ANNALES

DE

L'INSTITUT FOURIER

Kazuko MATSUMOTO & Takeo OHSAWA

On the real analytic Levi flat hypersurfaces in complex tori of dimension two

Tome 52, nº 5 (2002), p. 1525-1532.

<http://aif.cedram.org/item?id=AIF_2002__52_5_1525_0>

Ann. Inst. Fourier, Grenoble
52, 5 (2002), 1525–1532

ON THE REAL ANALYTIC LEVI FLAT HYPERSURFACES IN COMPLEX TORI OF DIMENSION TWO

by K. MATSUMOTO & T. OHSAWA

Introduction.

Let X be a complex manifold of dimension n and let M be a real hypersurface of X. M is called Levi flat if it locally separates X into two Stein domains, i.e. if M is locally pseudoconvex from both sides. In recent works of Lins-Neto [LN] and the second named author [O] it was proved that $\mathbb{P}^n$, complex projective space of dimension n, contains no compact real analytic Levi flat hypersurfaces if $n \geqslant 2$ (for the smooth case see [S]).

The purpose of the present article is to extend this reasoning by studying the geometry of Levi flat hypersurfaces in complex tori. Let Γ be a lattice of $\mathbb{C}^n$, let $T = \mathbb{C}^n/\Gamma$, and let $\pi : \mathbb{C}^n \longrightarrow T$ be the canonical projection. Unlike the case of $\mathbb{P}^n$ $(n \geqslant 2)$, T contains infinitely many compact Levi flat hypersurfaces $\pi(\oplus_{j=1}^{2n-1}\mathbb{R}u_j + u)$, where u_j $(j = 1, \dots, 2n-1)$ are $\mathbb{R}$-linearly independent vectors in Γ and $u \in \mathbb{C}^n$. Therefore the best thing one can hope is the following.

CONJECTURE. — *Let M be a compact Levi flat hypersurface of T. Then $\pi^{-1}(M)$ is a union of complex affine hyperplanes. If moreover T contains no proper complex tori of positive dimension, M is flat, i.e. M is of the form $\pi(\oplus_{j=1}^{2n-1}\mathbb{R}u_j + u)$.*

Keywords: Levi flat – Complex torus.
Math. classification: 32V40.

We shall give a partial answer to this question by proving

THEOREM. — *Let M, T and π be as above. If M is real analytic and* $\dim T = 2$, *then $\pi^{-1}(M)$ is a union of complex affine lines. Moreover, if M does not contain any elliptic curve, M is flat.*

For the proof we combine the method of extending the analytic normal bundle of M and its roots from a neighbourhood of M to the whole space with an explicit computation of the Levi form of $-\log\delta(z)$ for the euclidean distance function $\delta(z)$ from z to a nonsingular complex curve in $\mathbb{C}^2$.

1. The key lemma.

Let M be a compact Levi flat hypersurface in a complex torus T $(=\mathbb{C}^n/\Gamma)$, and let $\delta_M(z)$ be the distance from $z \in T$ to M with respect to the euclidean metric. Since $T\setminus M$ is locally Stein by assumption, $-\log\delta_M$ is a continuous plurisubharmonic exhaustion function on $T\setminus M$. A finer property of this function is derived from the following.

LEMMA. — *Let C be a complex hypersurface in $\mathbb{C}^2$ defined by*

$$C = \{(t, f(t)) \mid t \in V\}$$

for open $V \subset \mathbb{C}$ and holomorphic f. Then for any $p \in C$ there exists a neighbourhood U $(\subset \mathbb{C}^2)$ of p such that

$$\sum_{i,j=1}^{2} \frac{\partial^2(-\log\delta_C)}{\partial z_i \partial \bar{z}_j}(z_1,z_2)\xi_i\bar{\xi}_j$$

$$= \left.\frac{\left|\frac{\partial^2 f}{\partial t^2}\right|^2 \left|\xi_1 + \frac{\partial \bar{f}}{\partial \bar{t}}\xi_2\right|^2}{2\left(\left|\frac{\partial f}{\partial t}\right|^2+1\right)^2\left\{\left(\left|\frac{\partial f}{\partial t}\right|^2+1\right)^2 - \left|\frac{\partial^2 f}{\partial t^2}\right|^2 |z_2 - f(t)|^2\right\}}\right|_{t=t(z_1,z_2)}$$

for any $(z_1,z_2) \in U\setminus C$ and for any $(\xi_1,\xi_2) \in \mathbb{C}^2$. Here $\delta_C(z_1,z_2)$ denotes the euclidean distance from (z_1,z_2) to C and $t = t(z_1,z_2)$ is the solution of

$$z_1 - t + \frac{\partial \bar{f}}{\partial \bar{t}}\{z_2 - f(t)\} = 0.$$

Proof. — If we put

$$\varphi(z_1,z_2,t) := |z_1 - t|^2 + |z_2 - f(t)|^2$$

for $(z_1, z_2) \in \mathbb{C}^2$ and $t \in V$, then

$$\frac{\partial \varphi}{\partial t} = -(\overline{z_1 - t}) - \frac{\partial f}{\partial t}\{\overline{z_2 - f(t)}\}$$

and

$$H(z_1, z_2, t) := \det \begin{pmatrix} \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} & \frac{\partial^2 \varphi}{\partial t^2} \\ \frac{\partial^2 \varphi}{\partial t^2} & \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \end{pmatrix} = \left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)^2 - \left|\frac{\partial^2 f}{\partial t^2}\right|^2 |z_2 - f(t)|^2.$$

Since $H(t, f(t), t) \neq 0$ for $t \in V$, it follows by the implicit function theorem that one can find a C^ω function $t = t(z_1, z_2)$ defined in some neighbourhood U of $p \in C$ which satisfies

$$\frac{\partial \varphi}{\partial t}(z_1, z_2, t(z_1, z_2)) = \frac{\partial \varphi}{\partial \bar{t}}(z_1, z_2, t(z_1, z_2)) = 0. \tag{1}$$

Then

$$\delta_C(z_1, z_2)^2 = \varphi(z_1, z_2, t(z_1, z_2))$$

for any $(z_1, z_2) \in U$.

We put

$$\psi(z_1, z_2) := \varphi(z_1, z_2, t(z_1, z_2)) = \left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)|z_2 - f(t)|^2$$

for simplicity. Applying (1) we have

$$\frac{\partial \psi}{\partial \bar{z}_i} = \frac{\partial \varphi}{\partial \bar{z}_i} + \frac{\partial \varphi}{\partial t}\frac{\partial t}{\partial \bar{z}_i} + \frac{\partial \varphi}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial \bar{z}_i} = \frac{\partial \varphi}{\partial \bar{z}_i}$$

for $i = 1, 2$. Therefore we obtain

$$\begin{cases} \dfrac{\partial \psi}{\partial \bar{z}_1} = \dfrac{\partial \varphi}{\partial \bar{z}_1} = z_1 - t = -\dfrac{\partial \bar{f}}{\partial \bar{t}}\{z_2 - f(t)\} \\ \dfrac{\partial \psi}{\partial \bar{z}_2} = \dfrac{\partial \varphi}{\partial \bar{z}_2} = z_2 - f(t) \end{cases} \tag{2}$$

and

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial z_1 \partial \bar{z}_1} = 1 - \dfrac{\partial t}{\partial z_1} \\ \dfrac{\partial^2 \psi}{\partial z_1 \partial \bar{z}_2} = -\dfrac{\partial f}{\partial t}\dfrac{\partial t}{\partial z_1} \\ \dfrac{\partial^2 \psi}{\partial z_2 \partial \bar{z}_2} = 1 - \dfrac{\partial f}{\partial t}\dfrac{\partial t}{\partial z_2}. \end{cases} \tag{3}$$

Moreover by differentiating (1) we have

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t \partial z_i} + \dfrac{\partial^2 \varphi}{\partial t^2}\dfrac{\partial t}{\partial z_i} + \dfrac{\partial^2 \varphi}{\partial t \partial \bar{t}}\dfrac{\partial \bar{t}}{\partial z_i} = 0 \\ \dfrac{\partial^2 \varphi}{\partial t \partial \bar{z}_i} + \dfrac{\partial^2 \varphi}{\partial t^2}\dfrac{\partial t}{\partial \bar{z}_i} + \dfrac{\partial^2 \varphi}{\partial t \partial \bar{t}}\dfrac{\partial \bar{t}}{\partial \bar{z}_i} = 0 \end{cases}$$

for $i = 1, 2$, and hence

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial t\partial\bar{t}} & \frac{\partial^2\varphi}{\partial\bar{t}^2} \\ \frac{\partial^2\varphi}{\partial t^2} & \frac{\partial^2\varphi}{\partial t\partial\bar{t}} \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial z_1} & \frac{\partial t}{\partial z_2} \\ \frac{\partial\bar{t}}{\partial z_1} & \frac{\partial\bar{t}}{\partial z_2} \end{pmatrix} = -\begin{pmatrix} \frac{\partial^2\varphi}{\partial\bar{t}\partial z_1} & \frac{\partial^2\varphi}{\partial\bar{t}\partial z_2} \\ \frac{\partial^2\varphi}{\partial t\partial z_1} & \frac{\partial^2\varphi}{\partial t\partial z_2} \end{pmatrix}.$$

Since

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial t\partial\bar{t}} & \frac{\partial^2\varphi}{\partial\bar{t}^2} \\ \frac{\partial^2\varphi}{\partial t^2} & \frac{\partial^2\varphi}{\partial t\partial\bar{t}} \end{pmatrix} = \begin{pmatrix} \left|\frac{\partial f}{\partial t}\right|^2 + 1 & -\frac{\partial^2\bar{f}}{\partial\bar{t}^2}\{z_2 - f(t)\} \\ -\frac{\partial^2 f}{\partial t^2}\{\overline{z_2 - f(t)}\} & \left|\frac{\partial f}{\partial t}\right|^2 + 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial\bar{t}\partial z_1} & \frac{\partial^2\varphi}{\partial\bar{t}\partial z_2} \\ \frac{\partial^2\varphi}{\partial t\partial z_1} & \frac{\partial^2\varphi}{\partial t\partial z_2} \end{pmatrix} = \begin{pmatrix} -1 & -\frac{\partial\bar{f}}{\partial\bar{t}} \\ 0 & 0 \end{pmatrix}$$

it follows that

$$\begin{pmatrix} \frac{\partial t}{\partial z_1} & \frac{\partial t}{\partial z_2} \\ \frac{\partial\bar{t}}{\partial z_1} & \frac{\partial\bar{t}}{\partial z_2} \end{pmatrix} = \frac{1}{H} \begin{pmatrix} \left|\frac{\partial f}{\partial t}\right|^2 + 1 & \frac{\partial\bar{f}}{\partial\bar{t}}\left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right) \\ \frac{\partial^2 f}{\partial t^2}\{\overline{z_2 - f(t)}\} & \frac{\partial\bar{f}}{\partial\bar{t}}\frac{\partial^2 f}{\partial t^2}\{\overline{z_2 - f(t)}\} \end{pmatrix}.$$

Hence we obtain

$$(4)\qquad \begin{cases} \dfrac{\partial^2\psi}{\partial z_1\partial\bar{z}_1} = 1 - \dfrac{1}{H}\left(\left|\dfrac{\partial f}{\partial t}\right|^2 + 1\right) \\ \dfrac{\partial^2\psi}{\partial z_1\partial\bar{z}_2} = -\dfrac{1}{H}\dfrac{\partial f}{\partial t}\left(\left|\dfrac{\partial f}{\partial t}\right|^2 + 1\right) \\ \dfrac{\partial^2\psi}{\partial z_2\partial\bar{z}_2} = 1 - \dfrac{1}{H}\left|\dfrac{\partial f}{\partial t}\right|^2\left(\left|\dfrac{\partial f}{\partial t}\right|^2 + 1\right). \end{cases}$$

We put

$$A := -\log\psi = -\log{\delta_C}^2$$

on $U \setminus C$. Then we have

$$\partial\bar{\partial}A = \frac{-\partial\bar{\partial}\psi}{\psi} + \frac{\partial\psi\wedge\bar{\partial}\psi}{\psi^2},$$

or

$$\frac{\partial^2 A}{\partial z_i\partial\bar{z}_j} = \frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial z_i}\frac{\partial\psi}{\partial\bar{z}_j} - \psi\frac{\partial^2\psi}{\partial z_i\partial\bar{z}_j}\right).$$

Combining this with (2) and (4) we obtain

$$\begin{pmatrix} \frac{\partial^2 A}{\partial z_1\partial\bar{z}_1} & \frac{\partial^2 A}{\partial z_1\partial\bar{z}_2} \\ \frac{\partial^2 A}{\partial z_2\partial\bar{z}_1} & \frac{\partial^2 A}{\partial z_2\partial\bar{z}_2} \end{pmatrix} = \frac{\left|\frac{\partial^2 f}{\partial t^2}\right|^2}{\left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)^2 H} \begin{pmatrix} 1 & \frac{\partial f}{\partial t} \\ \frac{\partial\bar{f}}{\partial\bar{t}} & \left|\frac{\partial f}{\partial t}\right|^2 \end{pmatrix}.$$

In other words the Levi form of A is written as

$$\begin{aligned}
&\sum_{i,j=1}^{2} \frac{\partial^2 A}{\partial z_i \partial \bar{z}_j}(z_1, z_2)\xi_i \bar{\xi}_j \\
&= \frac{\left|\frac{\partial^2 f}{\partial t^2}\right|^2}{\left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)^2 H} \left(|\xi_1|^2 + \frac{\partial f}{\partial t}\xi_1\bar{\xi}_2 + \frac{\partial \bar{f}}{\partial \bar{t}}\xi_2\bar{\xi}_1 + \left|\frac{\partial f}{\partial t}\right|^2 |\xi_2|^2\right) \\
&= \frac{\left|\frac{\partial^2 f}{\partial t^2}\right|^2 \left|\xi_1 + \frac{\partial \bar{f}}{\partial \bar{t}}\xi_2\right|^2}{\left(\left|\frac{\partial f}{\partial t}\right|^2 + 1\right)^2 H},
\end{aligned}$$

which proves the lemma. □

2. Proof of Theorem.

First we note that the lemma implies the following.

PROPOSITION. — *Let M be a compact Levi flat hypersurface of class C^2 in a complex torus T of dimension 2. Suppose that there exists a complex line in $\mathbb{C}^2$ whose image in T by the canonical projection osculates M but is not contained in M. Then $T \setminus M$ is a Stein open subset of T.*

Proof. — By assumption there exists a point $p \in M$ such that the germ of a complex curve passing through p and contained in M does not inflect at p. By the lemma, ${\delta_C}^{-1}$ $(= e^{-\log \delta_C})$ is strictly plurisubharmonic on $U \setminus M$ for some neighbourhood $U \ni p$. Since the set of such points p is open and dense in M, we can replace U by a smaller neighbourhood of p, if necessary, in such a way that ${\delta_M}^{-1}$ is also strictly plurisubharmonic on $U \setminus M$. Hence, since T is homogeneous, $T \setminus M$ is Stein by a theorem of Michel [M] and the Kontinuitätssatz of Docquier-Grauert [DG]. □

Let us suppose now that M is a compact Levi flat hypersurface of class C^ω in T, where $\dim T = 2$. We shall prove the theorem by contradiction. If we assume the contrary to the assertion, M would contain a nonlinear complex curve. Then by the above proposition $T \setminus M$ is Stein. On the other hand, by the real analyticity of M the Levi foliation of M, the foliation defined by the CR tangent bundle of M, is uniquely extendable to a tubular neighbourhood say Ω of M, as a complex analytic foliation.

Then, by the Steinness of $T \setminus M$ (together with $\dim T \geqslant 2$), the foliation is extendable complex analytically to the complement of a finite subset of T, say to T'. Call this extended foliation $\mathfrak{F}$. Let Θ be the holomorphic tangent bundle of T, let $\Theta' = \Theta \,|\, T'$ and let S be the subbundle of Θ' tangent to $\mathfrak{F}$.

We put $L = \Theta'/S$. Then L admits at least two linearly independent global holomorphic sections, say s_0 and s_1, because so does Θ' and $\mathfrak{F}$ is nonlinear.

Hence we have a meromorphic map $(s_0 : s_1)$ from T' to $\mathbb{P}^1$.

Since $\dim T = 2$, a meromorphic map from T' to $\mathbb{P}^1$ cannot admit any essential singularity at $T \setminus T'$, $(s_0 : s_1)$ extends to a meromorphic map from T to $\mathbb{P}^1$. In particular, by a well known algebraicity criterion for the complex tori, T is algebraic.

Let m be any positive integer. Then there exists a holomorphic line bundle $L_{(m)}$ over a neighbourhood of M such that $L_{(m)}^{\otimes(2m-1)} \simeq L$ there. This is simply because one can choose a system of transition functions of L near M so that they are real valued on M.

Let G_m be the group of $(2m-1)$-th roots of unity. Then for any $p \in M$ and for any homomorphism $\rho : \pi_1(M) \longrightarrow G_m$ we have a (holomorphic) line bundle

$$F_\rho = \widetilde{M} \times \mathbb{C}/ \sim_\rho \longrightarrow M$$

where $\widetilde{M}$ denotes the universal cover of M, and the equivalence relation $\sim_\rho$ is defined by

$$\begin{aligned}(x,\zeta) \sim_\rho (x',\zeta') \Longleftrightarrow\ & \text{There exists a covering transformation}\\ & \sigma : \widetilde{M} \longrightarrow \widetilde{M} \text{ such that } \sigma(x) = x'\\ & \text{and } \rho(\sigma^{-1})(\zeta) = \zeta'.\end{aligned}$$

Let us denote the canonical extensions of F_ρ to a tubular neighbourhood of M by the same symbol.

We note that

$$(L_{(m)} \otimes F_\rho)^{\otimes(2m-1)} \simeq L \qquad \text{near } M.$$

Choosing s_0 and s_1 in advance from the image of $H^0(T, \Theta)$ $(\cong \mathbb{C}^2)$, we may assume that $(s_0 : s_1)$ has no points of indeterminancy on M. We then put

$$T'' = T' \setminus \{p \in T' \mid s_0(p) = s_1(p) = 0\}$$

and consider the diagram

$$\begin{array}{ccccc} X := T'' \times_{\mathbb{P}^1} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \ni & z \\ \downarrow \varpi & & \downarrow & \downarrow \\ T'' & \xrightarrow{(s_0:s_1)} & \mathbb{P}^1 \ni & z^{2m-1} \end{array}$$

Here $T'' \times_{\mathbb{P}^1} \mathbb{P}^1$ denotes the fiber product of T'' and $\mathbb{P}^1$ over $\mathbb{P}^1$ with respect to the morphisms $(s_0 : s_1)$ and z^{2m-1}. Then the map $\varpi : X \longrightarrow T''$ is a branched $(2m-1)$ to 1 holomorphic map.

Take any point $q \in s_0^{-1}(0)$ and fix a single valued branch of $s_0^{2/(2m-1)}$ on a neighbourhood of $\varpi^{-1}(q)$. Then, by continuing it analytically we have a holomorphic section of $\varpi^*(L_{(m)}^{\otimes 2} \otimes F_\rho)$ for some ρ, defined on a neighbourhood of M. Note that this is possible because $L^{\otimes 2}$ is defined by a system of positive defining functions on M. In fact we have only to put

$$\rho(\sigma) = \exp\Big(\frac{\sqrt{-1}}{2m-1}\int_\sigma d\big(\arg\frac{s_0}{s_1} - \arg s_0^2\big)\Big).$$

This implies that $\varpi^*(L_{(m)}^{\otimes 2} \otimes F_\rho)$ is isomorphic to $[\,|\varpi^{-1}(s_0^{-1}(0))|\,]^{\otimes 2}$ on a neighbourhood of $\varpi^{-1}(M)$. Here $|\varpi^{-1}(s_0^{-1}(0))|$ denotes the support of the divisor $\varpi^{-1}(s_0^{-1}(0))$ and $[\,|\varpi^{-1}(s_0^{-1}(0))|\,]$ denotes the line bundle over X associated to $|\varpi^{-1}(s_0^{-1}(0))|$. Therefore $\varpi^*(L_{(m)}^{\otimes 2} \otimes F_\rho)$ is analytically extendable to X. Moreover the locally free sheaf $\varpi_*([\,|\varpi^{-1}(s_0^{-1}(0))|\,])$ over T'' is extendable to T as a coherent analytic sheaf because so is L. Hence $L_{(m)}^{\otimes 2} \otimes F_\rho$ is a subbundle of a holomorphic vector bundle $\varpi_*(\varpi^*(L_{(m)}^{\otimes 2} \otimes F_\rho))$ which is extendable to T as a coherent analytic sheaf.

Since $\varpi_*(\varpi^*(L_{(m)}^{\otimes 2} \otimes F_\rho))$ is extendable to T coherently, its projectification is extendable as a complex analytic fiber bundle over a projective algebraic manifold which is birationally equivalent to T. The subbundle $L_{(m)}^{\otimes 2} \otimes F_\rho$ then induces a holomorphic section of that projective bundle say P, over a neighbourhood of M. Since P is projective algebraic by Kodaira's well known theorem, the section corresponding $L_{(m)}^{\otimes 2} \otimes F_\rho$ extends to a meromorphic section over T. This means that $L_{(m)}^{\otimes 2} \otimes F_\rho$ is extendable to a line bundle $L_m \longrightarrow T \setminus E_m$ for some finite subset E_m of T. (Actually E_m can be chosen to be empty.)

Now take any compact complex curve $C \subset T'' \setminus \cup_{m=2}^{\infty} E_m$ which is not contained in any fiber of $(s_0 : s_1)$. Then $\deg(L \,|\, C) > 0$ because $(s_0 : s_1)$ is nonconstant on C. However, $L^{\otimes 2} \,|\, C \simeq L_m^{\otimes(2m-1)} \,|\, C$ must hold because $L \simeq (L_{(m)} \otimes F_\rho)^{\otimes(2m-1)}$ near M and $T \setminus M$ is Stein.

Thus we obtain

$$\deg(L^{\otimes 2} \,|\, C) = (2m-1)\deg(L_m \,|\, C)$$

which is an absurdity. □

Acknowledgement. The authors thank the referee for valuable criticism.

Added in proof. Unfortunately the proof of Theorem turned out to be incorrect, so that the Steinness assertion for $T \setminus M$ only remains true.

BIBLIOGRAPHY

[D-G] F. DOCQUIER and H. GRAUERT, Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann., 140 (1960), 94–123.

[LN] A. LINS-NETO, A note on projective Levi-flats and minimal sets of algebraic functions, Ann. Inst. Fourier, 49-4 (1999), 1369–1385.

[M] D. MICHEL, Sur les ouverts pseudoconvexes des espaces homogènes, C. R. Acad. Sci. Paris, Sér. A, 283 (1976), 779–782.

[O] T. OHSAWA, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J., 158 (2000), 95–98.

[S] Y.-T. SIU, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension $\geqslant 3$, Ann. of Math., 151 (2000), 1217–1243.

Manuscrit reçu le 14 mai 2001,
accepté le 13 novembre 2001.

Kazuko MATSUMOTO,
Osaka Women's University
Department of Applied Mathematics
Daisen-cho
Sakai 590-0035 (Japan).
kazuko@appmath.osaka-wu.ac.jp
Takeo OHSAWA,
Nagoya University
Graduate School of Mathematics
Chikusa-ku
Nagoya 464-8602 (Japan).