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COMPLEXITY AND GROWTH FOR POLYGONAL BILLIARDS

by J. CASSAIGNE, P. HUBERT & S. TROUBETZKOY

1. Introduction.

A billiard ball, i.e. a point mass, moves inside a polygon $Q \subset \mathbb{R}^2$ with unit speed along a straight line until it reaches the boundary ∂Q, then instantaneously changes direction according to the mirror law "the angle of incidence is equal to the angle of reflection," and continues along the new line.

How complex is the game of billiards in a polygon? The first results in this direction, proven independently by Sinai [S] and Boldrighini, Keane and Marchetti [BKM] is that the metric entropy with respect to the invariant phase volume is zero. Sinai's proof in fact shows more, the "metric complexity" grows at most polynomially. Furthermore, it is known that the topological entropy (in various senses) is zero [K], [GKT], [GH].

To prove finer results there are two natural quantities one can count, one is the number of generalized diagonals, that is (oriented) orbit segments which begin and end in a vertex of the polygon and contain no vertex of the polygon in their interior. The number of links of a generalized diagonal is called its combinatorial length while its geometric length is simply the sum of the lengths of the segments. Let $N_g(t)$ (resp. $N_c(n)$) be the number of generalized diagonals of geometric (resp. combinatorial) length at most t (resp. n). Katok has shown that $N_g(t)$ grows slower than any exponential [K]. Masur has shown that for rational polygons, that is for polygons all of whose inner angles are commensurable with π, $N_g(t)$ has

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quadratic upper and lower bounds [M1], [M2]. By elementary reasoning there is a constant $B > 1$ such that $B^{-1} \leq N_c(t)/N_g(t) \leq B$, thus all of these results easily extend to the quantity $N_c(n)$. Furthermore, Veech has shown that there is a special class of polygons now commonly referred to as Veech polygons, for example regular polygons, such that the quantity $N_g(t)/t^2$ admits a limit as t tends to infinity [V], [V1].

To introduce the second natural quantity which can be counted, label the sides of Q by symbols from a finite alphabet $\mathcal{A}$ whose cardinality is equal to the number of sides of Q. We code the orbit by the sequence of sides it hits. Consider the set $\mathcal{L}(n)$ of all words of length n which arise via this coding. Let $p(n) = \#\mathcal{L}(n)$, this is called the complexity function of the language $\mathcal{L}(\cdot)$. The only general results known about the complexity function is that it grows slower than any exponential [K] and at least quadratically [Tr]. For billiards in a square the complexity function has been explicitly calculated, albeit for a slightly different coding (the alphabet consists of two symbols, one for vertical sides one for horizontal sides) [Mi], [BP]. For this coding of the square the collection of codes which appear are known as the Sturmian sequences. In fact it is not hard to relate the complexity functions for the two different codings, the relationship is $p_4(n) = 4p_2(n) - 4$. There are some related results on the complexity when one restricts to certain initial conditions: for rational polygons the "directional complexity" in each direction is known explicitly [H1], while for general polygons there are polynomial upper bounds for the directional complexity [GT].

There are several good surveys of billiards in polygons, in these surveys one can find more details about the definitions and more precise statements of the above mentioned results. We refer the reader to [Gu1], [Gu2], [MT], [T].

Our main theorem shows that $p(n)$ and $N_c(n)$ are related.

THEOREM 1.1. — *For any convex*[(1)] *polygon*

$$p(n) = \sum_{j=0}^{n-1} N_c(j).$$

Here we remark that $N_c(0)$ is the number of vertices of the polygon while the sides of Q are not counted as generalized diagonals. Applying the above mentioned results of Masur's [M1], [M2] we immediately conclude

[(1)] Recently N. Bedaride found a proof of Theorem 1.1 without the assumption of convexity.

Corollary 1.2. — *If Q is a rational convex polygon then there are positive constants D_1, D_2 such that*

$$D_1 < \frac{p(n)}{n^3} < D_2$$

for all $n \in \mathbb{N} \setminus \{0\}$.

Next we exhibit several examples where there are exact asymptotics. We show

Theorem 1.3. — *If Q is the square, the isosceles right triangle or the equilateral triangle then*

$$\lim_{n\to\infty} \frac{p(n)}{n^3} \tag{1}$$

exists. The following table expresses the limit:

Square : $\frac{4}{\pi^2}$	$\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$*-triangle* : $\frac{2}{3\pi^2}$	*Equilateral triangle* : $\frac{3}{4\pi^2}$

The proof of Theorem 1.1 is split into two parts. The first part is combinatorial. It uses the notion of bispecial words which was developed by Cassaigne [C]. The second part is geometric and uses a counting argument based on Euler's formula.

Remark. — It is known that for n sufficiently large the complexity of each aperiodic individual word is $4(n+1)$ for the square, $3(n+2)$ for the equilateral triangle, $4(n+2)$ for the isosceles right triangle and $6(n+2)$ for the half equilateral triangle [H], [H1], [H2]. For the square the complexity is four times larger than that of Sturmian sequences, thus the fact that $p(n)$ is asymptotically four times the number of Sturmian words of length n is not surprising [Mi], [BP].

Any infinite word of eventual complexity $3(n+2)$ whose language (i.e. the collection of finite factors) is invariant under cyclic permutations of the letters arises as the coding of a billiard trajectory in the equilateral triangle [H]. The third entry of the table in Theorem 1.3 gives the asymptotic growth rates of the number of all such words.

Two interesting tiling cases remaining to evaluate the limit (1) are the $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{6}\pi)$-triangle and the hexagon. In this triangular case the methods developed for the other three cases allow us to conclude that this limit exists and to calculate it explicitly. Since the combinatorics of this case is more complicated than the others, we leave its explicit computation to the dedicated reader.

The hexagonal case remains open since the corresponding lattice point counting problem seems not to have been investigated.

It would be interesting to know if the limit (1) exists in the case of Veech polygons and also to exhibit cases when it does not exist.

2. A combinatorial lemma.

Let $p(0) := 0$ and for any $n \geq 1$ let $s(n) := p(n+1) - p(n)$. For $u \in \mathcal{L}(n)$ let

$$m_\ell(u) := \#\{a \in \mathcal{A} : au \in \mathcal{L}(n+1)\},$$

$$m_r(u) := \#\{b \in \mathcal{A} : ub \in \mathcal{L}(n+1)\},$$

$$m_b(u) := \#\{(a,b) \in \mathcal{A}^2 : aub \in \mathcal{L}(n+2)\}.$$

We remark that all three of these quantities are larger than or equal to one. A word $u \in \mathcal{L}(n)$ is called *left special* if $m_\ell(u) > 1$, *right special* if $m_r(u) > 1$ and *bispecial* if it is left and right special. Let

$$\mathcal{BL}(n) := \{u \in \mathcal{L}(n) : u \text{ is bispecial}\}.$$

In this section we show that

THEOREM 2.1. — *For any polygon Q*

$$s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} \big(m_b(v) - m_\ell(v) - m_r(v) + 1\big).$$

Remark. — There is no assumption of convexity for this theorem, in fact it is not necessary that the language arises from the coding of a polygonal billiard.

Proof. — Since for every $u \in \mathcal{L}(n+1)$ there exist $b \in \mathcal{A}$ and $v \in \mathcal{L}(n)$ such that $u = vb$ we have

$$s(n) = \sum_{u \in \mathcal{L}(n)} \big(m_r(u) - 1\big).$$

Thus

$$s(n+1) - s(n) = \sum_{v \in \mathcal{L}(n+1)} \big(m_r(v) - 1\big) - \sum_{u \in \mathcal{L}(n)} \big(m_r(u) - 1\big).$$

For $u \in \mathcal{L}(n+1)$ we can write $u = av$ where $a \in \mathcal{A}$ and $v \in \mathcal{L}(n)$, thus

$$s(n+1) - s(n) = \sum_{v \in \mathcal{L}(n)} \Big[\sum_{av \in \mathcal{L}(n+1)} \big(m_r(av) - 1\big) - \big(m_r(v) - 1\big)\Big].$$

For any word $v \in \mathcal{L}(n)$ and $av \in \mathcal{L}(n+1)$ any legal prolongation to the right of av is a legal prolongation to the right of v as well thus if $m_r(v) = 1$ then $m_r(av) = 1$. Thus words with $m_r(v) = 1$ do not contribute to the above sum. Thus $s(n+1) - s(n)$ is equal to the above sum restricted to those v such that $m_r(v) > 1$. If furthermore $m_\ell(v) = 1$ then there is only a single a such that $av \in \mathcal{L}(n+1)$. For this a we have $m_r(av) = m_r(v)$ thus such words do not contribute to the sum either. Thus we can restrict the sum to bispecial words, yielding

$$s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} \Big[\sum_{av \in \mathcal{L}(n+1)} \big(m_r(av) - 1\big) - \big(m_r(v) - 1\big)\Big].$$

The lemma follows since for any $v \in \mathcal{BL}(n)$ we have

$$m_b(v) = \sum_{av \in \mathcal{L}(n+1)} m_r(av) \quad \text{and} \quad m_\ell(v) = \sum_{av \in \mathcal{L}(n+1)} 1. \qquad \square$$

3. Proof of Theorem 1.1.

Let $X := \{(s,v) : s \in \partial Q$ and v is an inner pointing unit vector$\}$ and P the "partition" of X induced by the sides of Q. The ambiguity of the definition of P at the vertices plays no role in our discussion. Let $T : X \to X$ be the billiard ball map. An element of the partition $P \vee T^{-1}P \vee \ldots \vee T^{-n+1}P$ is called an n-cell. The code of every point in a n-cell has the same prefix of length n, thus there is a bijection between the set of n-cells and the language $\mathcal{L}(n)$.

If the footpoint of $T^n x$ is a vertex then we say that x belongs to a discontinuity of order n. A discontinuity (of any order) is locally a curve whose endpoints lie on the boundary of X or on a discontinuity of lower order. We call each piece between such endpoints a smooth branch of the discontinuity.

For $v \in \mathcal{BL}(n)$ let $\mathrm{gd}(v)$ be the number of generalized diagonals of length $n+1$ such that the code of (the nonsingular part of) the generalized diagonal is v (see Figure 1). Let

$$I_\ell(v) := m_\ell(v) - 1 \quad \text{and} \quad I_r(v) := m_r(v) - 1.$$

For short we call $(I_\ell(v), I_r(v), \mathrm{gd}(v))$ the *index* of v.

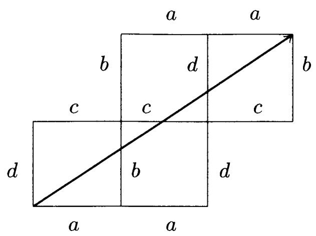


Figure 1. A generalized diagonal of combinatorial length 4 with code bcd.

LEMMA 3.1. — *Suppose that Q is a convex polygon. For any $v \in \mathcal{BL}(n)$*

$$m_b(v) = I_\ell(v) + I_r(v) + \mathrm{gd}(v) + 1.$$

Proof. — We consider the n-cell C with bispecial code v. Note that a n-cell is a convex polygon [K], thus geometrically the number $m_b(v)$ corresponds to the number of pieces C is cut into by the discontinuities of order -1 and n.

Let r be the number of sides of Q. There are $I_\ell(v) \leq r$ vertices of Q which produce the splitting on the left, they cut C via singularities of T^{-1}. Similarly there are $I_r(v) \leq r$ vertices which produce the splitting on the right, they correspond to cutting C via singularities of T^n.

Suppose the index of v is (i, j, k). The cell C is cut by $i + j$ singularities with k intersections inside the interior of C. We claim that since Q is convex, each of these k intersections consists of an intersection of exactly two smooth branches of the singularities. Consider an intersection point x. Its forward orbit arrives at a vertex in say $m > 0$ steps and ends. Thus x belongs to the interior of a discontinuity of order m. The forward orbit hits no other vertex before time m, and by definition ends at time m, thus x belongs to the interior of no other discontinuity of positive order. There are two possible continuations by continuity of the orbit of x. If either of these continuations is a generalized diagonal or tangent to a side of Q then x is an end point of another singularity of positive order. In the second case the order of this additional singularity is also m, while in the first case it is strictly larger than m. We note that the second possibility can only happen if Q is not convex. Similarly, considering the backwards orbit we see that x belongs to the interior of a single discontinuity of negative order. If Q is not convex then it is not the end point of any negative discontinuity of greater order. The claim is proved.

Next we will use Euler's formula to conclude our lemma. Let F, E, V stand for the number of faces, edges and vertices respectively of the partition of the interior of C by the discontinuities of order -1 and n. We have $E := i+j+2k$ and $V := k$. By Euler's formula we have $V-E+F = 1$ thus $F = 1 - V + E = 1 + i + j + k$. As discussed above $m_b(v) = F$. □

Proof of Theorem 1.1. — The theorem follows immediately from Lemma 3.1 and Theorem 2.1 since $N_c(n) = \sum_{j=0}^{n-1} \sum_{v \in \mathcal{BL}(j)} \mathrm{gd}(v)$. □

4. Proof of Theorem 1.3.

It is well known that if the images of Q under the action $A(Q)$ tile the plane, then Q is the square, the equilateral triangle, the right isosceles triangle or the half equilateral triangle (i.e. the triangle with angles $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{6}\pi)$). We will use this tiling to calculate $N_c(n)$.

4.1. The square.

The tiling is the usual square grid. Fix a corner of the square and call it the origin of the grid. Consider all the generalized diagonals in the grid of combinatorial length at most n which *start* from this corner and are in the first quadrant.

From Figure 2 it is clear that the number $M_c(n)$ of such generalized diagonals is

$$\#\{(i,j) \in \mathbb{N}^2 : i + j \leq n + 1 \text{ and } \langle i, j \rangle = 1\}$$

where $\langle i, j \rangle$ is the gcd of i and j. The condition $\langle i, j \rangle = 1$ arises since generalized diagonals stop as soon as they hit a vertex, thus if a line through the origin hits several vertices (for example the line $y = x$), it corresponds to only one generalized diagonal (starting at the origin). Thus we only count it once. Since there are four possible starting corners we have $N_c(n) = 4M_c(n)$.

The asymptotics of this quantity is well known by Mertens' formula and its generalizations [HW], [N]:

$$N_c(n) \sim \frac{12}{\pi^2} n^2.$$

Applying Theorem 1.1 we have

$$p(n) \sim \frac{12}{\pi^2} \sum_{k=1}^{n} k^2 \sim \frac{4}{\pi^2} n^3.$$

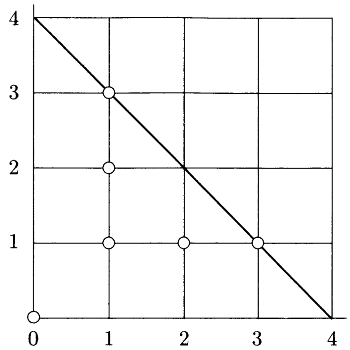


Figure 2. Counting the generalized diagonals starting at the origin of combinatorial length at most 3 in the square.

4.2. The equilateral triangle.

We consider the images of Q under the action of $A(G)$ specifying that one of the vertices is at the origin and another at the point $(1,0)$. We transform this grid to the grid in Figure 3 via the affine mapping which fixes the vector $\binom{1}{0}$ and takes the vector $\binom{\cos(\frac{1}{3}\pi)}{\sin(\frac{1}{3}\pi)}$ to $\binom{0}{1}$.

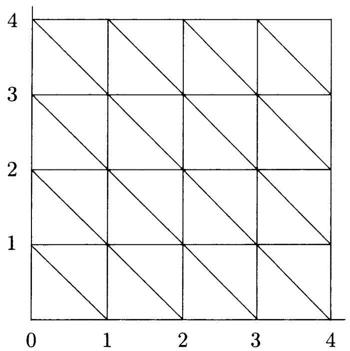


Figure 3. The affinely transformed grid of the equilateral triangle.

Consider all the generalized diagonals of combinatorial length at most n which start from the origin and are in the first quadrant. Let $M_c(n)$ be the cardinality of this set. Since there are 3 vertices we have $N_c(n) = 3M_c(n)$. From Figure 3 one sees that $M_c(2n) = M_c(2n+1)$ since in traversing a square in the tiling one alway crosses exactly two consecutive copies of the fundamental triangle. From the figure it is also clear that

$$M_c(2n) = \#\left\{(i,j) \in \mathbb{N}^2 : i+j \leq n+1 \text{ and } \langle i,j\rangle = 1\right\}.$$

By Mertens' formula [HW], [N]:

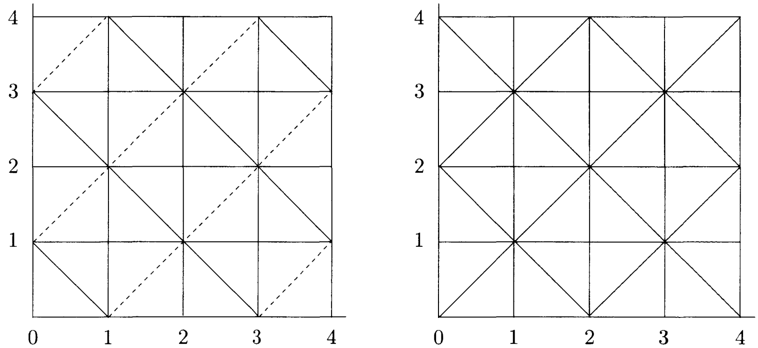
$$N_c(2n) = N_c(2n+1) \sim \frac{9}{\pi^2} n^2.$$

Applying Theorem 1.1 we have

$$p(n) \sim \frac{3}{4\pi^2} n^3.$$

4.3. The right isosceles triangle.

There are two different quantities which we must count. First we consider all the generalized diagonals of combinatorial length at most n which start from the origin of the grid in Figure 4 (a) and are in the first quadrant. Let $M_1(n)$ be the cardinality of this set. We also consider all the generalized diagonals of combinatorial length at most n which start from the origin of the grid in Figure 4 (b) and are in the first octant. Let $M_2(n)$ be the cardinality of this set. There are two vertices of our triangle with angle $\frac{1}{4}\pi$ thus $N_c(n) = M_1(n) + 2M_2(n)$.



4 (a) 4 (b)

Figure 4. The grid of the right isosceles triangle.

The number $2M_2(n)$ can also be interpreted as the cardinality of the set of all the generalized diagonals of combinatorial length at most n which start from the origin of the grid in Figure 4 (b) and are in the first quadrant. With this interpretation, if we overlay the grids from figures 4 (a) and 4 (b) we see that each generalized diagonal counted in $M_1(n)$ is also a generalized diagonal counted in $2M_2(n+1)$ and each generalized diagonal counted in $2M_2(n)$ is counted in $M_1(n+1)$. Thus the asymptotics of $N_n(c)$ is the same as the asymptotics of $2M_1(n)$.

We want to count $M_1(n)$. All generalized diagonals in the argument below will start at the origin of the grid pictured in Figure 4 (a) and all lengths will be combinatorial lengths. We count first the solid lines which the generalized diagonal crosses, we will deal later with the dashed lines. For most of the argument it will not matter whether the generalized diagonal is simple or not (i.e. contains no vertices in its interior), we will restrict to the set of simple generalized diagonals only in the last step of the proof.

Let $\ell(i,j)$ be the true combinatorial length of the generalized diagonal starting at the origin with end point (i,j) for any $(i,j) \in \mathbb{N}^2$. We view this length as the sum of the solid lines and the dashed lines it crosses plus one. If $i > j$ then the number of dashed lines it crosses is $\lfloor \frac{1}{2}(i-j) \rfloor$.

On the other hand the number of solid lines it crosses is characterized by the following statements. Suppose that $n = 3k$, then if it crosses $n-1$ solid lines then $i + j = \frac{2}{3}n + 1 = 2k + 1$. Inversely, supposing that $i + j = 2k + 1$, then it crosses $n - 1 = 3k - 1$ solid lines.

Combining these two facts we have if

$$(i,j) \in \mathbb{N}^2 \text{ and } i > j \text{ and } i + j = 2k + 1 \tag{2}$$

then

$$\ell(i,j) = n + \left\lfloor \frac{i-j}{2} \right\rfloor.$$

We need to calculate the region $\mathcal{R}(n)$ consisting of all $(i,j) \in \mathbb{N}^2$ such that $i > j$ and $\ell(i,j) \le n$. To do this fix (i,j) as in (2) and a natural number $m \le i - 1$. We compare how many fewer dashed lines are crossed by the generalized diagonal ending at $(i-m,j)$ than by the generalized diagonal ending at (i,j). This comparison yields

$$\ell(i-m,j) = n + \left\lfloor \frac{1}{2}(i-j) \right\rfloor - 2m + \varepsilon_m$$

where $\varepsilon_m = m \bmod 2$. Thus $\ell(i-m,j) \le n$ if and only if $n + \lfloor \frac{1}{2}(i-j) \rfloor - 2m + \varepsilon_m \le n$. A simple computation yields the following two implications:

$$m > \frac{i-j}{4} \implies \ell(i-m,j) \le n,$$
$$m \le \frac{i-j}{4} - \frac{1}{2} \implies \ell(i-m,j) \ge n+1.$$

Let

$$m_0 : = \min\{m : \ell(i-m, j) \le n\}.$$

From the above implications we have

$$m_0 \le \frac{1}{4}(i-j) + 1 \quad \text{and} \quad m_0 \ge \frac{1}{4}(i-j) - \frac{1}{2}\cdot$$

Let $\mathcal{D}(n)$ be the line $x = -\frac{1}{2}y + \frac{1}{2}n$. This line is the "ideal boundary" of the region $\mathcal{R}(n)$. The following computation shows that the distance of the true boundary from the ideal boundary is uniformly bounded:

$$\begin{aligned} d\big((i-m_0, j), \mathcal{D}(n)\big) &\le d\Big((i-m_0, j), \Big(-\frac{j}{2} + \frac{n}{2}, j\Big)\Big) \\ &= \Big|i + \frac{j}{2} - \frac{n}{2} - m_0\Big| \le \frac{5}{4}\cdot \end{aligned}$$

Let $\Delta^+(n)$ be the triangle whose boundaries are the x-axis, the line $y = x$ and the line $\mathcal{D}(n)$. By symmetry we also define a region $\Delta^-(n)$ in the second octant (i.e. we consider $i < j$). Let $\widetilde{M_1}(n)$ be the number of simple generalized diagonals starting at the origin whose other end is in the region $\Delta(n) : = \Delta^+(n) \cup \Delta^-(n)$. Since the distance of the set $\Delta(n)$ from the set $\mathcal{R}$ is uniformly bounded (in n) the asymptotics of $\widetilde{M_1}(n)$ and $M_1(n)$ are the same. By symmetry $\widetilde{M_1}(n)$ is twice the number of relatively prime lattice points in the region $\Delta^+(n)$.

The area of $\Delta^+(n)$ is $\frac{1}{12}n^2$. Thus applying Mertens [HW], [N] yields

$$M_1(n) \sim \widetilde{M_1}(n) \sim 2 \times \frac{n^2}{2\pi^2}\cdot$$

Thus

$$N_c(n) = 2M_1(n) \sim \frac{2n^2}{\pi^2}$$

and applying Theorem 1.1 gives

$$p(n) \sim \frac{2}{3\pi^2}n^3.$$

4.4. The half equilateral triangle.

The procedure is along the same lines as the previous examples. We consider the affinely transformed grid similarly to the case of the equilateral triangle. Counting the generalized diagonals which start at the origin reduces to an application of Mertens' formula. The explicit description of the region to which Mertens' formula must be applied is more complicated than in the previous examples, thus we do not carry it out.

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BIBLIOGRAPHY

[BP] J. BERSTEL & M. POCCHIOLA, A geometric proof of the enumeration formula for Sturmian words, J. Alg. Comp., 3 (1993), 349–355.

[BKM] C. BOLDRIGHINI, M. KEANE & F. MARCHETTI, Billiards in polygons, Ann. Prob., 6 (1978), 532–540.

[C] J. CASSAIGNE, Complexité et facteurs spéciaux, Bull. Belgian Math. Soc., 4 (1997), 67–88.

[CT] N. CHERNOV & S. TROUBETZKOY, Ergodicity of billiards in polygons with pockets, Nonlinearity, 11 (1998), 1095–1102.

[GKT] G. GALPERIN, T. KRÜGER & S. TROUBETZKOY, Local instability of orbits in polygonal and polyhedral billiards, Comm. Math. Phys., 169 (1995), 463–473.

[Gu1] E. GUTKIN, Billiards in polygons, Physica D, 19 (1986), 311–333.

[Gu2] E. GUTKIN, Billiards in polygons: survey of recent results, J. Stat. Phys., 174 (1995), 43–56.

[GuH] E. GUTKIN & N. HAYDN, Topological entropy of generalized interval exchanges, Bull. AMS, 32 (1995), 50–57.

[GuT] E. GUTKIN & S. TROUBETZKOY, Directional flows and strong recurrence for polygonal billiards, in Proceedings of the International Congress of Dynamical Systems, Montevideo, Uruguay, Pitman Research Notes in Math 362, Longman, Essex, 1996.

[HW] G.H. HARDY & E.M. WRIGHT, An introduction to the theory of numbers, Oxford Univ. Press, 1964.

[H] P. HUBERT, Dynamique symbolique des billards polygonaux rationnels, Thèse, Université d'Aix–Marseille II, 1995.

[H1] P. HUBERT, Complexité des suites définies par des billards rationnels, Bull. Soc. Math. France, 123 (1995), 257–270.

[H2] P. HUBERT, Propriétés combinatoires des suites définies par le billard dans les triangles pavants, Theoret. Comput. Sci., 164 (1996), 165–183.

[K] A. KATOK, The growth rate for the number of singular and periodic orbits for a polygonal billiard, Comm. Math. Phys., 111 (1987), 151–160.

[M1] H. MASUR, The growth rate of a quadratic differential, Ergod. Th. Dyn. Sys., 10 (1990), 151–176.

[M2] H. MASUR, Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential, In Holomorphic functions and moduli, vol. 1, D. Drasin ed., Springer-Verlag, 1988.

[MT] H. MASUR & S. TABACHNIKOV, Rational billiards and flat structures, preprint Max Planck Institut, 1999.

[Mi] F. MIGNOSI, On the number of factors of Sturmian words, Theor. Comp. Sci., 82 (1991), 71–84.

[N] A. NOGUEIRA, Orbit distribution on $\mathbb{R}^2$ under the natural action of SL(2,$\mathbb{Z}$), IML preprint 21, 2000.

[S] Ya. G. SINAI, Introduction to ergodic theory, Princeton Univ. Press, 1976.

[T] S. TABACHNIKOV, Billiards, Panoramas et Synthèses, Soc. Math. France, 1995.

[Tr] S. TROUBETZKOY, Complexity lower bounds for polygonal billiards, Chaos, 8 (1998), 242–244.

[V] W. VEECH, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards, Invent. Math., 97 (1989), 553–583.

[V1] W. VEECH, The billiard in a regular polygon, Geom. Func. Anal., 2 (1992), 341–379.

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