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# ON $p$ -ADIC ABELIAN STARK CONJECTURES AT $s = 1$

by David SOLOMON

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## 1. Introduction.

In the 1970's and 80's Harold Stark [St] made a series of conjectures concerning the values at  $s = 1$  and  $s = 0$  of complex Artin  $L$ -series attached to Galois extensions of number fields  $K/k$ . Subsequently, these conjectures have been extended and generalised in various ways, with particular attention paid recently to certain refined versions in the case where  $K/k$  is abelian ([Ru], [Po]). The main aim of this paper is to give a new formulation of an analogous  $p$ -adic conjecture in this case.

Since Stark's pioneering work, most authors have concentrated on the values (or leading terms) at  $s = 0$ . In any case, the complex conjectures at  $s = 1$  are equivalent to those at  $s = 0$ , thanks to the functional equation satisfied by the  $L$ -functions. However, the analogous  $p$ -adic  $L$ -functions are not known to satisfy a similar equation relating  $s = 1$  to  $s = 0$  in  $\mathbb{Z}_p$ . Consequently one must choose at which point to study their values. Tate's book [Ta] mentions two corresponding and independent  $p$ -adic Stark conjectures: one at  $s = 0$  ascribed to Gross, and one at  $s = 1$  ascribed to Serre. Unfortunately, the exposition of the latter is faulty and Serre's original paper actually only hints at a conjecture. (See Subsection 3.3 for a

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brief discussion.) We shall take a new approach to the abelian case of the  $p$ -adic conjecture at  $s = 1$ . This is based upon a reformulation developed in [So2] of a refined complex abelian conjecture at  $s = 0$  originally made by Rubin in [Ru]. The central idea of this reformulation was to replace (complex)  $L$ -functions at  $s = 0$  by what we called ‘twisted zeta-functions’ at  $s = 1$ . In Section 2 of this paper, the definitions and basic properties of the latter functions are reviewed and extended. Then we introduce and briefly study the analogous  $p$ -adic twisted zeta-functions which will serve to formulate the  $p$ -adic conjecture.

Section 3 contains the statements of all our conjectures, studies their properties and relates them to other Stark conjectures. Analogous complex and  $p$ -adic versions of the conjectures are stated in parallel. The natural setting of the  $p$ -adic versions being that in which  $K$  is (totally) real, we have restricted attention to this case.

One feature that should be mentioned is that for fixed real  $k$ , instead of ‘parametrizing’ our conjectures by the abelian extension field  $K$ , the use of twisted zeta-functions makes it more natural to use another parameter  $\mathfrak{f}$  which is a proper ideal of  $\mathcal{O}_k$  (as well as a further set of primes ideals, denoted  $T$ ). The field  $K$  is then always taken to be the ray-class field of  $k$  modulo  $\mathfrak{f}$ . Since  $\mathfrak{f}$  is not assumed to be a conductor, we may get different (though related) conjectures with the same extension  $K/k$ . (This corresponds to different choices of the set ‘ $S$ ’ in [Ru] *etc.*) Of course, every real abelian extension of  $k$  is contained in some such ray-class field.

Briefly, the conjectures take the following form: First we assemble all the complex (*resp.*  $p$ -adic) twisted zeta-functions for given  $\mathfrak{f}$  and  $T$  into a single group-ring-valued function  $\Phi_{\mathfrak{f},T}(s)$  (*resp.*  $\Phi_{\mathfrak{f},T,p}(s)$ ), assuming that  $T$  contains the primes above  $p$ ). The value of the latter at  $s = 1$  is then conjectured to be equal to the complex (*resp.*  $p$ -adic) group-ring-valued regulator of a certain element  $\eta_{\mathfrak{f},T}$  (*resp.*  $\eta_{\mathfrak{f},T,p}$ ), multiplied by an explicit algebraic constant. In general terms, both  $\eta_{\mathfrak{f},T}$  and  $\eta_{\mathfrak{f},T,p}$  are required to belong to a certain exterior power of the group of  $S$ -units of  $K$ , tensored with some subring  $\mathcal{R}$ , say, of  $\mathbb{Q}$ . (Here  $S$  consists of the infinite primes and those dividing  $\mathfrak{f}$ .) For general values of the parameters, only ‘basic’ conjectures are formulated for which we take  $\mathcal{R}$  to be  $\mathbb{Q}$  itself. If, however, the primes in  $T$  do not divide  $\mathfrak{f}$ , then there is a way to render the elements  $\eta_{\mathfrak{f},T}$  and (assuming Leopoldt’s conjecture)  $\eta_{\mathfrak{f},T,p}$  unique. In this situation we formulate a pair of ‘refined’ conjectures, essentially taking  $\mathcal{R}$  to be  $\mathbb{Z}[1/[K : k]]$ . The complex version is therefore similar to (but weaker than)

that of Rubin as reformulated at  $s = 1$  in [So2]. Not only are all our  $p$ -adic conjectures entirely analogous to their complex versions but, with appropriate normalisations, they can be combined with them: In essence, we postulate that  $\eta_{f,T}$  equals  $\eta_{f,T,p}$  for fixed  $k, f$  and  $T$  and every eligible  $p$ . We thus arrive at the main focus of the paper which we call the ‘Weak Refined Combined Conjecture’ (Conjecture 3.6). The paper closes with a study of two special cases of this conjecture and some remarks on the possibility of refining it, e.g. by taking  $\mathcal{R} = \mathbb{Z}$ .

A forthcoming sequel [RS] to the present paper is devoted to the numerical investigation of the Weak Refined Combined Conjecture in a variety of cases where  $k$  is real quadratic. For selected  $k$  and  $f$  we verify both the complex and  $p$ -adic parts of the conjecture to high precision, using new techniques to compute  $\Phi_{f,T,p}(1)$  for a number of different primes  $p$ .

## 2. Preliminaries.

Let  $k \subset \mathbb{C}$  be any number field of finite degree over  $\mathbb{Q}$ . Its ring of integers will be denoted  $\mathcal{O}$ . We first recall the definitions of the complex twisted zeta-functions over  $k$  and the function  $\Phi_{m,T}(s)$ , introduced in the paper [So2] (which may be consulted for more details and proofs).

### 2.1. Twisted Zeta-Functions and $\Phi_{m,T}(s)$ .

Let  $I$  be any fractional ideal of  $k$  and  $\xi$  any character on (the additive group of)  $I$  with values in  $\mu(\mathbb{C})$ , the complex roots of unity. The *annihilator* of  $\xi$  is the ideal  $\mathfrak{f} \triangleleft \mathcal{O}$  given by  $\mathfrak{f} = \{a \in \mathcal{O} : \xi(ax) = 1 \ \forall x \in I\}$ . Suppose that  $\mathfrak{z}$  is the formal product of some subset of the real places of  $k$ . Then, with the usual conventions, we write  $\mathfrak{m}$  for the *cycle* that is the formal product  $\mathfrak{f}\mathfrak{z}$ . We denote by  $E_{\mathfrak{m}}$  the subgroup of finite index in  $\mathcal{O}^\times$  consisting of the units that are congruent to 1 modulo  $\mathfrak{m}$  in the usual sense. For any finite set  $T$  of finite places (prime ideals) of  $\mathcal{O}$ , the group  $E_{\mathfrak{m}}$  acts by multiplication on the following subset of  $I$ :

$$S(I, \mathfrak{z}, T) := \{a \in I : a \in k_{\mathfrak{z}}^\times \text{ and } (aI^{-1}, T) = 1\}$$

where  $k_{\mathfrak{z}}^\times$  denotes the elements of  $k^\times$  which are positive at all places dividing  $\mathfrak{z}$  and the notation  $(J, T) = 1$  indicates that an ideal  $J$  of  $\mathcal{O}$  has support disjoint from  $T$ . The functions  $\xi|_{S(I, \mathfrak{z}, T)}$  and  $a \mapsto |I : (a)|$  are constant on the orbits of  $E_{\mathfrak{m}}$  acting on the above set and for  $s \in \mathbb{C}$ ,  $\Re(s) > 1$

we consider the absolutely convergent Dirichlet series, called the ‘twisted zeta-function’ for these data, defined by

$$\begin{aligned}
 Z_T(s; \xi, I, \mathfrak{m}) &:= \sum_{a \in \mathcal{S}(I, \mathfrak{z}, T)/E_{\mathfrak{m}}} \frac{\xi(a)}{|I : (a)|^s} = \sum_{a \in \mathcal{S}(I, \mathfrak{z}, T)/E_{\mathfrak{m}}} \frac{\xi(a)}{N(aI^{-1})^s} \\
 (1) \qquad \qquad \qquad &= N I^s \sum_{a \in \mathcal{S}(I, \mathfrak{z}, T)/E_{\mathfrak{m}}} \xi(a) |N_{k/\mathbb{Q}}(a)|^{-s}.
 \end{aligned}$$

(As the notation indicates, the sums are taken over any set of orbit representatives of  $E_{\mathfrak{m}}$  on  $\mathcal{S}(I, \mathfrak{z}, T)$ .) Now suppose that  $I'$  is another fractional ideal, and that  $\xi' : I' \rightarrow \mu(\mathbb{C})$  a character of  $I'$  such that  $I = cI'$  and  $\xi' = \xi \circ c$  for some  $c \in k_{\mathfrak{z}}^{\times}$ . This relation between  $(\xi, I)$  and  $(\xi', I')$  clearly implies that  $\xi'$  also has annihilator  $\mathfrak{f}$ . In fact, it is easy to see that it is an equivalence relation (depending on  $\mathfrak{z}$ ) on the set  $W_{\mathfrak{f}}$  of all pairs  $(\psi, J)$ , where  $\psi$  is a character of annihilator  $\mathfrak{f}$  on a fractional ideal  $J$ . The quotient set of  $W_{\mathfrak{f}}$  by this equivalence relation will be denoted  $\mathfrak{W}_{\mathfrak{m}}$ . Now, by ‘transport of structure’, it is clear that the above relation also implies that  $Z_T(s; \xi, I, \mathfrak{m})$  equals  $Z_T(s; \xi', I', \mathfrak{m})$ . Thus, for any equivalence class  $\mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$  we can unambiguously define

$$(2) \qquad Z_T(s; \mathfrak{w}) := Z_T(s; \xi, I, \mathfrak{m}) \quad \text{for any } (\xi, I) \in \mathfrak{w}.$$

For any non-zero integral ideal  $\mathfrak{a}$  of  $\mathcal{O}$  and any pair  $w = (\xi, I) \in W_{\mathfrak{f}}$ , we write  $\mathfrak{a} \cdot w$  for the pair  $(\xi|_{\mathfrak{a}I}, \mathfrak{a}I)$  which lies in  $W_{\tilde{\mathfrak{f}}}$ , where  $\tilde{\mathfrak{f}} = \mathfrak{f}(\mathfrak{f}, \mathfrak{a})^{-1}$ . The map  $w \mapsto \mathfrak{a} \cdot w$  thus defined respects the above equivalence relation and so descends to a map of classes  $\mathfrak{w} \mapsto \mathfrak{a} \cdot \mathfrak{w}$  from  $\mathfrak{W}_{\mathfrak{m}}$  to  $\mathfrak{W}_{\tilde{\mathfrak{m}}}$  where  $\tilde{\mathfrak{m}} = \tilde{\mathfrak{f}}\mathfrak{z}$ . Now let  $\text{Cl}_{\mathfrak{m}}(k)$  denote the ray-class group of  $k$  modulo  $\mathfrak{m}$ . Thus  $\text{Cl}_{\mathfrak{m}}(k) := \mathcal{I}_{\mathfrak{f}}(k)/P_{\mathfrak{m}}(k)$  where  $\mathcal{I}_{\mathfrak{f}}(k)$  denotes the group of fractional ideals prime to  $\mathfrak{f}$  and  $P_{\mathfrak{m}}(k)$  the subgroup consisting of those of the form  $(a)$  for some  $a \in k^{\times}$ ,  $a \equiv 1 \pmod{\mathfrak{m}}$ . We write  $[\mathfrak{a}]_{\mathfrak{m}}$  for the class in  $\text{Cl}_{\mathfrak{m}}(k)$  of a fractional ideal  $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(k)$ . If  $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(k)$  is an *integral* ideal then the map  $\mathfrak{w} \mapsto \mathfrak{a} \cdot \mathfrak{w}$  sends  $\mathfrak{W}_{\mathfrak{m}}$  into itself. It is easy to check that this determines a well-defined action of  $\text{Cl}_{\mathfrak{m}}(k)$  on  $\mathfrak{W}_{\mathfrak{m}}$  by setting  $[\mathfrak{a}]_{\mathfrak{m}} \cdot \mathfrak{w} := \mathfrak{a} \cdot \mathfrak{w}$  for any integral  $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(k)$ . Moreover, this action is free and transitive (see [So2]). We use it to combine all the twisted zeta-functions for  $\mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$  (and fixed  $T$ ) into a single function  $\Phi_{\mathfrak{m}, T}$ , as follows. Let  $\mathcal{D} \triangleleft \mathcal{O}$  denote the absolute different of  $k$  and write  $\xi_{\mathfrak{f}}^0$  for the character on  $\mathfrak{f}^{-1}\mathcal{D}^{-1}$  which sends  $a$  to  $\exp(2\pi i \text{Tr}_{k/\mathbb{Q}}(a))$ . The pair  $(\xi_{\mathfrak{f}}^0, \mathfrak{f}^{-1}\mathcal{D}^{-1})$  clearly lies in  $W_{\mathfrak{f}}$  and we write  $\mathfrak{w}_{\mathfrak{m}}^0$  for its image in  $\mathfrak{W}_{\mathfrak{m}}$ . Let  $k(\mathfrak{m}) \subset \mathbb{C}$  be the *ray-class field* over  $k$  modulo  $\mathfrak{m}$  as described by Global Class-Field Theory. Then the Galois

group  $G_{\mathfrak{m}} := \text{Gal}(k(\mathfrak{m})/k)$  is isomorphic to  $\text{Cl}_{\mathfrak{m}}(k)$ , via the Artin map sending  $\mathfrak{c} \in \text{Cl}_{\mathfrak{m}}(k)$  to  $\sigma_{\mathfrak{c}} \in G_{\mathfrak{m}}$ , say. (Given a fractional ideal  $\mathfrak{a} \in \mathcal{I}_{\mathfrak{f}}(k)$ , we shall often write  $\sigma_{\mathfrak{a},\mathfrak{m}}$  in place of  $\sigma_{[\mathfrak{a}]_{\mathfrak{m}}}$ .) We let  $\mathbb{C}G_{\mathfrak{m}}$  denote the complex group-ring of  $G_{\mathfrak{m}}$  and make the

DEFINITION 2.1. — For any cycle  $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$  for  $k$  and any finite set  $T$  of prime ideals of  $\mathcal{O}$ , we write  $\Phi_{\mathfrak{m},T}$  for the function

$$\begin{aligned} \Phi_{\mathfrak{m},T} : \{s \in \mathbb{C} : \Re(s) > 1\} &\longrightarrow \mathbb{C}G_{\mathfrak{m}} \\ s &\longmapsto \sum_{\mathfrak{c} \in \text{Cl}_{\mathfrak{m}}(k)} Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) \sigma_{\mathfrak{c}}^{-1}. \end{aligned}$$

The basic properties of  $\Phi_{\mathfrak{m},T}(s)$  were laid out in [So2, §3]. However, in that paper, both  $\Phi_{\mathfrak{m},T}(s)$  and the twisted zeta-functions  $Z_T(\mathfrak{w}; s)$  were only defined under the additional hypothesis that  $(\mathfrak{f}, T) = 1$ . This was motivated by an application to the reformulation of K. Rubin’s refinement (in [Ru]) of the complex abelian Stark Conjectures, for which he imposes an equivalent condition. In the present paper, however, we shall need to *p*-adically interpolate  $\Phi_{\mathfrak{m},T}(s)$  and since this requires that  $T$  contain all the prime ideals dividing  $p$  in  $\mathcal{O}$ , the above condition would exclude from treatment all cycles  $\mathfrak{m}$  not prime to  $p$ . It is in order to avoid such a restriction that we have here abandoned the assumption that  $(\mathfrak{f}, T) = 1$ , at least for the present. So far, this generalisation has cost us nothing. In particular, our definitions of  $Z_T(\mathfrak{w}; s)$  and  $\Phi_{\mathfrak{m},T}(s)$  still make perfect sense. The important Theorems 3.2, 3.3 and 3.4 of [So2, §3] will, however, require some modifications, which we now indicate.

First we prove a lemma that allows us to remove one at a time the primes in  $T$  that divide  $\mathfrak{f}$  (compare [So2, Thm. 3.1]). For any cycle  $\tilde{\mathfrak{m}}$  dividing  $\mathfrak{m}$  we use the notation  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}$  to denote the natural homomorphism  $\text{Cl}_{\mathfrak{m}}(k) \rightarrow \text{Cl}_{\tilde{\mathfrak{m}}}(k)$  and also for the restriction homomorphism  $G_{\mathfrak{m}} \rightarrow G_{\tilde{\mathfrak{m}}}$  which corresponds to the former by the Artin isomorphisms. The symbol  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}$  will also be used for the linear extension of either of these maps to a homomorphism of group-rings. We write  $\nu_{\tilde{\mathfrak{m}},\mathfrak{m}} : \mathbb{C}G_{\tilde{\mathfrak{m}}} \rightarrow \mathbb{C}G_{\mathfrak{m}}$  for the  $\mathbb{C}$ -linear homomorphism which sends  $\tilde{g} \in G_{\tilde{\mathfrak{m}}}$  to the average of its inverse images under  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}$ , that is:  $\nu_{\tilde{\mathfrak{m}},\mathfrak{m}}(\tilde{g}) = |\ker(\pi_{\mathfrak{m},\tilde{\mathfrak{m}}})|^{-1} \sum_{\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}(g)=\tilde{g}} g \in \mathbb{C}G_{\mathfrak{m}}$ . Thus  $\nu_{\tilde{\mathfrak{m}},\mathfrak{m}}$  induces an isomorphism between  $\mathbb{C}G_{\tilde{\mathfrak{m}}}$  and the ring of  $\ker(\pi_{\mathfrak{m},\tilde{\mathfrak{m}}})$ -invariant elements in  $\mathbb{C}G_{\mathfrak{m}}$  such that  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}} \circ \nu_{\tilde{\mathfrak{m}},\mathfrak{m}}$  is the identity.

LEMMA 2.1. — With notations  $k, \mathfrak{m} = \mathfrak{f}\mathfrak{z}$  etc. as above, we suppose that  $\mathfrak{q}$  is a prime ideal in  $T$  which also divides  $\mathfrak{f}$ . Write  $\mathfrak{f} = \mathfrak{q}\tilde{\mathfrak{f}}$  and  $\tilde{\mathfrak{m}}$  for

the cycle  $\tilde{f}_3$ . Then, for all  $s \in \mathbb{C}$ ,  $\Re(s) > 1$  we have

$$(3) \quad \Phi_{\mathfrak{m}, T}(s) = \Phi_{\mathfrak{m}, T \setminus \{q\}}(s) - \begin{cases} (Nq^{1-s} - Nq^{-s})\nu_{\tilde{\mathfrak{m}}, \mathfrak{m}}(\Phi_{\tilde{\mathfrak{m}}, T \setminus \{q\}}(s)) & \text{if } q \nmid \tilde{f}, \text{ and} \\ Nq^{1-s}\nu_{\tilde{\mathfrak{m}}, \mathfrak{m}}(\Phi_{\tilde{\mathfrak{m}}, T \setminus \{q\}}(s)) & \text{if } q \mid \tilde{f}. \end{cases}$$

*Proof.* — Let  $\mathfrak{c}$  be any element of  $\text{Cl}_{\mathfrak{m}}(k)$  and suppose that  $\mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0$  is represented by the pair  $(\xi, I) \in W_{\mathfrak{m}}$ . Since  $\mathcal{S}(I, \mathfrak{z}, T \setminus \{q\})$  is the disjoint union of  $\mathcal{S}(I, \mathfrak{z}, T)$  with  $\mathcal{S}(qI, \mathfrak{z}, T \setminus \{q\})$  we find, by Equations (1) and (2) that

$$Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) = Z_{T \setminus \{q\}}(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) - \sum_{a \in \mathcal{S}(qI, \mathfrak{z}, T \setminus \{q\})/E_{\mathfrak{m}}} \frac{\xi(a)}{N(aI^{-1})^s}.$$

Now, each term in the sum on the R.H.S. can be written as  $Nq^{-s}\xi(a)/N(a(qI)^{-1})^s$ . Moreover, since the restriction  $\xi|_{qI}$  has annihilator  $\tilde{f}$ , this term depends only on the  $E_{\tilde{\mathfrak{m}}}$ -orbit of  $a$  in  $\mathcal{S}(qI, \mathfrak{z}, T \setminus \{q\})$ , which is a union of precisely  $|E_{\tilde{\mathfrak{m}}} : E_{\mathfrak{m}}|$  orbits for  $E_{\mathfrak{m}}$ . Therefore, since  $q \cdot (\mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0)$  lies in  $\mathfrak{W}_{\tilde{\mathfrak{m}}}$ , we have

$$\sum_{a \in \mathcal{S}(qI, \mathfrak{z}, T \setminus \{q\})/E_{\mathfrak{m}}} \frac{\xi(a)}{N(aI^{-1})^s} = Nq^{-s}|E_{\tilde{\mathfrak{m}}} : E_{\mathfrak{m}}|Z_{T \setminus \{q\}}(s; q \cdot (\mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0)).$$

In fact, it is clear from the definitions that  $q \cdot (\mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) = \pi_{\mathfrak{m}, \tilde{\mathfrak{m}}}(\mathfrak{c}) \cdot (q \cdot \mathfrak{w}_{\mathfrak{m}}^0) = \pi_{\mathfrak{m}, \tilde{\mathfrak{m}}}(\mathfrak{c}) \cdot \mathfrak{w}_{\tilde{\mathfrak{m}}}^0$ , so the last two equations give

$$(4) \quad Z_T(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) = Z_{T \setminus \{q\}}(s; \mathfrak{c} \cdot \mathfrak{w}_{\mathfrak{m}}^0) - Nq^{-s}|E_{\tilde{\mathfrak{m}}} : E_{\mathfrak{m}}|Z_{T \setminus \{q\}}(s; \pi_{\mathfrak{m}, \tilde{\mathfrak{m}}}(\mathfrak{c}) \cdot \mathfrak{w}_{\tilde{\mathfrak{m}}}^0).$$

Now the exact sequence (see for example [So2, §6])

$$1 \rightarrow E_{\mathfrak{m}} \rightarrow E_{\mathfrak{z}} \rightarrow (\mathcal{O}/\mathfrak{f})^{\times} \rightarrow \text{Cl}_{\mathfrak{m}}(k) \rightarrow \text{Cl}_{\mathfrak{z}}(k) \rightarrow 1$$

and a similar one with  $\tilde{\mathfrak{m}}$  and  $\tilde{f}$  in place of  $\mathfrak{m}$  and  $f$ , give the equalities

$$|E_{\mathfrak{z}} : E_{\mathfrak{m}}| = |(\mathcal{O}/\mathfrak{f})^{\times}| |\text{Cl}_{\mathfrak{m}}(k)|^{-1} |\text{Cl}_{\mathfrak{z}}(k)|$$

$$\text{and } |E_{\tilde{\mathfrak{z}}} : E_{\tilde{\mathfrak{m}}}| = |(\mathcal{O}/\tilde{f})^{\times}| |\text{Cl}_{\tilde{\mathfrak{m}}}(k)|^{-1} |\text{Cl}_{\tilde{\mathfrak{z}}}(k)|.$$

Dividing, and using the fact that  $\pi_{\mathfrak{m}, \tilde{\mathfrak{m}}}$  is surjective gives

$$(5) \quad |E_{\tilde{\mathfrak{m}}} : E_{\mathfrak{m}}| = |(\mathcal{O}/\mathfrak{f})^{\times}| |(\mathcal{O}/\tilde{f})^{\times}|^{-1} |\text{Cl}_{\mathfrak{m}}(k)|^{-1} |\text{Cl}_{\tilde{\mathfrak{m}}}(k)|$$

$$= \begin{cases} (Nq - 1) |\ker(\pi_{\mathfrak{m}, \tilde{\mathfrak{m}}})|^{-1} & \text{if } q \nmid \tilde{f}, \text{ and} \\ Nq |\ker(\pi_{\mathfrak{m}, \tilde{\mathfrak{m}}})|^{-1} & \text{if } q \mid \tilde{f}. \end{cases}$$

Finally, substituting this equation into Equation (4), multiplying by  $\sigma_{\mathfrak{c}}^{-1}$  and summing over  $\mathfrak{c} \in \text{Cl}_{\mathfrak{m}}(k)$  gives (3). □

The following result generalises Theorem 3.2 of [So2].

**THEOREM 2.1.** — *Let  $\mathfrak{m} = \mathfrak{f}\tilde{\mathfrak{z}}$  be a cycle for  $k$  and  $\tilde{\mathfrak{m}} = \tilde{\mathfrak{f}}\tilde{\mathfrak{z}}$  any cycle dividing  $\mathfrak{m}$ . Thus  $\tilde{\mathfrak{z}}|\mathfrak{z}$ ,  $\tilde{\mathfrak{f}}|\mathfrak{f}$  and we write*

$$\mathfrak{f} = \mathfrak{g}\tilde{\mathfrak{f}} = \mathfrak{g}_0\mathfrak{g}'\tilde{\mathfrak{f}}$$

where  $\mathfrak{g} = \mathfrak{g}_0\mathfrak{g}'$ , the (integral) ideal  $\mathfrak{g}_0$  is supported on primes in  $T$  and  $(\mathfrak{g}', T) = 1$ . Define also

$$\hat{T} = T \setminus \{\mathfrak{p} : \mathfrak{p}|\mathfrak{g}\} = T \setminus \{\mathfrak{p} : \mathfrak{p}|\mathfrak{g}_0\}$$

and suppose that  $s$  lies in  $\mathbb{C}$  with  $\Re(s) > 1$ .

If  $\mathfrak{g}_0$  is square-free and prime to  $\tilde{\mathfrak{f}}$  then

(6)

$$\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m},T}(s)) = N\mathfrak{g}'^{1-s} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}|\mathfrak{g}', \mathfrak{p}\nmid\tilde{\mathfrak{f}}}} (1 - N\mathfrak{p}^{s-1}\sigma_{\mathfrak{p},\tilde{\mathfrak{m}}}^{-1}) \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}|\mathfrak{g}_0}} (N\mathfrak{p}^{-s} - \sigma_{\mathfrak{p},\tilde{\mathfrak{m}}}^{-1})\Phi_{\tilde{\mathfrak{m}},\hat{T}}(s).$$

Otherwise,  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m},T}(s)) = 0$ .

*Proof.* — Theorem 3.2 of [So2] is simply the special case of this result under the hypothesis  $(\mathfrak{f}, T) = 1$ , which is equivalent to the two conditions  $\mathfrak{g}_0 = \mathcal{O}$  and  $(\tilde{\mathfrak{f}}, \hat{T}) = 1$ . However, an examination of the proof in [So2, §6] reveals that it only uses the fact that  $\mathfrak{p} \notin T$  for any prime ideal  $\mathfrak{p}|\tilde{\mathfrak{f}}^{-1}\mathfrak{f} = \mathfrak{g}$ . It therefore goes across word for word to prove the Theorem in the special case  $\mathfrak{g}_0 = \mathcal{O}$  (thus  $\hat{T} = T$  but  $(\mathfrak{f}, T) = (\tilde{\mathfrak{f}}, \hat{T})$  may or may not be 1). For the general case, we write  $\mathfrak{f}_1$  for the ideal  $\mathfrak{g}_0\tilde{\mathfrak{f}}$  and  $\mathfrak{m}_1$  for the cycle  $\mathfrak{f}_1\tilde{\mathfrak{z}} = \mathfrak{g}_0\tilde{\mathfrak{m}}$ . Thus  $\tilde{\mathfrak{m}}|\mathfrak{m}_1|\mathfrak{m}$  and  $\pi_{\mathfrak{m},\tilde{\mathfrak{m}}} = \pi_{\mathfrak{m}_1,\tilde{\mathfrak{m}}} \circ \pi_{\mathfrak{m},\mathfrak{m}_1}$ . Since  $\mathfrak{f} = \mathfrak{g}'\mathfrak{f}_1$ , the above-mentioned special case (with  $\mathfrak{f}_1$  and  $\mathfrak{m}_1$  and in place of  $\tilde{\mathfrak{f}}$  and  $\tilde{\mathfrak{m}}$ ) gives

$$\pi_{\mathfrak{m},\mathfrak{m}_1}(\Phi_{\mathfrak{m},T}(s)) = N\mathfrak{g}'^{1-s} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}|\mathfrak{g}', \mathfrak{p}\nmid\mathfrak{f}_1}} (1 - N\mathfrak{p}^{s-1}\sigma_{\mathfrak{p},\mathfrak{m}_1}^{-1})\Phi_{\mathfrak{m}_1,T}(s).$$

Now clearly, for any  $\mathfrak{p}|\mathfrak{g}'$  the conditions  $\mathfrak{p} \nmid \mathfrak{f}_1$  and  $\mathfrak{p} \nmid \tilde{\mathfrak{f}}$  are equivalent, and if they hold then  $\pi_{\mathfrak{m}_1,\tilde{\mathfrak{m}}}(\sigma_{\mathfrak{p},\mathfrak{m}_1}) = \sigma_{\mathfrak{p},\tilde{\mathfrak{m}}}$ . Therefore, the full result will follow on applying  $\pi_{\mathfrak{m}_1,\tilde{\mathfrak{m}}}$  to the above equation once we have proven that

(7)

$$\pi_{\mathfrak{m}_1,\tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m}_1,T}(s)) = \begin{cases} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}|\mathfrak{g}_0}} (N\mathfrak{p}^{-s} - \sigma_{\mathfrak{p},\tilde{\mathfrak{m}}}^{-1})\Phi_{\tilde{\mathfrak{m}},\hat{T}}(s) & \text{if } \mathfrak{g}_0 \text{ is square-free and} \\ & \text{prime to } \tilde{\mathfrak{f}}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

CLAIM 2.1. — Suppose that  $\mathfrak{p}$  is a prime dividing  $\mathfrak{g}_0$  and set  $\mathfrak{f}_2 := (\mathfrak{p}^{-1}\mathfrak{g}_0)\tilde{\mathfrak{f}}$ ,  $\mathfrak{m}_2 := \mathfrak{f}_2\tilde{\mathfrak{f}}$  so that  $\mathfrak{f}_1 = \mathfrak{p}\mathfrak{f}_2$  and  $\mathfrak{m}_1 = \mathfrak{p}\mathfrak{m}_2$ . Then

$$(8) \quad \pi_{\mathfrak{m}_1, \mathfrak{m}_2}(\Phi_{\mathfrak{m}_1, T}(s)) = \begin{cases} (N\mathfrak{p}^{-s} - \sigma_{\mathfrak{p}, \mathfrak{m}_2}^{-1})\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}\}}(s) & \text{if } \mathfrak{p} \nmid \mathfrak{f}_2, \text{ and} \\ 0 & \text{if } \mathfrak{p} | \mathfrak{f}_2. \end{cases}$$

Assuming this Claim for the moment, we use it to prove Equation (7). We can suppose that  $\mathfrak{g}_0 \neq \mathcal{O}$  (otherwise Equation (7) is trivial), then choose  $\mathfrak{p} = \mathfrak{p}_1 | \mathfrak{g}_0$ , say, and define  $\mathfrak{m}_2$  as in the Claim. Suppose first that  $\mathfrak{g}_0$  is square-free and prime to  $\tilde{\mathfrak{f}}$ . Then  $\mathfrak{p}_1 \nmid \mathfrak{f}_2$ , so applying  $\pi_{\mathfrak{m}_2, \tilde{\mathfrak{m}}}$  to Equation (8) gives  $\pi_{\mathfrak{m}_1, \tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m}_1, T}(s))$  on the L.H.S. and on the R.H.S. it gives

$$\begin{aligned} \pi_{\mathfrak{m}_2, \tilde{\mathfrak{m}}}(N\mathfrak{p}_1^{-s} - \sigma_{\mathfrak{p}_1, \mathfrak{m}_2}^{-1})\pi_{\mathfrak{m}_2, \tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}_1\}}(s)) \\ = (N\mathfrak{p}_1^{-s} - \sigma_{\mathfrak{p}_1, \tilde{\mathfrak{m}}}^{-1})\pi_{\mathfrak{m}_2, \tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}_1\}}(s)). \end{aligned}$$

But  $\mathfrak{p}_1^{-1}\mathfrak{g}_0$  is also square-free and prime to  $\tilde{\mathfrak{f}}$ . So, replacing  $\mathfrak{m}_1$  by  $\mathfrak{m}_2$ ,  $\mathfrak{g}_0$  by  $\mathfrak{p}_1^{-1}\mathfrak{g}_0$  and  $T$  by  $T \setminus \{\mathfrak{p}_1\}$ , Equation (7) follows in this case by induction on the number of primes dividing  $\mathfrak{g}_0$ . If, on the other hand,  $\mathfrak{g}_0$  is either not square-free or not prime to  $\tilde{\mathfrak{f}}$ , then either the same will be true of  $\mathfrak{p}_1^{-1}\mathfrak{g}_0$  or  $\mathfrak{p}_1$  must divide  $\mathfrak{f}_2$ . In either case, the equation  $\pi_{\mathfrak{m}, \tilde{\mathfrak{m}}}(\Phi_{\mathfrak{m}, T}(s)) = 0$  follows by applying  $\pi_{\mathfrak{m}_2, \tilde{\mathfrak{m}}}$  to Equation (8) (and using (7) inductively if necessary).

It remains to prove the Claim. For this we use Lemma 2.1, taking ‘ $\mathfrak{m}$ ’ to be  $\mathfrak{m}_1 = \mathfrak{g}_0\tilde{\mathfrak{f}}$  and ‘ $\mathfrak{q}$ ’ to be  $\mathfrak{p}$  so that ‘ $\tilde{\mathfrak{m}}$ ’ is actually  $\mathfrak{m}_2$ . Applying  $\pi_{\mathfrak{m}_1, \mathfrak{m}_2}$  to Equation (3) with these substitutions gives

$$(9) \quad \begin{aligned} \pi_{\mathfrak{m}_1, \mathfrak{m}_2}(\Phi_{\mathfrak{m}_1, T}(s)) \\ = \pi_{\mathfrak{m}_1, \mathfrak{m}_2}(\Phi_{\mathfrak{m}_1, T \setminus \{\mathfrak{p}\}}(s)) - \begin{cases} (N\mathfrak{p}^{1-s} - N\mathfrak{p}^{-s})\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}\}}(s) & \text{if } \mathfrak{p} \nmid \mathfrak{f}_2 \\ N\mathfrak{p}^{1-s}\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}\}}(s) & \text{if } \mathfrak{p} | \mathfrak{f}_2. \end{cases} \end{aligned}$$

But the case  $\mathfrak{g}_0 = \mathcal{O}$  of the current Theorem (already proven!) gives

$$\pi_{\mathfrak{m}_1, \mathfrak{m}_2}(\Phi_{\mathfrak{m}_1, T \setminus \{\mathfrak{p}\}}(s)) = \begin{cases} (N\mathfrak{p}^{1-s} - \sigma_{\mathfrak{p}, \mathfrak{m}_2}^{-1})\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}\}}(s) & \text{if } \mathfrak{p} \nmid \mathfrak{f}_2, \text{ and} \\ N\mathfrak{p}^{1-s}\Phi_{\mathfrak{m}_2, T \setminus \{\mathfrak{p}\}}(s) & \text{if } \mathfrak{p} | \mathfrak{f}_2 \end{cases}$$

and substituting into Equation (9) gives (8). This completes the proof of the Claim and hence of the Theorem. □

Let  $\chi : \text{Cl}_{\mathfrak{m}}(k) \rightarrow \mu(\mathbb{C})$  be any ray-class character modulo  $\mathfrak{m}$ . For us, the conductor of  $\chi$  will be the unique minimal cycle  $\mathfrak{m}(\chi) = \mathfrak{f}(\chi)\mathfrak{z}(\chi)$  dividing  $\mathfrak{m}$  such that  $\chi$  factors through  $\pi_{\mathfrak{m}, \mathfrak{m}(\chi)}$ . The primitive form  $\tilde{\chi}$  of  $\chi$  is then the unique character on  $\text{Cl}_{\mathfrak{m}(\chi)}(k)$  such that  $\chi = \tilde{\chi} \circ \pi_{\mathfrak{m}, \mathfrak{m}(\chi)}$  and

we say that  $\chi$  is *primitive (modulo  $\mathfrak{m}$ )* if and only if  $\chi = \tilde{\chi}$ . The Artin isomorphism allows us to regard  $\chi$  as a character on  $G_{\mathfrak{m}}$  and then, by  $\mathbb{C}$ -linear extension, as a ring-homomorphism from  $\mathbb{C}G_{\mathfrak{m}}$  to  $\mathbb{C}$ , and similarly for  $\tilde{\chi}$ , *mutatis mutandis*. If  $\mathfrak{a}$  is an ideal in  $\mathcal{I}_f(k)$ , we shall sometimes write  $\chi(\mathfrak{a})$  instead of  $\chi([\mathfrak{a}]_{\mathfrak{m}}) = \chi(\sigma_{\mathfrak{a},\mathfrak{m}})$ , and similarly for  $\tilde{\chi}$ . If  $\chi$  is *primitive modulo  $\mathfrak{m}$*  ( $= \mathfrak{m}(\chi)$ ), and  $s \in \mathbb{C}$ ,  $\Re(s) > 1$ , then the  $L$ -function of  $\chi$  is given by the absolutely convergent expressions

$$L(s, \chi) = \sum_{\substack{I \in \mathcal{I}_f(k) \\ I \subset \mathcal{O}}} \frac{\chi(I)}{NI^s} = \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \nmid f}} \left( 1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \right)^{-1}.$$

Under the Hypothesis that  $(f, T) = 1$ , Theorem 3.3 of [So2] provides a relation between  $\Phi_{\mathfrak{m},T}$  and the  $L$ -functions of the characters  $\chi$  of  $\text{Cl}_{\mathfrak{m}}(k)$ . Without this hypothesis we have the following generalisation.

**THEOREM 2.2.** — *For  $\mathfrak{m} = f\mathfrak{z}$ , any  $T$  and any character  $\chi$  of  $\text{Cl}_{\mathfrak{m}}(k)$  as above, we write*

$$f = \mathfrak{g}f(\chi) = \mathfrak{g}_0\mathfrak{g}'f(\chi)$$

where the ideal  $\mathfrak{g}_0$  is supported on primes in  $T$  and  $(\mathfrak{g}', T) = 1$ . Suppose that  $s$  lies in  $\mathbb{C}$  with  $\Re(s) > 1$ .

Suppose that  $\mathfrak{g}_0$  is square-free, a product of  $t$  distinct primes, none of which divides  $f(\chi)$ . Then

$$(10) \quad \chi(\Phi_{\mathfrak{m},T}(s)) = (-1)^t \tilde{\chi}^{-1}(\mathfrak{g}_0)g_{\mathfrak{m}(\chi)}(\tilde{\chi})N\mathfrak{g}'^{1-s} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \nmid f', \mathfrak{p} \nmid f(\chi)}} \left( 1 - \frac{\tilde{\chi}^{-1}(\mathfrak{p})}{N\mathfrak{p}^{1-s}} \right) \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \in T, \mathfrak{p} \nmid f(\chi)}} \left( 1 - \frac{\tilde{\chi}(\mathfrak{p})}{N\mathfrak{p}^s} \right) L(s, \tilde{\chi})$$

where  $g_{\mathfrak{m}(\chi)}(\tilde{\chi}) \neq 0$  is the Gauss sum attached to  $\tilde{\chi}$ . (See [SO2, §6.4] for a definition.)

Otherwise,  $\chi(\Phi_{\mathfrak{m},T}(s)) = 0$ .

*Proof.* — Write  $\chi$  as  $\tilde{\chi} \circ \pi_{\mathfrak{m},\mathfrak{m}(\chi)}$  and use Theorem 2.1 to evaluate  $\tilde{\chi}(\pi_{\mathfrak{m},\mathfrak{m}(\chi)}(\Phi_{\mathfrak{m},T}(s)))$ . If  $\mathfrak{g}_0$  does not satisfy the stated condition, then we get zero. Otherwise, a little rearrangement gives

$$(11) \quad \chi(\Phi_{\mathfrak{m},T}(s)) = (-1)^t \tilde{\chi}^{-1}(\mathfrak{g}_0)N\mathfrak{g}'^{1-s} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \nmid f', \mathfrak{p} \nmid f(\chi)}} \left( 1 - \frac{\tilde{\chi}^{-1}(\mathfrak{p})}{N\mathfrak{p}^{1-s}} \right) \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \nmid \mathfrak{g}_0}} \left( 1 - \frac{\tilde{\chi}(\mathfrak{p})}{N\mathfrak{p}^s} \right) \tilde{\chi}(\Phi_{\mathfrak{m}(\chi),T}(s))$$

where  $\hat{T} := T \setminus \{\mathfrak{p} : \mathfrak{p} | \mathfrak{g}_0\}$ . Let also  $T' := \{\mathfrak{p} \in \hat{T} : \mathfrak{p} \nmid f(\chi)\}$ . Now if  $\mathfrak{p}$  lies in  $\hat{T} \setminus T'$  then it divides  $f(\chi)$  and the primitivity of  $\tilde{\chi}$  implies that this character is non-trivial on  $\ker \pi_{\mathfrak{m}(\chi), \mathfrak{n}}$ , where  $\mathfrak{n}$  denotes the cycle  $\mathfrak{p}^{-1}\mathfrak{m}(\chi)$ . From this, it follows easily that  $\tilde{\chi} \circ \nu_{\mathfrak{n}, \mathfrak{m}(\chi)} = 0$ , so Lemma 2.1 gives  $\tilde{\chi}(\Phi_{\mathfrak{m}(\chi), \hat{T}}(s)) = \tilde{\chi}(\Phi_{\mathfrak{m}(\chi), \hat{T} \setminus \{\mathfrak{p}\}}(s))$ . By iterating this procedure, we can remove from  $\hat{T}$  all the primes that are not in  $T'$  and since  $(f(\chi), T') = 1$ , we can then apply Theorem 3.3 of [So2] to get

$$\tilde{\chi}(\Phi_{\mathfrak{m}(\chi), \hat{T}}(s)) = \tilde{\chi}(\Phi_{\mathfrak{m}(\chi), T'}(s)) = g_{\mathfrak{m}(\chi)}(\tilde{\chi}) \prod_{\mathfrak{p} \in T'} \left(1 - \frac{\tilde{\chi}(\mathfrak{p})}{N\mathfrak{p}^s}\right) L(s, \tilde{\chi}).$$

Substituting into Equation (11) yields Equation (10) since the condition on  $\mathfrak{g}_0$  implies that the set  $\{\mathfrak{p} \in T : \mathfrak{p} \nmid f(\chi)\}$  is the disjoint union of the set  $\{\mathfrak{p} : \mathfrak{p} | \mathfrak{g}_0\}$  with  $T'$ . □

We write  $\text{Cl}_{\mathfrak{m}}(k)^*$  for the dual group of  $\text{Cl}_{\mathfrak{m}}(k)$ , i.e. the group of all ray-class characters modulo  $\mathfrak{m}$ , and identify it as above with the dual group  $G_{\mathfrak{m}}^*$  of  $G_{\mathfrak{m}}$ . We have a character decomposition, for any  $s \in \mathbb{C}$ ,  $\Re(s) > 1$ :

$$(12) \quad \Phi_{\mathfrak{m}, T}(s) = \sum_{\chi \in G_{\mathfrak{m}}^*} \chi(\Phi_{\mathfrak{m}, T}(s)) e_{\chi}$$

(where  $e_{\chi}$  denotes the idempotent  $\frac{1}{|G_{\mathfrak{m}}|} \sum_{\sigma \in G_{\mathfrak{m}}} \chi(\sigma) \sigma^{-1}$ ). Thus the behaviour of the primitive  $L$ -functions  $\{L(s, \tilde{\chi}) : \chi \in G_{\mathfrak{m}}^*\}$  determines that of  $\Phi_{\mathfrak{m}, T}(s)$  via Theorem 2.2. For instance, each  $L(s, \tilde{\chi})$  possesses a meromorphic continuation to the whole of  $\mathbb{C}$ , which is actually holomorphic unless  $\tilde{\chi}$  is trivial ( $\Leftrightarrow \chi$  is trivial), in which case  $L(s, \tilde{\chi}) = \zeta_k(s)$  has a simple pole at  $s = 1$  with known residue (see e.g. [Ta, §I.1]). Combining this information with Equation (12) and Theorem 2.2 yields:

**THEOREM 2.3.** — *For any  $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$  and  $T$ , the function  $\Phi_{\mathfrak{m}, T}$  extends to a meromorphic function on  $\mathbb{C}$  with values in  $\mathbb{C}G_{\mathfrak{m}}$ . If  $\mathfrak{f}$  is not a product of distinct primes lying in  $T$ , then this extension is holomorphic. Otherwise it has a unique, simple pole at  $s = 1$ . In all cases  $x\Phi_{\mathfrak{m}, T}(s)$  is holomorphic for any  $x$  in the augmentation ideal  $I(\mathbb{C}G_{\mathfrak{m}})$  of  $\mathbb{C}G_{\mathfrak{m}}$ . □*

*Remark 2.1.* — Note that the product mentioned in the statement of the Theorem could be empty; thus  $\Phi_{\mathfrak{m}, T}(s)$  has a pole if  $\mathfrak{f} = \mathcal{O}$ . Indeed, if we restrict to  $(\mathfrak{f}, T) = 1$  then this is the only case in which there is a pole, and Theorem 3.4 of [So2] (the analogue of Theorem 2.3 with this restriction) gives the residue. The equivalent, generalised residue formula in Theorem 2.3 is left as an exercise.

We shall henceforth make use of its meromorphic continuation to regard  $\Phi_{m,T}(s)$  as a function on the whole of  $\mathbb{C}$  (or of  $\mathbb{C} \setminus \{1\}$ ). In so doing, identities such as (3), (3.5), (10) and (12), automatically become valid over the whole of these domains, not just for  $\Re(s) > 1$ .

**2.2. *p*-adic interpolation.**

Let  $\bar{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$  within  $\mathbb{C}$ . Thus  $\bar{\mathbb{Q}}$  contains the values of all the additive and multiplicative characters mentioned so far. In addition, the field  $k$ , along with all its abelian extensions are to be considered as subfields of  $\bar{\mathbb{Q}}$ . For the purposes of *p*-adic interpolation, we need  $k$  to be *totally real*.

We choose a prime number  $p$ , write  $\mathbb{C}_p$  for the completion of an algebraic closure of the field  $\mathbb{Q}_p$  of *p*-adic numbers and fix an embedding  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ . We write  $\mu(\mathbb{Q}_p)$  for the group of roots of unity in  $\mathbb{Q}_p$ . Every element  $x$  of  $\mathbb{Z}_p^\times$  can be written uniquely as a product

$$x = \omega(x)\langle x \rangle$$

where  $\omega(x)$  lies in  $\mu(\mathbb{Q}_p)$  and  $\langle x \rangle$  lies in  $1 + p\mathbb{Z}_p$  for  $p$  odd and in  $1 + 4\mathbb{Z}_2$  for  $p = 2$ . ( $\omega$  is the so-called Teichmüller character.) Let

$$w_p := \text{ord}(\omega) = |\mu(\mathbb{Q}_p)| = \begin{cases} p - 1 & \text{if } p \neq 2, \text{ and} \\ 2 & \text{if } p = 2 \end{cases}$$

and consider the set of rational integers defined by

$$\mathcal{M}(p) := \{m \in \mathbb{Z} : m \leq 0, m \equiv 1 \pmod{w_p}\}.$$

Thus the closure of  $\mathcal{M}(p)$  in  $\mathbb{Z}_p$  is the *p*-adic disc  $D(p) := \mathbb{Z}_p$  for  $p$  odd,  $D(p) := 1 + 2\mathbb{Z}_2$  for  $p = 2$ . We also write  $D^0(p)$  for the punctured disc  $D(p) \setminus \{1\} \supset \mathcal{M}(p)$ .

We first state a result on the existence of *p*-adic *L*-functions attached to primitive *p*-adic valued ray-class characters. The latter are, of course, precisely the characters of the form  $j \circ \psi$  for some primitive complex ray-class character  $\psi$  and, roughly speaking, the associated *p*-adic *L*-function ‘interpolates’ the function  $L(s, \psi)$  at the points  $s \in \mathcal{M}(p)$ . The details we require are set out in the

**THEOREM 2.4** (*p*-adic *L*-functions).— *Let  $k$  be totally real and  $p, j$  as above. For any primitive complex ray-class character  $\psi$  modulo  $\mathfrak{m}(\psi) = \mathfrak{f}(\psi)\mathfrak{z}(\psi)$  we have*

(i) For every  $m \in \mathbb{Z}_{\leq 0}$ , the value  $L(m, \psi)$  lies in the field  $\mathbb{Q}(\psi) \subset \mathbb{C}$  generated by the values of  $\psi$ .

(ii) There exists a unique ( $p$ -adically) continuous function

$$L_p(\cdot, j \circ \psi) : D^0(p) \longrightarrow \mathbb{C}_p$$

called the  $p$ -adic  $L$ -function associated to  $j \circ \psi$ , satisfying the interpolation condition

$$L_p(m, j \circ \psi) = \prod_{\substack{p \text{ prime} \\ p|p, \mathfrak{p} \in \mathfrak{f}(\psi)}} \left( 1 - \frac{(j \circ \psi)(\mathfrak{p})}{N\mathfrak{p}^m} \right) j(L(m, \psi)) \quad \forall m \in \mathcal{M}(p).$$

(iii)  $L_p(s, j \circ \psi)$  is a  $p$ -adic meromorphic function on  $D(p)$ . If  $\psi$  (equivalently,  $j \circ \psi$ ) is not the trivial character, then  $L_p(s, j \circ \psi)$  is actually analytic on this domain. Otherwise it has at most simple pole at  $s = 1$ .

(iv) Suppose that  $\psi$  is totally even, i.e.  $\mathfrak{z}(\psi)$  is trivial or, equivalently, the ray-class field  $K := k(\mathfrak{m}(\psi))$  is totally real. If Leopoldt's conjecture holds for  $K$  at  $p$  then  $L_p(1, j \circ \psi) \neq 0$  if  $\psi$  is non-trivial, and  $\text{res}_{s=1} L_p(s, j \circ \psi) \neq 0$  if  $\psi$  is trivial.

(v) If  $\psi$  is not totally even then  $L_p(s, j \circ \psi) = 0 \forall s$ .

*Proof.* — Siegel [Si] and Klengen showed—and Shintani [Sh, Cor. to Thm. 1] re-proved—that the partial zeta-function of  $k$  associated to any  $\mathfrak{c} \in \text{Cl}_{\mathfrak{m}}(k)$  takes rational values on  $\mathbb{Z}_{\leq 0}$ . Part (i) follows on multiplying by  $\psi(\mathfrak{c})$  and summing over  $\mathfrak{c}$ . Parts (iii) and (v) follow from [CN, Thm. 1.4] (see also Thm. 2.2, Ch. VI of [Ta]). For part (v), see Remark 2.3, Ch. VI of [Ta]). The main theorem of [Col] implies that if Leopoldt's Conjecture holds for  $K$  and  $p$  then the  $p$ -adic Dedekind zeta-function  $\zeta_{K,p}(s)$  has a simple pole at  $s = 1$ . Part (iv) therefore follows from (iii) and the formula expressing  $\zeta_{K,p}(s)$  as a product of  $L_p(s, j \circ \tilde{\chi})$  for  $\tilde{\chi} \in \text{Cl}_{\mathfrak{m}(\psi)}(k)^*$  including  $\psi$ . (This formula follows by interpolation from the analogous one for  $\zeta_K(s)$ .)  $\square$

We now apply Theorem 2.4 to the  $p$ -adic interpolation of  $\Phi_{\mathfrak{m},T}(s)$ .

**THEOREM/DEFINITION 2.1** ( $p$ -adic interpolation of  $\Phi_{\mathfrak{m},T}$ ). — Under the hypotheses of Theorem 2.4, we let  $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$  be any cycle for  $k$  and  $T$  any finite set of prime ideals of  $k$  including all those dividing  $p$ . We extend  $j$  to a homomorphism from  $\bar{\mathbb{Q}}G_{\mathfrak{m}}$  to  $\mathbb{C}_p G_{\mathfrak{m}}$  by defining  $j(\sum_{\sigma \in G_{\mathfrak{m}}} a_{\sigma} \sigma) := \sum_{\sigma \in G_{\mathfrak{m}}} j(a_{\sigma}) \sigma$ . Then

(i) For every  $m \in \mathbb{Z}_{\leq 0}$ , the value  $\Phi_{\mathfrak{m},T}(m)$  lies in  $\bar{\mathbb{Q}}G_{\mathfrak{m}}$ .

(ii) There exists a unique (*p*-adically) continuous function  $\Phi_{m,T,p} : D^0(p) \rightarrow \mathbb{C}_p G_m$  satisfying the interpolation condition

$$(13) \quad \Phi_{m,T,p}(m) = j(\Phi_{m,T}(m)) \quad \forall m \in \mathcal{M}(p).$$

We shall write  $\Phi_{m,T,p}^{(j)}$  instead of  $\Phi_{m,T,p}$  whenever its dependence on *j* (via Equation (13)) needs to be made explicit.

(iii)  $\Phi_{m,T,p}$  is a *p*-adic meromorphic function on  $D(p)$ . If *f* is not a product of distinct primes lying in *T*, then  $\Phi_{m,T,p}$  is actually analytic on this domain. Otherwise it has at most a unique, simple pole at  $s = 1$ . In all cases  $x\Phi_{m,T,p}(s)$  is analytic in  $D(p)$  for any *x* in the augmentation ideal  $I(\mathbb{C}_p G_m)$  of  $\mathbb{C}_p G_m$ .

(iv) For any  $s \in D^0(p)$  we have

$$(14) \quad \Phi_{m,T,p}^{(j)}(s) = \sum_{\chi \in G_m^*} A_{m,T,p}^{(j)}(s, \chi) j(e_\chi) = \sum_{\chi \in G_m^*} A_{m,T,p}^{(j)}(s, \chi) e_{j \circ \chi}$$

where for every  $\chi \in G_m^*$  the *p*-adic function  $A_{m,T,p}^{(j)}(s, \chi)$  is defined as follows. We first define  $\mathfrak{g}_0 = \mathfrak{g}_0(\chi)$ ,  $\mathfrak{g}' = \mathfrak{g}'(\chi)$  and  $g_{m(\chi)}(\tilde{\chi})$  as in Theorem 2.2. If  $\mathfrak{g}_0$  is squarefree, a product of *t* distinct primes none of which divides *f*( $\chi$ ), we set

$$(15) \quad A_{m,T,p}^{(j)}(s, \chi) := (-1)^t j(\tilde{\chi}^{-1}(\mathfrak{g}_0)g_{m(\chi)}(\tilde{\chi})) \langle N\mathfrak{g}' \rangle^{1-s} \times \prod_{\substack{p \text{ prime} \\ p|\mathfrak{g}', p \nmid f(\chi)}} \left( 1 - \frac{(j \circ \tilde{\chi})^{-1}(\mathfrak{p})}{\langle N\mathfrak{p} \rangle^{1-s}} \right) \prod_{\substack{p \text{ prime} \\ p \in T, p \nmid f(\chi)}} \left( 1 - \frac{(j \circ \tilde{\chi})(\mathfrak{p})}{\omega(N\mathfrak{p})\langle N\mathfrak{p} \rangle^s} \right) L_p(s, j \circ \tilde{\chi})$$

for all  $s \in D^0(p)$ .

Otherwise, we set  $A_{m,T,p}^{(j)}(s, \chi) := 0$  for all  $s \in D^0(p)$ .

*Proof.* — Since  $g_{m(\chi)}(\tilde{\chi})$  is algebraic for each  $\chi \in \text{Cl}_m(k)^*$ , Part (i) follows from Theorem 2.2 and Equation (12) together with Part (i) of Theorem 2.4 with  $\psi = \tilde{\chi}$ . On the other hand, for any  $x \in 1 + p\mathbb{Z}_p$  (or  $x \in 1 + 4\mathbb{Z}_4$  if  $p = 2$ ), the function  $s \mapsto x^s$  is analytic and non-zero on  $\mathbb{Z}_p$ . It follows from Part (iii) of Theorem 2.4 ( $\psi = \tilde{\chi}$ ) that, for all  $\chi \in \text{Cl}_m(k)^*$ , the function  $A_{m,T,p}^{(j)}(s, \chi)$  is analytic on  $D^0(p)$  except perhaps for a simple pole at  $s = 1$  when  $\chi$  is trivial and *f* is a product of distinct primes in *T*. Therefore, if we use Equation (14) to define  $\Phi_{m,T,p}$ , it will certainly satisfy both Parts (iii) and (iv). To check that it also has the interpolation property (13), we apply  $j \circ \chi$  to both sides and see that it suffices to show  $A_{m,T,p}^{(j)}(m, \chi) = j(\chi(\Phi_{m,T}(m)))$  for all  $\chi \in G_m^*$  and  $m \in \mathcal{M}(p)$ . But

this follows from the definition of  $A_{m,T,p}^{(j)}(s, \chi)$ , Part (ii) of Theorem 2.4 ( $\psi = \tilde{\chi}$ ) and the fact that  $m \equiv 1 \pmod{w_p}$ , together with Theorem 2.2. Also,  $\Phi_{m,T,p}$  is continuous since analytic on  $D^0(p)$  and the uniqueness in Part (ii) follows as before by density.  $\square$

*Remark 2.2.* — Lemma 3.3 below gives a sharpening of part (i). Similarly, we refer to Lemma 3.4 for the ‘field of definition’ of  $\Phi_{m,T,p}^{(j)}$  and to Corollary 3.1 for its dependence on  $j$ .

From a metamathematical —*i.e.* a logical and even historical— viewpoint, one could argue that it is somewhat perverse to construct  $\Phi_{m,T,p}$  by means of  $p$ -adic  $L$ -functions, as we have done above, rather than the other way around. The latter are, of course, much better known than the former. However, their construction by interpolation in [CN] passes first by that of certain functions  $Z(L, \xi, s)$  (see *ibid.*, Thm. 22) which are much closer to our twisted zeta-functions. And these in turn are the coefficients of  $\Phi_{m,T}$ . It might therefore be logically preferable, as it is certainly possible, to *start* by  $p$ -adically interpolating the twisted zeta-functions. From the numerical viewpoint too, the computation of  $\Phi_{m,T,p}$  at integer arguments can be carried out more efficiently by applying methods (derived from those of [CN], [Sh], [La] and [Ka]) directly to the twisted zeta-functions themselves rather than passing *via* the  $L$ -functions (See *e.g.* [RS, Theorems 3.2, 3.3]). If desired, one can then recover the  $p$ -adic  $L$ -functions by means of Theorem 2.2.

For any two cycles  $m$  and  $\tilde{m}$  with  $\tilde{m}|m$  we use the symbols  $\pi_{m,\tilde{m}}$  and  $\nu_{\tilde{m},m}$  to denote the  $\mathbb{C}_p$ -algebra homomorphisms between  $\mathbb{C}_p G_m$  and  $\mathbb{C}_p G_{\tilde{m}}$  defined in an entirely analogous way to the corresponding homomorphisms between  $\mathbb{C}G_m$  and  $\mathbb{C}G_{\tilde{m}}$ . The following corollary shows that the functions  $\Phi_{m,T,p}$  ‘really belong to cycles with trivial infinite part’.

**COROLLARY 2.1.** — *Let  $k, p, m$  and  $T$  be as in Theorem/Definition 2.1. Then*

$$\Phi_{m,T,p}(s) = \nu_{f,m}(\Phi_{f,T,p}(s)) \quad \text{for all } s \in D(p) \text{ (or } D^0(p)\text{)}.$$

*Proof.* — Part (v) ( $\psi = \tilde{\chi}$ ) of Theorem 2.4 shows that  $A_{m,T,p}^{(j)}(s, \chi)$  vanishes unless  $\chi = \phi \circ \pi_{m,f}$  for some  $\phi \in G_f^*$ , in which case it is easily seen to equal  $A_{f,T,p}^{(j)}(s, \phi)$ . Therefore, Equation (14) gives

$$\begin{aligned} \Phi_{m,T,p}(s) &= \sum_{\phi \in G_f^*} A_{f,T,p}^{(j)}(s, \phi) e_{j \circ \phi \circ \pi_{m,f}} = \sum_{\phi \in G_f^*} A_{f,T,p}^{(j)}(s, \phi) \nu_{f,m}(e_{j \circ \phi}) \\ &= \nu_{f,m}(\Phi_{f,T,p}(s)). \end{aligned} \quad \square$$

### 3. Conjectures.

#### 3.1. Statements of the basic conjectures.

We shall first state two parallel, ‘rational forms’ of the Stark Conjecture, one complex and one *p*-adic. They will be formulated side by side for the same field *k* (of degree *r* say, over  $\mathbb{Q}$ ), cycle  $\mathfrak{m} = \mathfrak{f}\mathfrak{z}$  and set *T*, but in terms of  $\Phi_{\mathfrak{m},T}(1)$  and  $\Phi_{\mathfrak{m},T,p}(1)$  respectively. We make the

HYPOTHESIS 3.1.

- (i) *k* is totally real,
- (ii)  $\mathfrak{f}$  is not a product of distinct primes lying in *T* (in particular,  $\mathfrak{f}$  is not trivial), and
- (iii)  $\mathfrak{z}$  is trivial, i.e.  $\mathfrak{m} = \mathfrak{f}$ .

Hypothesis 3.1 will be assumed from now on unless the contrary is explicitly stated, so that, in general, we can write  $\Phi_{\mathfrak{f},T}$  etc. in place of  $\Phi_{\mathfrak{m},T}$  etc. We shall also write *K* for the ray-class field  $k(\mathfrak{m}) = k(\mathfrak{f}) \subset \mathbb{C}$  (necessarily totally real) and *G* for  $G_{\mathfrak{f}} = \text{Gal}(K/k)$ . Let  $S_{\infty}$  and  $S_0 = S_0(\mathfrak{f})$  denote respectively the set of infinite (real) places of *k* and the set of finite places dividing  $\mathfrak{f}$  in *k*. We let  $S = S_{\infty} \cup S_0$ . The notations  $S_{\infty}(K)$ ,  $S_0(K)$  and  $S(K)$  refer to the sets of places of *K* dividing those in these three sets. We abbreviate to  $U_S(K)$  or  $U_S$  the group  $U_{S(K)}(K)$  of  $S(K)$ -units of *K*. Let  $\iota_1, \dots, \iota_r$  denote the embeddings of *k* into  $\mathbb{Q}$ . For each  $i = 1, \dots, r$  we choose an embedding  $\tilde{\iota}_i : K \rightarrow \mathbb{Q}$  extending  $\iota_i$ . We write  $\iota_{i,p}$  for the *p*-adic embedding  $j \circ \iota_i$  of *k* into  $\mathbb{C}_p$ , and also  $\tilde{\iota}_{i,p}$  for its extension  $j \circ \tilde{\iota}_i : K \rightarrow \mathbb{C}_p$ . We then define logarithmic maps  $\lambda_i : U_S \rightarrow \mathbb{R}G$  and  $\lambda_{i,p} : U_S \rightarrow \mathbb{C}_pG$  by setting

$$\lambda_i(u) := \sum_{\sigma \in G} \log |\tilde{\iota}_i \circ \sigma(u)| \sigma^{-1}$$

and 
$$\lambda_{i,p}(u) := \sum_{\sigma \in G} \log_p(\tilde{\iota}_{i,p} \circ \sigma(u)) \sigma^{-1} \text{ for all } u \in U_S,$$

where ‘ $\log_p$ ’ denotes Iwasawa’s *p*-adic logarithm. Both  $\lambda_i$  and  $\lambda_{i,p}$  are clearly  $\mathbb{Z}G$ -linear and so ‘extend’ by  $\mathbb{Q}$ -linearity to  $\mathbb{Q}U_S := \mathbb{Q} \otimes_{\mathbb{Z}} U_S$ . (Henceforth, we shall often write  $\mathcal{R}A$  in place of  $\mathcal{R} \otimes_{\mathbb{Z}} A$  considered as an  $\mathcal{R}$ -module, for any commutative ring  $\mathcal{R}$  and abelian group *A*.) These extensions in turn define unique,  $\mathbb{Q}G$ -linear, group-ring-valued ‘regulator

maps'  $R$  and  $R_p$  sending the  $r$ th exterior power  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S \cong \mathbb{Q} \otimes \bigwedge_{\mathbb{Z}G}^r U_S$  into  $\mathbb{R}G$  and  $\mathbb{C}_p G$  respectively and satisfying

$$R(u_1 \wedge \dots \wedge u_r) = \det(\lambda_i(u_l))_{i,l=1}^r$$

and  $R_p(u_1 \wedge \dots \wedge u_r) = \det(\lambda_{i,p}(u_l))_{i,l=1}^r \quad \forall u_1, \dots, u_r \in \mathbb{Q}U_S.$

We shall use an additive notation for  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  as  $\mathbb{Z}G$ -module and write  $\lambda_{i,p}^{(j)}$  and  $R_p^{(j)}$  instead of  $\lambda_{i,p}$  and  $R_p$  whenever their dependence on  $j$  (via the  $\tilde{v}_{i,p}$ ) needs to be made explicit. Finally, we let  $\sqrt{d_k} \in \mathbb{R}$  denote the positive square-root of the (positive) absolute discriminant  $d_k$  of  $k$ . We make the

**CONJECTURE 3.1** (Basic complex conjecture). — *If Hypothesis 3.1 holds then there exists  $\eta_{f,T} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  such that*

$$(16) \quad \Phi_{f,T}(1) = \frac{2^r}{\sqrt{d_k} \prod_{p \in T} N\mathfrak{p}} R(\eta_{f,T}) \quad \text{in } \mathbb{C}G.$$

Now suppose that  $p$  is a prime number and consider the condition

(17)  $T$  contains the set  $T_p$  of all primes dividing  $p$  in  $\mathcal{O}$ .

This condition is necessary to define  $\Phi_{f,T,p}(s)$ . We make the

**CONJECTURE 3.2** (Basic  $p$ -adic conjecture). — *If Hypothesis 3.1 and condition (17) hold then, choosing an embedding  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ , there exists  $\eta_{f,T,p} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  such that*

$$(18) \quad \Phi_{f,T,p}^{(j)}(1) = \frac{2^r}{j(\sqrt{d_k}) \prod_{p \in T} N\mathfrak{p}} R_p^{(j)}(\eta_{f,T,p}) \quad \text{in } \mathbb{C}_p G.$$

### 3.2. A study of the basic conjectures.

First note that we could of course do away with the factor  $2^r (\prod_{p \in T} N\mathfrak{p})^{-1}$  in the statements of both conjectures since we are free to multiply the elements  $\eta_{f,T}$  and  $\eta_{f,T,p}$  of  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  by any  $a \in \mathbb{Q}^\times$ . This is what we mean by calling the form of these basic conjectures 'rational'. In contrast, the 'integral' forms to be discussed later will not be insensitive to such multiplications. The introduction of the above factor simply anticipates these refinements.

It is easy to check that changing the choice and/or ordering of the extensions  $\tilde{l}_i, i = 1, \dots, r$  would have the effect of multiplying both  $R(\eta_{f,T})$  and  $R_p^{(j)}(\eta_{f,T,p})$  by  $\pm\sigma$  for some  $\sigma \in G$ . Clearly, these changes could be nullified by replacing  $\eta_{f,T}$  and  $\eta_{f,T,p}$  by  $\pm\sigma^{-1}\eta_{f,T}$  and  $\pm\sigma^{-1}\eta_{f,T,p}$  respectively. Consequently, this choice/ordering is immaterial to the veracity of Conjectures 3.1 and 3.2.

The choice of  $j$  in Conjecture 3.2, turns out to be irrelevant too, but in the stronger sense that it doesn't even affect  $\eta_{f,T,p}$ . To prove this statement, we first need some facts about the 'fields of definition' and the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on certain values of  $\Phi_{f,T}$  and  $\Phi_{f,T,p}$ . We shall suspend until further notice Hypothesis 3.1 and return to writing  $k(\mathfrak{m}), G_{\mathfrak{m}}$  and  $\Phi_{\mathfrak{m},T}$  etc. instead of  $K, G$  and  $\Phi_{f,T}$  etc.

LEMMA 3.1. — Suppose that  $\mathfrak{z}$  is trivial and let  $'*'$  denote the complex conjugation extended coefficientwise to  $\mathbb{C}G_{\mathfrak{m}}$ . Then  $\Phi_{\mathfrak{m},T}(s^*)^* = \Phi_{\mathfrak{m},T}(s)$  for all  $s \in \mathbb{C}$  (or in  $\mathbb{C} \setminus \{1\}$ ). In particular  $\Phi_{\mathfrak{m},T}(s)$  takes  $\mathbb{R}$  (or  $\mathbb{R} \setminus \{1\}$ ) into  $\mathbb{R}G_{\mathfrak{m}}$ .

Proof. — By analytic continuation, it suffices to show that  $\Phi_{\mathfrak{m},T}(s^*)^* = \Phi_{\mathfrak{m},T}(s)$  whenever  $\Re(s) > 1$ . But for such an  $s$  and any  $\mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$ , represented by a pair  $(\xi, J) \in W_f$ , say, the definitions show that  $Z_T(s^*; \mathfrak{w})^* = Z_T(s^*; \xi, J, \mathfrak{m})^* = Z_T(s; \xi^*, J, \mathfrak{m})$ . But  $\xi^* = \xi \circ (-1)$  and  $J = -J$ , so, by definition, the pair  $(\xi^*, J)$  is equivalent to  $(\xi, J)$ , relative to a trivial  $\mathfrak{z}$ . Thus  $Z_T(s^*; \mathfrak{w})^* = Z_T(s; \mathfrak{w})$  from which the result follows easily.  $\square$

Next we define a unique integer  $f > 0$  by setting  $f \cap \mathbb{Z} = f\mathbb{Z}$ . We write  $\mu_f$  for the group of  $f$ th roots of unity in  $\mathbb{C}$ . For any pair  $(\xi, J) \in W_f$ , the image of  $\xi$  in  $\mu(\mathbb{C})$  is precisely  $\mu_f$ , and we have

LEMMA 3.2. — If  $k$  is totally real and  $m \in \mathbb{Z}_{\leq 0}$ , then  $Z_T(m; \xi, J, \mathfrak{m})$  lies in  $\mathbb{Q}(\mu_f)$  and  $\Phi_{\mathfrak{m},T}(m)$  in  $\mathbb{Q}(\mu_f)G_{\mathfrak{m}}$ . Moreover, for every  $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , the pair  $(\alpha \circ \xi, J)$  lies in  $W_f$  and

$$(19) \quad Z_T(m; \xi, J, \mathfrak{m})^\alpha = Z_T(m; \alpha \circ \xi, J, \mathfrak{m}).$$

Proof. — It is clear that  $(\xi, J) \in W_f$  implies  $(\alpha \circ \xi, J) \in W_f$  so it suffices to show that  $Z_T(m; \xi, J, \mathfrak{m})$  lies in  $\mathbb{Q}(\mu_f)$  and that (19) holds. First reduce to the case where  $\mathfrak{z}$  is maximal and  $T = \emptyset$  using successively Theorem 2.1 and Equation (4) above and Theorem 3.1 of [So2]. Two approaches are now possible. One is two rewrite  $Z_\emptyset(s; \xi, J, \mathfrak{m})$  as a linear combination of partial zeta-functions with coefficients in  $\mathbb{Q}(\mu_f)$  (cf. [So2] proof of Theorem 3.3) and then use the rationality result of Siegel/Klingen/Shintani

cited in the proof of Theorem 2.4. Alternatively, one can apply the methods of Shintani directly to  $Z_\varnothing(s; \xi, J, \mathfrak{m})$ : First write it in terms of simpler Dirichlet series involving the character  $\xi$  by means of a ‘cone decomposition’ of a fundamental domain for the action of  $E_{\mathfrak{m}}$  on  $\mathcal{S}(J, \mathfrak{z}, \varnothing)$  (cf. [Sh, Prop. 4]). Then apply [Sh, Prop. 1] to each of these series. For notes and details of the second procedure the reader can consult [R-S], Theorem 3.1 and Example 3.1 where it is carried out explicitly in the case  $r = 2$ .  $\square$

Let  $\infty$  denote the unique real place of  $\mathbb{Q}$ . If the infinite part  $\mathfrak{z}$  of  $\mathfrak{m}$  is not (resp. is) trivial we shall denote by  $\mathfrak{m}_{\mathbb{Q}}$  the cycle  $f\mathbb{Z}.\infty$  (resp.  $f\mathbb{Z}$ ) for  $\mathbb{Q}$ , so that the ray-class field  $\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})$  is equal to  $\mathbb{Q}(\mu_f)$  (resp. to  $\mathbb{Q}(\mu_f)^+ = \mathbb{Q}(\cos(2\pi/f))$ ), the maximal real subfield of  $\mathbb{Q}(\mu_f)$ ). Lemmas 3.1 and 3.2 combine to give

LEMMA 3.3. — *If  $k$  is totally real then  $\Phi_{\mathfrak{m}, T}(m)$  lies in  $\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})G_{\mathfrak{m}}$  for every  $m \in \mathbb{Z}_{<0}$ .*  $\square$

For any prime  $p$ , we shall (abusively) denote by  $\mathbb{Q}_p(\mu_f)^+$  the field that is the (topological) closure of any embedding of  $\mathbb{Q}(\mu_f)^+$  in  $\mathbb{C}_p$ . The following is a  $p$ -adic analogue of Lemma 3.1.

LEMMA 3.3. — *If  $k$  is totally real and condition (17) holds then, for any choice of  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ , the function  $\Phi_{\mathfrak{m}, T, p}^{(j)}$  maps  $D^0(p)$  (or  $D(p)$ ), under Hypothesis 3.1 (ii) into  $\mathbb{Q}_p(\mu_f)^+G_{\mathfrak{m}}$ .*

*Proof.* — By Corollary 2.1 we can assume w.l.o.g. that  $\mathfrak{z}$  is trivial, i.e.  $\mathfrak{m} = \mathfrak{f}$ . Then Lemma 3.3 and Equation (13) tell us that  $\Phi_{\mathfrak{m}, T, p}^{(j)}(m)$  lies in  $j(\mathbb{Q}(\mu_f)^+)G_{\mathfrak{m}}$  for all  $m \in \mathcal{M}(p)$ . The result follows by density and continuity.  $\square$

The extension of ideals from  $\mathcal{I}_{f\mathbb{Z}}(\mathbb{Q})$  to  $\mathcal{I}_{\mathfrak{f}}(k)$  induces a homomorphism (denoted  $\mathfrak{t}_{\mathfrak{m}}$ ) from  $\text{Cl}_{\mathfrak{m}_{\mathbb{Q}}}(\mathbb{Q})$  to  $\text{Cl}_{\mathfrak{m}}(k)$ . Let  $\mathbb{Q}^{\text{ab}}$  and  $k^{\text{ab}}$  denote the maximal abelian extensions of  $\mathbb{Q}$  and  $k$  respectively inside  $\bar{\mathbb{Q}} \subset \mathbb{C}$ . We write simply Ver for the transfer homomorphism from the abelianisation  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{\text{ab}}$  (identified with  $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ ) to  $\text{Gal}(\bar{\mathbb{Q}}/k)^{\text{ab}}$  (identified with  $\text{Gal}(k^{\text{ab}}/k)$ ). It is a well-known fact from Class-Field Theory (see e.g. Ch. VII, §8 of [Se2]) that the outer rectangle of the diagram (20) commutes. (The horizontal isomorphisms are the maps  $\sigma_{\mathfrak{c}} \mapsto \mathfrak{c}$ , i.e. the inverses of the respective Artin maps. The other horizontal maps are the restrictions.) We define the map  $V_{\mathfrak{m}}$  to be the composite of Ver with the restriction  $\text{Gal}(k^{\text{ab}}/k) \rightarrow G_{\mathfrak{m}}$ , so that the whole diagram commutes. We can now prove

$$\begin{array}{ccccc}
 \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) & \longrightarrow & \text{Gal}(\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})/\mathbb{Q}) & \xrightarrow{\simeq} & \text{Cl}_{\mathfrak{m}_{\mathbb{Q}}}(\mathbb{Q}) \\
 \downarrow \text{Ver} & \searrow V_{\mathfrak{m}} & & & \downarrow \mathfrak{t}_{\mathfrak{m}} \\
 \text{Gal}(k^{\text{ab}}/k) & \longrightarrow & \text{Gal}(k(\mathfrak{m})/k) = G_{\mathfrak{m}} & \xrightarrow{\simeq} & \text{Cl}_{\mathfrak{m}}(k)
 \end{array}$$

(20)

PROPOSITION 3.1 (Galois action on  $\Phi_{\mathfrak{m},T}(m)$ ,  $m \in \mathbb{Z}_{\leq 0}$ ). — If  $k$  is totally real then for any  $m \in \mathbb{Z}_{\leq 0}$  and any  $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  we have

$$\Phi_{\mathfrak{m},T}(m)^{\alpha} = V_{\mathfrak{m}}(\alpha|_{\mathbb{Q}^{\text{ab}}})\Phi_{\mathfrak{m},T}(m) \quad \text{in } \mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})G_{\mathfrak{m}}.$$

(Note that  $\alpha$  acts coefficientwise on the L.H.S. and that the R.H.S. is a product in the group-ring  $\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})G_{\mathfrak{m}}$ .)

Proof. — We fix  $m$  and  $\alpha$  and choose (as we may)  $d \in \mathbb{Z}_{>0}$ ,  $(d, f) = 1$  such that

$$(21) \quad \alpha|_{\mathbb{Q}(\mu_f)} = \sigma_{d\mathbb{Z}, f\mathbb{Z}\infty} \quad \text{in } \text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q}) = \text{Gal}(\mathbb{Q}(f\mathbb{Z}\infty)/\mathbb{Q}).$$

Then explicit class-field theory for  $\mathbb{Q}$  tells us that  $\zeta^{\alpha} = \zeta^d$  for every  $\zeta \in \mu_f$ . Thus, if  $\mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$  is represented by some pair  $(\xi, J) \in W_f$ , we have, by Lemma 3.2,

$$Z_T(m; \mathfrak{w})^{\alpha} = Z_T(m; \xi, J, \mathfrak{m})^{\alpha} = Z_T(m; \alpha \circ \xi, J, \mathfrak{m}) = Z_T(m; \xi \circ d, J, \mathfrak{m}).$$

But, since  $d$  is positive, the pair  $(\xi \circ d, J)$  is clearly equivalent to the pair  $(\xi|_{dJ}, dJ)$  relative to  $\mathfrak{z}$  and so lies in the class  $[d\mathcal{O}]_{\mathfrak{m}} \cdot \mathfrak{w} \in \mathfrak{W}_{\mathfrak{m}}$  by definition. Therefore  $Z_T(m; \mathfrak{w})^{\alpha} = Z_T(m; [d\mathcal{O}]_{\mathfrak{m}} \cdot \mathfrak{w})$ , from which it follows that

$$\Phi_{\mathfrak{m},T}(m)^{\alpha} = \sigma_{d\mathcal{O}, \mathfrak{m}}\Phi_{\mathfrak{m},T}(m).$$

Now on the one hand  $[d\mathcal{O}]_{\mathfrak{m}}$  equals  $\mathfrak{t}_{\mathfrak{m}}([d\mathbb{Z}]_{\mathfrak{m}_{\mathbb{Q}}})$ . On the other, restricting both sides of Equation (21) to  $\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})$ , we find  $\alpha|_{\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})} = (\alpha|_{\mathbb{Q}^{\text{ab}}})|_{\mathbb{Q}(\mathfrak{m}_{\mathbb{Q}})} = \sigma_{d\mathbb{Z}, \mathfrak{m}_{\mathbb{Q}}}$ . It therefore follows from Diagram (20) that  $\sigma_{d\mathcal{O}, \mathfrak{m}} = V_{\mathfrak{m}}(\alpha|_{\mathbb{Q}^{\text{ab}}})$ , as required.  $\square$

If  $j, j' : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$  are two embeddings then  $j' = j \circ \alpha$  for some  $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ .

**COROLLARY 3.1** (Dependence of  $\Phi_{\mathfrak{m},T,p}$  on  $j$ ). — *Suppose that  $k$  is totally real and that condition (17) holds. Then, for any  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$  and  $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ ,*

$$\Phi_{\mathfrak{m},T,p}^{(j \circ \alpha)}(s) = V_{\mathfrak{m}}(\alpha|_{\mathbb{Q}^{\text{ab}}})\Phi_{\mathfrak{m},T,p}^{(j)}(s)$$

for all  $s$  in  $D^0(p)$  (or in  $D(p)$ , under Hypothesis 3.1 (ii)).

*Proof.* — The interpolation property of  $\Phi_{\mathfrak{m},T,p}$  gives, for each  $m \in D^0(p)$ ,

$$\Phi_{\mathfrak{m},T,p}^{(j \circ \alpha)}(m) = j(\Phi_{\mathfrak{m},T}(m)^\alpha) = V_{\mathfrak{m}}(\alpha|_{\mathbb{Q}^{\text{ab}}})\Phi_{\mathfrak{m},T,p}^{(j)}(m)$$

by Proposition 3.1 and the result follows by density and the continuity of  $\Phi_{\mathfrak{m},T,p}^{(j \circ \alpha)}$  and  $\Phi_{\mathfrak{m},T,p}^{(j)}$ . □

We now reimpose Hypothesis 3.1 (so  $\mathfrak{m} = \mathfrak{f}$ ) and again write  $K, G$  and  $\Phi_{\mathfrak{f},T}$  etc instead of  $k(\mathfrak{m}), G_{\mathfrak{m}}$  and  $\Phi_{\mathfrak{m},T}$  etc. In particular, we can specialise Lemmas 3.1 and 3.4 and Corollary 3.1 at  $s = 1$  and deduce the properties

$$(22) \quad \Phi_{\mathfrak{f},T}(1) \text{ lies in } \mathbb{R}G$$

$$(23) \quad \Phi_{\mathfrak{f},T,p}^{(j)}(1) \text{ lies in } \mathbb{Q}_p(\mu_f)^+G$$

and

$$(24) \quad \Phi_{\mathfrak{f},T,p}^{(j \circ \alpha)}(1) = V_{\mathfrak{f}}(\alpha|_{\mathbb{Q}^{\text{ab}}})\Phi_{\mathfrak{f},T,p}^{(j)}(1) \text{ for all } \alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Notice that, for all  $\eta \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S, R(\eta)/\sqrt{d_k}$  actually lies in  $\mathbb{R}G$ . Hence Conjecture 3.1 is compatible with (22).

We shall now prove similar compatibilities between Conjecture 3.2 and both properties (23) and (24), by showing that the quantity  $R_p^{(j)}(u)/j(\sqrt{d_k})$  satisfies relations analogous to these. Let us further extend the embeddings  $\tilde{l}_1, \dots, \tilde{l}_r : K \rightarrow \bar{\mathbb{Q}}$  to automorphisms  $\{\beta_1, \dots, \beta_r\}$  of  $\bar{\mathbb{Q}}$  constituting a complete set of representatives for the coset space  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/\text{Gal}(k/\mathbb{Q})$ . Given any  $\alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ , there must exist  $\gamma_1, \dots, \gamma_r \in \text{Gal}(\bar{\mathbb{Q}}/k)$  and a permutation  $\pi_\alpha$  of  $\{1, \dots, r\}$  such that

$$\alpha \circ \beta_i = \beta_{\pi_\alpha(i)} \circ \gamma_i \text{ for } i = 1, \dots, r.$$

By definition,  $\text{Ver}(\alpha|_{\mathbb{Q}^{\text{ab}}})$  equals the image of  $\gamma_1 \dots \gamma_r$  in  $\text{Gal}(\bar{\mathbb{Q}}/k)^{\text{ab}}$ , so that

$$(25) \quad V_f(\alpha|_{\mathbb{Q}^{\text{ab}}}) = \gamma_1|_K \dots \gamma_r|_K \quad \text{in } G.$$

Now, for any  $u \in U_S$  and each  $i = 1, \dots, r$ , the definition of  $\lambda_{i,p}(u)$  gives

$$\begin{aligned} \lambda_{i,p}^{(j \circ \alpha)}(u) &= \sum_{\sigma \in G} \log_p(j \circ \alpha \circ \beta_i \circ \sigma(u)) \sigma^{-1} \\ &= \sum_{\sigma \in G} \log_p(j \circ \beta_{\pi_\alpha(i)} \circ \gamma_i \circ \sigma(u)) \sigma^{-1} = \lambda_{\pi_\alpha(i),p}^{(j)}(u) \gamma_i|_K \end{aligned}$$

in  $\mathbb{C}_p G$ , from which it follows by the definition of  $R_p^{(j)}(u_1 \wedge \dots \wedge u_r)$ , Equation (25) and  $\mathbb{Q}$ -linearity that

$$(26) \quad R_p^{(j \circ \alpha)}(\eta) = \text{sign}(\pi_\alpha) V_f(\alpha|_{\mathbb{Q}^{\text{ab}}}) R_p^{(j)}(\eta) \quad \text{for all } \eta \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S.$$

On the other hand, for any  $\mathbb{Z}$ -base  $\{a_l\}_{l=1}^r$  of  $\mathcal{O}$ , we have  $\sqrt{d_k} = \pm \det(\beta_i(a_l))_{i,l=1}^r$  so that

$$(27) \quad \begin{aligned} j \circ \alpha(\sqrt{d_k}) &= \pm j(\det(\beta_{\pi_\alpha(i)} \circ \gamma_i(a_l))) \\ &= \pm \text{sign}(\pi_\alpha) j(\det(\beta_i(a_l))) = \text{sign}(\pi_\alpha) j(\sqrt{d_k}). \end{aligned}$$

PROPOSITION 3.2. — For any  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$  and any  $\eta \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ , the element  $\frac{R_p^{(j)}(\eta)}{j(\sqrt{d_k})}$  of  $\mathbb{C}_p G$  actually lies in  $\mathbb{Q}_p(\mu_f)^+ G$ . Moreover

$$(28) \quad \frac{R_p^{(j \circ \alpha)}(\eta)}{j \circ \alpha(\sqrt{d_k})} = V_f(\alpha|_{\mathbb{Q}^{\text{ab}}}) \frac{R_p^{(j)}(\eta)}{j(\sqrt{d_k})} \quad \text{for all } \alpha \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

*Proof.* — Equation (28) is immediate from Equations (26) and (27). It is clear from the definitions that  $j(\sqrt{d_k})$  and  $R_p^{(j)}(\eta)$  lie in  $\bar{\mathbb{Q}}_p^\times$  and  $\bar{\mathbb{Q}}_p G$  respectively, where  $\bar{\mathbb{Q}}_p$  denotes the algebraic closure of  $\mathbb{Q}_p$  in  $\mathbb{C}_p$ . For the first statement of the Proposition it therefore suffices to show that  $R_p^{(j)}/j(\sqrt{d_k})$  is fixed by any element  $\rho$  of  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_f)^+)$ , extended coefficientwise to  $\bar{\mathbb{Q}}_p G$ . But such a  $\rho$  is continuous, so commutes with  $\log_p : \bar{\mathbb{Q}}_p^\times \rightarrow \bar{\mathbb{Q}}_p$ . It follows that  $\rho \circ \lambda_{i,p}^{(j)} = \lambda_{i,p}^{(\rho \circ j)} \forall i$  and since  $\rho \circ j$  equals  $j \circ \delta$  for some  $\delta \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\mu_f)^+)$ , it follows from Equation (28) that

$$\rho \left( \frac{R_p^{(j)}(\eta)}{j(\sqrt{d_k})} \right) = \frac{R_p^{(\rho \circ j)}(\eta)}{\rho \circ j(\sqrt{d_k})} = V_f(\delta|_{\mathbb{Q}^{\text{ab}}}) \frac{R_p^{(j)}(\eta)}{j(\sqrt{d_k})}.$$

But Diagram (20) shows that  $V_f$  factors through the restriction to  $\mathbb{Q}(f_{\mathbb{Q}}) = \mathbb{Q}(\mu_f)^+$  on which  $\delta$  is trivial, so  $V_f(\delta|_{\mathbb{Q}^{ab}}) = 1_G$  and the first statement follows.  $\square$

The first statement of Proposition 3.2 demonstrates (as we wished) that Conjecture 3.2 is compatible with (23). Furthermore, a comparison of Equations (28) and (24) yields

PROPOSITION 3.3. — *Suppose that  $\eta_{f,T,p} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  satisfies Conjecture 3.2 for some embedding  $j : \mathbb{Q} \rightarrow \mathbb{C}_p$ . Then it also satisfies the same conjecture for any other such embedding.*  $\square$

In order to discuss the dependence of the basic conjectures on  $T$ , we suppose that  $\mathfrak{q}$  is a prime in  $T$  and abbreviate the set  $T \setminus \{\mathfrak{q}\}$  to  $\tilde{T}$ . First, suppose that  $\mathfrak{q} \nmid f$ . Then we have

$$(29) \quad \Phi_{f,T}(s) = (1 - N\mathfrak{q}^{-s}\sigma_{\mathfrak{q},f})\Phi_{f,\tilde{T}}(s) \quad \forall s \in \mathbb{C}.$$

Indeed, this follows by analytic continuation from Theorem 3.1 of [So2]. (It is there assumed that  $(f, T) = 1$  but the proof uses only the condition  $(f, \mathfrak{q}) = 1$  and not  $(f, \tilde{T}) = 1$ .) Specialising at  $s = 1$ , comparing with (16) and noting that  $(N\mathfrak{q} - \sigma_{\mathfrak{q},f})$  is a unit of  $\mathbb{Q}G$ , we obtain

PROPOSITION 3.4. — *Suppose  $\mathfrak{q} \nmid f$ . Then  $\eta_{f,\tilde{T}} = \eta$  satisfies Conjecture 3.1 for  $k, f$  and  $\tilde{T}$ , if and only if  $\eta_{f,T} = (N\mathfrak{q} - \sigma_{\mathfrak{q},f})\eta$  satisfies Conjecture 3.1 for  $k, f$  and  $T$ .*  $\square$

Now, given a prime  $p$ , the analogue of Proposition 3.4 for Conjecture 3.2 requires the assumption of condition (17) for  $\tilde{T}$ , hence that  $\mathfrak{q}$  does not divide  $p$ . If we make these assumptions, then for any choice of  $j$ , it follows by density and Equation (13) together with Equation (29) for  $s \in \mathcal{M}(p)$  that

$$(30) \quad \Phi_{f,T,p}(s) = (1 - \omega(N\mathfrak{q})^{-1}\langle N\mathfrak{q} \rangle^{-s}\sigma_{\mathfrak{q},f})\Phi_{f,\tilde{T},p}(s) \quad \forall s \in D(p).$$

Specialising at  $s = 1$  again, we get

PROPOSITION 3.5. — *Suppose  $T \supset T_p$  and  $\mathfrak{q} \nmid pf$ . Then  $\eta_{f,\tilde{T},p} = \eta$  satisfies Conjecture 3.2 for  $k, f, \tilde{T}$  and  $p$ , if and only if  $\eta_{f,T,p} := (N\mathfrak{q} - \sigma_{\mathfrak{q},f})\eta$  satisfies Conjecture 3.2 for  $k, f, T$  and  $p$ .*  $\square$

In particular, Propositions 3.4 and 3.5 clearly show that for fixed  $k$  totally real and fixed  $f$  different from  $\mathcal{O}$  (resp. not a product of distinct

primes in  $T_p$ ), Conjecture 3.1 for  $T = \emptyset$  (resp. Conjecture 3.2 for  $p$  and  $T = T_p$ ) is equivalent to Conjecture 3.1 for any  $T$  such that  $(\mathfrak{f}, T) = 1$  (resp. to Conjecture 3.2 for any  $T \supset T_p$  such that  $(\mathfrak{f}, T \setminus T_p) = 1$ ).

The situation is a little more complicated if we allow  $T$  to vary by primes dividing  $\mathfrak{f}$ . Let  $k, \mathfrak{f}, \mathfrak{q} \in T$  and  $\tilde{T}$  be as before but suppose now that  $\mathfrak{q}|\mathfrak{f}$ . We write  $\mathfrak{f}'$  for  $\mathfrak{q}^{-1}\mathfrak{f}$  and  $K', G', S', U_{S'}, \tilde{v}'_i, \lambda'_i$  and  $R'$  respectively for  $k(\mathfrak{f}')$ ,  $G_{\mathfrak{f}'}, S_\infty \cup S(\mathfrak{f}')$  (contained in  $S$ ),  $U_{S'(K')}$  (contained in  $U_S$ ),  $\tilde{v}_i|_{K'}$  and the maps from  $U_{S'}$  and  $\bigwedge_{\mathbb{Q}G'}^r \mathbb{Q}U_{S'}$  to  $\mathbb{C}G'$  defined using  $\tilde{v}'_i$  for  $i = 1, \dots, r$ . It follows that  $k$  and  $\mathfrak{m}' := \mathfrak{f}'$  satisfy Hypothesis 3.1 since  $k$  and  $\mathfrak{m} = \mathfrak{f}$  do. Moreover Lemma 2.1 (continued to  $s = 1$ ) together with Equation (5) gives

$$(31) \quad \Phi_{\mathfrak{f}, T}(1) = \Phi_{\mathfrak{f}, \tilde{T}}(1) - N\mathfrak{q}^{-1}|E_{\mathfrak{f}'} : E_{\mathfrak{f}}| |\ker \pi_{\mathfrak{f}, \mathfrak{f}'}| \nu_{\mathfrak{f}, \mathfrak{f}'}(\Phi_{\mathfrak{f}', \tilde{T}}(1)).$$

On the other hand, for any  $u' \in U_{S'}$  we find that

$$\lambda'_i(u') = |\ker \pi_{\mathfrak{f}, \mathfrak{f}'}| \nu_{\mathfrak{f}, \mathfrak{f}'}(\lambda_i(u')) \quad \forall i,$$

and hence

$$(32) \quad R(\eta') = |\ker \pi_{\mathfrak{f}, \mathfrak{f}'}|^r \nu_{\mathfrak{f}, \mathfrak{f}'}(R'(\eta')) \quad \forall \eta' \in \bigwedge_{\mathbb{Q}G'}^r \mathbb{Q}U_{S'} \subset \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S.$$

Together with Equation (31) this gives

PROPOSITION 3.6. — *Suppose  $\mathfrak{q}|\mathfrak{f}$  and let  $\mathfrak{f}' = \mathfrak{q}^{-1}\mathfrak{f}$ . If  $\eta_{\mathfrak{f}, \tilde{T}} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  and  $\eta_{\mathfrak{f}', \tilde{T}} \in \bigwedge_{\mathbb{Q}G'}^r \mathbb{Q}U_{S'}$  satisfy Conjecture 3.1 for  $k, \mathfrak{f}, \tilde{T}$  and  $k, \mathfrak{f}', \tilde{T}$  respectively, then the element  $\eta_{\mathfrak{f}, T} := (N\mathfrak{q})\eta_{\mathfrak{f}, \tilde{T}} - |E_{\mathfrak{f}'} : E_{\mathfrak{f}}| |\ker \pi_{\mathfrak{f}, \mathfrak{f}'}|^{(1-r)}\eta_{\mathfrak{f}', \tilde{T}}$  satisfies Conjecture 3.1 for  $k, \mathfrak{f}$  and  $T$ .  $\square$*

To round off the discussion we note the *p*-adic analogue of Proposition 3.6. Assuming that  $\tilde{T}$  contains  $T_p$ , the analogue of Equation (32) follows by density from the continuation of Lemma 2.1 to  $\mathcal{M}(p)$  (again using Equation (5)). Equation (32) also has a natural analogue, replacing  $R$  and  $R'$  by  $R_p$  and  $R'_p$ , defined in the obvious way. The result is

PROPOSITION 3.7. — *Suppose  $T \supset T_p, \mathfrak{q}|\mathfrak{f}$  and  $\mathfrak{q} \nmid p\mathcal{O}$ . Let  $\mathfrak{f}' = \mathfrak{q}^{-1}\mathfrak{f}$ . If  $\eta_{\mathfrak{f}, \tilde{T}, p} \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  and  $\eta_{\mathfrak{f}', \tilde{T}, p} \in \bigwedge_{\mathbb{Q}G'}^r \mathbb{Q}U_{S'}$  satisfy Conjecture 3.2 for  $k, \mathfrak{f}, \tilde{T}, p$  and  $k, \mathfrak{f}', \tilde{T}, p$  respectively, then  $\eta_{\mathfrak{f}, T, p} := (N\mathfrak{q})\eta_{\mathfrak{f}, \tilde{T}, p} - |E_{\mathfrak{f}'} : E_{\mathfrak{f}}| |\ker \pi_{\mathfrak{f}, \mathfrak{f}'}|^{(1-r)}\eta_{\mathfrak{f}', \tilde{T}, p}$  satisfies Conjecture 3.2 for  $k, \mathfrak{f}, T$  and  $p$ .  $\square$*

Note that Propositions 3.4 and 3.6 (resp. 3.5 and 3.7) together allow one to reduce Conjecture 3.1 for fixed totally real  $k$  (resp. Conjecture 3.2

for fixed totally real  $k$  and  $p$ ) to the case  $T = \emptyset$  (resp. the case  $T = T_p$ ), provided that  $f$  is not fixed but made to vary subject to Hypothesis 3.1 (ii) for this minimal choice of  $T$ .

### 3.3. Relations between conjectures.

We can reduce Conjecture 3.1 to the case  $(f, T) = 1$  as above. Suppose first that no prime dividing  $f$  splits completely in  $K$  (e.g. if  $f$  is a conductor) Conjecture 3.1 then becomes essentially a special case of Conjecture 5.2 of [So2]. The choice of the field denoted ' $k$ ' must there be restricted to our totally real  $k$ , ' $K$ ' to our ray-class field  $k(f)$  and the more general set ' $S$ ' to our  $S = S_\infty \cup S_0(f)$ . The requirement ' $|S| \geq \rho(S) + 1$ ' of Conjecture 5.2 amounts to our insistence that  $f \neq \mathcal{O}$  in this case, and condition (ii) becomes precisely our condition (16) with  $\eta_{f,T} = \eta_{S,T}$ . (Condition (i) and the uniqueness requirement on  $\eta_{S,T}$  in Conjecture 5.2, as well as condition 20(c) of [So2] can all be dropped for this 'rational form', cf. *ibid.* §4.2.) Now suppose that there is a split prime  $\mathfrak{q}$  dividing  $f$ . Then the ' $\rho(S)$ ' of Conjecture 5.2 is at least equal to  $r + 1$  and so, to fulfil the requirement ' $|S| \geq \rho(S) + 1$ ',  $f$  must have at least two prime factors. If these conditions hold then equation (10) shows that  $\chi(\Phi_{f,T}(s))$  vanishes at  $s = 1$  for each  $\chi \in \text{Cl}_f(k)^*$ , hence so must  $\Phi_{f,T}(s)$  and Conjecture 3.1 is trivial. (In this case, Conjecture 5.2 of [So2] concerns higher derivatives of  $\Phi_{m,T}$  at  $s = 1$ .) Finally, if  $f = \mathfrak{q}^l$  for some  $\mathfrak{q}$  split in  $K$  and  $l > 0$ , then  $|S|$  and ' $\rho(S)$ ' both equal  $r + 1$ , so Conjecture 5.2 does not apply. However, in this case, both Conjecture 3.1 and Conjecture 3.2 can be proven directly (see Subsection 3.5).

In [So2], the functional equations of the complex  $L$ -functions were used to show that the full Conjecture 5.2 is itself equivalent to Rubin's Conjecture A', hence also to his Conjecture A, in [Ru]. Both of these latter Conjectures are 'rational forms'—or, as Rubin says, forms 'over  $\mathbb{Q}$ '—of his more refined, integral conjectures. They relate to the (higher) derivatives at  $s = 0$  of the  $L$ -functions for  $\chi \in G^*$  and Proposition 2.3 of [Ru] shows that they are in turn simply equivalent to Stark's original conjecture at  $s = 0$  (Conjecture I.5.1 of [Ta]) for an appropriate subset of these functions. There are also conjectures at  $s = 0$  for  $p$ -adic  $L$ -functions. These are due to Gross (see [Gro1], [Ta, §§VI.3-4] for the original conjectures and [Gro2] and [Ha] for refinements and developments). Since no functional equation is known for these functions, Gross' conjectures do not translate to  $s = 1$  and an independent  $p$ -adic conjecture at  $s = 1$  is therefore formulated as

Conjecture VI.5.1 in [Ta], where it is attributed to J-P. Serre. In fact, it appears to be Tate’s elaboration of a remark in [Se]. ‘Serre’s Conjecture’ does not relate directly to our Conjecture 3.2 but takes the form of a *comparison* between the values of complex and *p*-adic *L*-functions at  $s = 1$ . Unfortunately, the formulation given in [Ta] is defective. (The intended meanings of the notations ‘ $\log(U)$ ’ and ‘ $\mu_p$ ’ for instance, are unclear and there seems to be no entirely coherent way to resolve them.) It is not hard to furnish a corrected conjecture along similar lines, although for reasons of space we shall not state it here. Suffice it to say that under certain hypotheses one can prove the implication

$$\begin{aligned} &(\text{Corrected form of Serre’s Conjecture}) + (\text{Conjecture (3.1) with } \eta_{f,T} = \eta) \\ &\implies (\text{Corrected form of Serre’s Conjecture}) \\ &\qquad\qquad\qquad + (\text{Conjecture 3.2 with } \eta_{f,T,p} = \eta) \end{aligned}$$

and that the converse implication also holds at least if we assume Leopoldt’s Conjecture for  $K$  and  $p$ .

In any event, this strongly suggests the

**CONJECTURE 3.3** (Basic combined conjecture). — *If Hypothesis 3.1 and condition (17) hold then, there exists  $\eta \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  such that both (16) and (18) (for any  $j$ ) hold with  $\eta_{f,T} = \eta_{f,T,p} = \eta$ .*

Proposition 3.3 and the comments at the start of Subsection 3.2 again show that the veracity of Conjecture 3.3 is independent of the choice of  $j$  or on the choice and/or ordering of the  $\tilde{l}_i$ .

### 3.4. Refined conjectures.

These are either ‘integral forms’ of the basic conjectures (forms ‘over  $\mathbb{Z}$ ’ in the terminology of [Ta] and [Ru]) that require  $\eta_{f,T}$  and  $\eta_{f,T,p}$  to lie in certain  $\mathbb{Z}G$ -lattices inside  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ , or they may just ‘control the denominators’ of these elements with respect to such lattices. To begin the discussion, we introduce

**HYPOTHESIS 3.2.** — *Hypothesis 3.1 holds and  $(f, T) = 1$ .*

It would be interesting to formulate refined conjectures in the case  $(f, T) > 1$ . However, its exclusion here simultaneously achieves several simplifications: Part (ii) of Hypothesis 3.1 becomes simply ‘ $f \neq \mathcal{O}$ ’, the

‘denominators’ in  $\eta_{f,T}, \eta_{f,T,p}$  coming from Propositions 3.6, 3.7 (for  $r > 1$ ) are avoided and these elements also acquire a uniqueness property which we now explain.

For any  $\chi$  in  $G^*$  (identified with  $\text{Cl}_f(k)^*$ ) we set  $r(S, \chi) := \dim_{\mathbb{C}}(e_{\chi} \mathbb{C}U_S)$ . Let  $\chi_0 \in G^*$  denote the trivial character and for any place  $v$  of  $k$  let  $G(v)$  denote the decomposition subgroup of  $G$  associated to each of the places  $w$  of  $K$  dividing  $v$ . It is well known that

$$(33) \quad r(S, \chi) = \begin{cases} r + |\{\mathfrak{q} : \mathfrak{q}|f, \chi|_{G(\mathfrak{q})} = 1\}| & \text{if } \chi \neq \chi_0 \\ r - 1 + |\{\mathfrak{q} : \mathfrak{q}|f\}| & \text{if } \chi = \chi_0 \end{cases} \\ = r + \text{ord}_{s=1}(\chi(\Phi_{m,T}(s))).$$

The first equation follows from the logarithmic embedding of  $U_S$  (cf. Ch. I, §4 of [Ta]) and the second from (10), since  $(f, T) = 1$ . Because  $f \neq \mathcal{O}$ , it follows that  $r(S, \chi) \geq r$  for every  $\chi \in G^*$  and we set

$$(34) \quad e_{S,r} := \sum_{\substack{\chi \in G^* \\ r(S,\chi)=r}} e_{\chi} \quad \text{and} \quad e_{S,>r} := 1 - e_{S,r} = \sum_{\substack{\chi \in G^* \\ r(S,\chi)>r}} e_{\chi}.$$

*A priori*, these are mutually orthogonal idempotents of  $\bar{\mathbb{Q}}G$ . But  $r(S, \chi)$  obviously depends only on the  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -conjugacy class of  $\chi$  so they actually lie in  $\mathbb{Q}G$ . Henceforth we let  $g$  denote the cardinality of  $G$ . Thus  $\tilde{e}_{S,r} := ge_{S,r}$  and  $\tilde{e}_{S,>r} := ge_{S,>r}$  clearly lie in  $\mathbb{Z}G$ . For any  $\mathbb{Z}G$ -module  $A$ , we shall write  $A^{[S,r]} := \ker \tilde{e}_{S,>r}|_A$  so that  $A^{[S,r]} \supset \tilde{e}_{S,r}A \supset gA^{[S,r]}$ .

PROPOSITION 3.8. — *Suppose that Hypothesis 3.2 is satisfied.*

(i) *If Conjecture 3.1 has a solution  $\eta_{f,T} = \eta$  in  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  then it has a unique solution  $\eta_{f,T} = \eta'$  in  $(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S)^{[S,r]}$ .*

(ii) *Suppose also that  $T \supset T_p$ . If Conjecture 3.2 has a solution  $\eta_{f,T} = \eta$  in  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  then it has a solution  $\eta_{f,T} = \eta'$  in  $(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S)^{[S,r]}$  which is unique if Leopoldt’s conjecture holds for  $K$  at  $p$ .*

*Proof.* —  $\Phi_{f,T}(1)$  lies in  $\mathbb{C}G^{[S,r]}$  by Equation (33). Similarly  $\Phi_{f,T,p}^{(j)}(1)$  lies in  $\mathbb{C}_p G^{[S,r]}$  for any  $j$ , by (14), (15) (with  $(f, T) = 1$ ) and part (iii) of Theorem 2.4. Since  $R$  and  $R_p^{(j)}$  are  $\mathbb{Q}G$ -linear, the existence of  $\eta'$  (given  $\eta$ ) in parts (i) and (ii) follows on setting  $\eta' = e_{S,r}\eta$ . Uniqueness is proven for both parts just as for ‘ $\epsilon_{S,T}$ ’ in [So2, §4.2]: Briefly,  $(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S)^{[S,r]} = \bigwedge_{\mathbb{Q}G}^r ((\mathbb{Q}U_S)^{[S,r]})$  is free of rank 1 over  $e_{S,r}\mathbb{Q}G$  which is a product of fields. Again, Equation (33) (resp. (14), (15), Theorem 2.4 part (iv) and the assumption of Leopoldt’s Conjecture) implies that the

same is true of the  $\mathbb{Q}G$ -module generated by  $\Phi_{f,T}(1)$  in  $\mathbb{C}G^{[S,r]}$  (resp. by  $\Phi_{f,T,p}^{(j)}(1)$  in  $\mathbb{C}_pG^{[S,r]}$ ). But the existence of  $\eta'$  implies that the image by the map  $R$  (resp.  $R_p^{(j)}$ ) of one module contains the other. Therefore this map induces an isomorphism between them and  $\eta'$  must be unique.  $\square$

COROLLARY 3.2. — *Suppose that Hypothesis 3.2 is satisfied, that  $T \supset T_p$  and Conjecture 3.3 holds. If  $\eta$  lies in  $(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S)^{[S,r]}$  and satisfies Equation (16) then it also satisfies Equation (18) for any  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ .*

*Proof.* — By Conjecture 3.3 there exists some  $\eta'' \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  which satisfies Equation (16) as well as Equation (18) for any  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ . The proof of the Proposition shows that  $e_{S,r}\eta''$  also satisfies these equations and the uniqueness in Part (i) shows that  $\eta = e_{S,r}\eta''$ .  $\square$

For any  $\mathbb{Z}G$ -submodule  $M$  of  $U_S$ , we denote by  $\overline{\bigwedge_{\mathbb{Z}G}^r M}$  the image of the exterior power  $\bigwedge_{\mathbb{Z}G}^r M$  in  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ . Now suppose that  $M$  is of finite index in  $U_S$  so that  $\overline{\bigwedge_{\mathbb{Z}G}^r M}$  is a lattice of full rank in  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$ . Then elements  $\phi_1, \dots, \phi_r$  of  $\text{Hom}_{\mathbb{Z}G}(M, \mathbb{Z}G)$  extend uniquely to maps  $\phi_i : \mathbb{Q}U_S \rightarrow \mathbb{Q}G$  and together define a  $\mathbb{Q}G$ -linear map  $\phi_1 \wedge \dots \wedge \phi_r$  from  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S$  to  $\mathbb{Q}G$  by setting  $(\phi_1 \wedge \dots \wedge \phi_r)(u_1 \wedge \dots \wedge u_r) = \det(\phi_i(u_l))_{i,l=1}^r$  for any  $u_1, \dots, u_r \in \mathbb{Q}U_S$ . See [Ru, §1.2] for the following

PROPOSITION/DEFINITION 3.1. — *For  $M$  as above, we set*

$$\bigwedge_0^r M := \{ \eta \in \bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S : (\phi_1 \wedge \dots \wedge \phi_r)(\eta) \in \mathbb{Z}G \ \forall \phi_1, \dots, \phi_r \in \text{Hom}_{\mathbb{Z}G}(M, \mathbb{Z}G) \}.$$

*If  $r = 1$  then  $\bigwedge_0^1 M = \overline{\bigwedge_{\mathbb{Z}G}^1 M} = \text{image of } M \text{ in } \mathbb{Q}U_S$ . More generally  $\bigwedge_0^r M$  is a  $\mathbb{Z}G$ -lattice containing  $\overline{\bigwedge_{\mathbb{Z}G}^r M}$  with finite index supported on  $g$ .*

$\square$

(For  $n \in \mathbb{Z}_{>0}$ , we say that  $a \in \mathbb{Q}$  is ‘supported on  $n$ ’ iff  $\text{ord}_p(a) = 0$  for every prime  $p \nmid n$ .) Given any  $T$  we write  $T(K)$  for the set of primes in  $K$  lying above those in  $T$  and, if  $(f, T) = 1$  then we set  $U_{S,T} := \{ \eta \in U_S : \eta \equiv 1 \pmod{\mathfrak{P}} \ \forall \mathfrak{P} \in T(K) \}$ , which is a  $\mathbb{Z}G$ -submodule of finite index in  $U_S$ .

CONJECTURE 3.4 (Refined complex conjecture). — *Suppose that Hypothesis 3.2. holds and that  $U_{S,T}$  is  $\mathbb{Z}$ -torsionfree. Then there exists a unique  $\eta_{f,T} \in (\bigwedge_0^r U_{S,T})^{[S,r]}$  satisfying Equation (16).*

Let  $p$  be a prime number.

CONJECTURE 3.5 (Refined  $p$ -adic conjecture). — *Suppose that Hypothesis 3.2 and condition (17) hold and that  $U_{S,T}$  is  $\mathbb{Z}$ -torsionfree. Then there exists  $\eta_{f,T,p} \in (\bigwedge_0^r U_{S,T})^{[S,r]}$  satisfying Equation (18) for any  $j: \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ .*

Of course, Corollary 3.2 shows that Conjecture 3.5 would follow from Conjectures 3.3 and 3.4. The uniqueness of  $\eta_{f,T}$  in Conjecture 3.4 is automatic by Proposition 3.8 part (i) and that of  $\eta_{f,T,p}$  in Conjecture 3.5 follows from Proposition 3.8 part (ii) if we assume Leopoldt's Conjecture for  $K$  at  $p$ . A few further remarks on the above two conjectures mostly echo those on Conjectures 3.1 and 3.2 (see Subsections 3.2 and 3.3): Again, the choice and/or ordering of the extensions  $\tilde{L}_i$  is immaterial. So long as  $f \neq \mathfrak{q}^l$  for any  $l > 0$  and prime ideal  $\mathfrak{q}$  *splitting* in  $K$  then the integral Conjecture 3.4 is either trivial or a special case of the integral Conjecture 5.1 of the paper [So2]. The latter conjecture is there shown to be equivalent to Conjecture B' hence also to Conjecture B of [Ru]. If on the other hand  $f = \mathfrak{q}^l$  for some  $\mathfrak{q}$  split in  $K$ , then Conjectures 3.4 and 3.5 can be proven directly (see Remark 3.3).

In order to discuss the role of  $T$  (prime to  $f$ ) in the refined conjectures, we suppose as before that  $\mathfrak{q}$  is a prime in  $T$ . Now Popescu has already shown in [Po, Prop. 5.2.1] that if Rubin's Conjecture B' holds for  $T \setminus \{\mathfrak{q}\}$  then it holds for  $T$ . (The other data for the conjecture are more general than ours but are taken to be the same for  $T \setminus \{\mathfrak{q}\}$  and  $T$ .) The core of his proof is essentially the non-obvious fact that if  $U_{S,T \setminus \{\mathfrak{q}\}}$  is  $\mathbb{Z}$ -torsionfree then  $\eta \in \bigwedge_0^r U_{S,T \setminus \{\mathfrak{q}\}} \Rightarrow (N\mathfrak{q} - \sigma_{\mathfrak{q},f})\eta \in \bigwedge_0^r U_{S,T}$ . By combining this implication with Propositions 3.4 and 3.5 and iterating we arrive at the

PROPOSITION 3.9. — *Suppose that  $\tilde{T} \subset T$  and that the two sets of data  $(k, f, \tilde{T})$  and  $(k, f, T)$  (resp.  $(p, k, f, \tilde{T})$  and  $(p, k, f, T)$ ) obey the hypotheses of Conjecture 3.4 (resp. Conjecture 3.5). Let  $\eta$  be an element of  $\bigwedge_{\mathbb{Q}_G}^r \mathbb{Q}U_S$ . Then  $\eta$  satisfies this conjecture with the first set of data if and only if  $\prod_{\mathfrak{q} \in T \setminus \tilde{T}} (N\mathfrak{q} - \sigma_{\mathfrak{q},f})\eta$  satisfies it with the second set of data.  $\square$*

This allows us to reduce the refined conjectures to the cases where  $T$  is minimal satisfying the appropriate conditions. In our situation  $K$  is totally real so  $U_{S,T}$  is  $\mathbb{Z}$ -torsionfree if and only if it doesn't contain  $-1$ . Thus, for Conjectures 3.4, 3.5 ( $p \neq 2$ ) and for Conjecture 3.5 ( $p = 2$ ), these minimal sets  $T$  are precisely those of form  $\{\mathfrak{q}\}$ ,  $T_p$  and  $T_2 \cup \{\mathfrak{q}\}$  respectively, where  $\mathfrak{q}$  is any prime ideal of  $k$  not dividing  $2f$ .

In [Po] Popescu formulates an integral, complex conjecture that slightly weakens Rubin's Conjecture B' but appears to behave better under

change of the base field *k*. We shall shortly state a ‘combined’ conjecture (Conjecture 3.6) that is considerably weaker than Conjectures 3.4 and 3.5 (although still much stronger than Conjecture 3.3) but it is better adapted for numerical verification (as in [RS]). Instead of seeking ‘solutions’ lying in a specific lattice inside  $\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S^{[S,r]}$  it merely controls the primes dividing their denominators relative to the lattice  $\overline{\bigwedge_{\mathbb{Z}G}^r U_S^{[S,r]}}$ . For certain purposes this will allow us to replace  $U_S$  with the true unit group  $E(K)$  of  $K$ , as we now explain. For any subgroup  $H < G$  we set  $N_H := \sum_{\sigma \in H} \sigma \in \mathbb{Z}G$ . In particular,  $N_G = ge_{\chi_0}$  and we define

$$(35) \quad \tilde{e}'_{S,r} := g \sum_{\substack{\chi \in G^* \setminus \{\chi_0\} \\ r(S,\chi)=r}} e_\chi = \begin{cases} \tilde{e}_{S,r} - N_G & \text{if } r(S, \chi_0) = r \\ \tilde{e}_{S,r} & \text{otherwise.} \end{cases}$$

Note that  $\tilde{e}'_{S,r}$  lies in the augmentation ideal  $I(\mathbb{Z}G)$  of  $\mathbb{Z}G$  and  $(\tilde{e}'_{S,r})^2 = g\tilde{e}'_{S,r}$ . Note also that  $r(S, \chi_0) = r$  if and only if  $\mathfrak{f}$  is a (non-trivial) power of a prime ideal, a condition we shall denote simply ‘ $\mathfrak{f} = \mathfrak{q}^l$ ’. If this is not the case then  $r(S, \chi_0) > r$  and we shall write ‘ $\mathfrak{f} \neq \mathfrak{q}^l$ ’.

LEMMA 3.5. — *In the above notations*

- (i)  $\tilde{e}'_{S,r} \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r U_S} = \tilde{e}'_{S,r} \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r E(K)}$ .
- (ii) *For any  $x \in I(\mathbb{Z}G)$ , and even for  $x = 1$  if  $\mathfrak{f} \neq \mathfrak{q}^l$ , we have  $x \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r U_S^{[S,r]}} = x \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r E(K)^{[S,r]}}$ .*

*Proof.* — In part (i), the containment of the R.H.S. in the L.H.S. is obvious. Thus it suffices to show that if  $\varepsilon_1, \dots, \varepsilon_r$  are in  $U_S$  then  $\tilde{e}'_{S,r} \overline{\varepsilon_1 \wedge \dots \wedge \varepsilon_r}$  lies in  $\tilde{e}'_{S,r} \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r E(K)}$  (bars denote images in  $\bigwedge_{\mathbb{Q}G} \mathbb{Q}U_S$ ). Let  $\mathfrak{q}$  be any prime dividing  $\mathfrak{f}$ . For every  $\chi \in G^* \setminus \{\chi_0\}$  such that  $r(S, \chi) = r$ , Equation (33) implies that  $\chi|_{G(\mathfrak{q})} \neq 1$ . It follows that  $N_{G(\mathfrak{q})} \tilde{e}'_{S,r} = 0$  so that, for every prime  $\mathfrak{Q}$  above  $\mathfrak{q}$  in  $K$  and every  $\varepsilon \in U_S$ , we have  $\text{ord}_{\mathfrak{Q}}(\tilde{e}'_{S,r} \varepsilon) = |G(\mathfrak{q})|^{-1} \text{ord}_{\mathfrak{Q}}(N_{G(\mathfrak{q})} \tilde{e}'_{S,r} \varepsilon) = 0$ . Letting  $\mathfrak{q}$  and  $\mathfrak{Q}$  vary with  $\mathfrak{q}|\mathfrak{f}$  and  $\mathfrak{Q}|\mathfrak{q}$ , we deduce that  $\tilde{e}'_{S,r} \varepsilon$  is an actual unit, i.e. that  $\tilde{e}'_{S,r} U_S \subset E(K)$ , hence

$$\tilde{e}'_{S,r} \overline{\varepsilon_1 \wedge \dots \wedge \varepsilon_r} = \frac{1}{g^r} \tilde{e}'_{S,r} \overline{(\tilde{e}'_{S,r} \varepsilon_1) \wedge \dots \wedge (\tilde{e}'_{S,r} \varepsilon_r)} \in \tilde{e}'_{S,r} \overline{\mathbb{Z}[1/g] \bigwedge_{\mathbb{Z}G}^r E(K)}$$

as required. Now, for any  $\mathbb{Z}G$ -module  $M$ , we have  $\mathbb{Z}[1/g](M^{[S,r]}) = (\mathbb{Z}[1/g]M)^{[S,r]} = \tilde{e}_{S,r} \overline{\mathbb{Z}[1/g]M}$ . On the other hand, Equation (35) shows that  $x \tilde{e}_{S,r} = x \tilde{e}'_{S,r}$  for any  $\mathfrak{f}$  and all  $x \in I(\mathbb{Z}G)$ , and even for  $x = 1$  if  $\mathfrak{f} \neq \mathfrak{q}^l$ . Using these two facts, part (ii) follows easily from (i). □

CONJECTURE 3.6 (Weak refined combined conjecture). — Suppose that  $k$  is totally real and  $\mathfrak{f} \neq \mathcal{O}$  is any proper integral ideal. Then, in the above notations, there exists a unique element  $\eta_{\mathfrak{f}}$  of  $(\bigwedge_{\mathbb{Q}_G}^r \mathbb{Q}U_S)^{[S,r]}$  with the following properties:

(i)

$$(36) \quad \frac{2^r}{\sqrt{d_k}} R(\eta_{\mathfrak{f}}) = \Phi_{\mathfrak{f}, \varnothing}(1).$$

(ii) For every prime number  $p$  with  $(p, \mathfrak{f}) = 1$  and for one (hence any) embedding  $j : \mathbb{Q} \rightarrow \mathbb{C}_p$  we have

$$(37) \quad \prod_{\mathfrak{p} \in T_p} (1 - N\mathfrak{p}^{-1}\sigma_{\mathfrak{p}, \mathfrak{f}}) \frac{2^r}{j(\sqrt{d_k})} R_p^{(j)}(\eta_{\mathfrak{f}}) = \Phi_{\mathfrak{f}, T_p, p}^{(j)}(1).$$

(iii) If  $\mathfrak{f} \neq \mathfrak{q}^l$  then

$$(38) \quad \eta_{\mathfrak{f}} \in \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]} = \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r E(K)}^{[S,r]}.$$

(iv) If  $\mathfrak{f} = \mathfrak{q}^l$  then

$$(39) \quad \eta_{\mathfrak{f}} \in \frac{1}{2} \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$$

and

$$(40) \quad I(\mathbb{Z}G)\eta_{\mathfrak{f}} \subset \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}.$$

*Remark.* — Taken by itself, Condition (40) on an element  $\eta_{\mathfrak{f}} \in (\bigwedge_{\mathbb{Q}_G}^r \mathbb{Q}U_S)^{[S,r]}$  has many equivalent reformulations. For example, it is equivalent to the containment of  $\mathbb{Z}[1/g]I(\mathbb{Z}G)\eta_{\mathfrak{f}}$  in  $\mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$  or (since  $\mathbb{Z}[1/g]G = \mathbb{Z}[1/g]I(\mathbb{Z}G) \times \mathbb{Z}[1/g]N_G$ ) in  $\mathbb{Z}[1/g]I(\mathbb{Z}G) \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$  which equals  $\mathbb{Z}[1/g]I(\mathbb{Z}G) \overline{\bigwedge_{\mathbb{Z}G}^r E(K)}^{[S,r]}$  by Lemma 3.4.5. It may therefore also be rewritten

$$(41) \quad I(\mathbb{Z}G)\eta_{\mathfrak{f}} \subset \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r E(K)}^{[S,r]}.$$

*Remark.* — Once Condition (36) is known to be satisfied, we can calculate  $N_G \eta_{\mathfrak{f}}$  explicitly and hence show that (40) implies (39). Indeed, suppose that  $\mathfrak{f} = \mathfrak{q}^l$  for some prime ideal  $\mathfrak{q}$  and let  $\mathcal{O}_{k, \{\mathfrak{q}\}}$  denote the

ring of  $\mathfrak{q}$ -integers of  $k$ , namely the subring  $\{a \in k : v_{\mathfrak{p}}(a) \geq 0 \ \forall \mathfrak{p} \neq \mathfrak{q}\}$  of  $k$ . Thus  $\mathcal{O}_{k,\{\mathfrak{q}\}}^{\times} = U_S(k) = U_S(K)^G$ . Let us denote by  $\eta_{k,\{\mathfrak{q}\}}$  the element  $\overline{u_1 \wedge \dots \wedge u_r}$  of  $\overline{\bigwedge_{\mathbb{Z}G}^r U_S}^G \subset \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$  where  $u_1, \dots, u_r$  are any  $r$  elements of  $\mathcal{O}_{k,\{\mathfrak{q}\}}^{\times}$  constituting a  $\mathbb{Z}$ -base modulo  $\{\pm 1\}$ . Thus  $\eta_{k,\{\mathfrak{q}\}}$  is defined up to sign and we fix it completely by insisting in addition that  $\det(\log |\nu_i(u_i)|)_{i,l=1}^r$  be positive hence equal to the regulator  $R_{k,\{\mathfrak{q}\}}$  of  $\mathcal{O}_{k,\{\mathfrak{q}\}}^{\times}$ . It then follows easily from the definitions that

$$(42) \quad R(\eta_{k,\{\mathfrak{q}\}}) = g^{r-1} R_{k,\{\mathfrak{q}\}} N_G.$$

Now suppose that  $\eta_{\mathfrak{f}} \in \left(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S\right)^{[S,r]}$  satisfies Equation (36). Applying  $N_G$  to both sides and using Theorem 2.2 with  $\mathfrak{m} = \mathfrak{f} = \mathfrak{q}^l$ ,  $T = \emptyset$  and  $\chi = \chi_0$ , we obtain

$$(43) \quad \frac{2^r}{\sqrt{d_k}} R(N_G \eta_{\mathfrak{f}}) = \chi_0(\Phi_{\mathfrak{f},\emptyset})(1) N_G = \left( \left(1 - \frac{1}{N \mathfrak{q}^{s-1}}\right) \zeta_k(s) \right) \Big|_{s=1} N_G \\ = - \frac{2^r}{\sqrt{d_k}} \frac{h_{k,\{\mathfrak{q}\}} R_{k,\{\mathfrak{q}\}}}{2} N_G$$

where  $h_{k,\{\mathfrak{q}\}}$  denotes the class-number of  $\mathcal{O}_{k,\{\mathfrak{q}\}}$  and the last equality follows by a standard calculation from the ‘Analytic Class Number Formula’ for  $\text{res}_{s=1} \zeta_k(s)$  (see Thm. 1.1 and the method of Cor. 2.2 in [Ta, Ch I], for example). But  $R$  is injective on  $\overline{\bigwedge_{\mathbb{Z}G}^r U_S}^G$  (by Dirichlet’s Theorem for  $U_S(k)$ , for instance) so, comparing Equations (42) and (43), we find that

$$\frac{1}{g} N_G \eta_{\mathfrak{f}} = - \frac{h_{k,\{\mathfrak{q}\}}}{2g^r} \eta_{k,\{\mathfrak{q}\}} \in \frac{1}{2} \mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}.$$

Now, writing  $\eta_{\mathfrak{f}}$  as  $\frac{1}{g}(g - N_G)\eta_{\mathfrak{f}} + \frac{1}{g}N_G\eta_{\mathfrak{f}}$ , we see that (40) implies (39). Since  $\mathbb{Z}[1/g]I(\mathbb{Z}G) = \mathbb{Z}[1/g]G(g - N_G)$ , it is also equivalent to the containment of the element  $\eta_{\mathfrak{f}} + \frac{h_{k,\{\mathfrak{q}\}}}{2g^r} \eta_{k,\{\mathfrak{q}\}}$  in  $\mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$  (or in  $\mathbb{Z}[1/g] \overline{\bigwedge_{\mathbb{Z}G}^r E(K)}^{[S,r]}$ , by the previous Remark).

We now show that Conjecture 3.6 follows from the previous conjectures. The following result is Lemma 1.1, Ch. IV of [Ta]. (The notation ‘ $\langle \dots \rangle_{\mathbb{Z}}$ ’ means ‘the  $\mathbb{Z}$ -submodule generated by’.)

LEMMA 3.6. — *Let  $M/L$  be any abelian extension of number fields with group  $G$  and let  $\mathcal{Q}_{M/L}$  denote the set of prime ideals of  $\mathcal{O}_L$  which are unramified in  $M/L$  and do not divide  $|\mu(M)|$ . Then, for any subset  $\mathcal{Q} \subset \mathcal{Q}_{M/L}$  with  $|\mathcal{Q}_{M/L} \setminus \mathcal{Q}| < \infty$ , we have*

$$\text{ann}_{\mathbb{Z}G}(\mu(M)) = \langle N\mathfrak{q} - \text{Frob}_{\mathfrak{q}}(M/L) : \mathfrak{q} \in \mathcal{Q} \rangle_{\mathbb{Z}}. \quad \square$$

We apply this lemma with  $M/L = K/k$ . It shows that,  $\text{ann}_{\mathbb{Z}G}(\mu_2)$  is generated over  $\mathbb{Z}$  by the elements  $N\mathfrak{q} - \sigma_{\mathfrak{q},\mathfrak{f}}$  as  $\mathfrak{q}$  runs through the set  $\mathcal{Q}(2\mathfrak{f}) := \{\mathfrak{q} : \mathfrak{q} \nmid 2\mathfrak{f}\} \subset \mathcal{Q}_{K/k}$ . It follows, of course, that there are finite subsets of  $\mathcal{Q}(2\mathfrak{f})$  for which the same is true.

**PROPOSITION 3.10.** — *Let  $\mathcal{Q}_0$  be any subset of  $\mathcal{Q}(2\mathfrak{f})$  such that  $\text{ann}_{\mathbb{Z}G}(\mu_2)$  equals  $\langle N\mathfrak{q} - \sigma_{\mathfrak{q},\mathfrak{f}} : \mathfrak{q} \in \mathcal{Q}_0 \rangle_{\mathbb{Z}}$  (for example,  $\mathcal{Q}_0 = \mathcal{Q}(2\mathfrak{f})$ ). Then Conjecture 3.6 is a consequence of Conjecture 3.3 for  $k, \mathfrak{f}$  and  $T_p$  for all  $p$  such that  $(p, \mathfrak{f}) = 1$ , together with Conjecture 3.4 for  $k, \mathfrak{f}$  and  $T = \{\mathfrak{q}\}$  for all  $\mathfrak{q} \in \mathcal{Q}_0$ .*

*Proof.* — If Conjecture 3.3 holds for  $k, \mathfrak{f}$  and  $T_p$  for some  $p$  such that  $(p, \mathfrak{f}) = 1$ , then so does Conjecture 3.1 and hence, by Proposition 3.4 (iterated) the latter also holds for  $k, \mathfrak{f}$  and  $T = \emptyset$ . By Proposition 3.8 (i) there is a unique element  $\eta_{\mathfrak{f}} := \eta_{\mathfrak{f}, \emptyset} \in \left(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S\right)^{[S,r]}$  satisfying (36). For any  $p$  such that  $(p, \mathfrak{f}) = 1$ , Proposition 3.4 shows that the element  $\prod_{p \in T_p} (Np - \sigma_{p,\mathfrak{f}})\eta_{\mathfrak{f}}$  of  $\left(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S\right)^{[S,r]}$  satisfies (16) for  $k, \mathfrak{f}$  and  $T = T_p$ . Therefore, by Corollary 3.2 it also satisfies (18) with these data (for any  $j$ ) and Property (ii) of Conjecture 3.6 follows. Now suppose that Conjecture 3.4 holds for  $k, \mathfrak{f}$  and  $T = \{\mathfrak{q}\}$  for a given  $\mathfrak{q} \in \mathcal{Q}_0$ . Then, by Proposition 3.4 and uniqueness, the element  $\overline{\eta_{\mathfrak{f},\{\mathfrak{q}\}}} = (N\mathfrak{q} - \sigma_{\mathfrak{q},\mathfrak{f}})\eta_{\mathfrak{f}}$  must lie in  $\left(\bigwedge_0^r U_{S,\{\mathfrak{q}\}}\right)^{[S,r]}$ , a fortiori in  $\mathbb{Z}[1/g]\overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$ , by Proposition/Definition 3.1. Therefore, letting  $\widehat{\eta}_{\mathfrak{f}}$  denote the image of  $\eta_{\mathfrak{f}}$  in the quotient  $\left(\bigwedge_{\mathbb{Q}G}^r \mathbb{Q}U_S\right)^{[S,r]} / \mathbb{Z}[1/g]\overline{\bigwedge_{\mathbb{Z}G}^r U_S}^{[S,r]}$  we find

$$\text{ann}_{\mathbb{Z}G}(\widehat{\eta}_{\mathfrak{f}}) \supset \langle N\mathfrak{q} - \sigma_{\mathfrak{q},\mathfrak{f}} : \mathfrak{q} \in \mathcal{Q}_0 \rangle_{\mathbb{Z}} = \text{ann}_{\mathbb{Z}}(\mu_2) = \langle 2, I(\mathbb{Z}G) \rangle_{\mathbb{Z}}.$$

Conditions (39) and (40) now follow. Moreover, if  $\mathfrak{f} \neq \mathfrak{q}^l$  then  $g\eta_{\mathfrak{f}} = \tilde{e}_{S,r}\eta_{\mathfrak{f}} = \tilde{e}'_{S,r}\eta_{\mathfrak{f}} \in I(\mathbb{Z}G)\eta_{\mathfrak{f}}$  so the membership in (38) also follows (the equality is from Lemma 3.5 (ii)). □

### 3.5. Two special cases of conjecture.

The first special case in which we can prove this conjecture is that in which the set  $\{\chi \in G^* : r(S, \chi) = r\}$  consists precisely of the trivial character  $\chi_0$ . This means that  $\mathfrak{f} = \mathfrak{q}^l$  where  $\mathfrak{q}$  splits completely in  $K = k(\mathfrak{q}^l)$ , which implies in turn that  $K$  must also be the Hilbert Class Field  $k(\emptyset)$  of  $k$ , so that  $G$  is isomorphic to  $\text{Cl}(k)$  in which the class of  $\mathfrak{q}$  must be trivial. Using the notation of Remark 3.2, it follows that  $g = h_k = h_{k,\{\mathfrak{q}\}}$

and we set

$$\eta_{\mathfrak{f}} := -\frac{h_{k,\{\mathfrak{q}\}}}{2g^r} \eta_{k,\{\mathfrak{q}\}} = -\frac{1}{2g^{r-1}} \eta_{k,\{\mathfrak{q}\}} \in \frac{1}{2} \mathbb{Z}[1/g] \overline{\wedge}_{\mathbb{Z}G}^r U_S^{-G} = \frac{1}{2} \mathbb{Z}[1/g] \overline{\wedge}_{\mathbb{Z}G}^r U_S^{[S,r]}.$$

The second and third equalities in (43) together with Equation (42) give

$$(44) \quad \Phi_{\mathfrak{f},\emptyset}(1) = \chi_0(\Phi_{\mathfrak{f},\emptyset})(1)e_{\chi_0} = -\frac{2^r}{\sqrt{d_k}} \frac{h_{k,\{\mathfrak{q}\}} R_{k,\{\mathfrak{q}\}}}{2} e_{\chi_0} = \frac{2^r}{\sqrt{d_k}} R(\eta_{\mathfrak{f}})$$

which establishes condition (36) for this choice of  $\eta_{\mathfrak{f}}$ .

Analogous arguments establish the *p*-adic condition (37) for any prime *p* such that  $(p, \mathfrak{q}) = 1$ . Indeed, using Theorem/Definition 2.1 and Theorem 2.4, we find that

$$(45) \quad \Phi_{\mathfrak{f},T_p,p}^{(j)}(1) = -\log_p \langle N\mathfrak{q} \rangle \left( \lim_{s \rightarrow 1} (s-1) \zeta_{k,p}(s) \right) e_{j \circ \chi_0} \\ = -\log_p \langle N\mathfrak{q} \rangle 2^{r-1} \prod_{\mathfrak{p} \in T_p} \left( 1 - \frac{1}{N\mathfrak{p}} \right) \frac{R_{p,k} h_k}{d_k^{1/2}} e_{j \circ \chi_0}$$

(where  $\zeta_{k,p}(s)$  equals  $L_p(s, j \circ \widetilde{\chi}_0)$  in our notation). The second equality above comes from an evaluation of the limit analogous to that of the complex case. This is the principal result of [Col] where it is noted that although the *p*-adic regulator  $R_{p,k} \in \mathbb{C}_p$  of *k* and  $d_k^{1/2} \in \mathbb{C}_p$  are only defined up to sign, yet their quotient appearing in (45) is to be understood as it is uniquely prescribed in [A-F]. Let us assume (as we may) that the  $u_1, \dots, u_{r-1}$  appearing in the definition of  $\eta_{k,\{\mathfrak{q}\}}$  (see Remark 3.2) lie in  $E(k)$  and satisfy  $\det(\log |\iota_i(u_l)|)_{i,l=1}^{r-1} > 0$ . With this assumption, the prescribed value for the quotient  $R_{p,k}/d_k^{1/2}$  is easily seen to be  $\det(\log_p(j \circ \iota_i(u_l)))_{i,l=1}^{r-1} / j(\sqrt{d_k})$ . Since also  $\det(\log |\iota_i(u_l)|)_{i,l=1}^r > 0$ , a simple argument with complex determinants shows that  $u_r$  must generate the principal ideal  $\mathfrak{q}$  and an analogous *p*-adic calculation yields

$$\frac{R_p^{(j)}(\eta_{\mathfrak{f}})}{j(\sqrt{d_k})} = -\frac{1}{2} \log_p(N_k/\mathbb{Q}u_r) \frac{\det(\log_p(j \circ \iota_i(u_l)))_{i,l=1}^{r-1}}{j(\sqrt{d_k})} N_G \\ = -\frac{1}{2} g \log_p \langle N\mathfrak{q} \rangle \frac{R_{p,k}}{d_k^{1/2}} e_{j \circ \chi_0}.$$

Combining this with (45) establishes part (ii) of Conjecture 3.6 in this case. (The above arguments also show that if  $R_{k,p} = 0$ , i.e. Leopoldt’s Conjecture

fails for  $k$  at  $p$ , then Equation (37) will simply read  $0 = 0$ .) Finally, we note that Equations (39) and (40) are trivially satisfied, completing the verification of Conjecture 3.6 in this case.

*Remark.* — The above arguments can easily be adapted to prove the refined complex Conjecture 3.4 in this case for any  $T$  such that  $(\mathfrak{q}, T) = 1$  and  $U_{S,T}$  is torsionfree. After calculating  $\Phi_{f,T}(1)$  by means of Equations (29) and (44), the interested reader can see how to modify the definition of  $\eta_{f,T}$  and so conclude the proof by referring to Rubin’s Proposition 3.1 and his exact sequence (1) in [Ru]. (The Proposition actually establishes Rubin’s Conjecture B’ in an analogous but more general case.) Also, if  $T$  contains  $T_p$  then the  $p$ -adic arguments above can be similarly adapted using (30), (45) *etc.* to show that Conjecture 3.5 is satisfied with  $\eta_{f,T,p}$  equal to the same  $\eta_{f,T}$ .

We now sketch the proof of Conjecture 3.6 in our second special case, namely that in which  $k = \mathbb{Q}$ . Suppose that  $f = f\mathbb{Z}$  for some  $f \in \mathbb{Z}_{>1}$  (not necessarily a conductor) so that  $K = \mathbb{Q}(f) = \mathbb{Q}(\zeta_f + \zeta_f^{-1})$  where  $\zeta_f := \exp(2\pi i/f)$ . We introduce the notation  $\hat{K}$  for  $\mathbb{Q}(f) = \mathbb{Q}(\zeta_f)$  and  $\hat{G}$  for  $\text{Gal}(\hat{K}/\mathbb{Q}) \cong \text{Cl}_{f+}(\mathbb{Q}) \cong (\mathbb{Z}/f\mathbb{Z})^\times$ . We shall identify  $\text{Cl}_f(\mathbb{Q})$  with  $(\mathbb{Z}/f\mathbb{Z})^\times / \{\pm 1\}$ . To a class  $[\bar{b}]$  in the latter group ( $b \in \mathbb{Z}$  and  $(b, f) = 1$ ) the Artin map associates  $s_b = s_{b,f} := \sigma_{[\bar{b}],f}$  which is the image in  $G$  of the automorphism  $\hat{s}_b = \hat{s}_{b,f} \in \hat{G}$  sending  $\zeta_f$  to  $\zeta_f^b$ . In this case, we have the following explicit formulae:

$$(46) \quad \Phi_{f,\emptyset}(1) = \begin{cases} -\log(2) & \text{if } f = 2 \ (\Rightarrow G = \{1\}) \\ -\sum_{[\bar{b}] \in (\mathbb{Z}/f\mathbb{Z})^\times / \{\pm 1\}} \log |(1 - \zeta_f^b)(1 - \zeta_f^{-b})| s_b^{-1} & \text{if } f > 2 \end{cases}$$

and, for any prime  $p \nmid f$  and any embedding  $j : \bar{\mathbb{Q}} \rightarrow \mathbb{C}_p$ ,

$$(47) \quad \Phi_{f,\{p\},p}^{(j)}(1) = \begin{cases} -(1 - \frac{1}{p}) \log_p(2) & \text{if } f = 2 \\ -(1 - \frac{1}{p} s_p) \sum_{[\bar{b}] \in (\mathbb{Z}/f\mathbb{Z})^\times / \{\pm 1\}} \log_p |(1 - j(\zeta_f)^b)(1 - j(\zeta_f)^{-b})| s_b^{-1} & \text{if } f > 2. \end{cases}$$

There are two ways in which these formulae can be obtained. Firstly, one can treat them character by character, thus reducing by Theorem 2.2 (*resp.* by Theorem/Definition 2.1) to formulae for the value/residue at  $s = 1$  of  $L(s, \psi)$  (*resp.* of  $L_p(s, j \circ \psi)$ ) for even, primitive Dirichlet characters  $\psi$  of conductor  $\tilde{f} \geq 1$  dividing  $f$ . (See *e.g.* [Wa] for such formulae.) For

this approach one also needs well-known ‘norm relations’ which relate  $N_{\mathbb{Q}(\zeta_f)/\mathbb{Q}(\zeta_{\tilde{f}})}(1 - \zeta_f)$  to  $1 - \zeta_{\tilde{f}}$  (see for example [So1, Lemma 2.1] in the case  $f, \tilde{f} \not\equiv 2 \pmod{4}$ ) and note that  $(1 - \zeta_{4n+2}) = (1 - \zeta_{2n+1})^{(1 - \tilde{s}_{2,2n+1}^{-1})}$  for  $n > 0$ ).

The second approach is more direct, proving (46) and (47) from scratch, coefficient by coefficient. For example, if we first write  $\Phi_{f,\varnothing}(1)$  as  $\pi_{f+,f}(\Phi_{f+,\varnothing}(1))$  by Theorem 2.1, then it suffices to note that the coefficient of  $\hat{s}_b$  in  $\Phi_{f+,\varnothing}(s)$  is a twisted zeta-function that can be written out explicitly for  $\Re(s) > 1$ :

$$Z_{\varnothing}(s; [b\mathbb{Z}] \cdot \xi_{f\mathbb{Z}}^0, bf^{-1}\mathbb{Z}, f\mathbb{Z}+) = \sum_{n=1}^{\infty} \frac{\zeta_f^{bn}}{n^s}$$

and whose value at  $s = 1$  equals  $-\log(1 - \zeta_f^b)$ , by Abel’s lemma. (This is of course the very calculation that lies at the heart of the above-mentioned formulae for  $L(1, \psi)$ !) A similar method exists for (47): The coefficient of  $\hat{s}_b$  in  $\Phi_{f+,\{p\},p}^{(j)}(s)$  is a function of  $s$  which interpolates the values  $j(Z_{\{p\}}(m; [b\mathbb{Z}] \cdot \xi_{f\mathbb{Z}}^0, bf^{-1}\mathbb{Z}, f\mathbb{Z}+))$  for  $m \in \mathcal{M}(p)$  and for  $\Re(s) > 1$ ,

$$Z_{\{p\}}(s; [b\mathbb{Z}] \cdot \xi_{f\mathbb{Z}}^0, bf^{-1}\mathbb{Z}, f\mathbb{Z}+) = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \frac{\zeta_f^{bn}}{n^s}.$$

The value (in  $\mathbb{C}_p$ ) of this coefficient at  $s = 1 \in \mathbb{Z}_p$  can be determined by methods similar to, but simpler than those of [RS]. (The key point is that the analogues of the power-series  $H$  and  $U \bullet H$  appearing in Lemma 3.3 *ibid.* can now be written out explicitly in terms of the formal logarithmic series. Details are left to the reader. See also Ch. 4, §§1-3 of [La]).

Equations (46) and (47) show that conditions (36) and (37) can be satisfied in this case by letting  $i_1$  be the inclusion  $\mathbb{Q}(\zeta_f + \zeta_f^{-1}) \hookrightarrow \bar{\mathbb{Q}} \subset \mathbb{C}$  and  $\eta_f$  the following element of  $\mathbb{Q} \otimes \mathbb{Q}(\zeta_f + \zeta_f^{-1})^\times$ :

$$\eta_f := \begin{cases} -\frac{1}{2} \otimes 2 & \text{if } f = 2 \\ -\frac{1}{2} \otimes (1 - \zeta_f)(1 - \zeta_f^{-1}) & \text{if } f > 2. \end{cases}$$

To study the remaining parts of the Conjecture we can omit all exterior powers (since  $r = 1$ ) and introduce the following notation for  $F = K$  or  $\hat{K}$ :

$$V(F) := \begin{cases} E(F) & \text{if } f \neq q^l, q \text{ a prime number, } l > 0 \\ U_S(F) = \mathcal{O}_{F,\{q\}}^\times & \text{if } f = q^l. \end{cases}$$

It is well-known that  $1 - \zeta_f$  lies in  $V(\hat{K})$ . Supposing first that  $f \neq 2$  and working inside  $\mathbb{Q}V(\hat{K})$  (images in which are also denoted by a bar) it follows that  $\eta_f$  lies in  $\frac{1}{2}\overline{V(K)}$ . Furthermore, the above-mentioned norm relations can be used to show that  $\tilde{e}_{S, > 1}\eta_f = 0$ , so that  $\eta_f$  lies in  $\frac{1}{2}\overline{V(K)}^{[S, 1]}$  and Condition (39) is satisfied in all cases.

But  $\eta_f$  also equals  $\overline{(1 - \zeta_f)^{-1}}$  and so lies in  $\overline{V(\hat{K})}^H$  where  $H := \text{Gal}(\hat{K}/K) = \{\hat{s}_{\pm 1}\}$ . Consider the exact sequence of  $\hat{G}$ -modules

$$0 \rightarrow \mu(\hat{K}) \rightarrow V(\hat{K}) \rightarrow \overline{V(\hat{K})} \rightarrow 0.$$

Taking  $H$ -invariants, we get an exact sequence of  $G$ -modules

$$0 \rightarrow \overline{V(K)} \rightarrow \overline{V(\hat{K})}^H \xrightarrow{\delta} H^1(H, \mu(\hat{K})) \cong \mu(\hat{K})/\mu(\hat{K})^2 \cong \mathbb{Z}/2\mathbb{Z}$$

where  $\delta(\bar{v})$  is the class of  $\hat{s}_{-1}(v)/v$  in  $\mu(\hat{K})/\mu(\hat{K})^2$  for any  $v \in V(\hat{K})$  such that  $\bar{v} \in \overline{V(\hat{K})}^H$ . In particular  $\delta(\eta_f)$  is the class of  $-\zeta_f$  which is trivial if and only if  $f \equiv 2 \pmod{4}$  (i.e.  $f$  is not a conductor). In this case  $f \neq q^l$  and  $\eta_f$  lies in  $\overline{V(K)}$ . Thus

$f \neq 2, f \equiv 2 \pmod{4} \iff \eta_f \in \overline{E(K)}^{[S, 1]} \implies$  Condition (38) is satisfied.

If  $f \not\equiv 2 \pmod{4}$  and also  $f \neq q^l$  then  $\eta_f \in \frac{1}{2}\overline{E(K)}$  and  $g = \phi(f)/2$  must be even ( $\phi$ =Euler's function) so that (38) is still satisfied. In the general case (and in particular if  $f = q^l$ ) we have  $\delta(x\eta_f) = x\delta(\eta_f) = 0$  for any  $x \in I(\mathbb{Z}G)$ . Thus  $I(\mathbb{Z}G)\eta_f \subset \overline{V(K)}^{[S, 1]}$  so condition (40) certainly holds. In fact more is true:  $2(x\eta_f) = x(2\eta_f)$  lies in  $x\overline{V(K)} \subset \overline{E(K)}$ . But  $\overline{U_S(K)}/\overline{E(K)} \cong U_S(K)/E(K)$  is torsionfree, so for all  $f \neq 2$  we actually have

$$(48) \quad I(\mathbb{Z}G)\eta_f \subset \overline{E(K)}^{[S, 1]}.$$

Finally, Conditions (39) and (40) (and (48)) hold trivially for  $f = 2$ .

*Remark 3.4.* — An interesting question is when and under what conditions the factor  $\mathbb{Z}[1/g]$  can be suppressed where it occurs in Conditions (38), (39) and (40) without losing the corresponding memberships of  $\eta_f$  or the containment of  $I(\mathbb{Z}G)\eta_f$ . We look first at (38) (in the case  $f \neq q^l$ ). For  $k = \mathbb{Q}$ , since  $r = 1$  and  $\overline{U_S}/\overline{E_K}$  is clearly torsionfree, the equality in (38) can be strengthened to  $\overline{U_S}^{[S, 1]} = \overline{E(K)}^{[S, 1]}$  whenever  $f \neq q^l$ . We have shown that in this case, the factor  $\mathbb{Z}[1/g]$  can be suppressed in one (hence

both) occurrences if  $f \equiv 2 \pmod{4}$  and that otherwise 2 divides  $g$  and a factor of  $1/2$  is necessary in both. Nevertheless, the proof shows that this factor is in a certain sense, explained by the (cohomology of) roots of unity in  $U_S$ , hence, ultimately, by our abandoning the condition that  $U_{S,T}$  be torsionfree in our ‘weak’ refined conjecture.

In all the examples with  $f \neq \mathfrak{q}^l$  which are checked numerically in [RS], one finds that Equation (38) can be strengthened to  $\eta_f \in \overline{\Lambda_{\mathbb{Z}G}^r E(K)^{[S,r]}}$ . (In fact, one can show that  $\overline{\Lambda_{\mathbb{Z}G}^r U_S^{[S,r]}} = \overline{\Lambda_{\mathbb{Z}G}^r E(K)^{[S,r]}}$  in these examples.) However, a counterexample to a conjecture of Sands discussed by Rubin in [Ru, §4.2] shows that this is not always the case. There one has  $k = \mathbb{Q}(\sqrt{2})$ ,  $K = k(\zeta_7)^+ \subset \mathbb{Q}(\mu_{56})^+$ ,  $G \cong \mathbb{Z}/3\mathbb{Z}$ . It is an easy exercise to check that  $K = k(\mathfrak{f})$  where  $\mathfrak{f} = 7\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2$  ( $\mathfrak{p}_1 \neq \mathfrak{p}_2$ ). Since  $K/\mathbb{Q}$  is abelian we can evaluate  $\Phi_{\mathfrak{f},\varrho}(1)$  directly. Indeed, by [So2, Thm. 5.1] it equals  $\frac{4}{\sqrt{d_k}} \Theta_{S,\varrho}^{(2)}(0)$  in Rubin’s notation. His calculations therefore imply that, for appropriately chosen  $\tilde{v}_1, \tilde{v}_2$ , Conditions (36) and (37) are satisfied with

$$\eta_f = \alpha \wedge \beta \text{ where}$$

$$\alpha := -\frac{1}{2} \otimes (1 - \zeta_7)(1 - \zeta_7^{-1}), \beta := -\frac{1}{2} \otimes N_{\mathbb{Q}(\mu_{56})^+/\mathbb{Q}}(1 - \zeta_{56})(1 - \zeta_{56}^{-1}).$$

It is easy to check that (in an additive notation)  $2\alpha$  and  $2\beta$  lie in the (images of)  $U_S$  and  $E(K)$  respectively and that  $e_{S,>2}\beta = N_G\beta = 0$ . Therefore  $4\eta_f$  lies in  $\overline{\Lambda_{\mathbb{Z}G}^2 U_S^{[S,2]}}$ . On the other hand, Equation (48) shows that  $(3 - N_G)\alpha$  and  $3\beta = (3 - N_G)\beta$  also lie in the image of  $E(K)$ . We deduce that  $\beta$  does too and that  $3\eta_f = (3 - N_G)\eta_f$  lies in  $\overline{\Lambda_{\mathbb{Z}G}^2 E(K)^{[S,2]}} \subset \overline{\Lambda_{\mathbb{Z}G}^2 U_S^{[S,2]}}$ . Finally, therefore,  $\eta_f$  must lie in both  $\overline{\Lambda_{\mathbb{Z}G}^2 U_S^{[S,2]}}$  and  $\frac{1}{3}\overline{\Lambda_{\mathbb{Z}G}^2 E(K)^{[S,2]}}$ . But Rubin shows that  $\eta_f \notin \overline{\Lambda_{\mathbb{Z}G}^2 E(K)^{[S,2]}}$ .

Turning to (39) in the case  $\mathfrak{f} = \mathfrak{q}^l$ , the factor  $\mathbb{Z}[1/g]$  in this equation made its verification easy in the first special case of Conjecture 3.6 dealt with in this subsection. However, it is less clear whether its presence was necessary there. In the second special case we saw that the factor  $\mathbb{Z}[1/g]$  is unnecessary in (39) for  $k = \mathbb{Q}$  and  $\mathfrak{f} = \mathfrak{q}^l$ , but that the factor  $1/2$  is always necessary. Exactly the same situation occurs in all the examples verified numerically in [RS].

As regards Equation (40), in both special cases we have seen that, in fact,

$$(49) \quad I(\mathbb{Z}G)\eta_f \subset \overline{\Lambda_{\mathbb{Z}G}^r U_S^{[S,r]}}$$

whether or not  $\mathfrak{f} = \mathfrak{q}^l$ . This also holds in Rubin's example above and in all those treated in [RS]. The strengthened Condition (49) appeals not only by its uniformity but also because it eliminates the trivial character which is the source of so many difficulties and exceptions. In fact, one can check that in all of our examples and those mentioned above, Equation (49) can even be further strengthened to

$$(50) \quad I(\mathbb{Z}G)\eta_{\mathfrak{f}} \subset \overline{\bigwedge_{\mathbb{Z}G}^r E(K)}^{[S,r]}.$$

(That is, for every  $\mathfrak{f}$ , Equation (41) holds without the factor  $\mathbb{Z}[1/g]$ ). The author would be interested in any counterexample to Equation (49) or (50).

*Remark 3.5.* — The above discussion suggests the possibility of a new Combined Conjecture with conditions similar to (36)–(39) but multiplied by an arbitrary element  $x \in I(\mathbb{Z}G)$ . Without entering into the details, let us note that the hypothesis  $\mathfrak{f} \neq \mathcal{O}$  could then be removed since both  $(x\Phi_{\mathcal{O},\mathcal{O}})(1)$  and  $(x\Phi_{\mathcal{O},T_p,p})(1)$  are well-defined, by Theorem 2.3 and Theorem/Definition 2.1 part (iii).

Regarding the numerical verification of such a conjecture, it is true that the methods of [RS] require  $\mathfrak{f} \neq \mathcal{O}$  in order to calculate  $\Phi_{\mathfrak{f},T_p,p}(s)$ . But one could, for instance, make use of the  $p$ -adic interpolation of Equation (6) which expresses  $((1 - \sigma_{\tau,\mathcal{O}})\Phi_{\mathcal{O},T_p,p})(1)$  as  $\pi_{\tau,\mathcal{O}}(\Phi_{\tau,T_p,p}(1))$  for any prime ideal  $\tau \nmid p$ . Moreover, if the set of classes  $\{\mathfrak{r}_i\}_{i=1}^n$  generates  $\text{Cl}_{\mathcal{O}}(k) = \text{Cl}(k)$  then the elements  $\{1 - \sigma_{\tau_i,\mathcal{O}}\}_{i=1}^n$  generate  $I(\mathbb{Z}G)$  over  $\mathbb{Z}G$ .

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