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A FORMULA FOR THE RATIONAL LS-CATEGORY OF CERTAIN SPACES

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1. Introduction.

The Lusternik-Schnirelmann category is an old and well-known numerical invariant of the homotopy type of spaces which may be defined as follows: A space S has category n if this is the least integer for which S can be covered by $n+1$ open sets contractible in S. This invariant is hard to compute even for 1-connected rational spaces where the algebraic machinery of rational homotopy theory may the reader think it would be easy. Indeed, in [2], Félix and Halperin developed a deep approach, within rational homotopy theory, for computing the LS-category. Later on, and also concerning this hard task, Félix, Halperin and Lemaire [3] showed that for Poincaré duality spaces (and hence for elliptic spaces) the rational LS-category coincide with the Toomer invariant which, at first sight, may look easier to compute. In this paper we shall find a formula (or a bound in some other cases) for this invariant which generalizes and in some cases it complements previous results in [1], [2], [5], [6]. Let us very briefly introduce the basic notions on rational homotopy theory and Sullivan minimal models we shall use, and the definition of this invariant. For the reader non familiar with this the basic reference is [4].

The minimal model of the 1-connected space of finite type S is a commutative differential graded algebra $(\Lambda V, d)$ which algebraically models

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the rational homotopy type of the space. By ΛV we mean the free commutative algebra generated by the graded vector space V, i.e., $\Lambda V = TV/I$ where TV denotes the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|v||w|} w \otimes v$, $v, w \in V$. The differential d of any element of V is a "polynomial" in ΛV with no linear term. A model $(\Lambda V, d)$ is *elliptic* if both V and $H^*(\Lambda V, d)$ are finite dimensional spaces. From now on we consider only models for which $\dim V < \infty$, and we set $\dim V^{\text{even}} = n$, $\dim V^{\text{odd}} = m$.

The Toomer invariant of a minimal model $e_0(\Lambda V, d)$ (which equals the rational LS-category of the space which it represents if it is elliptic [3]) may be defined as the biggest integer s for which there is a non trivial cohomology class in $H^*(\Lambda V, d)$ represented by a cycle in $\Lambda^{\geq s} V$. As usual, $\Lambda^s V$ denotes the elements in ΛV of "wordlength" s. Given a model $(\Lambda V, d)$ we denote by $(\Lambda V, d_\sigma)$ its associated pure model, i.e., d_σ is the component of d which satisfies $d_\sigma\, V^{\text{even}} = 0$ and $d_\sigma\, V^{\text{odd}} \subset \Lambda V^{\text{even}}$.

Let $(\Lambda V, d)$ be a minimal model and let k be the biggest integer for which we may write $d = \sum_{i \geq k} d_k$ with $d_i(V) \subset \Lambda^i V$. Thus d_k induces a differential in ΛV and our main result which generalizes the one in [2] for coformal spaces, reads:

THEOREM 1. — *If $(\Lambda V, d_k)$ is elliptic then*

$$\operatorname{cat}(\Lambda V, d) = \operatorname{cat}(\Lambda V, d_k) = \operatorname{cat}(\Lambda V, d_{k\sigma}) = n(k-2) + m.$$

In the next section we prove this result and point out its relation with previous results on this subject.

2. A formula for the Toomer invariant.

The cohomology of an elliptic model is a Poincaré duality algebra so the cycle of maximal wordlength of a non trivial class occurs precisely in the fundamental class.

We shall be using a process for determining the fundamental class of the cohomology of a minimal model given in [7]. We recall it here for completeness: Let $(\Lambda V, d)$ be a pure model. Choose homogeneous basis $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_m\}$ of V^{even} and V^{odd} respectively, and write

$$dy_j = a_j^1 x_1 + a_j^2 x_2 + \cdots + a_j^{n-1} x_{n-1} + a_j^n x_n, \qquad j = 1, \ldots, m,$$

where each a^i_j is a polynomial in the variables $x_i, x_{i+1}, \dots, x_n$. For any $1 \le j_1 < \dots < j_n \le m$, denote by $P_{j_1 \dots j_n}$, the determinant of the matrix of order n:

$$\begin{pmatrix} a^1_{j_1} & \dots & a^n_{j_1} \\ \vdots & \ddots & \vdots \\ a^1_{j_n} & \dots & a^n_{j_n} \end{pmatrix}$$

Then (see [8]) the element $w_0 \in \Lambda V$,

$$w_0 = \sum_{1 \le j_1 < \dots < j_n \le m} (-1)^{j_1 + \dots + j_n} P_{j_1 \dots j_n}\, y_1 \dots \hat{y}_{j_1} \dots \hat{y}_{j_n} \dots y_m,$$

is a cycle representing a class, which is the fundamental class if the model is elliptic, or null otherwise.

If $(\Lambda V, d)$ is a non necessarily pure elliptic model, consider $(\Lambda V, d_\sigma)$ its associated pure model and $w_0 \in \Lambda V$ obtained as before. Note that this cycle lives in $(\Lambda V^{\text{even}} \otimes \Lambda^{m-n} V^{\text{odd}})^N$ in which $m = \dim V^{\text{odd}}$, $n = \dim V^{\text{even}}$ and N is the formal dimension.

Write $M^i_j = (\Lambda V^{\text{even}} \otimes \Lambda^j V^{\text{odd}})^i$, $p = m - n$, and observe that

$$dw_0 = \alpha^0_1 + \alpha^0_3 + \dots + \alpha^0_r, \quad \alpha^0_i \in M^{N+1}_{p+i}, \quad r \leqslant N/3 - p.$$

Since $d^2 w_0 = 0$ it follows that $d_\sigma \alpha^0_1 = 0$ (indeed this is the only summand of $d^2 w_0$ in M^*_{p+1}). Hence α^0_1 is a d_σ-boundary: $d_\sigma \beta_1 = \alpha^0_1$, $\beta_1 \in M^N_{p+2}$. Consider $w_1 = w_0 - \beta_1$ and note that

$$dw_1 = \alpha^1_3 + \alpha^1_5 + \dots + \alpha^1_r, \quad \alpha^1_i \in M^{N+1}_{p+i}, \quad r \leqslant N/3 - p.$$

Again, for the same reason, $d_\sigma \alpha^1_3 = 0$, so there exists $\beta_2 \in M^N_{p+4}$ such that $d_\sigma \beta_2 = \alpha^1_3$. Hence we define inductively elements $w_j, \beta_j \in (\Lambda V)^N$ satisfying $w_j = w_{j-1} - \beta_j$ and $dw_j \in \sum_{i=2j+1}^k M^{N+1}_{p+i}$. Hence, for the first j_o such that $2j_o + 1 > r$ this process stops and w_{j_o} is a d-cycle which we denote by w. Then w represents the fundamental class of $(\Lambda V, d)$ [7]. We shall use a similar process to prove our main results.

Before that, note that the formula for the fundamental class of an elliptic pure model given above already gives us a bound for e_0 in some cases (some of them already known by other methods):

Remark 2. — Let $(\Lambda V, d)$ be an elliptic model for which $dV \in \Lambda^k V$ for some k. Hence $d(\Lambda^n V) \in \Lambda^{n+k-1} V$ and then the differential d is of bidegree (k-1,1) with respect to the gradation $(\Lambda^p V)^q$. Note that this induces a bigradation in the cohomology algebra and therefore all the cycles representing the fundamental class have the same wordlength $e_0(\Lambda V, d)$.

PROPOSITION 3. — *Let $(\Lambda V, d)$ be an elliptic pure model in which, for each $y_j \in V^{\mathrm{odd}}$, dy_j is homogeneous of wordlength l_j. Then*

$$e_0(\Lambda V, d) \geq \sum_{i=1}^{n}(l_{j_i} - 1) + m - n.$$

If $l_j = l$ for all j then the equality holds, i.e, $e_0(\Lambda V, d) = n(l-2) + m$.

Proof. — Indeed, observe that the length of w_o obtained as in the formula above is precisely the requested bound. If dy_j has the same length for all j, then the equality holds by Remark 2. □

Remark 4. — Observe that for models for which $\chi_\pi(\Lambda V, d) = 0$, i.e., $\dim V^{\mathrm{even}} = \dim V^{\mathrm{odd}} = n$, any cycle of the fundamental class will have the same length. Therefore

$$e_0(\Lambda V, d) = \sum_j l_j - n.$$

A similar result is already remarked in [5].

The following two results clearly imply Theorem 1:

THEOREM 5. — *Let $(\Lambda V, d)$ be a model and let k be the biggest integer for which $dV \subset \Lambda^{\geq k} V$. If $(\Lambda V, d_k)$ is elliptic then*

$$e_0(\Lambda V, d) = e_0(\Lambda V, d_k).$$

Proof. — First note that, in view of the Milnor-Moore spectral sequence, $(\Lambda V, d)$ is elliptic since $(\Lambda V, d_k)$ is so, and both have the same formal dimension.

Call $s = e_0(\Lambda V, d_k)$ and observe, by Remark 2, that any cycle w_0 representing the fundamental class of $(\Lambda V, d_k)$ lives in $\Lambda^s V$. Then

$$dw_0 = d_k w_0 + a_1^0 + a_2^0 + \cdots + a_r^0 = a_1^0 + a_2^0 + \cdots + a_r^0,$$

with $a_i^0 \in \Lambda^{s+k-1+i} V$. Note also that r is a fix integer. Indeed the degree of a_r^0 is greater or equal than $2(s + k - 1 + r)$ and it coincides with $N + 1$ being N the formal dimension. Hence $r \leq (N + 3 - 2s - 2k)/2$.

Since $d^2 w_0 = 0$, by wordlength reasons, $d_k a_1^0 = 0$. Hence, a_1^0 is a d_k boundary, i.e., there is $b_1 \in \Lambda^{s+1} V$ such that $d_k b_1 = a_1^0$. Consider $w_1 = w_0 - b_1$ and observe that

$$dw_1 = a_2^1 + a_3^1 + \cdots + a_r^1, \qquad a_i^1 \in \Lambda^{s+k-1+i} V.$$

Again, $d_k a_2^1 = 0$ so there exists $b_2 \in \Lambda^{s+2}V$ for which $d_k b_2 = a_2^1$. We continue this process defining inductively $w_j = w_{j-1} - b_j$, $b_j \in \Lambda^{s+j}V$, $j \leq r$. Finally observe that $dw_r = 0$ and that w_r cannot be a d-boundary. Indeed if $w_r = w_0 - b_1 - \cdots - b_r$ were a d-boundary, by wordlength reasons, w_0 would be a d_k-boundary. Thus w_r is a non trivial cycle of the formal dimension and therefore it represents the fundamental class. Since $w_r \in \Lambda^{\geq s}V$ we have $e_0(\Lambda V, d) \geq e_0(\Lambda V, d_k)$.

On the other hand, call $p = e_0(\Lambda V, d)$ and let w be a cycle representing the fundamental class of $H^*(\Lambda V, d)$. Write

$$w = w_0 + w_1 + \cdots + w_r, \qquad w_i \in \Lambda^{p+i}V.$$

Since $dw = 0$, by wordlength reasons, it follows that $d_k w_0 = 0$. If w_0 were a d_k-boundary, i.e., $w_0 = d_k b$, then $w - db \in \Lambda^{>p}$ which contradicts the fact $p = e_0(\Lambda V, d)$. Hence w_0 represents the fundamental class of $H^*(\Lambda V, d_k)$ and thus $e_0(\Lambda V, d) \leq e_0(\Lambda V, d_k)$, which finishes the proof of the theorem.□

THEOREM 6. — *Let $(\Lambda V, d)$ be an elliptic model for which the differential is homogeneous of wordlength k, i.e., $dV \in \Lambda^k V$. If $(\Lambda V, d_k)$ is elliptic then*

$$e_0(\Lambda V, d) = e_0(\Lambda V, d_\sigma) = n(k-2) + m.$$

Proof. — First note that, if w_0 is the cycle given in the formula above representing the fundamental class of $(\Lambda V, d_\sigma)$, its length is $n(k-2)+m$. Thus, by Remark 2, this is precisely $e_0(\Lambda V, d_\sigma)$. We now show that $e_0(\Lambda V, d) = e_0(\Lambda V, d_\sigma)$.

Call $s = e_0(\Lambda V, d_\sigma)$ and let $w_0 \in \Lambda^s V$ be any cycle representing the fundamental class of $(\Lambda V, d_\sigma)$. Consider now

$$w = w_0 - \beta_1 - \cdots - \beta_{j_o} \tag{1}$$

the representative of the fundamental class of $(\Lambda V, d)$ given by the method above. In this case, since the differential has homogeneous wordlength degree k, $\beta_j \in \Lambda^s$ for all j. Therefore $w \in \Lambda^s V$ and $e_0(\Lambda V, d) \geq e_0(\Lambda V, d_\sigma)$.

On the other hand the equation (1) tells us that $e_0(\Lambda V, d_\sigma) \geq e_0(\Lambda V, d)$ and the theorem holds. □

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