and its version for small domains of CR manifolds, proved in [Br], by considering Whitney cohomology on locally closed subsets and cohomology with supports for currents. With this approach we obtain a global result for CR manifolds generically embedded in a Stein manifold. Namely a necessary condition for global embedding into an open subset of a Stein manifold is that the $\bar{\partial}_M$-cohomology groups must be either zero or infinite dimensional.

1. An abstract Laufer Theorem.

Let X be a Stein manifold of complex dimension N. Let F be a locally closed subset of X. This means that F is a closed subset of an open submanifold Y of X. We denote by $\mathfrak{W}_F$ the space of *Whitney functions* on F. With $\mathfrak{F}(Y;F)$ denoting the subspace of the space $\mathcal{E}(Y)$ of (complex valued) smooth functions on Y that vanish of infinite order at each point of F, the space $\mathfrak{W}_F$ is defined by the exact sequence:

$$0 \longrightarrow \mathfrak{F}(Y;F) \longrightarrow \mathcal{E}(Y) \longrightarrow \mathfrak{W}_F \longrightarrow 0.$$

Keywords: $\bar{\partial}$-operator – Tangential CR operator – Embedding of CR manifolds.
Math. classification: 32V05 – 32V30.

Note that the space $\mathfrak{W}_F$ can be intrinsically defined in terms of jets and turns out to be independent of the choice of the open neighborhood Y of F in X. We also consider the space $\mathfrak{W}_F^{\text{comp}}$ of *Whitney functions with compact support in* F, which can be defined by the exact sequence:

$$0 \longrightarrow \mathfrak{J}(Y;F) \cap \mathcal{D}(Y) \longrightarrow \mathcal{D}(Y) \longrightarrow \mathfrak{W}_F^{\text{comp}} \longrightarrow 0,$$

where $\mathcal{D}(Y)$ is the standard notation for the space of $f \in \mathcal{E}(Y)$ having compact support in Y.

Likewise we shall consider the spaces $\mathcal{D}'_F$ (resp. $\mathcal{E}'_F$) of distributions in Y with support (resp. compact support) contained in F.

The Dolbeault complexes on Y define, by passing to sub-complexes and quotients, $\bar{\partial}$-complexes on *Whitney forms* with closed (or compact) supports in F and on currents with closed (or compact) supports contained in F. We denote by $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F)$, $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\text{comp}})$, $H^{p,q}_{\bar{\partial}}(\mathcal{D}'_F)$, $H^{p,q}_{\bar{\partial}}(\mathcal{E}'_F)$ the corresponding cohomology groups (see, for more details, [N1], [N2], [NV]). More generally, if Φ is a paracompactifying family of supports in Y (see [B]), we can consider the cohomology groups $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\Phi})$ for Whitney forms on F with supports in Φ, which are quotients of smooth forms in Y with supports in Φ, and $H^{p,q}_{\bar{\partial}}(\mathcal{D}'^{\Phi}_F)$ for currents with supports in the intersection of F and closed sets of Φ.

Note that when $F = Y$ is open, these are the usual Dolbeault cohomology groups.

THEOREM 1.1. — *Let F be a locally closed subset of a Stein manifold X. Let Y be an open neighborhood of F in X, with $\overline{F} \cap Y = F$, and Φ a paracompactifying family of supports in Y. Then, for all $0 \leqslant p, q \leqslant N$, each of the cohomology groups $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F)$, $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\text{comp}})$, $H^{p,q}_{\bar{\partial}}(\mathcal{D}'_F)$, $H^{p,q}_{\bar{\partial}}(\mathcal{E}'_F)$, $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\Phi})$, $H^{p,q}_{\bar{\partial}}(\mathcal{D}'^{\Phi}_F)$, is either 0 or infinite dimensional.*

Proof. — The proof follows the argument in [L].

Denote by $\mathbf{H}$ one of the groups $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F)$, $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\text{comp}})$, $H^{p,q}_{\bar{\partial}}(\mathcal{D}'_F)$, $H^{p,q}_{\bar{\partial}}(\mathcal{E}'_F)$, $H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\Phi})$, $H^{p,q}_{\bar{\partial}}(\mathcal{D}'^{\Phi}_F)$, and assume that $\mathbf{H}$ is finite dimensional. Our goal is to show that $\mathbf{H} = \{0\}$.

The multiplication of a Whitney form or of a current by a function $f \in \mathcal{O}(X)$ is a linear map, commuting with $\bar{\partial}$, and preserving supports. Thus, by passing to the quotient, we obtain on $\mathbf{H}$ the structure of an $\mathcal{O}(X)$-module.

Let $\mathcal{I} = \{f \in \mathcal{O}(X) \mid f\mathbf{H} = \{0\}\}$ be the ideal in the ring $\mathcal{O}(X)$ of functions that annihilate $\mathbf{H}$. We want to show that $1 \in \mathcal{I}$.

Fix an embedding $X \hookrightarrow \mathbb{C}^{2N+1}$ of X into an Euclidean space. The coordinates $z_1, \ldots, z_{2N+1}$ on $\mathbb{C}^{2N+1}$ define functions $z_1^*, \ldots, z_{2N+1}^*$ in $\mathcal{O}(X)$. Fix $j \in \{1, \ldots, 2N+1\}$, and let $[u_1], \ldots, [u_m]$ be a basis of $\mathbf{H}$. Then the finite dimensionality of $\mathbf{H}$ implies that there exist nontrivial polynomials $P_{[u_i]}$ such that $P_{[u_i]}(z_j^*)[u_i] = \{0\}$, $i = 1, \ldots, m$. Consider $P = P_{[u_1]} \cdot \ldots \cdot P_{[u_m]}$. Then one has $P(z_j^*)\mathbf{H} = \{0\}$. Thus for each $j = 1, \ldots, 2N+1$ there is a polynomial $P_j(z_j) \in \mathbb{C}[z_j] \setminus \{0\}$ of minimal degree such that $P_j(z_j^*)\mathbf{H} = \{0\}$. This shows on the one hand that $\mathcal{I} \neq \{0\}$ and on the other hand that the set V of common zeros in X of the functions in $\mathcal{I}$ is finite, being contained in the inverse image by the embedding $X \hookrightarrow \mathbb{C}^{2N+1}$ of the finite set $\{z \in \mathbb{C}^{2N+1} \mid P_j(z_j) = 0 \text{ for } j = 1, \ldots, 2N+1\}$. By the Nullstellensatz $1 \in \mathcal{I}$ if and only if $V = \emptyset$. To show this, we prove first the following:

LEMMA 1.2. — *Let $f \in \mathcal{I}$ and let λ be a holomorphic vector field on X. Then $\lambda(f) \in \mathcal{I}$.*

Proof. — We recall the formula for the Lie derivative L_λ of an exterior differential form α:

$$L_\lambda \alpha = d(\lambda \rfloor \alpha) + \lambda \rfloor d\alpha.$$

If f is a smooth function, then

$$L_\lambda (f \cdot \alpha) = (\lambda(f)) \cdot \alpha + f \cdot L_\lambda \alpha.$$

Thus we obtain

$$(*) \qquad (\lambda(f)) \cdot \alpha = d(\lambda \rfloor (f \cdot \alpha)) + \lambda \rfloor d(f \cdot \alpha) - f \cdot d(\lambda \rfloor \alpha) - f \cdot \lambda \rfloor d\alpha.$$

Assume now that λ is a holomorphic vector field, that f is a holomorphic function on X and α is a form (or a current) of bidegree (p, q) in Y. Since the left hand side of $(*)$ is then of bidegree (p, q), keeping on the right hand side only the summands which are homogeneous of bidegree (p, q) we obtain

$$(**) \qquad (\lambda(f)) \cdot \alpha = \partial(\lambda \rfloor (f \cdot \alpha)) + \lambda \rfloor \partial(f \cdot \alpha) - f \cdot \partial(\lambda \rfloor \alpha) - f \cdot \lambda \rfloor \partial\alpha.$$

Assume now that α is $\bar\partial$-closed. Then each term on the right hand side is $\bar\partial$-closed. By considering subspaces and quotients, we note that $(**)$ is valid and that the summands on the right hand side are $\bar\partial$-closed also when α is a Whitney form on F or a current with support in F.

Assume now that $f \in \mathcal{I}$ and that α is the representative of an element of $\mathbf{H}$. Then the last two summands are cohomologous to zero because $f \in \mathcal{I}$. Moreover, $f \cdot \alpha = \bar{\partial}\beta$, again because $f \in \mathcal{I}$. Thus we have

$$\begin{array}{rcccl} \partial(\lambda\rfloor(f\cdot\alpha)) & = & \partial(\lambda\rfloor\bar{\partial}\beta) & = & \bar{\partial}\left(\partial(\lambda\rfloor\beta)\right), \\ \lambda\rfloor\partial(f\cdot\alpha) & = & \lambda\rfloor(\partial\bar{\partial}\beta) & = & \bar{\partial}\left(\lambda\rfloor\partial\beta\right), \end{array}$$

showing that also the first two summands are cohomologous to zero. This proves our contention.

End of the proof of Theorem 1.1. — We prove by contradiction that $V = \emptyset$. In fact, assume that $x \in V \neq \emptyset$. Then $\mathcal{I}$ contains a nonzero f having a zero of minimal order $\mu > 0$ at x. This means that, for holomorphic coordinates $\zeta_1, \ldots, \zeta_N$ centered at x, we have an expansion

$$f = \sum_{h=\mu}^{\infty} f_h(\zeta)$$

of f as a convergent series of homogeneous polynomials f_h of degree h, with $f_\mu \neq 0$. Then there is a coordinate ζ_j such that $\frac{\partial f_\mu}{\partial \zeta_j} \neq 0$. By Cartan's Theorem B, since the sheaf of germs of holomorphic vector fields is coherent, there is a holomorphic vector field λ on X such that $\lambda_x = \left.\frac{\partial}{\partial \zeta_j}\right|_0$. By Lemma 1.2 we have $\lambda(f) \in \mathcal{I}$. But this gives a contradiction because $\lambda(f) \neq 0$ has a zero of order $\mu - 1$ in x.

This completes the proof of the theorem.

Next we consider the following situation: F is a locally closed subset of a complex manifold X, and S a subset of F which is closed in F. If Y is an open neighborhood of F in X with $\overline{F} \cap Y = F$, then also $\overline{S} \cap Y = S$. The inclusion map $\iota : S \hookrightarrow F$ naturally induces maps: $\iota^* : H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F) \to H^{p,q}_{\bar{\partial}}(\mathfrak{W}_S)$, $\iota_* : H^{p,q}_{\bar{\partial}}(\mathfrak{W}_S^{\text{comp}}) \to H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\text{comp}})$, $\iota_* : H^{p,q}_{\bar{\partial}}(\mathcal{D}'_S) \to H^{p,q}_{\bar{\partial}}(\mathcal{D}'_F)$, $\iota_* : H^{p,q}_{\bar{\partial}}(\mathcal{E}'_S) \to H^{p,q}_{\bar{\partial}}(\mathcal{E}'_F)$ for every $0 \leqslant p, q \leqslant N$. We can also consider an open subset ω of Y and, corresponding to the inclusion $\sigma : \omega \cap F \to F$, the maps in cohomology: $\sigma^* : H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F) \to H^{p,q}_{\bar{\partial}}(\mathfrak{W}_{\omega\cap F})$, $\sigma_* : H^{p,q}_{\bar{\partial}}(\mathfrak{W}_{F\cap\omega}^{\text{comp}}) \to H^{p,q}_{\bar{\partial}}(\mathfrak{W}_F^{\text{comp}}))$, $\sigma^* : H^{p,q}_{\bar{\partial}}(\mathcal{D}'_F) \to H^{p,q}_{\bar{\partial}}(\mathcal{D}'_{\omega\cap F})$, $\sigma_* : H^{p,q}_{\bar{\partial}}(\mathcal{E}'_{\omega\cap F}) \to H^{p,q}_{\bar{\partial}}(\mathcal{E}'_F)$ (for $0 \leqslant p, q \leqslant N$). Denote by $\mathbf{K}$ any of the images of the maps in cohomology considered above. Then we obtain, just by repeating the argument of the proof of Theorem 1.1:

THEOREM 1.2. — *With the notation above, if X is a Stein manifold then $\mathbf{K}$ is either zero or infinite dimensional.*

2. Obstructions to generic embeddings of CR manifolds.

Let M be a smooth (abstract) CR manifold of type (n, k) and let $\bar{\partial}_M$ be the tangential Cauchy-Riemann operator on M. Fix a paracompactifying family Ψ of supports in M and consider, for $0 \leqslant p \leqslant n+k$, $0 \leqslant q \leqslant n$, the $\bar{\partial}_M$-cohomology groups for smooth differential forms with support in Ψ, denoted by $H^{p,q}_{\bar{\partial}_M}([\mathcal{E}]^{\Psi}(M))$, and the corresponding groups for currents with supports in Ψ, denoted by $H^{p,q}_{\bar{\partial}_M}([\mathcal{D}']^{\Psi}(M))$.

THEOREM 2.1. — *If for some (p, q), with $0 \leqslant p \leqslant n+k$, $0 \leqslant q \leqslant n$, and a paracompactifying family of supports Ψ in M, any one of the groups $H^{p,q}_{\bar{\partial}_M}([\mathcal{E}]^{\Psi}(M))$, $H^{p,q}_{\bar{\partial}_M}([\mathcal{D}']^{\Psi}(M))$ is finite dimensional and different from zero, then there does not exist a generic CR embedding of M into any open subset Y of a Stein manifold X.*

Proof. — Assume that M can be generically embedded into an open subset Y of a Stein manifold X. The complex dimension of X is $n+k$ and M, being a closed subset of Y, is locally closed in X.

The family Φ of closed subsets S of Y such that $S \cap M \in \Psi$ is a paracompactifying family in Y.

In this situation it is a well-known consequence of the formal Cauchy-Kowalewski theorem (and its dual version) (see [AFN], [AHLM], [HN2], [N1], [NV]) that

$$(\clubsuit) \qquad H^{p,q}_{\bar{\partial}_M}([\mathcal{E}]^{\Psi}(M)) \simeq H^{p,q}_{\bar{\partial}}(\mathfrak{W}_{\mathfrak{F}})$$

and

$$(\diamondsuit) \qquad H^{p,q}_{\bar{\partial}_M}([\mathcal{D}']^{\Psi}(M)) \simeq H^{p,q+k}_{\bar{\partial}}(\mathcal{D}'^{\Phi}_{M}).$$

Thus we obtain the conclusion using Theorem 1.1.

Remark. — If M has a *non-generic CR* embedding as a closed submanifold of an open subset Y of a Stein manifold X, then the groups in the right hand side of ($\clubsuit$) and ($\diamondsuit$) are either zero or infinite dimensional. But the isomorphism fails, and in fact the conclusion of Theorem 2.1 is false, as it will be shown by some examples in the next section.

3. Applications.

1. In particular let Ω be any domain having a smooth boundary $M = \partial\Omega$ in an N-dimensional Stein manifold X. Then for $0 \leqslant p \leqslant N$ and $0 < q \leqslant N-1$, $H^{p,q}([\mathcal{E}](M))$ and $H^{p,q}([\mathcal{D}'](M))$ cannot be finite dimensional without being zero. They are clearly infinite dimensional for $q = 0$; and if $\Omega \subset\subset X$, we know they are also infinite dimensional for $q = N-1$ by [HN1].

2. In [Br] it was shown that, if D is a sufficiently small open subset of a generic CR submanifold M of some open set Ω in $\mathbb{C}^N$, then $H^{p,q}([\mathcal{E}](D))$ is either zero or infinite dimensional. This follows from Theorem 2.1, without any assumption on D, as far as the embedding $M \hookrightarrow \Omega$ is generic.

Dropping the genericity assumption, the result is still valid for small open D's because an appropriate holomorphic projection into an affine $\mathbb{C}^{n+k}$ will produce a local generic CR embedding.

3. In [Br] it was also pointed out that there exist compact strictly pseudoconvex CR manifolds M of hypersurface type $(n,1)$, with $n \geqslant 2$, which are non-generically CR embedded into some $\mathbb{C}^N$, with $0 < \dim H^{0,1}([\mathcal{E}](M)) < \infty$. By Theorem 2, such an M has no generic CR embedding into any Stein manifold. But by [HN1] we know that the top cohomology groups $H^{p,n}([\mathcal{E}](M))$ are infinite dimensional for $0 \leqslant p \leqslant n+1$, due to the fact that M is embedded, even non-generically, into $\mathbb{C}^N$. Hence there can be no example of the type pointed out in [Br] with $\dim_{\mathbb{R}} M = 3$.

4. Suppose M is a compact CR manifold of any type (n,k), $n,k \geqslant 1$, which has a non-generic CR embedding in some Stein manifold X. Then for $0 \leqslant p \leqslant n+k$ the bottom and the top cohomology groups $H^{p,0}([\mathcal{E}](M))$ and $H^{p,n}([\mathcal{E}](M))$ are infinite dimensional, according to [HN1]. Hence finite dimensionality of some bottom or top group obstructs even non-generic embeddings. In this situation the existence of any nonzero but finite dimensional intermediate cohomology group $H^{p,q}([\mathcal{E}](M))$, $0 \leqslant p \leqslant n+k$, $0 < q < n$, would obstruct any attempt to make the non-generic embedding generic. In particular this means that, for such an M, no matter how we embed the Stein manifold X into some $\mathbb{C}^N$, the M becomes so positioned in $\mathbb{C}^N$ as not to have any one-to-one holomorphic projection into any affine $\mathbb{C}^{n+k}$ contained in the $\mathbb{C}^N$.

5. Consider a compact smooth orientable CR manifold M of hypersurface type $(n,1)$, $n \geqslant 1$, which has a non-generic CR embedding in some $\mathbb{C}^N$. Then by [HL] there is a holomorphic chain C whose boundary is M in

the sense of currents. Let V denote the support of C and set $F = M \cup V$. Then F is a closed set in $Y = \mathbb{C}^N$. Hence by Theorem 1, we have that for $0 \leqslant p \leqslant N$ and $0 < q \leqslant N$, the cohomology groups $H^{p,q}(\mathfrak{W}_F)$ and $H^{p,q}(\mathcal{D}'_F)$ are either zero or infinite dimensional.

Suppose $N = n + 2$ (so M has real codimension 3), $n > 1$, M is strictly pseudoconvex, and V has only isolated hypersurface singularities. Then $H^{p,q}_{\bar\partial_M}([\mathcal{E}](M))$ is nonzero and finite dimensional, for $p + q = n$ and $0 < q < n$, see [Y]. Thus there are many examples like the one pointed out by Brinkschulte in [Br]. Moreover this phenomenon starts to occur as soon as the embedding is in just one complex dimension too high to be generic, so that the embedding would be generic, if the embedding dimension were to be reduced by one.

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Manuscrit reçu le 26 juin 2001,
accepté le 11 juillet 2002.

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