Kazuko MATSUMOTO & Takeo OHSAWA

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ON THE REAL ANALYTIC LEVI FLAT HYPERSURFACES IN COMPLEX TORI OF DIMENSION TWO

by K. MATSUMOTO & T. OHSAWA

Introduction.

Let $X$ be a complex manifold of dimension $n$ and let $M$ be a real hypersurface of $X$. $M$ is called Levi flat if it locally separates $X$ into two Stein domains, i.e. if $M$ is locally pseudoconvex from both sides. In recent works of Lins-Neto [LN] and the second named author [O] it was proved that $\mathbb{P}^n$, complex projective space of dimension $n$, contains no compact real analytic Levi flat hypersurfaces if $n \geq 2$ (for the smooth case see [S]).

The purpose of the present article is to extend this reasoning by studying the geometry of Levi flat hypersurfaces in complex tori. Let $\Gamma$ be a lattice of $\mathbb{C}^n$, let $T = \mathbb{C}^n/\Gamma$, and let $\pi : \mathbb{C}^n \longrightarrow T$ be the canonical projection. Unlike the case of $\mathbb{P}^n$ $(n \geq 2)$, $T$ contains infinitely many compact Levi flat hypersurfaces $\pi(\bigoplus_{j=1}^{2n-1} \mathbb{R}u_j + u)$, where $u_j$ $(j = 1, \ldots, 2n-1)$ are $\mathbb{R}$-linearly independent vectors in $\Gamma$ and $u \in \mathbb{C}^n$. Therefore the best thing one can hope is the following.

**CONJECTURE.** — Let $M$ be a compact Levi flat hypersurface of $T$. Then $\pi^{-1}(M)$ is a union of complex affine hyperplanes. If moreover $T$ contains no proper complex tori of positive dimension, $M$ is flat, i.e. $M$ is of the form $\pi(\bigoplus_{j=1}^{2n-1} \mathbb{R}u_j + u)$.

Keywords: Levi flat – Complex torus.
We shall give a partial answer to this question by proving

**THEOREM.** — Let $M$, $T$ and $\pi$ be as above. If $M$ is real analytic and $\dim T = 2$, then $\pi^{-1}(M)$ is a union of complex affine lines. Moreover, if $M$ does not contain any elliptic curve, $M$ is flat.

For the proof we combine the method of extending the analytic normal bundle of $M$ and its roots from a neighbourhood of $M$ to the whole space with an explicit computation of the Levi form of $-\log \delta(z)$ for the euclidean distance function $\delta(z)$ from $z$ to a nonsingular complex curve in $\mathbb{C}^2$.

**1. The key lemma.**

Let $M$ be a compact Levi flat hypersurface in a complex torus $T (= \mathbb{C}^n / \Gamma)$, and let $\delta_M(z)$ be the distance from $z \in T$ to $M$ with respect to the euclidean metric. Since $T \setminus M$ is locally Stein by assumption, $-\log \delta_M$ is a continuous plurisubharmonic exhaustion function on $T \setminus M$. A finer property of this function is derived from the following.

**LEMMA.** — Let $C$ be a complex hypersurface in $\mathbb{C}^2$ defined by

$$C = \{(t, f(t)) \mid t \in V\}$$

for open $V \subset \mathbb{C}$ and holomorphic $f$. Then for any $p \in C$ there exists a neighbourhood $U (\subset \mathbb{C}^2)$ of $p$ such that

$$\sum_{i,j=1}^2 \frac{\partial^2 (-\log \delta_C)}{\partial z_i \partial \bar{z}_j}(z_1, z_2)\xi_i \xi_j = \frac{\left| \frac{\partial^2 f}{\partial t^2} \right|^2 \left| \xi_1 + \frac{\partial f}{\partial t} \xi_2 \right|^2}{2\left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right)^2 \left\{ \left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right)^2 - \left| \frac{\partial^2 f}{\partial t^2} \right|^2 \right\}} \bigg|_{t=t(z_1, z_2)}$$

for any $(z_1, z_2) \in U \setminus C$ and for any $(\xi_1, \xi_2) \in \mathbb{C}^2$. Here $\delta_C(z_1, z_2)$ denotes the euclidean distance from $(z_1, z_2)$ to $C$ and $t = t(z_1, z_2)$ is the solution of

$$z_1 - t + \frac{\partial f}{\partial t} \{z_2 - f(t)\} = 0.$$

**Proof.** — If we put

$$\varphi(z_1, z_2, t) := |z_1 - t|^2 + |z_2 - f(t)|^2$$

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for \((z_1, z_2) \in \mathbb{C}^2\) and \(t \in V\), then
\[
\frac{\partial \varphi}{\partial t} = -(\bar{z}_1 - t) - \frac{\partial f}{\partial t}\{z_2 - f(t)\}
\]
and
\[
H(z_1, z_2, t) := \det \begin{pmatrix}
\frac{\partial^2 \varphi}{\partial t \partial \bar{z}_1} & \frac{\partial^2 \varphi}{\partial t \partial z_1} \\
\frac{\partial^2 \varphi}{\partial t \partial \bar{z}_2} & \frac{\partial^2 \varphi}{\partial t \partial z_2}
\end{pmatrix} = \left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)^2 - \left|\frac{\partial^2 f}{\partial t^2}\right|^2 \left|z_2 - f(t)\right|^2.
\]
Since \(H(t, f(t), t) \neq 0\) for \(t \in V\), it follows by the implicit function theorem that one can find a \(C^\omega\) function \(t = t(z_1, z_2)\) defined in some neighbourhood \(U\) of \(p \in C\) which satisfies
\[
\frac{\partial \varphi}{\partial t}(z_1, z_2, t(z_1, z_2)) = \frac{\partial \varphi}{\partial t}(z_1, z_2, t(z_1, z_2)) = 0.
\]
Then
\[
\delta_C(z_1, z_2)^2 = \varphi(z_1, z_2, t(z_1, z_2))
\]
for any \((z_1, z_2) \in U\).

We put
\[
\psi(z_1, z_2) := \varphi(z_1, z_2, t(z_1, z_2)) = \left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)|z_2 - f(t)|^2
\]
for simplicity. Applying (1) we have
\[
\frac{\partial \psi}{\partial \bar{z}_i} = \frac{\partial \varphi}{\partial \bar{z}_i} + \frac{\partial \varphi}{\partial t} \frac{\partial t}{\partial \bar{z}_i} + \frac{\partial \varphi}{\partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial \bar{z}_i} = \frac{\partial \varphi}{\partial \bar{z}_i}
\]
for \(i = 1, 2\). Therefore we obtain
\[
\begin{align*}
\frac{\partial \psi}{\partial \bar{z}_1} &= \frac{\partial \varphi}{\partial \bar{z}_1} = z_1 - t = -\frac{\partial f}{\partial t}\{z_2 - f(t)\} \\
\frac{\partial \psi}{\partial \bar{z}_2} &= \frac{\partial \varphi}{\partial \bar{z}_2} = z_2 - f(t)
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial^2 \psi}{\partial z_1 \partial \bar{z}_1} &= 1 - \frac{\partial t}{\partial z_1} \\
\frac{\partial^2 \psi}{\partial z_1 \partial \bar{z}_2} &= -\frac{\partial f}{\partial t} \frac{\partial t}{\partial z_1} \\
\frac{\partial^2 \psi}{\partial z_2 \partial \bar{z}_2} &= 1 - \frac{\partial f}{\partial t} \frac{\partial t}{\partial z_2}.
\end{align*}
\]
Moreover by differentiating (1) we have
\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial t \partial \bar{z}_i} + \frac{\partial^2 \varphi}{\partial t^2 \bar{z}_i} + \frac{\partial^2 \varphi}{\partial t \partial t \partial \bar{z}_i} &= 0 \\
\frac{\partial^2 \varphi}{\partial \bar{z}_i \partial \bar{z}_i} + \frac{\partial^2 \varphi}{\partial \bar{z}_i \partial \bar{z}_i} + \frac{\partial^2 \varphi}{\partial \bar{t} \partial \bar{z}_i} &= 0
\end{align*}
\]
for \( i = 1, 2 \), and hence
\[
\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial t \partial \zeta_1} & \frac{\partial^2 \varphi}{\partial t \partial \zeta_2} \\
\frac{\partial^2 \varphi}{\partial \zeta_1 \partial \zeta_2} & \frac{\partial^2 \varphi}{\partial \zeta_2 \partial \zeta_2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial t}{\partial \zeta_1} & \frac{\partial t}{\partial \zeta_2} \\
\frac{\partial \zeta_1}{\partial \zeta_1} & \frac{\partial \zeta_2}{\partial \zeta_2}
\end{pmatrix}
= - \begin{pmatrix}
\frac{\partial^2 \varphi}{\partial t \partial \zeta_1} & \frac{\partial^2 \varphi}{\partial t \partial \zeta_2} \\
\frac{\partial^2 \varphi}{\partial \zeta_1 \partial \zeta_2} & \frac{\partial^2 \varphi}{\partial \zeta_2 \partial \zeta_2}
\end{pmatrix}.
\]

Since
\[
\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial t \partial t} & \frac{\partial^2 \varphi}{\partial t \partial \zeta} \\
\frac{\partial^2 \varphi}{\partial \zeta \partial t} & \frac{\partial^2 \varphi}{\partial \zeta \partial \zeta}
\end{pmatrix}
= \begin{pmatrix}
| \frac{\partial f}{\partial t} |^2 + 1 & - \frac{\partial^2 f}{\partial t \partial \zeta} \{ z_2 - f(t) \} \\
- \frac{\partial^2 f}{\partial \zeta \partial t} \{ z_2 - f(t) \} & | \frac{\partial f}{\partial t} |^2 + 1
\end{pmatrix}
\]

and
\[
\begin{pmatrix}
\frac{\partial^2 \varphi}{\partial \zeta_1 \partial \zeta_1} & \frac{\partial^2 \varphi}{\partial \zeta_1 \partial \zeta_2} \\
\frac{\partial^2 \varphi}{\partial \zeta_2 \partial \zeta_1} & \frac{\partial^2 \varphi}{\partial \zeta_2 \partial \zeta_2}
\end{pmatrix}
= \begin{pmatrix}
-1 & - \frac{\partial f}{\partial t} \\
0 & 0
\end{pmatrix}
\]

it follows that
\[
\begin{pmatrix}
\frac{\partial t}{\partial \zeta_1} & \frac{\partial t}{\partial \zeta_2} \\
\frac{\partial \zeta_1}{\partial \zeta_1} & \frac{\partial \zeta_2}{\partial \zeta_2}
\end{pmatrix}
= \frac{1}{H} \begin{pmatrix}
| \frac{\partial f}{\partial t} |^2 + 1 & \frac{\partial f}{\partial t} \left( | \frac{\partial f}{\partial t} |^2 + 1 \right) \\
\frac{\partial^2 f}{\partial t \partial \zeta} \{ z_2 - f(t) \} & \frac{\partial^2 f}{\partial t \partial \zeta} \{ z_2 - f(t) \}
\end{pmatrix}.
\]

Hence we obtain
\[
\begin{align*}
\frac{\partial^2 \psi}{\partial \zeta_1 \partial \zeta_1} &= 1 - \frac{1}{H} \left( | \frac{\partial f}{\partial t} |^2 + 1 \right) \\
\frac{\partial^2 \psi}{\partial \zeta_1 \partial \zeta_2} &= - \frac{1}{H} \frac{\partial f}{\partial t} \left( | \frac{\partial f}{\partial t} |^2 + 1 \right) \\
\frac{\partial^2 \psi}{\partial \zeta_2 \partial \zeta_2} &= 1 - \frac{1}{H} \left( | \frac{\partial f}{\partial t} |^2 + 1 \right).
\end{align*}
\]

(4)

We put
\[
A := - \log \psi = - \log \delta_C^2
\]
on \( U \setminus C \). Then we have
\[
\partial \bar{\partial} A = - \partial \bar{\partial} \psi + \frac{\partial \psi \wedge \bar{\partial} \psi}{\psi^2},
\]
or
\[
\frac{\partial^2 A}{\partial \zeta_1 \partial \zeta_j} = \frac{1}{\psi^2} \left( \frac{\partial \psi}{\partial \zeta_1} \frac{\partial \psi}{\partial \zeta_j} - \psi \frac{\partial^2 \psi}{\partial \zeta_1 \partial \zeta_j} \right).
\]

Combining this with (2) and (4) we obtain
\[
\begin{pmatrix}
\frac{\partial^2 A}{\partial \zeta_1 \partial \zeta_1} & \frac{\partial^2 A}{\partial \zeta_1 \partial \zeta_2} \\
\frac{\partial^2 A}{\partial \zeta_2 \partial \zeta_1} & \frac{\partial^2 A}{\partial \zeta_2 \partial \zeta_2}
\end{pmatrix}
= \frac{\left( \frac{\partial^2 f}{\partial t \partial t} \right)^2}{\left( | \frac{\partial f}{\partial t} |^2 + 1 \right)^2} H \begin{pmatrix}
1 & \frac{\partial f}{\partial t} \\
\frac{\partial f}{\partial t} & | \frac{\partial f}{\partial t} |^2
\end{pmatrix}.
\]
In other words the Levi form of $A$ is written as
\[
\sum_{i,j=1}^{2} \frac{\partial^2 A}{\partial z_i \partial \bar{z}_j}(z_1, z_2) \xi_i \xi_j
\]
\[
= \left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right)^2 H \left( |\xi_1|^2 + \frac{\partial f}{\partial t} \xi_1 \xi_2 + \frac{\partial f}{\partial t} \xi_2 \bar{\xi}_1 + \left| \frac{\partial f}{\partial t} \right|^2 |\xi_2|^2 \right)
\]
\[
= \left( \left| \frac{\partial f}{\partial t} \right|^2 \right)^2 \left( |\xi_1|^2 + \frac{\partial f}{\partial t} \xi_2 \right)^2
\]
\[
\left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right)^2 H,
\]
which proves the lemma. \qed

2. Proof of Theorem.

First we note that the lemma implies the following.

**Proposition.** — Let $M$ be a compact Levi flat hypersurface of class $C^2$ in a complex torus $T$ of dimension 2. Suppose that there exists a complex line in $\mathbb{C}^2$ whose image in $T$ by the canonical projection osculates $M$ but is not contained in $M$. Then $T \setminus M$ is a Stein open subset of $T$.

**Proof.** — By assumption there exists a point $p \in M$ such that the germ of a complex curve passing through $p$ and contained in $M$ does not inflect at $p$. By the lemma, $\delta_C^{-1} (= e^{-\log \delta_C})$ is strictly plurisubharmonic on $U \setminus M$ for some neighbourhood $U \ni p$. Since the set of such points $p$ is open and dense in $M$, we can replace $U$ by a smaller neighbourhood of $p$, if necessary, in such a way that $\delta_M^{-1}$ is also strictly plurisubharmonic on $U \setminus M$. Hence, since $T$ is homogeneous, $T \setminus M$ is Stein by a theorem of Michel [M] and the Kontinuitätsatz of Docquier-Grauert [DG]. \qed

Let us suppose now that $M$ is a compact Levi flat hypersurface of class $C^\omega$ in $T$, where $\dim T = 2$. We shall prove the theorem by contradiction. If we assume the contrary to the assertion, $M$ would contain a nonlinear complex curve. Then by the above proposition $T \setminus M$ is Stein. On the other hand, by the real analyticity of $M$ the Levi foliation of $M$, the foliation defined by the CR tangent bundle of $M$, is uniquely extendable to a tubular neighbourhood say $\Omega$ of $M$, as a complex analytic foliation.

Then, by the Steinness of $T \setminus M$ (together with $\dim T \geq 2$), the foliation is extendable complex analytically to the complement of a finite
subset of $T$, say to $T'$. Call this extended foliation $\mathcal{F}$. Let $\Theta$ be the holomorphic tangent bundle of $T$, let $\Theta' = \Theta | T'$ and let $S$ be the subbundle of $\Theta'$ tangent to $\mathcal{F}$.

We put $L = \Theta'/S$. Then $L$ admits at least two linearly independent global holomorphic sections, say $s_0$ and $s_1$, because so does $\Theta'$ and $\mathcal{F}$ is nonlinear.

Hence we have a meromorphic map $(s_0 : s_1)$ from $T'$ to $\mathbb{P}^1$.

Since $\dim T = 2$, a meromorphic map from $T'$ to $\mathbb{P}^1$ cannot admit any essential singularity at $T \setminus T'$, $(s_0 : s_1)$ extends to a meromorphic map from $T$ to $\mathbb{P}^1$. In particular, by a well known algebraicity criterion for the complex tori, $T$ is algebraic.

Let $m$ be any positive integer. Then there exists a holomorphic line bundle $L_{(m)}$ over a neighbourhood of $M$ such that $L_{(m)} \cong L$ there. This is simply because one can choose a system of transition functions of $L$ near $M$ so that they are real valued on $M$.

Let $G_m$ be the group of $(2m-1)$-th roots of unity. Then for any $p \in M$ and for any homomorphism $\rho : \pi_1(M) \rightarrow G_m$ we have a (holomorphic) line bundle

$$F_\rho = \tilde{M} \times \mathbb{C} / \sim_\rho \rightarrow M$$

where $\tilde{M}$ denotes the universal cover of $M$, and the equivalence relation $\sim_\rho$ is defined by

$$(x, \zeta) \sim_\rho (x', \zeta') \iff \text{There exists a covering transformation}$$

$$\sigma : \tilde{M} \rightarrow \tilde{M} \text{ such that } \sigma(x) = x'$$

and $\rho(\sigma^{-1})(\zeta) = \zeta'$.

Let us denote the canonical extensions of $F_\rho$ to a tubular neighbourhood of $M$ by the same symbol.

We note that

$$(L_{(m)} \otimes F_\rho)^{\otimes (2m-1)} \cong L \quad \text{near } M.$$ 

Choosing $s_0$ and $s_1$ in advance from the image of $H^0(T, \Theta) (\cong \mathbb{C}^2)$, we may assume that $(s_0 : s_1)$ has no points of indeterminacy on $M$. We then put

$$T'' = T' \setminus \{p \in T' | s_0(p) = s_1(p) = 0\}$$

and consider the diagram

$$\begin{array}{ccc}
X := T'' \times_{\mathbb{P}^1} \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 \ni z \\
\downarrow \varpi & & \downarrow \\
T'' & \overset{(s_0:s_1)}{\rightarrow} & \mathbb{P}^1 \ni z^{2m-1}
\end{array}$$

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Here $T'' \times_{P_i} \mathbb{P}^1$ denotes the fiber product of $T''$ and $\mathbb{P}^1$ over $\mathbb{P}^1$ with respect to the morphisms $(s_0 : s_1)$ and $z^{2m-1}$. Then the map $\varpi : X \to T''$ is a branched $(2m - 1)$ to 1 holomorphic map.

Take any point $q \in S_0^{-1}(0)$ and fix a single valued branch of $s_0^{2/(2m-1)}$ on a neighbourhood of $\varpi^{-1}(q)$. Then, by continuing it analytically we have a holomorphic section of $\varpi^*(L^{\otimes 2}_{(m)} \otimes F_\rho)$ for some $\rho$, defined on a neighbourhood of $M$. Note that this is possible because $L^{\otimes 2}$ is defined by a system of positive defining functions on $M$. In fact we have only to put

$$\rho(\sigma) = \exp \left( \frac{1}{2m-1} \int_\sigma d(\arg \frac{s_0}{s_1} - \arg s_0^2) \right).$$

This implies that $\varpi^*(L^{\otimes 2}_{(m)} \otimes F_\rho)$ is isomorphic to $[|\varpi^{-1}(s_0^{-1}(0))|]^{\otimes 2}$ on a neighbourhood of $\varpi^{-1}(M)$. Here $|\varpi^{-1}(s_0^{-1}(0))|$ denotes the support of the divisor $\varpi^{-1}(s_0^{-1}(0))$ and $[|\varpi^{-1}(s_0^{-1}(0))|]$ denotes the line bundle over $X$ associated to $|\varpi^{-1}(s_0^{-1}(0))|$. Therefore $\varpi^*(L^{\otimes 2}_{(m)} \otimes F_\rho)$ is analytically extendable to $X$. Moreover the locally free sheaf $\varpi^*([|\varpi^{-1}(s_0^{-1}(0))|])$ over $T''$ is extendable to $T$ as a coherent analytic sheaf because so is $L$. Hence $L^{\otimes 2}_{(m)} \otimes F_\rho$ is a subbundle of a holomorphic vector bundle $\varpi^*(\varpi^*(L^{\otimes 2}_{(m)} \otimes F_\rho))$ which is extendable to $T$ as a coherent analytic sheaf.

Since $\varpi^*(\varpi^*(L^{\otimes 2}_{(m)} \otimes F_\rho))$ is extendable to $T$ coherently, its projectification is extendable as a complex analytic fiber bundle over a projective algebraic manifold which is birationally equivalent to $T$. The subbundle $L^{\otimes 2}_{(m)} \otimes F_\rho$ then induces a holomorphic section of that projective bundle say $P$, over a neighbourhood of $M$. Since $P$ is projective algebraic by Kodaira’s well known theorem, the section corresponding $L^{\otimes 2}_{(m)} \otimes F_\rho$ extends to a meromorphic section over $T$. This means that $L^{\otimes 2}_{(m)} \otimes F_\rho$ is extendable to a line bundle $L_m \to T \setminus E_m$ for some finite subset $E_m$ of $T$. (Actually $E_m$ can be chosen to be empty.)

Now take any compact complex curve $C \subset T'' \setminus \bigcup_{m=2}^{\infty} E_m$ which is not contained in any fiber of $(s_0 : s_1)$. Then $\deg(L|C) > 0$ because $(s_0 : s_1)$ is nonconstant on $C$. However, $L^{\otimes 2}|C \simeq L^{\otimes 2}_{(m)}|C$ must hold because $L \simeq (L^{\otimes 2}_{(m)} \otimes F_\rho)^{\otimes (2m-1)}$ near $M$ and $T \setminus M$ is Stein.

Thus we obtain

$$\deg(L^{\otimes 2}|C) = (2m - 1) \deg(L_m|C)$$

which is an absurdity. $\square$

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Added in proof. Unfortunately the proof of Theorem turned out to be incorrect, so that the Steinness assertion for $T \setminus M$ only remains true.

BIBLIOGRAPHY


