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Equivariant deformation quantization for the cotangent bundle of a flag manifold


<http://aif.cedram.org/item?id=AIF_2002__52_3_881_0>
1. Introduction.

In the context of algebraic geometry, the equivariant deformation quantization (EDQ) problem for cotangent bundles is to construct a graded $G$-equivariant star product $\ast$ on the symbol algebra $\mathcal{R} = R(T*X)$ where $X$ is a homogeneous space of a complex algebraic group $G$. Motivated by geometric quantization (GQ), we require that the specialization of $\ast$ at $t = 1$ produces the algebra $\mathcal{D} = \mathcal{D}^{1/2}_{alg}(X)$ of (linear) twisted differential operators for the (locally defined) square root of the canonical bundle $\mathcal{K}$ on $X$. (There are other interesting choices for the line bundle but we do not consider them in this paper.) Then $\ast$ corresponds to a quantization map $\mathfrak{q}$ from $\mathcal{R}$ onto $\mathcal{D}$; $G$-equivariance of $\ast$ amounts to $G$-equivariance of $\mathfrak{q}$. The choice of half-forms is naturally consistent with our requiring parity for $\ast$.

Suppose from now on that $G$ is a (connected) complex semisimple Lie group and $X$ is a flag manifold of $G$. Flag manifolds are the most familiar compact homogeneous spaces of $G$; they exemplify the phenomenon of a big symmetry group acting on a small space.

In this paper we solve the EDQ problem for $\mathcal{R}$ when the geometry of the moment map $\mu$ for the $G$-action on $T*X$ is “good” in the sense

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Keywords: Deformation quantization – Flag manifold – Unitary representation. 
of Borho-Brylinski. Goodness of $\mu$ amounts to $\mathcal{R}$ being generated by the momentum functions $\mu^x$ where $x$ lies in $\mathfrak{g} = \text{Lie}(G)$. Then $\mathcal{R} = S(\mathfrak{g})/I$ and $\mathcal{D} = U(\mathfrak{g})/J$ with $\text{gr} J = I$ for some (two-sided) ideals $I$ and $J$. The good case occurs, for instance, when $G = SL_n(\mathbb{C})$ or if $X$ is the full flag variety.

We solve the EDQ problem for $\mathcal{R}$ in Theorem 6.1, for the good case, by using representation theory to construct a preferred choice of $\star$. We prove the existence and uniqueness of a graded $G$-equivariant star product $\star$ such that the corresponding representation $\pi : \mathfrak{g} \oplus \mathfrak{g} \to \text{End} \mathcal{R}$, $(x, y) \mapsto \pi^{x,y}$, makes $\mathcal{R}$ into the Harish-Chandra module of a unitary representation of $G$. The operators $\pi^{x,y}$ are given by

$$\pi^{x,y}(\phi) = (\mu^x \star \phi - \phi \star \mu^y)_{t=1}.$$ 

In this way, we get a connection between deformation quantization and the orbit method in representation theory. In addition, motivation comes from the constructions in [LO] and [DLO] for certain real flag varieties.

We now outline our construction of $\star$. In fact, we do not directly construct $\star$ but instead we construct a preferred quantization map $q$ in the following way. Results in representation theory of Conze-Berline and Duflo ([C-BD]) and Vogan ([V]) give a canonical embedding $\Delta$ of $\mathcal{D}$ into the space of smooth half-densities on $X$ ($\S 8$); here we regard $X$ as a real manifold. We give a new geometric formula for $\Delta$ in (8.1). The natural pairing $\int_X \alpha \beta$ of half-densities induces a positive definite inner product $\gamma$ on $\mathcal{D}$. The $\gamma$-orthogonal splitting of the order filtration on $\mathcal{D}$ defines our $q$.

In this way, $\mathcal{R}$ acquires a positive definite inner product $\langle \phi | \psi \rangle = \gamma(q(\phi), q(\psi))$ where the grading of $\mathcal{R}$ is orthogonal. Then $\langle \cdot | \cdot \rangle$ is new even if $q$ was unique to begin with (so if the representation of $G$ on $\mathcal{R}$ is multiplicity free). The completion of $\mathcal{R}$ is a new Fock space type model of the unitary representation of $G$ on $L^2$ half-densities on $X$ ($\S 12$). So $\mathcal{R}$ is now the Harish-Chandra module of this unitary representation.

Now $q$ defines a preferred graded $G$-equivariant star product $\star$ on $\mathcal{R}$. We find in Corollary 9.3 that the star product $\mu^x \star \phi$ of a momentum function with an arbitrary function in $\mathcal{R}$ has the form $\mu^x \phi + \frac{1}{2} \{\mu^x, \phi\} t + \Lambda^x(\phi)t^2$ where $\Lambda^x$ is the $\langle \cdot | \cdot \rangle$-adjoint of ordinary multiplication by $\mu^{\sigma(x)}$ ($\sigma$ is a Cartan involution of $\mathfrak{g}$). This property that $\mu^x \star \phi$ is a three term sum uniquely determines $q$ (Proposition 11.1). The $\Lambda^x$ completely determine $\star$, but they are not differential operators in the known examples; see $\S 10$. Thus $\mu^x \star \phi$ is not local in $\phi$.

An important feature is that $\mathcal{D}$ has a natural trace functional $T$ (Proposition 8.4). We give a formula computing $T$ by integration in (8.4).
Then \( \langle \phi | \psi \rangle = \mathcal{T}(q(\phi)q(\psi^\sigma)) \) where \( \sigma \) is some anti-linear involution of \( \mathcal{R} \); see (11.2).

I thank Pierre Bieliavsky, Jean-Luc Brylinski, Michel Duflo, Christian Duval, Simone Gutt, Valentin Ovsienko, Stefan Waldmann, and Alan Weinstein for useful conversations in the summer of 2000. I especially thank David Vogan for discussions in November 1999 which led to this paper. Part of this work was carried out while I was Professeur Invité at the CPT and IML of the Université de la Méditerranée in Marseille, France.

2. Cotangent bundles of flag manifolds.

Let \( G \) be a connected complex semisimple Lie group \( G \). Let \( X \) be a (generalized) flag manifold of \( G \). Then \( X = G/P \) is a projective complex algebraic manifold. The classification of flag manifolds is well known. For example, if \( G = SL_n(\mathbb{C}) \) then the flag manifolds are \( X^d(\mathbb{C}) \) where \( d = (d_1, \ldots, d_s) \) with \( 1 \leq d_1 < \cdots < d_s \leq n-1 \). Here \( X^d(\mathbb{C}) \) parameterizes the flags \( V = (V_1 \subset \cdots \subset V_s) \) in \( \mathbb{C}^n \) where \( \dim V_j = d_j \). The simplest cases are the grassmannians of \( k \)-dimensional subspaces in \( \mathbb{C}^n \).

The cotangent bundle \( T^*X \) is a quasi-projective algebraic manifold. Let \( \mathcal{R} = R(T^*X) \) be the algebra of regular functions on \( T^*X \), in the sense of algebraic geometry. Each regular function is polynomial (of finite degree) on the cotangent fibers. Thus we have the algebra grading

\[
\mathcal{R} = \bigoplus_{d=0}^{\infty} \mathcal{R}^d
\]

by homogeneous degree along the fibers.

The canonical holomorphic symplectic form on \( T^*X \) is algebraic and thus defines a Poisson bracket \( \{ \cdot, \cdot \} \) on \( \mathcal{R} \). Then \( \mathcal{R} \) is a graded Poisson algebra where \{\( \phi, \psi \)\} is homogeneous of degree \( j + k - 1 \) if \( \phi \) and \( \psi \) are homogeneous of degrees \( j \) and \( k \). We have a Poisson algebra anti-automorphism \( \phi \mapsto \phi^\alpha \) given by \( \phi^\alpha = (-1)^d \phi \) if \( \phi \) is homogeneous of degree \( d \).

The action of \( G \) on \( X \) lifts canonically to a Hamiltonian action on \( T^*X \) with moment map \( \mu : T^*X \to g^* \). The moment map embeds the cotangent spaces of \( X \) into \( g^* \). In our example, the cotangent space of \( X^d(\mathbb{C}) \) at \( V \) identifies with the subspace of \( \mathfrak{s}l_n(\mathbb{C}) \) consisting of maps \( e : \mathbb{C}^n \to \mathbb{C}^n \) such that \( e(V_j) \subseteq V_{j-1} \).
The Hamiltonian action of $G$ on $T^*X$ defines a natural (complex linear) representation of $G$ on $\mathcal{R}$. Then $G$ acts on $\mathcal{R}$ by graded Poisson algebra automorphisms which commute with $\alpha$. The corresponding representation of $\mathfrak{g}$ on $\mathcal{R}$ is given by the operators $\{\mu^x, \cdot\}$, $x \in \mathfrak{g}$, where $\mu^x \in \mathcal{R}^1$ are the momentum functions.

$\mathcal{R}$ is the algebra of symbols for (linear) algebraic differential operators acting on sections of a line bundle over $X$.

### 3. Equivariant star product problem for $T^*X$.

Our problem is to construct a preferred graded $G$-equivariant star product (with parity) on $\mathcal{R}$. This means that we want an associative product $\ast$ on $\mathcal{R}[t]$ which makes $\mathcal{R}[t]$ into an algebra over $\mathbb{C}[t]$ in the following way. If $\phi, \psi \in \mathcal{R}$, then the product has the form

$$\phi \ast \psi = \phi \psi + \frac{1}{2} \{\phi, \psi\} t + \sum_{p=2}^{\infty} C_p(\phi, \psi)t^p$$

where the coefficients $C_p$ satisfy

1. $C_p(\phi, \psi) \in \mathcal{R}^{j+k+p}$ if $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^k$
2. $C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi)$
3. $\mu^x \ast \phi - \phi \ast \mu^x = t\{\mu^x, \phi\}$ for all $x \in \mathfrak{g}$.

Axiom (ii) is the parity axiom. (Dropping parity amounts to dropping (ii) and relaxing (3.1) from $= \phi \psi + \frac{1}{2} \{\phi, \psi\} t$ to $\phi \psi + \frac{1}{2} \{\phi, \psi\}$.) Axiom (iii) is often called strong invariance – we use the term “equivariant”. This is an important notion because it corresponds to equivariant quantization of symbols (see §4). Strong invariance implies the weaker notion of invariance, which means that the operators $C_p$ are $G$-invariant.

At $t = 1$, $\ast$ specializes to a noncommutative product on $\mathcal{B} = \mathcal{R}[t]/(t - 1)$. Then, because of axiom (i), $\mathcal{B}$ has an increasing algebra filtration (defined by the grading on $\mathcal{R}$) and the obvious vector space isomorphism $\mathcal{q} : \mathcal{R} \to \mathcal{B}$ induces a graded Poisson algebra isomorphism from $\mathcal{R}$ to $\text{gr}\mathcal{B}$. Via $\mathcal{q}$, the structures on $\mathcal{R}$ pass over to $\mathcal{B}$. Axiom (ii) implies that $\alpha$ defines a filtered algebra anti-involution $\beta$ on $\mathcal{B}$. By (iii), the map $\mathfrak{g} \to \mathcal{B}$ given by $x \mapsto \mathcal{q}(\mu^x)$ is a Lie algebra homomorphism and so we get a representation of $\mathfrak{g}$ on $\mathcal{B}$ by the operators $[\mathcal{q}(\mu^x), \cdot]$. Then $\mathcal{q}$ is $\mathfrak{g}$-equivariant. Consequently, the $\mathfrak{g}$-representation on $\mathcal{B}$ integrates to a locally finite representation of $G$ on $\mathcal{B}$ compatible with everything.
There is an obvious candidate for $B$, namely the algebra $D = D^1_{\text{alg}}(X)$ of algebraic twisted differential operators for the (locally defined) square root of the canonical bundle $K$ on $X$. Fortunately, $D$ already has all the structure discussed above. It has the order filtration and the principal symbol map identifies $\text{gr } D$ with $\mathcal{R}$. (The latter statement follows by [BoBr, Lem. 1.4] – their result goes through to the twisted case with the same proof.) There is a canonical $G$-invariant filtered algebra anti-involution $\beta$ of $D$ such that $\beta(\phi) = \phi$ for $\phi \in \mathcal{R}^0$ and $\beta(\eta_{1/2}) = -\eta_{1/2}$. Here $\eta_{1/2}$ is the Lie derivative of a vector field $\eta$ on $X$. Then $\beta$ induces $\alpha$ upon taking principal symbols. Let $x^\tau$ be the vector field on $X$ defined by $x$. The map

$$g \to D, \quad x \mapsto \eta_{1/2}^x$$

is a Lie algebra homomorphism. The corresponding $g$-representation on $D$ given by the operators $[\eta_{1/2}^x, \cdot]$ integrates to a locally finite representation of $G$ on $D$ compatible with everything.

**4. Quantizing symbols**

*into differential operators equivariantly.*

Now that we have decided upon $B = D$, we can reformulate our star product problem in terms of quantization maps. To begin with, we can axiomatize the properties of our vector space isomorphism $q : \mathcal{R} \to D$ from §3:

\begin{align}
(\text{i}) & \quad \text{if } \phi \in \mathcal{R}^d \text{ then the principal symbol of } q(\phi) \text{ is } \phi \\
(\text{ii}) & \quad q(\phi^\alpha) = q(\phi)^\beta \\
(\text{iii}) & \quad q(\mu^x) = \eta_{1/2}^x \text{ and } q(\{\mu^x, \phi\}) = [\eta_{1/2}^x, q(\phi)] \text{ if } x \in g.
\end{align}

In (iii), we used the semisimplicity of $g$ to get $q(\mu^x) = \eta_{1/2}^x$. Axiom (iii) means that $q$ is $g$-equivariant. This amounts to $G$-equivariance.

We call $q$ a $G$-equivariant quantization map. We can recover $\star$ from $q$ by the formula $\phi \star \psi = q^{-1}(q_\ell(\phi)q_\ell(\psi))$ where $q_\ell(\phi^p) = q(\phi)^{\ell^+p}$ if $\phi \in \mathcal{R}^j$. In this way, we get a bijection between graded equivariant star products on $\mathcal{R}$ and equivariant quantization maps (up to algebra automorphisms of $D$ which are compatible with principal symbols, the $G$-action, etc.).
5. The momentum algebra $\mathcal{R}_\mu$.

The momentum algebra $\mathcal{R}_\mu$ is the subalgebra of $\mathcal{R}$ generated by the momentum functions $\mu^x$, $x \in \mathfrak{g}$. Soon (§6 onwards) we will restrict to the case where $\mathcal{R} = \mathcal{R}_\mu$.

Clearly we may identify $\mathcal{R}_\mu = \mathcal{S}(\mathfrak{g})/I$ where $I$ is a graded ideal in $\mathcal{S}(\mathfrak{g})$. Let $\zeta : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}$ be the algebra homomorphism defined by $\zeta(x) = \eta^x_2$ for $x \in \mathfrak{g}$. Then by restriction we get, for each $p$, a map $\zeta_p$ from the space $\mathcal{U}_p(\mathfrak{g})$ (spanned by all $p$-fold products of elements of $\mathfrak{g}$) to the space $\mathcal{D}_p$ of operators or order at most $p$. Let $J$ be the kernel of $\zeta$; then $J$ is a two-sided ideal in $\mathcal{U}(\mathfrak{g})$.

**Lemma 5.1.** — The following are equivalent:

(i) $\mathcal{R} = \mathcal{R}_\mu$

(ii) $\zeta_p : \mathcal{U}_p(\mathfrak{g}) \rightarrow \mathcal{D}_p$ is surjective for all $p$

(iii) $\zeta : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}$ is surjective and gr $J = I$.

**Proof.** — The associated graded map gr $\zeta$ is the algebra homomorphism $\mathcal{S}(\mathfrak{g}) \rightarrow \mathcal{R}$ defined by $x \mapsto \mu^x$. The result easily follows. □

In the next section we find a preferred $G$-equivariant graded star product on $\mathcal{R}$. We do this under the hypothesis that $\mathcal{R} = \mathcal{R}_\mu$. This is a hypothesis on $(G, X)$ which is satisfied for instance if (i) $G = SL_n(\mathbb{C})$ and $X$ is arbitrary ([KP]), or (ii) $G$ is arbitrary but $X$ is the full flag manifold.

This hypothesis was important in [BoBr] in studying noncommutative analogs of $R(T^*X)$; it is equivalent ([BoBr, Th. 5.6]) to the condition that the moment map $\mu : T^*X \rightarrow \mathfrak{g}^*$ has good geometry in the sense that $\mu$ is generically 1-to-1 and its image in $\mathfrak{g}^*$ is a normal variety. These conditions have been studied a lot in geometric representation theory, especially since the image of $\mu$ is the closure of a single nilpotent coadjoint orbit $\mathcal{O}$ of $G$.

We note that the ideal $I$ contains all casimirs (i.e., $G$-invariants in $\bigoplus_{d=1}^\infty \mathcal{S}^d(\mathfrak{g})$). The casimirs generate $I$ if and only if $X$ is the full flag variety.
6. A preferred star product on $\mathcal{R}$.

Suppose $\phi \ast \psi$ is a graded $G$-equivariant star product on $\mathcal{R}$ (see §3). This defines a noncommutative associative product $\circ$ on $\mathcal{R}$ where $\phi \circ \psi$ is the specialization at $t = 1$ of $\phi \ast \psi$. Then we obtain a representation $\pi$ of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ given by $\pi^{x,y}(\phi) = \mu^x \circ \phi - \phi \circ \mu^y$. Notice that the equivariance axiom (3.2) (iii) says that the quantum operator $\pi^{x,x}$ coincides with the classical operator $\{\mu^x, \cdot\}$.

**THEOREM 6.1.** — Assume $\mathcal{R}$ is generated by $\mu^x$, $x \in \mathfrak{g}$. Suppose $\ast$ is a graded $G$-equivariant star product on $\mathcal{R}$ where $\ast$ corresponds to a $G$-equivariant quantization map $q : \mathcal{R} \to \mathcal{D}$. (Such maps $q$ always exist). Then

(I) The representation $\pi$ of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ is irreducible and unitarizable, i.e., there exists a unique positive definite invariant hermitian form $\langle \cdot | \cdot \rangle$ on $\mathcal{R}$ with $\langle 1 | 1 \rangle = 1$.

(II) There is a unique choice of $q$, and hence a unique choice of $\ast$, such that the grading (2.1) is orthogonal with respect to $\langle \cdot | \cdot \rangle$. Then

\[ \pi^{x,y}(\phi) = \mu^{x-y} \phi + \frac{1}{2} \{\mu^{x+y}, \phi\} + \Lambda^{x-y}(\phi) \]

where $\Lambda^x$, $x \in \mathfrak{g}$, are certain operators on $\mathcal{R}$.

**Proof.** — The proof occupies §7–9. \(\square\)

We now discuss what unitarizable means and introduce some notations. To begin with, the restriction of $\pi$ to $\mathfrak{g}^{\text{diag}} = \{(x,x) \mid x \in \mathfrak{g}\}$, i.e., the $\mathfrak{g}$-representation on $\mathcal{R}$ given by the operators $\pi^{x,x}$, corresponds to the natural $G$-representation on $\mathcal{R}$. Thus $\mathcal{R}$ is a $(\mathfrak{g} \oplus \mathfrak{g}, G)$-module in the sense of Harish-Chandra.

Now *unitarizability* of $\pi$ means that there is a positive definite hermitian inner product $\langle \cdot | \cdot \rangle$ on $\mathcal{R}$ which is invariant for $\mathfrak{g}^* = \{(x,\sigma(x)) \mid x \in \mathfrak{g}\}$, i.e., the operators $\pi^{x,\sigma(x)}$ are skew-hermitian. Here $\sigma$ is a fixed Cartan involution of $\mathfrak{g}$. Then $\sigma$ corresponds to a maximal compact subgroup $G_c$ with Lie algebra $\mathfrak{g}_c = \{x \in \mathfrak{g} \mid x = \sigma(x)\}$. E.g., if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then take $\sigma(x) = -x^*$ so that $\mathfrak{g}_c = \mathfrak{su}_n$.

By a theorem of Harish-Chandra, the operators $\pi^{x,\sigma(x)}$ then correspond to a unitary representation of $G$ on the Hilbert space completion $\hat{\mathcal{R}}$ of $\mathcal{R}$ with respect to $\langle \cdot | \cdot \rangle$. If the $\mathcal{R}_d$ are orthogonal, then $\hat{\mathcal{R}}$ is the Hilbert
space direct sum $\bigoplus_{d=0}^{\infty} \mathcal{R}^d$. Notice that we end up with two very different actions of $G$: the graded algebraic action on $\mathcal{R}$ corresponding to $g^{\text{diag}}$ and the unitary action on $\mathcal{R}$ corresponding to $g^u$.

7. Existence proof for $q$.

A $G$-equivariant quantization map $q$ is completely determined by the subspaces $\mathcal{F}^d = q(\mathcal{R}^d)$. This is immediate from (4.1)(i). Then the decomposition $\mathcal{D} = \bigoplus_{d=0}^{\infty} \mathcal{F}^d$ “splits the order filtration” in the sense that $\bigoplus_{d=0}^{\infty} \mathcal{F}^d = \mathcal{D}_p$. Referring to (4.1) again, we see that the spaces $\mathcal{F}^d$ are stable under $\beta$ and $g$ (which acts by $A \mapsto [\eta^{\frac{r}{2}}, A]$). Conversely, any such splitting corresponds to a choice of $q$.

**Lemma 7.1.** — We can always construct a $G$-equivariant quantization map $q : \mathcal{R} \to \mathcal{D}$. If the representation of $G$ on $\mathcal{R}$ is multiplicity free, there is only one choice for $q$.

*Proof.* — By complete reducibility, we can find a $g$-stable complement $\mathcal{G}^d$ to $\mathcal{D}_{d-1}$ inside $\mathcal{D}_d$. This gives a $g$-stable splitting of the order filtration; let $p$ be the corresponding quantization map. The spaces $\mathcal{G}^d$ may fail to be stable under $\beta$. To remedy this, we “correct” $p$ by putting $p'(\phi) = \frac{1}{2} (p(\phi) + p(\phi^\alpha)^{\beta})$. Now $p'$ is a valid choice for $q$.

If $\mathcal{R}$ is multiplicity free, then $\mathcal{G}^d$ is unique for each $d$, and so $p$ is the unique choice for $q$. Notice that uniqueness of $q$ does not require (ii)-(iii) in (4.1). \hfill $\square$

In the multiplicity free case, the method explained in Remark 9.4 gives a sort of formula for $q$. We note that $\mathcal{R}$ is multiplicity free whenever the parabolic subgroup $P$ (where $X = G/P$) has the property that its unipotent radical is abelian. For $G = SL_n(\mathbb{C})$, this happens when $X$ is a grassmannian. The full classification of multiplicity free cases is well known.

In general, there will be infinitely many choices for $q$.


The quantization map $q$ intertwines our representation $\pi$ of $g \oplus g$ on $\mathcal{R}$ with the representation $\Pi$ of $g \oplus g$ on $\mathcal{D}$ given by $\Pi^{x,y}(A) = \eta^{\frac{r}{2}} A - A\eta^{\frac{y}{2}}$. Indeed, $q(\phi \circ \psi) = q(\phi)q(\psi)$ and so $q(\pi^{x,y}(\phi)) = \Pi^{x,y}(q(\phi))$. 

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Therefore proving $\pi$ is irreducible and unitarizable amounts to proving $\Pi$ is irreducible and unitarizable. For this, we will need our hypothesis that $\mathcal{R}$ is generated by the $\mu^x$.

We can regard $X$ as a real manifold and then consider the algebra $\mathcal{D}_{\mathcal{E}}(X)$ of smooth differential operators on the space $\Gamma(X, \mathcal{E}^{1,1})$ of smooth half-densities on $X$. Notice that the half-density line bundle $\mathcal{E}^{1,1}$ is $G$-homogeneous, and so we get induced actions of $G$ on $\Gamma(X, \mathcal{E}^{1,1})$ and $\mathcal{D}_{\mathcal{E}}(X)$. There is a natural $G$-equivariant filtered algebra embedding $A \mapsto A^1$ of $\mathcal{D}$ into $\mathcal{D}_{\mathcal{E}}(X)$. We put $\xi^x = (\eta^x)^1$; these $\xi^x$ are twisted holomorphic vector fields on $X$.

Let $\delta$ be the unique $G_c$-invariant positive real density on $X$ such that $\int_X \delta = 1$. Let $\delta^{\frac{1}{2}}$ be the positive square root of $\delta$. We map $\mathcal{D}$ into $\Gamma(X, \mathcal{E}^{1,1})$ by

$$\Delta(A) = A^1(\delta^{\frac{1}{2}}).$$

Now $\mathcal{D}$ acquires the $G_c$-invariant hermitian pairing

$$\gamma(A, B) = \int_X \Delta(A)\overline{\Delta(B)}.$$

From now on we assume that the equivalent conditions of Lemma 5.1 are satisfied.

**Proposition 8.1.** — $\gamma$ is $\mathfrak{g}$-invariant and positive definite.

**Proof.** — $\mathfrak{g}$-invariance means that the operators $\Pi^{x,\sigma(x)}$ are skew-hermitian, or equivalently, the adjoint of $\Pi^{x,0}$ is $-\Pi^{0,\sigma(x)}$. So we want to show

$$\gamma(\eta^x, A, B) = \gamma(A, B\eta_{\frac{1}{2}}^{\sigma(x)}).$$

We have $\gamma(\eta^x, A, B) = \int_X (\xi^x \Delta(A)) \overline{\Delta(B)} = -\int_X \Delta(A) \overline{\Delta(B)}(\xi^x \Delta(B))$; the last equality holds because $\int_X \xi^x(\alpha\beta) = 0$ for any half-densities $\alpha, \beta$.

$G_c$-invariance of $\delta^{\frac{1}{2}}$ means that $\xi^x + \overline{\xi^x}$ kills $\delta^{\frac{1}{2}}$ if $x \in \mathfrak{g}_c$, or equivalently $\xi^x + \xi^{\sigma(x)}$ kills $\delta^{\frac{1}{2}}$ if $x \in \mathfrak{g}$. Using this and the commutativity of holomorphic and anti-holomorphic operators we find $\xi^x \overline{\Delta(B)} = -\overline{B^1}\xi^{\sigma(x)}(\delta^{\frac{1}{2}}) = -\Delta(B\eta_{\frac{1}{2}}^{\sigma(x)})$ and so we get (8.2).

For positive definiteness, we just need to show that $\Delta$ is 1-to-1 on $\mathcal{D}$. We expect there is a geometric proof of this, but we have not worked that out. Instead, we will use results from representation theory.
\( \Delta \) is \( G_c \)-equivariant and so \( \Delta \) maps \( \mathcal{D} \) into the space \( \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\text{fin}} \) of \( G_c \)-finite smooth half-densities. On the other hand we have the maps

\[
\mathcal{D} \xrightarrow{\Psi} \text{End}_{\mathfrak{g}-\text{fin}}(M_{\mathfrak{p},-\nu}) \xrightarrow{\Phi} \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\text{fin}}.
\]

Here \( \text{End}_{\mathfrak{g}-\text{fin}}(M_{\mathfrak{p},-\nu}) \) is the algebra of \( \mathfrak{g} \)-finite endomorphisms of the generalized Verma module \( M_{\mathfrak{p},-\nu} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_{-\nu} \) where \( X = G/P, \mathfrak{p} = \text{Lie}(P), \nu : \mathfrak{p} \rightarrow \mathbb{C} \) is the Lie algebra homomorphism defined by \( \nu(x) = -\frac{1}{2} \text{tr} \text{ad}_{\mathfrak{g}/\mathfrak{p}}(x) \). To define \( \Psi \), we consider the natural action of \( \mathcal{D} \) on \( \Gamma(X_o, \mathcal{K}^{\frac{1}{2}}) \). There is a non-degenerate \( \mathfrak{g} \)-invariant bilinear pairing

\[
\omega : \Gamma(X_o, \mathcal{K}^{\frac{1}{2}}) \times M_{\mathfrak{p},-\nu} \rightarrow \mathbb{C}.
\]

It follows that \( \mathcal{D} \) acts faithfully on \( M_{\mathfrak{p},-\nu} \) so that \( \omega(s, A(m)) = \omega(A^{\delta}(s), m) \) where \( s \) is a section and \( m \in M_{\mathfrak{p},-\nu} \). In this way we get a 1-to-1 algebra homomorphism \( \Psi \). Next, \( \Phi \) is the map defined by Conze-Berline and Duflo in [C-BD, §5.3]. (This is the “\( \pi_1 = \pi_2 = 0 \)” case in their notation.) Both maps \( \Psi \) and \( \Phi \) are \( \mathfrak{g} \oplus \mathfrak{g} \)-equivariant; here \( \mathfrak{g} \oplus \mathfrak{g} \) acts on \( \Gamma(X, \mathcal{E}^{\frac{1}{2}}) \) by the twisted vector fields \( \xi^x, \gamma = \xi^x + \xi^{\sigma(y)} \).

The map \( \Phi \) is an isomorphism. Indeed, \( M_{\mathfrak{p},-\nu} \) is irreducible by Vogan’s result [V, Prop. 8.5]. (This is the case “\( \lambda - \rho(l) = 0 \)” in his notation.) So [C-BD, Proposition 5.5] applies and says \( \Phi \) is an isomorphism.

Thus the composite map \( \Phi \Psi \) in (8.3) is 1-to-1. It is easy to compare \( \Phi \Psi \) with \( \Delta \). Both maps are \( \mathfrak{g} \oplus \mathfrak{g} \)-equivariant and send 1 to a non-zero multiple of \( \delta^{\frac{1}{2}} \). It follows, since by hypothesis \( \mathcal{D} \) is a quotient of \( \mathcal{U}(\mathfrak{g}) \) (cf. Lemma 5.1(iii)), that \( \Phi \Psi \) is just a scalar multiple of \( \Delta \). Consequently, \( \Delta \) is 1-to-1.

**Corollary 8.2.** — \( \Delta \) is an isomorphism, of \((\mathfrak{g} \oplus \mathfrak{g})\)-representations, from \( \mathcal{D} \) onto the Harish-Chandra module of the natural unitary representation of \( G \) on \( L^2(X, \mathcal{E}^{\frac{1}{2}}) \).

**Proof.** — The Harish-Chandra module is \( \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\text{fin}} \). We just established injectivity of \( \Delta \). Surjectivity follows because the source and target contain the same irreducible \( G_c \)-representations with the same multiplicities. Indeed, \( \mathcal{D} \simeq \mathcal{R} \simeq R(G/L) \) where \( L \) is any Levi factor of \( P \). We may choose \( L \) so that \( L_c = L \cap G_c \) is a compact form of \( L \). Then \( R(G/L) \simeq C^\infty(G_c/L_c)^{G_c-\text{fin}} \simeq \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\text{fin}} \).

**Corollary 8.3.** — There is a unique anti-linear algebra involution \( \theta \) of \( \mathcal{D} \) such that \( \theta(\eta^\gamma_{\frac{1}{2}}) = \eta^{\sigma(x)}_{\frac{1}{2}} \). Then \( \gamma(A, B) = \gamma(B^\theta A, 1) \).
Proof. — The formula $\Delta(B^\beta) = B^\theta(\delta^\frac{1}{2})$ defines an anti-linear map $B \mapsto B^\theta$. Then $B = \eta^{\frac{x_1}{2}} \ldots \eta^{\frac{x_m}{2}}$ gives $B^\theta = \eta^{\sigma(x_1)} \ldots \eta^{\sigma(x_m)}$. Since the $\eta^{\frac{x}{2}}$ generate $\mathcal{D}$ by hypothesis, it follows that $\theta$ is an anti-linear algebra involution. Now (8.2) gives $\gamma(A,B) = \gamma(B^\theta A,1)$. □

The formula $T(A) = \gamma(A,1)$ defines a linear functional $T$ on $\mathcal{D}$. Explicitly,

$$T(A) = \int_X A^!(\delta^\frac{1}{2})\delta^\frac{1}{2}. \tag{8.4}$$

Then $\gamma(A,B) = T(B^\theta A)$.

PROPOSITION 8.4. — $T$ is the unique $G_c$-invariant linear functional on $\mathcal{D}$ with $T(1) = 1$. Moreover $T$ is a trace.

Proof. — Clearly $T$ is $G_c$-invariant. Then $T : \mathcal{D} \to \mathbb{C}$ is the unique invariant linear projection because the $G_c$-action on $\mathcal{D}$ is completely reducible and the constants are the only $G_c$-invariants in $\mathcal{D}$ (since the constants are the only $G_c$-invariants in $\mathcal{R}$).

$T$ is $g^2$-invariant, i.e., $T([\eta^{\frac{x}{2}},A]) = 0$. We can write this as $T(\eta^{\frac{x}{2}}A) = T(A\eta^{\frac{x}{2}})$. Iteration gives $T(\eta^{x_1} \ldots \eta^{x_k}A) = T(A\eta^{x_1} \ldots \eta^{x_k})$. This proves $T(BA) = T(AB)$ since the $\eta^{\frac{x}{2}}$ generate $\mathcal{D}$ by hypothesis. □

Now we can show that $\gamma$ is the unique $g^2$-invariant hermitian form on $\mathcal{D}$ such that $\gamma(1,1) = 1$. Indeed suppose $\lambda$ is any such form. Then $\lambda(A,1) = T(A)$ by the uniqueness of $T$. So (8.2) gives $\lambda(A,B) = \lambda(B^\theta A,1) = T(B^\theta A) = \gamma(A,B)$. This uniqueness of $\gamma$ implies $\Pi$ is irreducible.

This completes the proof of Theorem 6.1(I). Once $q$ is chosen, $\langle \cdot | \cdot \rangle$ is given by

$$\langle \phi | \psi \rangle = \gamma(q(\phi),q(\psi)) = T(q(\phi)q(\psi)^\theta). \tag{8.5}$$

Finally we record

COROLLARY 8.5. — $\Pi$ is irreducible. Equivalently, $\mathcal{D}$ is a simple ring.

The graded pieces \( R^d \) are orthogonal with respect to \( \langle \cdot | \cdot \rangle \) iff their images \( q(R^d) \) are orthogonal with respect to \( \gamma \). So we have only one possible choice of \( q \), namely the one such that \( q(R^d) = V^d \) where \( \bigoplus_{d=0}^{\infty} V^d \) is the \( \gamma \)-orthogonal splitting of the order filtration of \( D \). According to §7, we need to check

**Lemma 9.1.** \( V^d \) is stable under \( \beta \) and \( g \).

**Proof.** This follows because \( \gamma \) is invariant under \( \beta \) and \( G_c \). We obtain \( \beta \)-invariance using \( T(A^\beta) = T(A) \), \( \beta \theta = \theta \beta \) (clear since \( D \) is a quotient of \( U(g) \)), and \( T(AB) = T(BA) \). \( \square \)

Thus \( \bigoplus_{d=0}^{\infty} V^d \) defines \( q \). Then \( q \) defines a graded \( G \)-equivariant star product \( * \) on \( R \); this is the only one for which the direct sum \( \bigoplus_{d=0}^{\infty} R^d \) is \( \langle \cdot | \cdot \rangle \)-orthogonal.

**Proposition 9.2.** This star product \( * \) satisfies

\[
R^j \ast R^k \subseteq R^{j+k} \oplus \cdots \oplus R^{l - k} t^{2 \min(j,k)}. \tag{9.1}
\]

**Proof.** Since \( * \) is graded, it suffices to consider \( \circ \). Let \( \ell(\phi) \) and \( r(\phi) \) denote respectively left and right \( \circ \)-multiplication by \( \phi \). The map \( \mu^x \mapsto \mu^\sigma(x) \) extends to a graded anti-linear algebra involution \( \phi \mapsto \phi^\sigma(x) \) of \( R \); this follows because the complex nilpotent orbit \( O \) (defined in §5) is \( \sigma \)-stable. We claim that the adjoint with respect to \( \langle \cdot | \cdot \rangle \) of \( \ell(\phi) \) is \( r(\phi^\sigma) \).

Using this we can show: if \( \phi \in R^j \), \( \psi \in R^k \) and \( \nu \in R^d \) occurs in \( \phi \circ \psi \), then \( j + k \geq d \geq |j - k| \). Indeed, the highest degree term in \( \phi \circ \psi \) is \( \phi \psi \) and this lies in \( R^{j+k} \). Now \( \nu \) occurs in \( \phi \circ \psi \) implies that \( \psi \) occurs in \( \nu \circ \phi^\sigma \) and so \( d + j \geq k \). Similarly \( d + k \geq j \).

To verify that \( r(\phi^\sigma) \) is adjoint to \( \ell(\phi) \), we will use our hypothesis (cf. Lemma 5.1(ii)) that, for each \( p, U_p(g) \) maps onto \( D_p \). With this, it follows that \( \theta \) preserves the filtration components \( D_p \) and moreover \( \theta \) induces \( \sigma \) on \( \text{gr } D = R \). Now (8.2) implies that \( r(\phi^\sigma) \) is adjoint to \( \ell(\phi) \). \( \square \)

**Corollary 9.3.** For \( x \in g \) and \( \phi \in R \) we have

\[
\mu^x \ast \phi = \mu^x \phi + \frac{1}{2} \{ \mu^x, \phi \} t + \Lambda^x(\phi) t^2. \tag{9.2}
\]
where $\Lambda^x$ is the adjoint with respect to $\langle \cdot | \cdot \rangle$ of ordinary multiplication by $\mu^x$.

Proof. Certainly (9.1) implies (9.2) where $\Lambda^x(\psi) = C_2(\mu^x, \psi) = C_2(\psi, \mu^x)$. Now suppose $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^{j+1}$. Because of orthogonality of the spaces $\mathcal{R}^d$ we find $\langle \phi | \Lambda^x(\psi) \rangle = \langle \phi | \psi \circ \mu^x \rangle = \langle \mu^{\sigma(x)} \circ \phi | \psi \rangle = \langle \mu^{\sigma(x)} \phi | \psi \rangle$.

Now (9.2) gives (6.1). This concludes the proof of Theorem 6.1.

Remark 9.4. We know another method for constructing a $G$-equivariant quantization map $r : \mathcal{R} \to \mathcal{D}$. We start with the positive definite $G$-invariant hermitian pairing $\lambda(f, g) = \partial_g(f)$ on $\mathcal{S}(\mathfrak{g})$, where $\partial_x$ is the constant coefficient vector field on $\mathfrak{g}$ defined by $\partial_x(y) = -\langle \sigma(x), y \rangle$ and $\partial_{g_1g_2} = \partial_{g_1}\partial_{g_2}$. Let $H$ be the $\lambda$-orthogonal complement to $I$ in $\mathcal{S}(\mathfrak{g})$ where $\mathcal{R} = \mathcal{S}(\mathfrak{g})/I$. Then $H = \oplus_{d=0}^{\infty} H^d$ is graded. We put $\mathcal{F}^d = \zeta(s(H^d))$, where $s : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is the symmetrization map. Then $\mathcal{D} = \oplus_{d=0}^{\infty} \mathcal{F}^d$ is a $\mathfrak{g}$-stable and $\beta$-stable splitting of the order filtration. So by §7 this splitting defines $r$.

Here is a formula for $r$: if we pick a basis $x_1, \ldots, x_m$ of $\mathfrak{g}$ and $\sum a_{i_1, \ldots, i_d} x_{i_1} \cdots x_{i_d}$ lies in $H^d$, then

$$
(9.3) \quad r \left( \sum a_{i_1, \ldots, i_d} \mu^{x_{i_1}} \cdots \mu^{x_{i_d}} \right) = \frac{1}{d!} \sum_{\tau} a_{i_1, \ldots, i_d} \eta^{x_{i_{\tau(1)}}}_{1/2} \cdots \eta^{x_{i_{\tau(d)}}}_{1/2}
$$

where we sum over all permutations $\tau$ of $\{1, \ldots, d\}$.

We conjecture that $\mathcal{F}^d = \mathcal{V}^d$, or equivalently, that $r = q$. This is obviously true in the multiplicity free case by uniqueness (Lemma 7.1). Analytic methods may well be needed to show $r = q$, just as we needed integration to establish the positivity of $\gamma$ (or even the weaker fact that $\gamma$ is non-degenerate on each space $\mathcal{D}_d$).

Suppose $X$ is the full flag manifold. Then $H$ is Kostant’s space of harmonic polynomials, and $r$ is simply a $\rho$-shifted version of the map constructed by Cahen and Gutt in [CG] for the principal nilpotent orbit case.

10. The operators $\Lambda^x$ on $\mathcal{R}$.

In Corollary 9.3 we saw that our star product $\ast$ produces operators $\Lambda^x, x \in \mathfrak{g}$, on $\mathcal{R}$. Conversely, the $\Lambda^x$ completely determine $\ast$. This follows
because if we know the $\Lambda^x$, then using associativity we can compute $(\mu^{x_1} \cdots \mu^{x_n}) \star \psi$ by induction on $n$. Here (9.2) provides the first step $n = 1$, and also it propels the induction.

Several nice properties follow from Corollary 9.3:

(i) $\Lambda^x$ is graded of degree $-1$, i.e., $\Lambda^x(\mathcal{R}^j) \subseteq \mathcal{R}^{j-1}$.

(ii) The $\Lambda^x$ commute and generate a graded subalgebra of $\text{End } \mathcal{R}$ isomorphic to $\mathcal{R}$.

(iii) The $\Lambda^x$ transform in the adjoint representation of $\mathfrak{g}$, i.e., $[\Phi^x, \Lambda^y] = \Lambda^{[x,y]}$ where $\Phi^x = \{\mu^x, \cdot\}$

(iv) The map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(x, y) \mapsto \Lambda^x(\mu^y)$, is a non-degenerate $\mathfrak{g}$-invariant symmetric complex bilinear pairing.

The $\Lambda^x$ are not differential operators on $\mathcal{R}$ in general. Indeed differentiability fails when $G = SL_{n+1}(\mathbb{C})$ and $X = \mathbb{CP}^k$. In that case $\Lambda^x$ is a reasonably nice operator as it is the left quotient of an algebraic differential operator $L^x$ (of order 4) on the closure on $\mathcal{O}$ by the invertible operator $(E + \frac{n}{2})(E + \frac{n}{2} + 1)$. Moreover $L^x$ extends to $T^*\mathbb{CP}^k$. See [AB2], [LO] and [B].

The $\Lambda^x$ determine $\star$ in a rather simple way, and so their failure to be differential should control the failure of $\star$ to be bidifferential.

We conjecture that $\Lambda^x$ is of the form $\Lambda^x = P^{-1}L^x$ where (i) $P$ and $L^x$ are algebraic differential operators on $T^*X$, (ii) $P$ is $G$-invariant and vertical so that $P$ “acts along the fibers of $T^*X \to X$” (iii) $P$ is invertible on $\mathcal{R}$, in fact $P$ is diagonalizable with positive spectrum and (iv) the formal order of $P^{-1}L^x$ is 2.

The case where $G = SL_{n+1}(\mathbb{C})$ and $X = \mathbb{CP}^k$ is an example where (i)-(iv) works. This example was part of a quantization program for minimal nilpotent orbits (see [AB1, §1]). In fact, our conjecture here arises from a larger program we have on quantization of general nilpotent orbits. A proof of our conjecture, coming most likely out of properties of $\langle \cdot | \cdot \rangle$, would give more evidence for our program.

11. The inner product $\langle \cdot | \cdot \rangle$ on $\mathcal{R}$.

In Theorem 6.1, the hermitian form $\langle \cdot | \cdot \rangle$ completely determines the star product $\star$, and vice versa. To show this, it suffices (see §10) to show
that knowing $\langle \cdot | \cdot \rangle$ is equivalent to knowing the $\Lambda^x$. Certainly $\langle \cdot | \cdot \rangle$ produces $\Lambda^x$, as $\Lambda^x$ is (Corollary 9.3) the adjoint of $\phi \mapsto \mu^{\sigma(x)}(\phi)$. Conversely, suppose we know the $\Lambda^x$. To produce $\langle \cdot | \cdot \rangle$, we only need to compute $\langle \phi | \psi \rangle$ for $\phi, \psi \in \mathcal{R}^d$, since $\mathcal{R}^j$ is orthogonal to $\mathcal{R}^k$ if $j \neq k$. By adjointness again we find

\[(\mu^{x_1} \cdots \mu^{x_d} | \psi) = \Lambda^{\sigma(x_1)} \cdots \Lambda^{\sigma(x_d)}(\psi), \quad \text{if } \psi \in \mathcal{R}^d.\]

The cleanest formula for $\langle \phi | \psi \rangle$ comes from (8.5). Let $\mathcal{T} : \mathcal{R} \to \mathbb{C}$ be the projection operator defined by the grading of $\mathcal{R}$. Notice that $\mathcal{T}$ is classical, i.e., we know it before we quantize anything. Recall the map $\phi \mapsto \phi^\sigma$ from the proof of Proposition 9.2; this is also classical. $\mathcal{T}$ and $\mathcal{T}$ correspond via $\mathcal{q}$ and so $\mathcal{T}$ is a $\sigma$-trace by Proposition 8.4; we view this as the “quantum analog” of the fact that $\mathcal{T}$ vanishes on Poisson brackets. So (8.5) gives

\[(\phi | \psi) = \mathcal{T}(\mathcal{q}(\phi)\mathcal{q}(\psi^\sigma)) = \mathcal{T}(\phi \circ \psi^\sigma), \quad \phi, \psi \in \mathcal{R}.\]

For $\phi, \psi \in \mathcal{R}^d$, this reduces to $\langle \phi | \psi \rangle = C^R_{2d}(\phi, \psi^\sigma)$ where $C^R_p$ are the coefficients of $\star$.

We can now characterize $\star$ without the explicit use of symmetry and unitarity.

**Proposition 11.1.** — *The preferred star product $\star$ on $\mathcal{R}$ we found in Theorem 6.1 is uniquely determined by just the two properties: (i) $\star$ corresponds to a $G$-equivariant quantization map $\mathcal{q} : \mathcal{R} \to \mathcal{D}$, and (ii) $\star$ satisfies (9.2) where the $\Lambda^x$ are any operators.*

**Proof.** — Suppose $\star$ satisfies (i) and (ii). Then, since $\mathcal{R} = \mathcal{R}_u$ by hypothesis, $\star$ satisfies (9.1) and so $\mathcal{T}(R^j \circ R^k) = 0$ for $j \neq k$. Equivalently, $\mathcal{T}(\mathcal{V}^j \mathcal{V}^k) = 0$ if $j \neq k$. We claim that this uniquely determines $\oplus_{d=0}^\infty \mathcal{V}^d$ among all $\mathfrak{g}$-stable splittings of the order filtration of $\mathcal{D}$. For it implies that the spaces $\mathcal{V}^d$ are orthogonal with respect to the symmetric bilinear pairing $\lambda(A, B) = \mathcal{T}(BA)$. But we know $\lambda$ is non-degenerate on $\mathcal{V}^d$; this follows because $\mathcal{V}^d$ is $\theta$-stable and $\lambda(A, \gamma^\theta A) = \gamma(A, A)$ is positive if $A \neq 0$. So there is only one $\lambda$-orthogonal splitting. This proves our claim.

**12. $\widehat{\mathcal{R}}$ is a Fock space type model of $L^2(X, \mathcal{E}^{1/2})$.**

Combining the discussion in §6 with our work in §8, we find Theorem 6.1 gives
Corollary 12.1. — The Hilbert space completion $\hat{\mathcal{R}} = \bigoplus_{d=0}^{\infty} \mathcal{R}^d$ of $\mathcal{R}$ with respect to $\langle \cdot , \cdot \rangle$ becomes a holomorphic model for the unitary representation of $G$ on $L^2(X, \mathcal{E}^{1/2})$. We have, for the Harish-Chandra modules, the explicit intertwining isomorphism

$$\mathcal{R} \xrightarrow{\alpha} \mathcal{D} \xrightarrow{\Lambda} \Gamma(X, \mathcal{E}^{1/2})^G_{c - \text{fin}}.$$  

While $L^2(X, \mathcal{E}^{1/2})$ is itself a Schrödinger type model, our $\hat{\mathcal{R}}$ is a generalization of the Fock space model of the oscillator representation of the metaplectic group. This follows for three reasons. First, $\hat{\mathcal{R}}$ is a Hilbert space of “polynomial” holomorphic functions. (We conjecture that $\hat{\mathcal{R}}$ is a Hilbert space of holomorphic functions on $T^*X$. This is proven when $G = \text{SL}_{n+1}(\mathbb{C})$ and $X = \mathbb{CP}^k$ in [AB2, Cor. 9.3].)

Second, the action of the skew-hermitian operators $\pi^{x, \sigma(x)}$ corresponding to the non-compact part of $\mathfrak{g}^\sharp$ is given by creation and annihilation operators. For the non-compact part of $\mathfrak{g}^\sharp$ is $\{(ix, -ix) \mid x \in \mathfrak{g}_c\}$ and (6.1) gives

$$\pi^{ix, -ix} = 2\mu^{ix} + 2\Lambda^{ix}.$$  

The multiplication operators $\mu^{ix}$ are “creation” operators mapping $\mathcal{R}^d$ to $\mathcal{R}^{d+1}$, while the $\Lambda^{ix}$ are “annihilation” operators mapping $\mathcal{R}^d$ to $\mathcal{R}^{d-1}$.

Third, the operators $\pi^{x,x}$ corresponding to the compact part $\{(x, x) \mid x \in \mathfrak{g}_c\}$ of $\mathfrak{g}^\sharp$ are just the derivations $\{\mu^x, \cdot \}$ and these map $\mathcal{R}^d$ to $\mathcal{R}^d$. Notice that the multiplication operators $\mu^{ix}$ and the derivations $\{\mu^x, \cdot \}$ are classical objects, while the $\Lambda^{ix}$ are quantum objects (which encode $\langle \cdot , \cdot \rangle$).

This gives new examples in the orbit method. For $\mathcal{R}$ identifies with the algebra of $G$-finite holomorphic functions on the complex nilpotent orbit $\mathcal{O}$ associated to $T^*X$ (see §5). We may regard $\mathcal{O}$ as a real coadjoint orbit of $G$. Then Theorem 6.1 and Corollary 12.1 give a quantization of $\mathcal{O}$ (with respect to a certain $G_c$-invariant complex polarization).

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Manuscrit reçu le 26 mars 2001,
accepté le 15 mai 2001.

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