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On the real analytic Levi flat hypersurfaces in complex tori of dimension two


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Introduction.

Let $X$ be a complex manifold of dimension $n$ and let $M$ be a real hypersurface of $X$. $M$ is called Levi flat if it locally separates $X$ into two Stein domains, i.e. if $M$ is locally pseudoconvex from both sides. In recent works of Lins-Neto [LN] and the second named author [O] it was proved that $\mathbb{P}^n$, complex projective space of dimension $n$, contains no compact real analytic Levi flat hypersurfaces if $n \geq 2$ (for the smooth case see [S]).

The purpose of the present article is to extend this reasoning by studying the geometry of Levi flat hypersurfaces in complex tori. Let $\Gamma$ be a lattice of $\mathbb{C}^n$, let $T = \mathbb{C}^n/\Gamma$, and let $\pi : \mathbb{C}^n \rightarrow T$ be the canonical projection. Unlike the case of $\mathbb{P}^n \ (n \geq 2)$, $T$ contains infinitely many compact Levi flat hypersurfaces $\pi(\bigoplus_{j=1}^{2n-1} \mathbb{R}u_j + u)$, where $u_j \ (j = 1, \ldots, 2n-1)$ are $\mathbb{R}$-linearly independent vectors in $\Gamma$ and $u \in \mathbb{C}^n$. Therefore the best thing one can hope is the following.

**Conjecture.** — Let $M$ be a compact Levi flat hypersurface of $T$. Then $\pi^{-1}(M)$ is a union of complex affine hyperplanes. If moreover $T$ contains no proper complex tori of positive dimension, $M$ is flat, i.e. $M$ is of the form $\pi(\bigoplus_{j=1}^{2n-1} \mathbb{R}u_j + u)$.

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We shall give a partial answer to this question by proving

**Theorem.** — Let \( M, T \) and \( \pi \) be as above. If \( M \) is real analytic and \( \dim T = 2 \), then \( \pi^{-1}(M) \) is a union of complex affine lines. Moreover, if \( M \) does not contain any elliptic curve, \( M \) is flat.

For the proof we combine the method of extending the analytic normal bundle of \( M \) and its roots from a neighbourhood of \( M \) to the whole space with an explicit computation of the Levi form of \( -\log \delta(z) \) for the euclidean distance function \( \delta(z) \) from \( z \) to a nonsingular complex curve in \( \mathbb{C}^2 \).

1. The key lemma.

Let \( M \) be a compact Levi flat hypersurface in a complex torus \( T (= \mathbb{C}^n / \Gamma) \), and let \( \delta_M(z) \) be the distance from \( z \in T \) to \( M \) with respect to the euclidean metric. Since \( T \setminus M \) is locally Stein by assumption, \( -\log \delta_M \) is a continuous plurisubharmonic exhaustion function on \( T \setminus M \). A finer property of this function is derived from the following.

**Lemma.** — Let \( C \) be a complex hypersurface in \( \mathbb{C}^2 \) defined by

\[
C = \{(t, f(t)) \mid t \in V\}
\]

for open \( V \subset \mathbb{C} \) and holomorphic \( f \). Then for any \( p \in C \) there exists a neighbourhood \( U (\subset \mathbb{C}^2) \) of \( p \) such that

\[
\sum_{i,j=1}^{2} \frac{\partial^2 (\log \delta_C)}{\partial z_i \partial \bar{z}_j} (z_1, z_2) \bar{\xi}_i \bar{\xi}_j
\]

\[
= \frac{\frac{\partial^2 f}{\partial t^2} \left| \xi_1 + \frac{\partial f}{\partial t} \xi_2 \right|^2}{2 \left( \left( \frac{\partial f}{\partial t} \right)^2 + 1 \right)^2 \left( \left( \frac{\partial^2 f}{\partial t^2} \right)^2 - \left( \frac{\partial^2 f}{\partial t^2} \right) \left| z_2 - f(t) \right| \right)} \bigg|_{t = t(z_1, z_2)}
\]

for any \( (z_1, z_2) \in U \setminus C \) and for any \( (\xi_1, \xi_2) \in \mathbb{C}^2 \). Here \( \delta_C(z_1, z_2) \) denotes the euclidean distance from \( (z_1, z_2) \) to \( C \) and \( t = t(z_1, z_2) \) is the solution of

\[
z_1 - t + \frac{\partial f}{\partial t} \{z_2 - f(t)\} = 0.
\]

**Proof.** — If we put

\[
\varphi(z_1, z_2, t) := |z_1 - t|^2 + |z_2 - f(t)|^2
\]
for \((z_1, z_2) \in \mathbb{C}^2\) and \(t \in V\), then

\[
\frac{\partial \varphi}{\partial t} = -(z_1 - t) - \frac{\partial f}{\partial t}\{z_2 - f(t)\}
\]

and

\[
H(z_1, z_2, t) := \det \left( \begin{array}{cc}
\frac{\partial^2 \varphi}{\partial t \partial \overline{t}} & \frac{\partial^2 \varphi}{\partial \overline{t}^2} \\
\frac{\partial^2 \varphi}{\partial \overline{t} \partial t} & \frac{\partial^2 \varphi}{\partial t^2}
\end{array} \right) = \left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right)^2 - \left| \frac{\partial^2 f}{\partial t^2} \right|^2 \left| z_2 - f(t) \right|^2.
\]

Since \(H(t, f(t), t) \neq 0\) for \(t \in V\), it follows by the implicit function theorem that one can find a \(C^\omega\) function \(t = t(z_1, z_2)\) defined in some neighbourhood \(U\) of \(p \in C\) which satisfies

\[
(1) \quad \frac{\partial \varphi}{\partial t}(z_1, z_2, t(z_1, z_2)) = \frac{\partial \varphi}{\partial t}(z_1, z_2, t(z_1, z_2)) = 0.
\]

Then

\[
\delta_C(z_1, z_2)^2 = \varphi(z_1, z_2, t(z_1, z_2))
\]

for any \((z_1, z_2) \in U\).

We put

\[
\psi(z_1, z_2) := \varphi(z_1, z_2, t(z_1, z_2)) = \left( \left| \frac{\partial f}{\partial t} \right|^2 + 1 \right) \left| z_2 - f(t) \right|^2
\]

for simplicity. Applying (1) we have

\[
\frac{\partial \psi}{\partial \overline{z}_i} = \frac{\partial \varphi}{\partial \overline{z}_i} + \frac{\partial \varphi}{\partial t} \frac{\partial t}{\partial \overline{z}_i} + \frac{\partial \varphi}{\partial \overline{t}} \frac{\partial \overline{t}}{\partial \overline{z}_i} = \frac{\partial \varphi}{\partial \overline{z}_i}
\]

for \(i = 1, 2\). Therefore we obtain

\[
(2) \quad \begin{cases}
\frac{\partial \psi}{\partial \overline{z}_1} = \frac{\partial \varphi}{\partial \overline{z}_1} = z_1 - t = -\frac{\partial f}{\partial t}\{z_2 - f(t)\} \\
\frac{\partial \psi}{\partial \overline{z}_2} = \frac{\partial \varphi}{\partial \overline{z}_2} = z_2 - f(t)
\end{cases}
\]

and

\[
(3) \quad \begin{cases}
\frac{\partial^2 \psi}{\partial z_1 \partial \overline{z}_1} = 1 - \frac{\partial t}{\partial z_1} \\
\frac{\partial^2 \psi}{\partial z_1 \partial \overline{z}_2} = -\frac{\partial f}{\partial \overline{t}} \frac{\partial t}{\partial z_1} \\
\frac{\partial^2 \psi}{\partial z_2 \partial \overline{z}_2} = 1 - \frac{\partial f}{\partial \overline{t}} \frac{\partial t}{\partial z_2}.
\end{cases}
\]

Moreover by differentiating (1) we have

\[
\begin{aligned}
&\left( \frac{\partial^2 \varphi}{\partial t \partial \overline{z}_i} \right) + \left( \frac{\partial^2 \varphi}{\partial \overline{t} \partial t} \right) + \left( \frac{\partial^2 \varphi}{\partial \overline{t} \partial \overline{z}_i} \right) = 0 \\
&\left( \frac{\partial^2 \varphi}{\partial t \partial z_i} \right) + \left( \frac{\partial^2 \varphi}{\partial t^2} \right) + \left( \frac{\partial^2 \varphi}{\partial t \partial \overline{z}_i} \right) = 0.
\end{aligned}
\]
for \(i = 1, 2\), and hence
\[
\begin{pmatrix}
\frac{\partial^2 \phi}{\partial \tau \partial \tau} & \frac{\partial^2 \phi}{\partial \tau \partial z_2} \\
\frac{\partial^2 \phi}{\partial \tau \partial z_1} & \frac{\partial^2 \phi}{\partial \tau \partial z_2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \tau}{\partial z_1} & \frac{\partial \tau}{\partial z_2} \\
\frac{\partial \tau}{\partial z_1} & \frac{\partial \tau}{\partial z_2}
\end{pmatrix}
= - \begin{pmatrix}
\frac{\partial^2 \phi}{\partial \tau \partial \tau} & \frac{\partial^2 \phi}{\partial \tau \partial z_2} \\
\frac{\partial^2 \phi}{\partial \tau \partial z_1} & \frac{\partial^2 \phi}{\partial \tau \partial z_2}
\end{pmatrix}.
\]
Since
\[
\begin{pmatrix}
\frac{\partial^2 \phi}{\partial \tau \partial \tau} & \frac{\partial^2 \phi}{\partial \tau \partial z_2} \\
\frac{\partial^2 \phi}{\partial \tau \partial z_1} & \frac{\partial^2 \phi}{\partial \tau \partial z_2}
\end{pmatrix}
= \begin{pmatrix}
|\frac{\partial f}{\partial \tau}|^2 + 1 & -\frac{\partial^2 f}{\partial \tau^2}\{z_2 - f(t)\} \\
-\frac{\partial^2 f}{\partial \tau^2}\{z_2 - f(t)\} & |\frac{\partial f}{\partial \tau}|^2 + 1
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
\frac{\partial^2 \phi}{\partial \tau \partial \tau} & \frac{\partial^2 \phi}{\partial \tau \partial z_2} \\
\frac{\partial^2 \phi}{\partial \tau \partial z_1} & \frac{\partial^2 \phi}{\partial \tau \partial z_2}
\end{pmatrix}
= \begin{pmatrix}
-1 & -\frac{\partial f}{\partial \tau} \\
0 & 0
\end{pmatrix}
\]
it follows that
\[
\begin{pmatrix}
\frac{\partial \tau}{\partial z_1} & \frac{\partial \tau}{\partial z_2} \\
\frac{\partial \tau}{\partial z_1} & \frac{\partial \tau}{\partial z_2}
\end{pmatrix}
= \frac{1}{H} \begin{pmatrix}
|\frac{\partial f}{\partial \tau}|^2 + 1 & \frac{\partial f}{\partial \tau}\left(|\frac{\partial f}{\partial \tau}|^2 + 1\right) \\
\frac{\partial^2 f}{\partial \tau^2}\{z_2 - f(t)\} & \frac{\partial f}{\partial \tau}\frac{\partial^2 f}{\partial \tau^2}\{z_2 - f(t)\}
\end{pmatrix}.
\]
Hence we obtain
\[
\begin{cases}
\frac{\partial^2 \psi}{\partial z_1 \partial z_1} = 1 - \frac{1}{H}\left(|\frac{\partial f}{\partial \tau}|^2 + 1\right) \\
\frac{\partial^2 \psi}{\partial z_1 \partial z_2} = -\frac{1}{H}\frac{\partial f}{\partial \tau}\left(|\frac{\partial f}{\partial \tau}|^2 + 1\right) \\
\frac{\partial^2 \psi}{\partial z_2 \partial z_2} = 1 - \frac{1}{H}\left|\frac{\partial f}{\partial \tau}\right|^2\left(|\frac{\partial f}{\partial \tau}|^2 + 1\right).
\end{cases}
\]
We put
\[
A := -\log \psi = -\log \delta_C^2
\]
on \(U \setminus C\). Then we have
\[
\partial \bar{\partial} A = -\frac{\partial \bar{\partial} \psi}{\psi} + \frac{\partial \psi \wedge \bar{\partial} \psi}{\psi^2},
\]
or
\[
\frac{\partial^2 A}{\partial z_1 \partial \bar{z}_j} = \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial z_1}\frac{\partial \psi}{\partial \bar{z}_j} - \psi\frac{\partial^2 \psi}{\partial z_1 \partial \bar{z}_j}\right).
\]
Combining this with (2) and (4) we obtain
\[
\begin{pmatrix}
\frac{\partial^2 A}{\partial z_1 \partial z_1} & \frac{\partial^2 A}{\partial z_1 \partial z_2} \\
\frac{\partial^2 A}{\partial z_2 \partial z_1} & \frac{\partial^2 A}{\partial z_2 \partial z_2}
\end{pmatrix}
= \frac{\left|\frac{\partial f}{\partial \tau}\right|^2}{\left(|\frac{\partial f}{\partial \tau}|^2 + 1\right)^2} H \begin{pmatrix}
1 & \frac{\partial f}{\partial \tau} \\
\frac{\partial f}{\partial \tau} & |\frac{\partial f}{\partial \tau}|^2
\end{pmatrix}.
\]
In other words the Levi form of $A$ is written as
\[
\sum_{i,j=1}^{2} \frac{\partial^2 A}{\partial z_i \partial \bar{z_j}}(z_1, z_2) \xi_i \xi_j
\]
which proves the lemma. 0

2. Proof of Theorem.

First we note that the lemma implies the following.

**Proposition.** — Let $M$ be a compact Levi flat hypersurface of class $C^2$ in a complex torus $T$ of dimension 2. Suppose that there exists a complex line in $\mathbb{C}^2$ whose image in $T$ by the canonical projection osculates $M$ but is not contained in $M$. Then $T \setminus M$ is a Stein open subset of $T$.

**Proof.** — By assumption there exists a point $p \in M$ such that the germ of a complex curve passing through $p$ and contained in $M$ does not inflect at $p$. By the lemma, $\delta_c^{-1} (= e^{-\log \delta_c})$ is strictly plurisubharmonic on $U \setminus M$ for some neighbourhood $U \ni p$. Since the set of such points $p$ is open and dense in $M$, we can replace $U$ by a smaller neighbourhood of $p$, if necessary, in such a way that $\delta_M^{-1}$ is also strictly plurisubharmonic on $U \setminus M$. Hence, since $T$ is homogeneous, $T \setminus M$ is Stein by a theorem of Michel [M] and the Kontinuitätssatz of Docquier-Grauert [DG]. 0

Let us suppose now that $M$ is a compact Levi flat hypersurface of class $C^\omega$ in $T$, where $\dim T = 2$. We shall prove the theorem by contradiction. If we assume the contrary to the assertion, $M$ would contain a nonlinear complex curve. Then by the above proposition $T \setminus M$ is Stein. On the other hand, by the real analyticity of $M$ the Levi foliation of $M$, the foliation defined by the CR tangent bundle of $M$, is uniquely extendable to a tubular neighbourhood say $\Omega$ of $M$, as a complex analytic foliation.

Then, by the Steinness of $T \setminus M$ (together with $\dim T \geq 2$), the foliation is extendable complex analytically to the complement of a finite
subset of $T$, say to $T'$. Call this extended foliation $\mathcal{F}$. Let $\Theta$ be the
holomorphic tangent bundle of $T$, let $\Theta' = \Theta | T'$ and let $S$ be the subbundle
of $\Theta'$ tangent to $\mathcal{F}$.

We put $L = \Theta' / S$. Then $L$ admits at least two linearly independent
global holomorphic sections, say $s_0$ and $s_1$, because so does $\Theta'$ and $\mathcal{F}$ is
nonlinear.

Hence we have a meromorphic map $(s_0 : s_1)$ from $T'$ to $\mathbb{P}^1$.

Since $\dim T = 2$, a meromorphic map from $T'$ to $\mathbb{P}^1$ cannot admit
any essential singularity at $T \setminus T'$, $(s_0 : s_1)$ extends to a meromorphic map
from $T$ to $\mathbb{P}^1$. In particular, by a well known algebraicity criterion for the
complex tori, $T$ is algebraic.

Let $m$ be any positive integer. Then there exists a holomorphic line
bundle $L_{(m)}$ over a neighbourhood of $M$ such that $L_{(m)} \otimes (2m-1) \simeq L$ there.
This is simply because one can choose a system of transition functions of
$L$ near $M$ so that they are real valued on $M$.

Let $G_m$ be the group of $(2m-1)$-th roots of unity. Then for any $p \in M$
and for any homomorphism $\rho : \pi_1(M) \to G_m$ we have a (holomorphic)
line bundle
$$F_{\rho} = \tilde{M} \times \mathbb{C} / \sim_\rho \to M$$
where $\tilde{M}$ denotes the universal cover of $M$, and the equivalence relation
\(\sim_\rho\) is defined by

$$\begin{align*}
(x, \zeta) \sim_\rho (x', \zeta') \iff \exists \text{ covering transformation } \\
\sigma : \tilde{M} \to \tilde{M} \text{ such that } \sigma(x) = x' \\
\text{and } \rho(\sigma^{-1})(\zeta) = \zeta'.
\end{align*}$$

Let us denote the canonical extensions of $F_{\rho}$ to a tubular neighbour-
hood of $M$ by the same symbol.

We note that
$$(L_{(m)} \otimes F_{\rho}) \otimes (2m-1) \simeq L \quad \text{near } M.$$
Here $T^\prime\prime \times_{P_1} \mathbb{P}^1$ denotes the fiber product of $T^\prime\prime$ and $\mathbb{P}^1$ over $\mathbb{P}^1$ with respect to the morphisms $(s_0 : s_1)$ and $z^{2m-1}$. Then the map $\varpi : X \to T^\prime\prime$ is a branched $(2m - 1)$ to $1$ holomorphic map.

Take any point $q \in s_0^{-1}(0)$ and fix a single valued branch of $s_0^{2/(2m-1)}$ on a neighborhood of $\varpi^{-1}(q)$. Then, by continuing it analytically we have a holomorphic section of $\varpi^*(L_{(m)}^\otimes F_\rho)$ for some $\rho$, defined on a neighborhood of $M$. Note that this is possible because $L^\otimes 2$ is defined by a system of positive defining functions on $M$. In fact we have only to put

$$\rho(\sigma) = \exp \left( \frac{-1}{2m - 1} \int_\sigma d(\arg \frac{s_0}{s_1} - \arg s_0^2) \right).$$

This implies that $\varpi^*(L_{(m)}^\otimes \otimes F_\rho)$ is isomorphic to $[|\varpi^{-1}(s_0^{-1}(0))|]^{\otimes 2}$ on a neighborhood of $\varpi^{-1}(M)$. Here $|\varpi^{-1}(s_0^{-1}(0))|$ denotes the support of the divisor $\varpi^{-1}(s_0^{-1}(0))$ and $[|\varpi^{-1}(s_0^{-1}(0))|]$ denotes the line bundle over $X$ associated to $|\varpi^{-1}(s_0^{-1}(0))|$. Therefore $\varpi^*(L_{(m)}^\otimes \otimes F_\rho)$ is analytically extendable to $X$. Moreover the locally free sheaf $\varpi^*(|\varpi^{-1}(s_0^{-1}(0))|)$ over $T^\prime\prime$ is extendable to $T$ as a coherent analytic sheaf because so is $L$. Hence $L_{(m)}^\otimes \otimes F_\rho$ is a subbundle of a holomorphic vector bundle $\varpi^*(L_{(m)}^\otimes \otimes F_\rho)$ which is extendable to $T$ as a coherent analytic sheaf.

Since $\varpi^*(\varpi^*(L_{(m)}^\otimes \otimes F_\rho))$ is extendable to $T$ coherently, its projectification is extendable as a complex analytic fiber bundle over a projective algebraic manifold which is birationally equivalent to $T$. The subbundle $L_{(m)}^\otimes \otimes F_\rho$ then induces a holomorphic section of that projective bundle say $P$, over a neighborhood of $M$. Since $P$ is projective algebraic by Kodaira’s well known theorem, the section corresponding $L_{(m)}^\otimes \otimes F_\rho$ extends to a meromorphic section over $T$. This means that $L_{(m)}^\otimes \otimes F_\rho$ is extendable to a line bundle $L_m \to T \setminus E_m$ for some finite subset $E_m$ of $T$. (Actually $E_m$ can be chosen to be empty.)

Now take any compact complex curve $C \subset T'' \setminus \cup_{m=2}^\infty E_m$ which is not contained in any fiber of $(s_0 : s_1)$. Then $\deg(L \mid C) > 0$ because $(s_0 : s_1)$ is nonconstant on $C$. However, $L^\otimes 2 \mid C \simeq L_{(m)}^{\otimes (2m-1)} \mid C$ must hold because $L \simeq (L_{(m)} \otimes F_\rho)^{\otimes (2m-1)}$ near $M$ and $T \setminus M$ is Stein.

Thus we obtain

$$\deg(L^\otimes 2 \mid C) = (2m - 1) \deg(L_m \mid C)$$

which is an absurdity. \qed

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Added in proof. Unfortunately the proof of Theorem turned out to be incorrect, so that the Steinness assertion for $T \setminus M$ only remains true.

BIBLIOGRAPHY


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